J. Appl. Numer. Optim. 5 (2023), No. 1, pp. 149-161
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.5.2023.1.10

LAGRANGE DUALITY OF VECTOR VARIATIONAL INEQUALITIES

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Abstract. In this paper, we investigate Lagrange dualities for a vector variational inequality problem. For a vector variational inequality problem with convex inclusion constraints, we develop an equivalent saddle point formulation via scalarization. For a vector variational inequality problem with linear constraints, we formulate a dual vector variational inequality via that of a linear multiobjective optimization problem and show that for a solution of a vector variational inequality problem, there is one corresponding solution for its dual. We give some examples to illustrate the results.

Keywords. vector variational inequality, Lagrange dual, saddle point formulation, dual linear multiobjective program.

2020 Mathematics Subject Classification. 90C29, 90C46.

1. Introduction

The vector variational inequality problem was introduced in [1]. This model has found various applications in vector optimization problems, vector complementarity problems, vector equilibriums, vector traffic equilibrium problems and Minty vector variational inequality; see [2, 3, 4, 5, 6, 7] and the references therein. The advantage of a vector variational inequality approach over the vector optimization one is that the equilibrium is not necessarily the extremum of a functional, so that no such functional must be supposed to exist.

The dual vector variational inequality in the sense of Mosco [8] was investigated in [9] in terms of vector conjugate functions studied in [10, 11]. One drawback of Mosco's type formulation is that the nonlinear mapping involved in the vector variational inequality problem is required to be one-to-one.

In this note, some dual vector variational inequality problems of Lagrangian type are formulated for vector variational inequality problems with convex inclusion constraints and convex polyhedral constraints, respectively, and their relations are discussed. For a vector variational inequality problem with convex inclusion constraints, we apply a scalarization technique to transform it to a scalar variational inequality problem. A saddle point problem or a primal-dual formulation is proposed via that of a scalar variational inequality in Banach spaces. Equivalent relations of a solution of a vector variational inequality problem and its saddle point formulation are obtained. We make use of the scalarization and saddle point results in [12, 13, 14] in this approach.

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Received March 22, 2022; Accepted November 4, 2022.

For vector variational inequality problems with convex polyhedral constraints (denoted by VVI), we formulate their duals (denoted by DVVI) via that (denoted by DMOP) of linear multiobjective optimization problems (denoted by MOP). Mature duality theories for linear multi-objective optimization problems and their duals are available in the literature; see [10, 14, 15], and will be employed in the study. We obtain relations between such vector variational inequality problems and their duals in finite dimensional Euclidean spaces. These relations show that for a solution of the vector variational inequality problem, there is a corresponding solution of its dual, but not vice versa in general. We summarize these relations among VVI, MOP, DMOP and DVVI (with a positive orthant ordering) in Figure 1. The outer clockwise arrows illustrate that for a given solution of VVI, a solution of DVVI can be found via MOP and DMOP, while the inner anticlockwise arrows show that for a given solution of DVVI, a solution of VVI may not be found unless the Pareto solution $x^{*\prime}$ corresponding to the solution of DVVI is equal to a prior solution, x^*, of VVI.

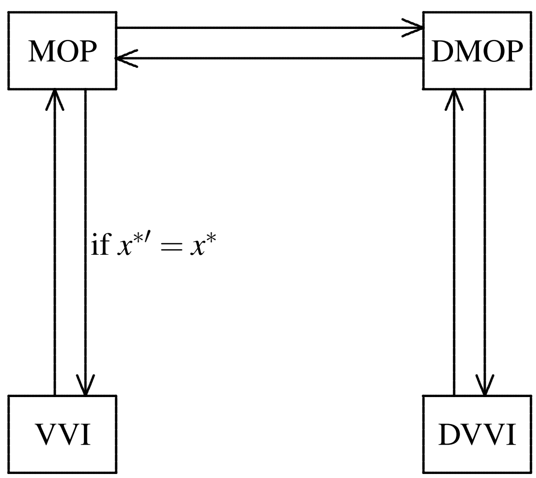

FIGURE 1. The relations among VVI, MOP, DMOP and DVVI

We also give some simple examples to illustrate the obtained dual relations in both cases.

Let X, Y and Z be Banach spaces, Y^* be the dual space of Y, and the dual pair between Y^* and Y be denoted by $\langle \cdot, \cdot \rangle$. Let $L(X,Y)$ be the space of linear continuous operators from X to Y. Let P be a nonempty closed and convex set of Y and $y_0 \in P$. The normal cone of P at y_0 is defined by

$$N_P(y_0) = \{y^* \in Y^* | \langle y^*, y - y_0 \rangle \geq 0, \forall y \in P\}.$$

The following definitions can be found in [14]. Let $D \subset Y$ be a convex cone with a nonempty interior $intD$, if necessary, and $l(D)$ denote the set $D \cap (-D)$. The following orderings are defined respectively:

$$x \leq_E y \quad \text{if and only if} \quad y - x \in E,$$
$$x \not\leq_E y \quad \text{if and only if} \quad y - x \notin E,$$

where E can be either D, or $intD$, or $D \setminus \{0\}$.

