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VALIDITY OF PRANDTL EXPANSIONS FOR STEADY MHD IN THE SOBOLEV FRAMEWORK*

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Abstract. This paper concerns the vanishing viscosity and magnetic resistivity limit for the two-dimensional steady incompressible MHD system on the half-plane with no-slip boundary conditions on velocity fields and perfectly conducting wall conditions on magnetic fields. We prove the nonlinear stability of shear flows of the Prandtl type with nondegenerate tangential magnetic fields but without any positivity or monotonicity assumption on velocity fields. It contrasts sharply with the steady Navier–Stokes equations and reflects the stabilization effect of magnetic fields. The main aims in the analysis are to design an intrinsic weight function to treat the incompatibility of the natural multiplier with boundary conditions and to establish critical Hardy-type inequalities and properly weighted estimates that lead to an almost optimal convergence rate.

Key words. 2D steady MHD system, Prandtl expansion, high Reynolds number limit, Sobolev spaces, stability

MSC codes. 76N20, 35A07, 35G31, 35M33

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1. Introduction. In this paper, we consider the vanishing viscosity and magnetic resistivity limit of the two-dimensional (2D) steady MHD system in the half-space $\Omega = \{(x,y) \mid x \in \mathbb{T}_\varrho, y > 0\}$:

$$
\begin{cases}
\mathbf{U}\cdot\nabla\mathbf{U} + \nabla P - \mathbf{H}\cdot\nabla\mathbf{H} - \mu\varepsilon\Delta\mathbf{U} = \mathbf{F_U},\\
\mathbf{U}\cdot\nabla\mathbf{H} - \mathbf{H}\cdot\nabla\mathbf{U} - \kappa\varepsilon\Delta\mathbf{H} = \mathbf{F_H},\\
\nabla\cdot\mathbf{U} = \nabla\cdot\mathbf{H} = 0.
\end{cases}
\tag{1.1}
$$

Here $\mathbf{U} = (u,v)$, $\mathbf{H} = (h,g)$, and P stand for the velocity field, magnetic field, and total pressure, respectively, and vectors $\mathbf{F_U} = (F_{1,\mathbf{U}}, F_{2,\mathbf{U}}), \mathbf{F_H} = (F_{1,\mathbf{H}}, F_{2,\mathbf{H}})$ are given external forces. The tangential variable x takes value in torus $\mathbb{T}_\varrho = \mathbb{R}/(2\pi\varrho)\mathbb{Z}$ with periodicity $2\pi\varrho$ and the normal variable $y > 0$ with the boundary $\{y = 0\}$. $\mu\varepsilon$ and $\kappa\varepsilon$ are viscosity and magnetic resistivity coefficients, respectively, with $\varepsilon \ll 1$ and μ, κ being two positive constants. We impose the steady MHD system with the following no-slip boundary condition on velocity fields and perfectly conducting wall condition on magnetic fields:

$$
\mathbf{U}|_{y=0} = (\partial_y h, g)|_{y=0} = \mathbf{0}.
\tag{1.2}
$$

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Moreover, it is natural to assume compatibility conditions for $\mathbf{F_H} = (F_{1,\mathbf{H}}, F_{2,\mathbf{H}})$:

$$\nabla \cdot \mathbf{F_H} = 0, \quad F_{2,\mathbf{H}}|_{y=0} = 0. \tag{1.3}$$

We are concerned with the asymptotic behavior of solutions $(\mathbf{U}, \mathbf{H})$ to (1.1)–(1.2) as $\varepsilon \to 0$. The problem of the high-Reynolds-number limit is fundamental in hydrodynamics. And it is very challenging in the presence of boundaries, especially when considering the no-slip boundary conditions for the velocity field. The key difficulty comes from the creation of the large vorticity for small viscosity near the boundary—the so-called boundary layer phenomenon. A powerful tool for studying these problems is the boundary layer theory, which was introduced by Prandtl [38] in 1904. According to Prandtl's view, the boundary layer of the system (1.1)–(1.2) is characteristic with the scale $\sqrt{\varepsilon}$, and the solutions should have the following asymptotic behavior:

- $(\mathbf{U}, \mathbf{H})(x,y) \sim (\mathbf{U}^I, \mathbf{H}^I)(x,y)$ away from the boundary, where $(\mathbf{U}^I, \mathbf{H}^I)$ satisfies the ideal MHD system with boundary conditions $\mathbf{U}^I \cdot \vec{n}|_{y=0} = \mathbf{H}^I \cdot \vec{n}|_{y=0} = 0$;
- $(\mathbf{U}, \mathbf{H})(x,y) \sim (u^p(x, \frac{y}{\sqrt{\varepsilon}}), \sqrt{\varepsilon} v^p(x, \frac{y}{\sqrt{\varepsilon}}), h^p(x, \frac{y}{\sqrt{\varepsilon}}), \sqrt{\varepsilon} g^p(x, \frac{y}{\sqrt{\varepsilon}}))$ near the boundary, where $(u^p, v^p, h^p, g^p)(x,Y)$ satisfies a Prandtl-type system with boundary conditions $(u^p, v^p, \partial_Y h^p, g^p)|_{y=0} = \mathbf{0}$ and far-field conditions $\lim_{Y\to+\infty}(u^p, h^p)(x,Y)$ matching the tangential part of $(\mathbf{U}^I, \mathbf{H}^I)$ on the boundary $\{y=0\}$.

Mathematically, it follows two fundamental problems:

- the well-posedness/ill-posedness of the Prandtl boundary layer system;
- the rigorous justification of the Prandtl expansion for small viscosities.

The first problem is, of course, very important and has a lot of results, while in the present paper, we mainly focus on the second problem corresponding to no-slip boundary conditions for the flow. For more on boundary layer theory, see the reviews [4, 35] and the references therein.

Before stating the main result of this paper, we review some mathematical results on the validity of Prandtl asymptotics. Let us first focus on the situations related to the classical unsteady Navier–Stokes equations with no-slip boundary conditions. To the best of our knowledge, the first rigorous verification of the Prandtl boundary layer theory was achieved in the analytic framework for both 2D and 3D cases by Sammartino and Caflisch in the celebrated paper [39]. One can also refer to [42] for a new proof for the 2D case based on a direct energy method. After that, a notable step forward in this direction was made by Maekawa [34], who justified rigorously the Prandtl ansatz in the inviscid limit for 2D Navier–Stokes equations with the initial vorticity supported away from the boundary. This result was generalized to three dimensions in [5]. Nguyen and Nguyen [37] established a direct proof of the inviscid limit for analytic data, without using the boundary layer expansion. Very recently, Kukavica, Vicol, and Wang [31] and Kukavica et al. [30] justified respectively the inviscid limit and the Prandtl expansion for the data that are analytic only near the boundary. In the absence of analytic regularity, the problem is very challenging due to a number of reasons, such as the reverse flow, the Tollmien–Schlichting wave, and so on; see the physics literature [3, 40]. The validity of Prandtl ansatz in the Sobolev framework is not expected mathematically, in particular in two dimensions, according to the instability of Prandtl asymptotics of shear flow type obtained in some recent papers. Precisely, Grenier and Nguyen established counterexamples to nonlinear stability of Prandtl boundary layer profiles with inflection points in [12, 15, 17]. Even for the monotonic and concave Prandtl boundary layer profiles, we may not expect the nonlinear stability of the Prandtl boundary layer in the Sobolev setting. In the notable work [14], the authors studied the linearized Navier–Stokes equations around generic stationary shear flows of the boundary layer type and constructed solutions with highly growing eigenmodes like $e^{t/\nu^{\frac{1}{4}}}$ (ν : viscosity) related to the $O(\nu^{-\frac{3}{8}})$ tangential frequency; see [13] for related statements and [16, 18, 19] for new progress. The result in [14] suggests somehow that one can only prove the validity of Prandtl boundary layer theory in the function spaces of Gevrey class, and recently there have been several interesting works in this direction; see [1, 9, 10].