Let $A \subset Y$. A point $a \in A$ is a Pareto minimal point of A with respect to D if $a \geq_D b$ for some $b \in A$, then $b \geq_D a$. The set of all Pareto minimal points of A with respect to D is denoted by $\mathrm{Min}(A|D)$. A point $a \in A$ is a weak Pareto minimal point of A with respect to D if $a \in \mathrm{Min}(A|\{0\} \cup intD)$. The set of all weak Pareto minimal points of A with respect to D is denoted by $\mathrm{WMin}(A|D)$. A point $a \in A$ is a proper Pareto minimal point of A with respect to D if there exists a convex cone K which is not the whole space and contains $D \setminus l(D)$ in its interior so that $a \in \mathrm{Min}(A|K)$. The set of all proper Pareto minimal points of A with respect to D is denoted by $\mathrm{PrMin}(A|D)$. It is clear that $\mathrm{PrMin}(A|D) \subset \mathrm{Min}(A|D)$.

A point $a \in A$ is a (respectively weak, proper) Pareto maximal point of A with respect to D if $a \in \mathrm{Min}(A|-D)$ (respectively $\mathrm{WMin}(A|-D)$, $\mathrm{PrMin}(A|-D)$).

Let $f : X \to Y$ be a mapping and $\mathscr{F} \subset X$ be a subset. Consider a vector optimization problem

$$\min_D f(x) \quad \text{s.t. } x \in \mathscr{F}. \tag{1.1}$$

A point $x^* \in \mathscr{F}$ is a (respectively weak, or proper) Pareto minimal solution of (1.1) if $f(x^*) \in \mathrm{Min}(f(\mathscr{F})|D)$ (respectively $f(x^*) \in \mathrm{WMin}(f(\mathscr{F})|D)$, or $\mathrm{PrMin}(f(\mathscr{F})|D)$).

A (respectively weak, or proper) Pareto maximal solution of (1.1) is defined similarly.

The dual cone and positive dual cone of D are defined respectively by

$$D^* = \{\tau \in Y^* | \langle \tau, y\rangle \geq 0, \ \forall y \in D\},$$
$$D^{*+} = \{\tau \in Y^* | \langle \tau, y\rangle > 0, \ \forall y \in D \setminus l(D)\}.$$

Let $K \subset Z$ be a closed and convex cone of Z. A function $g : X \to Z$ is said to be K-convex if

$$\alpha g(x_1) + (1-\alpha) g(x_2) - g(\alpha x_1 + (1-\alpha)x_2) \in K, \quad \forall \alpha \in [0,1], x_1, x_2 \in X.$$

2. Vector Variational Inequalities with Conic Constraints

Assume that Q is a nonempty closed and convex set of X, and K is a nonempty closed and convex cone of Z. Let $F : X \to L(X,Y)$ and $g : X \to Z$ be nonlinear mappings. Let

$$\mathscr{F}_1 = \{x \in Q | g(x) \in -K\}$$

be a feasible set. Assume that g is a continuously differentiable and K-convex function with Gâteaux derivative $Dg(x)$. Therefore $\mathscr{F}_1$ is a closed and convex set.

Consider the following weak vector variational inequality with convex inclusion constraints:

$$\text{(weak VVI)} \qquad x^* \in \mathscr{F}_1 : \ F(x^*)(x^* - x) \not\leq_{intD} 0, \quad \forall x \in \mathscr{F}_1, \tag{2.1}$$

and the following vector variational inequality with convex inclusion constraints:

$$\text{(VVI)} \qquad x^* \in \mathscr{F}_1 : \ F(x^*)(x^* - x) \not\leq_{D\setminus\{0\}} 0, \quad \forall x \in \mathscr{F}_1. \tag{2.2}$$

We will first proceed to discuss saddle point formulation of weak VVI (2.1) via scalarization first. By definition, the weak VVI (2.1) is equivalent to that x^* is a weak Pareto minimal solution of the following convex vector optimization problem

$$\min_D \ -F(x^*)x, \quad \text{s.t. } x \in \mathscr{F}_1. \tag{2.3}$$

As (2.3) is a convex problem, by [14, Chapter 4, Theorem 2.10], x^* is a weak Pareto minimal solution of vector optimization problem (2.3) if and only if there is a $\tau^* \in D^* \setminus \{0\}$ such that x^*

solves the following optimization problem

$$\min \ -\langle \tau^*, F(x^*)x \rangle, \quad \text{s.t. } x \in \mathscr{F}_1. \tag{2.4}$$

For fixed x^* and τ^*, a Lagrangian $L : X \times Z^* \to \mathbb{R}$ of optimization problem (2.4) is defined by

$$L(x,\beta;x^*,\tau^*) = -\langle \tau^*, F(x^*)x \rangle + \langle \beta, g(x) \rangle, \ (x,\beta) \in X \times Z^*,$$

and thus a Lagrangian dual problem is

$$\sup_{\beta \in K^*} \inf\{L(x,\beta;x^*,\tau^*) : x \in Q\}.$$

The Robinson's constraint qualification of optimization problem (2.4) is said to hold at x^* if

$$0 \in \operatorname{int}\{g(x^*) + Dg(x^*)(Q - x^*) + K\}. \tag{2.5}$$

Under the Robinson's constraint qualification (2.5), by [13, Theorem 3.9], if x^* solves the optimization problem (2.4), then there is a Lagrange multiplier $\beta^* \in Z^*$ such that

$$x^* \in \operatorname*{argmin}_{x \in Q} L(x,\beta^*;x^*,\tau^*), \ \beta^* \in N_K(g(x^*)). \tag{2.6}$$

Again as (2.3) is a convex problem, the conditions in (2.6) are equivalent to

$$0 \in D_x L(x^*,\beta^*;x^*,\tau^*) + N_Q(x^*), \ 0 \in \beta^* - N_K(g(x^*)).$$

That is,

$$0 \in -\langle \tau^*, F(x^*) \rangle + \langle \beta^*, \nabla g(x^*) \rangle + N_Q(x^*), \ 0 \in \beta^* - N_K(g(x^*)).$$

Note that $\beta^* \in N_K(g(x^*))$ is equivalent to $g(x^*) \in N_{K^*}(\beta^*)$. For $\beta \in K^*$, let

$$\begin{aligned} M(\beta) &:= \{x \in X : 0 \in -\langle \tau, F(x) \rangle + \langle \beta, \nabla g(x) \rangle + N_Q(x) \text{ for some } \tau \in D^* \setminus \{0\}\}, \\ G(\beta) &:= \{-g(x) : x \in M(\beta)\}, \\ T_D(\beta) &:= G(\beta) + N_{K^*}(\beta). \end{aligned}$$

The dual weak VVI of weak VVI (2.1) is formulated as

$$\beta^* \in K^*, y^* \in G(\beta^*) : \langle y^*, \beta - \beta^* \rangle \le 0, \ \forall \beta \in K^*,$$

which can also be written as

$$\beta^* \in K^* : \ 0 \in T_D(\beta^*). \tag{2.7}$$

A primal-dual formulation of weak VVI (2.1) can be defined as follows:

$$\text{(PDVVI)} \quad (x^*,\beta^*) \in X \times Z^* : (0,0) \in S_{\tau^*}(x^*,\beta^*), \ \text{ for some } \tau^* \in D^* \setminus \{0\}, \tag{2.8}$$

where the set-valued mapping S is defined on $X \times Y^*$ by

$$S_\tau(x,\beta) = \begin{cases} \left\{ (u,v) \in X^* \times Z^* \ \middle| \ \begin{pmatrix} u \\ v \end{pmatrix} \in \begin{pmatrix} D_x L(x,\beta;x,\tau) + N_Q(x) \\ \beta - N_K(g(x)) \end{pmatrix} \right\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } (x,\beta) \in \mathscr{F}_1 \times K^*, \\ \emptyset, \qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{cases}$$

If (x^*,β^*) is a solution of (2.8) and thus (2.6) holds, then by [13, Proposition 3.3], x^* is an optimal solution of (2.4), and thus a weak optimal solution of (2.3). Thus we obtain the following saddle point optimality condition.

Theorem 2.1. *Assume that the Robinson's constraint qualification (2.5) holds at x^*. Then x^* is a solution of weak VVI (2.1) if and only if there exist $\tau^* \in D^* \setminus \{0\}$ and $\beta^* \in Z^*$ such that (x^*, β^*) solves PDVVI (2.8).*

Assume that Y is reflexive. As (2.3) is a convex problem, by [14, Chapter 4, Theorem 2.11], x^* is a proper Pareto optimal solution of convex vector optimization problem (2.3) if and only if there is a $\tau^* \in D^{*+}$ such that x^* solves the optimization problem (2.4). Moreover a proper Pareto optimal solution is also a Pareto optimal solution. Thus we obtain the following result.

Theorem 2.2. *Assume that Y is reflexive. If there exist $\tau^* \in D^{*+}$ and $\beta^* \in Y^*$ such that (x^*, β^*) solves PDVVI (2.8), then x^* is a solution of VVI (2.2).*

Below we give an example to illustrate that a solution of DVVI (2.7) is not one for weak VVI (2.1).

Example 2.1. Let $\mathscr{F}_1 = [1, +\infty)$ and

$$F(x) = \begin{pmatrix} x^3 \\ x^3 - 12x \end{pmatrix}.$$

Consider the following weak VVI:

$$x^* \in \mathscr{F}_1 : F(x^*)(x^* - x) \not\leq_{int\mathbb{R}^2_+} 0, \quad \forall x \in \mathscr{F}_1. \tag{2.9}$$

Every $x^* \in [1,2]$ is a solution of weak VVI (2.9). Therefore, for each solution $x^* \in [1,2]$, it is equivalent that there exists $\tau^* \in [0,1]$ such that x^* is a solution of the following optimization problem:

$$\min -\begin{pmatrix} \tau^* \\ 1-\tau^* \end{pmatrix}^\top F(x^*)x, \quad \text{s.t. } x \in \mathscr{F}_1. \tag{2.10}$$

Now fix a $\tau^* \in [0,1]$. A Lagrangian $L : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ of primal problem (2.10) is defined by

$$L(x, \beta; x^*, \tau^*) = -\begin{pmatrix} \tau^* \\ 1-\tau^* \end{pmatrix}^\top F(x^*)x + \beta(-x+1), \; x \in \mathbb{R}, \beta \in \mathbb{R}_+,$$

and thus a Lagrangian dual problem is

$$\sup_{\beta \geq 0} \inf\{L(x, \beta; x^*, \tau^*) : x \in \mathscr{F}_1\}. \tag{2.11}$$