Now we turn to the steady Navier–Stokes case, and surprisingly the situation is more satisfactory than the unsteady case. The first rigorous result on the validity of steady Prandtl boundary layer profiles was proved by Guo and Nguyen in [21], in which they consider the steady Navier–Stokes equations in the domain $\{(x,y)\in[0,L]\times\mathbb{R}_+\}$ with a positive Dirichlet boundary condition for the tangential velocity (moving plate). They constructed general boundary layer expansions for small viscosity and proved their validity in the Sobolev framework for small L; see also some generalizations in [22, 23, 24, 25, 26]. Note that the moving plate condition is different with the no-slip boundary condition and avoids some difficulties from degeneracy on the boundary due to the vanishing tangential velocity. In [20], Guo and Iyer considered the case for homogeneous Dirichlet boundary conditions, the same as the no-slip boundary condition. And the boundary layer profiles in the Prandtl ansatz studied in [20] involve the famous Blasius flow. Very recently, Gao and Zhang gave a simplified proof of this result in [7]. In another important work [8], Gérard-Varet and Maekawa studied the steady Navier–Stokes equations with no-slip boundary condition and some additional source terms in the same domain as the present paper and obtained the H^1 stability of shear flows of Prandtl type. Last, we also mention the recent interesting work [27, 28] by Masmoudi and Iyer in which they characterized not only the boundary layer expansion in the vanishing viscosity limit but also the x asymptotics to the Blasius profile.

Back to the MHD system, its boundary layer theory is richer because of different choices of magnetic physical parameters; one can refer to [11, 43] for more details. In the 2D unsteady MHD system when viscosity and magnetic resistivity tend to zero at the same rate, the stabilization effect from the nondegenerate tangential magnetic field was discovered in [11, 32, 33], and the validity of Prandtl boundary layer theory was rigorously proved in [33], in sharp contrast with the unsteady Navier–Stokes system. As a further step in this direction, the purpose of this paper is to reveal the stability mechanism of a magnetic field for the steady system (1.1)–(1.2) in order to justify the stability of shear flows of Prandtl type in the Sobolev framework. Let us mention that in [2] the authors extended the result in [21] to the 2D steady MHD system with moving plate condition, and the stability mechanism is from the nondegenerate velocity field but not from the magnetic field, which is consistent with [21],

To state the main result in this paper, let us first introduce some notation and assumptions. Denote by

$$(\mathbf{U}_s,\mathbf{H}_s)(Y)=\big(U_s(Y),0,H_s(Y),0\big),\qquad Y:=\frac{y}{\sqrt{\varepsilon}},$$

a background shear flow with $U_s(0) = {H_s}'(0) = 0$. We can see that it is a special solution to (1.1) when the external forces

$$(\mathbf{F_U}, \mathbf{F_H}) = (\mathbf{F}_{\mathbf{U}_s}, \mathbf{F}_{\mathbf{H}_s}) := (-\varepsilon\mu\partial_y^2 U_s, 0, -\varepsilon\kappa\partial_y^2 H_s, 0) = \left(-\mu\partial_Y^2 U_s(Y), 0, -\kappa\partial_Y^2 H_s(Y), 0\right).$$

We are interested in a general class of shear flow that satisfies the following assumptions:

- $U_s, H_s \in C^3(\mathbb{R}_+) \cap C^1(\overline{\mathbb{R}_+})$ such that

  $$U_s(0) = 0,\ {H_s}'(0) = 0,\ \lim_{Y\to+\infty} U_s(Y) = U_E,\ \lim_{Y\to+\infty} H_s(Y) = H_E \neq 0, \tag{1.4}$$

  and

  $$\bar{M} := \sum_{1\le k\le 3} \sup_{Y\ge 0} (1+Y)^3 \left(|\partial_Y^k U_s(Y)| + |\partial_Y^k H_s(Y)|\right) < \infty. \tag{1.5}$$

- There are two positive constants $\underline{\gamma}, \bar{\gamma} > 0$ such that

  $$\underline{\gamma} \le |H_s(Y)| \le \bar{\gamma} \quad \text{for any } Y > 0. \tag{1.6}$$

  And setting $G_s(Y) := H_s^2(Y) - U_s^2(Y)$, it holds that

  $$\gamma_0 := \inf_{Y\ge 0} G_s(Y) = \inf_{Y\ge 0} \left(H_s^2(Y) - U_s^2(Y)\right) > 0. \tag{1.7}$$

Note that $\bar{M}$ measures the amplitude of perturbation of the boundary layer profile (U_s, H_s) around its far field (U_E, H_E), and (1.7) implies that the magnetic field dominates the velocity field.