It is clear that the Robinson's constraint qualification (2.5) holds at every $x^* \in [1,2]$. For every solution $x^* \in [1,2]$ of problem (2.11), there is $\beta^* \in \mathbb{R}_+$ such that

$$-\begin{pmatrix} \tau^* \\ 1-\tau^* \end{pmatrix}^\top F(x^*) - \beta = 0, \quad 0 \in \beta - N_{\mathbb{R}_+}(-x+1).$$

Let

$$\begin{aligned} M(\beta) &= \{x \in \mathbb{R} : \tau x^3 + (1-\tau)(x^3 - 12x) + \beta = 0, \quad \text{for some } \tau \in [0,1]\}, \\ G(\beta) &= \{-(-x+1) : x \in M(\beta)\}, \\ T_D(\beta) &= G(\beta) + N_{\mathbb{R}_+}(\beta). \end{aligned}$$

Thus

$$M(\beta) = [1, \sqrt[3]{\beta + 12}], \quad \forall \beta \geq 0.$$

The dual weak VVI is

$$\beta^* \geq 0 : 0 \in T_D(\beta^*). \tag{2.12}$$

A primal-dual formulation of weak VVI (2.1) can be defined as follows:

$$\text{(PDVVI)} \quad (x^*, \beta^*) \in \mathbb{R} \times \mathbb{R}_+ : (0,0) \in S_{\tau^*}(x^*, \beta^*), \qquad \text{for sme } \tau^* \in [0,1], \tag{2.13}$$

where set-valued mapping $S : \mathbb{R} \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$ by

$$S_\tau(x,\beta) = \begin{cases} \left\{ (u,v) \in \mathbb{R} \times \mathbb{R}_+ \;\middle|\; \begin{pmatrix} u \\ v \end{pmatrix} \in \begin{pmatrix} \tau x^3 + (1-\tau)(x^3 - 12x) + \beta \\ \beta - N_{\mathbb{R}_+}(-x+1) \end{pmatrix} \right\}, & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } (x,\beta) \in \mathbb{R} \times \mathbb{R}_+, & \\ \emptyset, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} & \end{cases}$$

As the Robinson's constraint qualification (2.5) holds at any point of the constraint set $\mathscr{F}_1$, we have: $x^* \in \mathscr{F}_1$ solves weak VVI (2.1) if and only if there exists $\tau^* \in [0,1], \beta^* \in \mathbb{R}_+$ such that (x^*, β^*) solves PDVVI (2.13), where

(i) when $x^* = 1$, we have $0 \leq \beta^* \leq 11$; and

(ii) when $1 < x^* \leq 2$, we have $1 - x^* \in N_{\mathbb{R}_+}(\beta^*)$, thus $\beta^* = 0$.

Let $\beta^* = 0$ be a solution of (2.12). Then (2.12) implies that $x \geq 1$ and $x \in M(0)$. So $1 \leq x = \sqrt{12} \in M(0)$ but $x = \sqrt{12}$ is not a solution of (2.9).

However for a constant VI with linear constraints, the scalar versions of weak VVI (2.1) and dual weak VVI (2.7) are a pair of duality relations by that of linear programs.

Example 2.2. Consider the following constant variational inequality problem with linear constraints

$$x^* \in \mathscr{F}_1 : \langle p, x^* - x \rangle \geq 0, \quad \forall x \in \mathscr{F}_1, \tag{2.14}$$

where $\mathscr{F}_1 = \{x \in \mathbb{R}^n : Ax \leq_{\mathbb{R}^m_+} b, 0 \leq_{\mathbb{R}^n_+} x\}$, $A = [a_1, \cdots, a_m]^\top$ is a matrix of $m \times n$ dimensions, $a_i \in \mathbb{R}^n (i = 1, \cdots, m)$, $b \in \mathbb{R}^m$ and $p \in \mathbb{R}^n$. Let $K = \mathbb{R}^m_+$, and

$$\begin{aligned} g_i(x) &= \begin{cases} a_i^\top x - b_i, & \text{if } i = 1, \cdots, m, x \in \mathscr{F}_1, \\ -x_{i-m}, & \text{if } i = m+1, \cdots, m+n, x \in \mathscr{F}_1, \end{cases} \\ g(x) &= (g_1(x), \cdots, g_{m+n}(x))^\top. \end{aligned}$$

For $(\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+$, let

$$\begin{aligned} M(\lambda,\mu) &= \left\{ x \in \mathbb{R}^n : 0 \in -p + \sum_{i=1}^m \lambda_i \partial g_i(x) + \sum_{i=m+1}^{m+n} \mu_i \partial g_i(x) \right\} \\ &= \{x \in \mathscr{F}_1 : 0 = -p + \sum_{i=1}^m \lambda_i a_i + \sum_{i=m+1}^{m+n} \mu_i(-e_{i-m})\} \\ &= \{x \in \mathscr{F}_1 : 0 = -p + A^\top \lambda - \mu\} \\ &= \begin{cases} \mathscr{F}_1, & \text{if } 0 = -p + A^\top \lambda - \mu, \\ \emptyset, & \text{if } 0 \neq -p + A^\top \lambda - \mu, \end{cases} \end{aligned}$$

where $e_i \in \mathbb{R}^n$ is the vector of 1 in ith component and 0 otherwise, and

$$\begin{aligned} G(\lambda,\mu) &= \{-g(x) : x \in M(\lambda,\mu)\} \\ &= \begin{cases} -g(\mathscr{F}_1), & \text{if } 0 = -p + A^\top\lambda - \mu, \\ \emptyset, & \text{if } 0 \neq -p + A^\top\lambda - \mu. \end{cases} \end{aligned}$$