In this paper, we will show the stability of $(\mathbf{U}_s, \mathbf{H}_s)$ in Sobolev spaces for the problem (1.1)–(1.2). Set

$$(\widetilde{\mathbf{U}}, \widetilde{\mathbf{H}}) = (\mathbf{U}, \mathbf{H}) - (\mathbf{U}_s, \mathbf{H}_s) \triangleq (\tilde{u}, \tilde{v}, \tilde{h}, \tilde{g})$$

to be the perturbation of $(\mathbf{U}_s, \mathbf{H}_s)$. From (1.1)–(1.2) the problem for $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{H}})$ is written as

$$\begin{cases} U_s\partial_x\widetilde{\mathbf{U}} + \tilde{v}\partial_y U_s\mathbf{e}_1 - H_s\partial_x\widetilde{\mathbf{H}} - \tilde{g}\partial_y H_s\mathbf{e}_1 + \nabla P - \mu\varepsilon\Delta\widetilde{\mathbf{U}} = -\widetilde{\mathbf{U}}\cdot\nabla\widetilde{\mathbf{U}} + \widetilde{\mathbf{H}}\cdot\nabla\widetilde{\mathbf{H}} + \mathbf{f_U}, \\ U_s\partial_x\widetilde{\mathbf{H}} + \tilde{v}\partial_y H_s\mathbf{e}_1 - H_s\partial_x\widetilde{\mathbf{U}} - \tilde{g}\partial_y U_s\mathbf{e}_1 - \kappa\varepsilon\Delta\widetilde{\mathbf{H}} = -\widetilde{\mathbf{U}}\cdot\nabla\widetilde{\mathbf{H}} + \widetilde{\mathbf{H}}\cdot\nabla\widetilde{\mathbf{U}} + \mathbf{f_H}, \\ \nabla\cdot\widetilde{\mathbf{U}} = \nabla\cdot\widetilde{\mathbf{H}} = 0, \\ \widetilde{\mathbf{U}}|_{y=0} = (\partial_y\tilde{h}, \tilde{g})|_{y=0} = \mathbf{0}, \end{cases} \tag{1.8}$$

where the vector $\mathbf{e}_1 = (1, 0)$, and the source term

$$(\mathbf{f_U}, \mathbf{f_H}) := (\mathbf{F_U}, \mathbf{F_H}) - (\mathbf{F}_{\mathbf{U}_s}, \mathbf{F}_{\mathbf{H}_s}) \triangleq (f_{1,\mathbf{U}}, f_{2,\mathbf{U}}, f_{1,\mathbf{H}}, f_{2,\mathbf{H}})$$

satisfying $\nabla\cdot\mathbf{f_H} = 0$, $f_{2,\mathbf{H}}|_{y=0} = 0$ by virtue of (1.3). Before stating the main result, we introduce the function spaces used in the paper. For any x-dependent function $f(x) \in L^2(\mathbb{T}_\varrho)$, we denote by f_n its nth Fourier coefficient, i.e.,

$$f_n = \frac{1}{2\pi\varrho}\int_0^{2\pi\varrho} e^{-i\tilde{n}x} f(x)\mathrm{d}x, \qquad n \in \mathbb{Z}, \quad \tilde{n} = \frac{n}{\varrho},$$

and by $\mathcal{P}_n f = f_n e^{i\tilde{n}x}$ the corresponding orthogonal projection on the nth Fourier mode. The divergence-free and boundary conditions in (1.8) imply

$$(\widetilde{\mathbf{U}}_0, \widetilde{\mathbf{H}}_0) = (\tilde{u}_0, 0, \tilde{h}_0, 0).$$

We further denote $\mathcal{Q}_0 f = (I - \mathcal{P}_0) f$ to be the projection on the nonzero Fourier modes. To study (1.8), we need to use a suitable solution space. Denote by H^s and $\dot{H}^s, s \in \mathbb{R}$, the inhomogeneous and homogeneous Sobolev spaces, respectively, and define the subspace of H^s:

$$H^s_\sigma := \left\{ \mathbf{U} = (U_1, U_2) \in H^s(\Omega) \,\middle|\, \nabla \cdot \mathbf{U} = 0,\, U_2|_{y=0} = 0 \right\}.$$