We have

$$\begin{aligned} dom(G) &= \{(\lambda,\mu) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+ : 0 = -p + A^\top\lambda - \mu\} \\ &= \{(\lambda,\mu) \in \mathbb{R}^m \times \mathbb{R}^n : A^\top\lambda \geq p, \lambda \geq 0, \mu \geq 0\}. \end{aligned}$$

DVI (2.7) is to find

$$(\lambda^*,\mu^*) \in dom(G), d^* \in G(\lambda^*,\mu^*) : \langle d^*, (\lambda,\mu) - (\lambda^*,\mu^*)\rangle \leq 0, \quad \forall(\lambda,\mu) \in dom(G). \tag{2.15}$$

Let

$$d^* = \begin{bmatrix} b - Ax^* \\ x^* \end{bmatrix} \in G(\lambda^*,\mu^*) \text{ with } x^* \in \mathscr{F}_1.$$

DVI (2.15) becomes

$$(\lambda^*,\mu^*) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+, x^* \in \mathscr{F}_1 : \ \langle b - Ax^*, \lambda - \lambda^*\rangle + \langle x^*, \mu - \mu^*\rangle \leq 0, \quad \forall(\lambda,\mu) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+. \tag{2.16}$$

We have, for $(\lambda^*,\mu^*), (\lambda,\mu) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+$,

$$\begin{aligned} & \langle b - Ax^*, \lambda - \lambda^*\rangle + \langle x^*, \mu - \mu^*\rangle \\ =\ & \langle b, \lambda - \lambda^*\rangle + \langle -x^*, A^\top\lambda - A^\top\lambda^*\rangle + \langle x^*, \mu - \mu^*\rangle \\ =\ & \langle b, \lambda - \lambda^*\rangle + \langle -x^*, A^\top\lambda - \mu\rangle + \langle x^*, A^\top\lambda^* - \mu^*\rangle \\ =\ & \langle b, \lambda - \lambda^*\rangle + \langle -x^*, p\rangle + \langle x^*, p\rangle \\ =\ & \langle b, \lambda - \lambda^*\rangle. \end{aligned}$$

Thus (2.16) is reduced to

$$\lambda^* \in \mathscr{H}^* : \langle b, \lambda - \lambda^*\rangle \leq 0, \quad \forall \lambda \in \mathscr{H}^*, \tag{2.17}$$

where $\mathscr{H}^* = \{\lambda : A^\top\lambda \geq p, \lambda \geq 0\}$.

It is clear by the duality results of linear programming that (2.14) and (2.17) are a pair of duals.

3. Vector Variational Inequalities with Linear Constraints

Let $A_{m\times n}$ be an $m \times n$ matrix, and $b \in \mathbb{R}^m$ be a vector. Let D, Q and M be pointed and convex polyhedral cones in $\mathbb{R}^k, \mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

Let a closed and convex feasible set be

$$\mathscr{F}_2 = \{x \in M \mid Ax \geq_Q b\}.$$

Let $F : \mathbb{R}^n \to \mathbb{R}^{k\times n}$ be a nonlinear mapping. We consider the following (respectively weak) vector variational inequality:

$$\text{((resp. weak) VVI)} \quad x^* \in \mathscr{F}_2 : \quad F(x^*)(x^* - x) \not\leq_{D\setminus\{0\}} 0, \ \text{(resp. } \not\leq_{intD}) \quad \forall x \in \mathscr{F}_2. \tag{3.1}$$

Let $x^* \in \mathscr{F}_2$ be fixed. We consider the following auxillary multiobjective optimization problem:

$$\text{(MOP)} \quad \max_D F(x^*)x \quad \text{s.t. } x \in \mathscr{F}_2. \tag{3.2}$$

Let $X \subset \mathbb{R}^n$ be a subset. The following positive polar of X is defined in [10] by

$$X^\circ = \{x^* \in \mathbb{R}^n : \langle x^*, x\rangle \geq 0, \ \forall x \in X\}.$$

For formulating a dual vector variational inequality problem, we consider the following dual feasible set:

$$\mathscr{G}_2 \ = \ \{U \in \mathbb{R}^{k\times m} | \ U^\top \rho \in Q^\circ \text{ and } A^\top U^\top \rho \geq_{M^\circ} F(x^*)^\top \rho \text{ for some } \rho \in \mathrm{int} D^\circ\}.$$

By using set $\mathscr{G}_2$, we consider the following nonlinear (respectively weak) dual vector variational inequality problem:

$$\text{((resp. weak) DVVI)} \quad U^* \in \mathscr{G}_2 : (U - U^*)b \not\leq_{D\setminus\{0\}} 0, \text{ (resp. } \not<_{intD}) \quad \forall U \in \mathscr{G}_2. \tag{3.3}$$

We also consider the following auxillary dual multiobjective optimization problem:

$$\text{(DMOP)} \quad \min_D Ub \quad \text{s.t. } U \in \mathscr{G}_2. \tag{3.4}$$

The pair of sets $\mathscr{F}_2$ and $\mathscr{G}_2$ and the pair of multiobjective optimization problems MOP (3.2) and DMOP (3.4) were considered in [10].

The following relations among VVI or weak VVI (3.1) and DVVI or weak DVVI (3.3) and MOP (3.2) and DMOP (3.4) are obvious by definition.

Lemma 3.1. *The following statements hold:*

(i) x^* solves (respectively weak) VVI (3.1) if and only if x^* is a (respectively weak) Pareto maximal solution of MOP (3.2);

(ii) U^* solves (respectively weak) DVVI (3.3) if and only if U^* is a (respectively weak) Pareto minimal solution of DMOP (3.4).

3.1. **Orderings being defined by convex polyhedral orthants.** In this subsection, we illustrate some relations between VVI or weak VVI (3.1) and DVVI or weak DVVI (3.3) by using the weak duality and strong duality between MOP (3.2) and DMOP (3.4).

We say that the weak duality of (respectively weak) Pareto maximal/minimal solutions between MOP (3.2) and DMOP (3.4) holds if

$$F(x^*)x \not\geq_{D\setminus\{0\}} Ub, \ \text{(respectively } \not\geq_{intD}) \quad \forall x \in \mathscr{F}_2, U \in \mathscr{G}_2. \tag{3.5}$$

We say that the weakened strong duality between MOP (3.2) and DMOP (3.4) holds if x^* is a (respectively weak) Pareto maximal/minimal solution of MOP (3.2), then the following is satisfied:

$$F(x^*)x^* \in \mathrm{Min}((\mathscr{G}_2)b|D) \quad \text{(respectively WMin}((\mathscr{G}_2)b|D)), \tag{3.6}$$

where $(\mathscr{G}_2)b = \{Ub | U \in \mathscr{G}_2\}$.

Lemma 3.2. *The following facts hold.*

(i) The weak duality (3.5) of (respectively weak) Pareto maximal/minimal solutions between MOP (3.2) and DMOP (3.4) holds.

(ii) The weakened strong duality (3.6) of (respectively weak) Pareto maximal/minimal solutions between MOP (3.2) and DMOP (3.4) holds.

Proof. The results (i) and (ii) in respect of Pareto maximal/minimal solutions were given in [10, Theorem 5.1.4]. Those in (i) and (ii) in respect of weak Pareto maximal/minimal solution follow from that for Pareto maximal/minimal solutions as the ordering $a \ngeq_{D\setminus\{0\}} b$ implies $a \not>_{intD} b$, and the inclusion $\mathrm{Min}((\mathscr{G}_2)b|D) \subseteq \mathrm{WMin}((\mathscr{G}_2)b|D)$ holds. □

Next we show that by virtue of weak duality and weakened strong duality results in Lemma 3.2, DVVI or weak DVVI (3.3) is a necessary condition of VVI or weak VVI (3.1).

Theorem 3.1. *x^* is a solution of (respectively weak) VVI (3.1) if and only if there is a feasible solution $U^* \in \mathscr{G}_2$ such that*

$$F(x^*)x^* = U^*b.$$

In this case, U^* is a solution of (respectively weak) DVVI (3.3).

Proof. If x^* is a solution of (respectively weak) VVI (3.1), then by Lemma 3.1 (i) x^* is a (respectively weak) Pareto maximal solution of MOP (3.2). By Lemma 3.2 (ii), there exists $U^* \in \mathscr{G}_2$ such that

$$F(x^*)x^* = U^*b \in \mathrm{Min}((\mathscr{G}_2)b|D) \quad (\text{respectively } \mathrm{WMin}((\mathscr{G}_2)b|D)).$$

That is

$$U^*b \ngeq_{D\setminus\{0\}} Ub, \quad (\text{respectively } \ngeq_{intD}) \quad \forall U \in \mathscr{G}_2.$$

Thus

$$(U - U^*)b \nleq_{D\setminus\{0\}} 0, \quad (\text{respectively } \nleq_{intD}) \quad \forall U \in \mathscr{G}_2.$$

Therefore U^* is a solution of (respectively weak) DVVI (3.3).

Conversely let a feasible solution $U^* \in \mathscr{G}_2$ such that $F(x^*)x^* = U^*b$. By Lemma 3.2 (i), we have,

$$F(x^*)x^* = U^*b \nleq_{D\setminus\{0\}} F(x^*)x, \ (\text{respectively } \nleq_{intD}) \quad \forall x \in \mathscr{F}_2.$$

Thus

$$F(x^*)(x^* - x) \nleq_{D\setminus\{0\}} 0, \ (\text{respectively } \nleq_{intD}) \quad \forall x \in \mathscr{F}_2.$$

Therefore x^* is a solution of VVI (respectively weak VVI) (3.1). □

3.2. **Orderings being defined by positive orthants.** In this subsection, we consider a special case of VVI or weak VVI (3.1). That is, let $D = \mathbb{R}^k_+, Q = \{0\}$ and $M = \mathbb{R}^n_+$ respectively.

Then sets $\mathscr{F}_2$ and $\mathscr{G}_2$ at the beginning of this section reduce to the following sets respectively

$$\begin{aligned} \mathscr{F}_3 &= \{x \in \mathbb{R}^n_+ \mid Ax = b\}, \\ \mathscr{G}_3 &= \{U \in \mathbb{R}^{k\times m} |\, A^\top U^\top \rho \geq_{\mathbb{R}^n_+} F(x^*)^\top \rho \text{ for some } \rho \in \mathbb{R}^k, \rho \geq_{int\mathbb{R}^k_+} 0\}, \end{aligned}$$

where the dual feasible set $\mathscr{G}_3$ was introduced in [16]. In [15], the following dual feasible set is considered:

$$\mathscr{G}_3' = \{U \in \mathbb{R}^{k\times m} |\, UAw \leq_{\mathbb{R}^k_+\setminus\{0\}} F(x^*)w \text{ for no } w \in \mathbb{R}^n_+\}.$$

Next we show that two sets $\mathscr{G}_3$ and $\mathscr{G}_3'$ are equal. However they may not be convex.

Lemma 3.3. *We have $\mathscr{G}_3 = \mathscr{G}_3'$.*

Proof. Let $U \in \mathscr{G}_3$. Thus $A^\top U^\top \rho \geq_{\mathbb{R}^n_+} F(x^*)^\top \rho$ for some $\rho \geq_{int\mathbb{R}^k_+} 0$. If $U \notin \mathscr{G}_3'$, then there exists $w \in \mathbb{R}^n_+$ such that $UAw \leq_{\mathbb{R}^k_+\setminus\{0\}} F(x^*)w$. Furthermore, as $\rho \geq_{int\mathbb{R}^k_+} 0$, it follows that $\rho^\top UAw < \rho^\top F(x^*)w$. As $w \in \mathbb{R}^n_+$, $A^\top U^\top \rho \geq_{\mathbb{R}^n_+} F(x^*)^\top \rho$ cannot be true, a contradiction. Therefore $U \in \mathscr{G}_3'$.

Conversely, let $U \in \mathscr{G}_3'$. Thus $UAw \leq_{\mathbb{R}^k_+\setminus\{0\}} F(x^*)w$ for no $w \in \mathbb{R}^n_+$. Then for any $\rho \geq_{int\mathbb{R}^k_+} 0$, $\rho^\top UAw < \rho^\top F(x^*)w$, for no $w \in \mathbb{R}^n_+$. Thus, there must be some $\rho \geq_{int\mathbb{R}^k_+} 0$ such that $A^\top U^\top \rho \geq_{\mathbb{R}^n_+} F(x^*)^\top \rho$. Therefore $U \in \mathscr{G}_3$. □

Below we only use the set $\mathscr{G}_3$. By using sets $\mathscr{F}_3$ and $\mathscr{G}_3$, we consider the following (respectively weak) vector variational inequality and its (respectively weak) dual vector variational inequality:

$$\text{((resp. weak) VVI)} \quad x^* \in \mathscr{F}_3 : F(x^*)(x^* - x) \not\leq_{\mathbb{R}^k\setminus\{0\}} 0, \text{ (resp. } \not\leq_{int\mathbb{R}^k}) \quad \forall x \in \mathscr{F}_3, \tag{3.7}$$

$$\text{((resp. weak) DVVI)} \quad U^* \in \mathscr{G}_3 : (U - U^*)b \not\leq_{\mathbb{R}^k\setminus\{0\}} 0, \text{ (resp. } \not\leq_{int\mathbb{R}^k}) \quad \forall U \in \mathscr{G}_3. \tag{3.8}$$

Let $x^* \in \mathscr{F}_3$ be fixed. We consider the following auxillary linear multiobjective optimization problem and dual multiobjective optimization problem:

$$\text{(MOP)} \qquad \max_D F(x^*)x \quad \text{s.t. } x \in \mathscr{F}_3, \tag{3.9}$$

$$\text{(DMOP)} \qquad \min_D Ub \quad \text{s.t. } U \in \mathscr{G}_3. \tag{3.10}$$

Unlike linear programs, it is noted that DMOP (3.10) cannot in general be derived as a Lagrange duality of MOP (3.9) in the sense of a linear optimization. See also the comments in [14]. However, we still call (3.10) is a Lagrange dual problem of (3.9). Furthermore, in such settings, there are complete weak duality and strong duality results of (weak) Pareto solutions in [16] (see also [15] for Pareto solutions), which can be employed to establish the relations between VVI or weak VVI (3.7) and DVVI or weak DVVI (3.8).

By using the existing weak duality and strong duality results of multiobjective optimization problems in [16] and Lemma 3.1, we obtain the following relations of VVI or weak VVI (3.7) and DVVI or weak DVVI (3.8).

Theorem 3.2. *We have the following.*

(i) x^* is a solution of (respectively weak) VVI (3.7) if and only if there is a feasible solution $U^* \in \mathscr{G}_3$ such that

$$F(x^*)x^* = U^*b. \tag{3.11}$$

In this case, U^* is then a solution of (respectively weak) DVVI (3.8);

(ii) U^* is a solution of (respectively weak) DVVI (3.8) if and only if there is a feasible solution $x{*\prime} \in \mathscr{F}_3$ such that

$$F(x^*)x^{*\prime} = U^*b, \quad (\textit{respectively } F(x^*)x^{*\prime} \geq_{\mathbb{R}^k_+} U^*b)$$

$x^{*\prime}$ is then a (respectively weak) Pareto maximal solution of MOP (3.9). Furthermore, if $x^{*\prime} = x^*$, then x^* is a solution of (respectively weak) VVI (3.7).

Proof. (i) is a special case of Theorem 3.1.

We now prove (ii). Let U^* be a solution of (respectively weak) DVVI (3.8), then by Lemma 3.1 (ii) U^* is a (respectively weak) Pareto minimal solution of DMOP (3.10). By Theorem

2.6 (ii) (respectively Theorem 2.6 (iii)), [16], there exists $x^{*\prime} \in \mathscr{G}_2$ such that $F(x^*)x^{*\prime} = U^*b$ (respectively $F(x^*)x^{*\prime} \geq_{\mathbb{R}^k_+} U^*b$). In this case, by the weak duality between MOP (3.9) and DMOP (3.10) in [16, Theorem 2.2], we have

$$F(x^*)x \not\geq_{\mathbb{R}^k_+\setminus\{0\}} U^*b = F(x^*)x^{*\prime}, \text{ (respectively } F(x^*)x \not\geq_{int\mathbb{R}^k_+} U^*b \leq_{\mathbb{R}^k_+} F(x^*)x^{*\prime}) \quad \forall x \in \mathscr{F}_3.$$

Thus

$$F(x^*)x \not\geq_{\mathbb{R}^k_+\setminus\{0\}} F(x^*)x^{*\prime}, \text{ (respectively } \not\geq_{int\mathbb{R}^k_+}) \quad \forall x \in \mathscr{F}_3.$$

Therefore $x^{*\prime}$ is a (respectively weak) Pareto maximal solution of MOP (3.9). If $x^{*\prime} = x^*$, then x^* is a (respectively weak) Pareto maximal solution of MOP (3.9). Thus by Lemma 3.1 (i), x^* is a solution of of (respectively weak) VVI (3.7).

Conversely let a feasible solution $x^{*\prime} \in \mathscr{F}_3$ such that $F(x^*)x^{*\prime} = U^*b$ (respectively $F(x^*)x^{*\prime} \geq_{\mathbb{R}^k_+} U^*b$). By the weak duality between MOP (3.9) and DMOP (3.10) of [16, Theorem 2.2], we have,

$$Ub \not\leq_{\mathbb{R}^k_+\setminus\{0\}} F(x^*)x^{*\prime} = U^*b, \text{ (respectively } Ub \not\leq_{int\mathbb{R}^k_+} F(x^*)x^{*\prime} \geq_{\mathbb{R}^k_+} U^*b) \quad \forall U \in \mathscr{G}_3.$$

Thus

$$(U - U^*)b \not\leq_{\mathbb{R}^k_+\setminus\{0\}} 0, \text{ (respectively } \not\leq_{int\mathbb{R}^k_+}) \quad \forall U \in \mathscr{G}_3.$$

Therefore U^* is a solution of (respectively weak) DVVI (3.8). □

Below we give an example to show that, for a solution of VVI, we can find a solution of DVVI that satisfies (3.11). Conversely, it may not be true.

Example 3.1. Let

$$\mathscr{F}_3 = \{(x_1,x_2) | x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0\} \text{ and } F(x) = \begin{bmatrix} -x_1 & 0 \\ 0 & -x_2 \end{bmatrix}.$$

Let $x^* = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. x^* is a solution of the following VVI:

$$F(x^*)(x^* - x) \not\leq_{\mathbb{R}^2_+\setminus\{0\}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \forall x \in \mathscr{F}_3. \tag{3.12}$$

We have the dual feasible set

$$\mathscr{G}_3 = \mathscr{G}_3' = \{(u_1,u_2) | u_1 + u_2 \geq -\frac{1}{2} \text{ or } u_1 > 0, \text{ or } u_2 > 0\}$$

and the following strong duality holds:

$$\mathrm{Max}(F(x^*)(\mathscr{F}_3)|\mathbb{R}^2_+) = \mathrm{Min}((\mathscr{G}_3)b|\mathbb{R}^2_+) = \left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \middle| u_1 + u_2 = -\frac{1}{2}, u_1 \leq 0, \text{and } u_2 \leq 0 \right\}.$$

Let U^* satisfy the following equation:

$$F(x^*)x^* = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix} = U^*b.$$

We obtain $U^* = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$ and it is clear that U^* is a solution of the following DVVI:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} \not\leq_{\mathbb{R}^2_+\setminus\{0\}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \forall \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathscr{G}_3. \tag{3.13}$$

Now let $U^* = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$ be a solution of the DVVI (3.13). Let $x^{*\prime}$ satisfy the following equation:

$$F(x^*)x^{*\prime} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} = U^*b.$$

We obtain $x^{*\prime} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $x^{*\prime} \neq x^*$ and this $x^{*\prime}$ is not a solution of VVI (3.12).

4. Conclusion

In this paper, we discussed saddle point relations between a vector variational inequality problem with convex inclusion constraints and its scalar primal-dual formulation by virtue of a scalarization technique in Banach spaces. We also developed duality relations between a vector variational inequality problem with convex polyhedral constraints and its duality problem by using that of a multiobjective linear optimization problem and its Lagrange dual problem.

Acknowledgment

The author would like to thank Professor Franco Giannessi for stimulating discussions and suggestions and for his encouragement during the development of this project. The project was supported in part by the Internal Grant of Hong Kong Polytechnic University (1-ZVWY).

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