Motivated by [8], we define the function space $\mathcal{X}$ for $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{H}}) = (\tilde{u}, \tilde{v}, \tilde{h}, \tilde{g})$ as follows:

$$\begin{aligned}\mathcal{X} = \Big\{ &(\widetilde{\mathbf{U}}, \widetilde{\mathbf{H}}) \big| \widetilde{\mathbf{U}}|_{y=0} = (\partial_y \tilde{h}, \tilde{g})|_{y=0} = \mathbf{0},\ \left(\widetilde{\mathbf{U}}_0, \widetilde{\mathbf{H}}_0\right) = (\tilde{u}_0, 0, \tilde{h}_0, 0) \in L^\infty(\mathbb{R}_+) \cap \dot{H}^1(\mathbb{R}_+), \\ &\left(\mathcal{Q}_0 \widetilde{\mathbf{U}}, \mathcal{Q}_0 \widetilde{\mathbf{H}}\right) \in H^1_\sigma(\Omega), \quad \|(\widetilde{\mathbf{U}}, \widetilde{\mathbf{H}})\|_{\mathcal{X}} < \infty \Big\},\end{aligned} \tag{1.9}$$

where the norm $\|(\widetilde{\mathbf{U}}, \widetilde{\mathbf{H}})\|_{\mathcal{X}}$ is given by

$$\begin{aligned}\|(\widetilde{\mathbf{U}}, \widetilde{\mathbf{H}})\|_{\mathcal{X}} := &\sum_n \|(\widetilde{\mathbf{U}}_n, \widetilde{\mathbf{H}}_n)\|_{L^\infty(\mathbb{R}_+)} \\ &+ \varepsilon^{\frac14} \|(\partial_y \tilde{u}_0, \partial_y \tilde{h}_0)\|_{L^2(\mathbb{R}_+)} + \|Z^{\frac12} (\partial_y \tilde{u}_0, \partial_y \tilde{h}_0)\|_{L^2(\mathbb{R}_+)} \\ &+ \varepsilon^{-\frac14} \|(\mathcal{Q}_0 \widetilde{\mathbf{U}}, \mathcal{Q}_0 \widetilde{\mathbf{H}})\|_{L^2(\Omega)} + \varepsilon^{-\frac12} \|Z^{\frac12} (\mathcal{Q}_0 \widetilde{\mathbf{U}}, \mathcal{Q}_0 \widetilde{\mathbf{H}})\|_{L^2(\Omega)} \\ &+ \varepsilon^{\frac14} \|(\nabla \mathcal{Q}_0 \widetilde{\mathbf{U}}, \nabla \mathcal{Q}_0 \widetilde{\mathbf{H}})\|_{L^2(\Omega)} + \|Z^{\frac12} (\nabla \mathcal{Q}_0 \widetilde{\mathbf{U}}, \nabla \mathcal{Q}_0 \widetilde{\mathbf{H}})\|_{L^2(\Omega)}.\end{aligned} \tag{1.10}$$

Here the weight function $Z^{\frac12}$ satisfies that $Z = Z(y) \in C^2(\mathbb{R}_+)$, $Z(y) \sim y$ for $y \in (0, 2)$ and remains constant for $y \geq 2$. We will specify it later in section 2. Moreover for simplicity we assume that $\mathbf{f_U} = \mathcal{Q}_0 \mathbf{f_U}$ and $\mathbf{f_H} = \mathcal{Q}_0 \mathbf{f_H}$, since one can extend our result to the general case by adding some shear flow profile, corresponding to the nonzero $(\mathbf{f}_{\mathbf{U},0}, \mathbf{f}_{\mathbf{H},0})$, to the solution $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{H}})$.

Our main result is presented as follows.

THEOREM 1.1. *Let* $(\mathbf{U}_s, \mathbf{H}_s)$ *be a given shear flow that satisfies assumptions* (1.4)–(1.7). *There exist positive constants* δ_1, δ_2, *and* ε_0 *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *and* $\eta > 0$, *if*

$$\varrho(\bar{M} + \bar{M}^4) \in (0, \delta_1) \tag{1.11}$$

and

$$\|(\mathbf{f_U}, \mathbf{f_H})\|_{L^2(\Omega)} + \varepsilon^{-\frac14} \|Z^{\frac12} (\mathbf{f_U}, \mathbf{f_H})\|_{L^2(\Omega)} \leq \frac{\delta_2 \varepsilon^{\frac34}}{|\log \varepsilon|^{3+\eta}},$$

then (1.8) *admits a unique solution* $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{H}}, \nabla P) : (\widetilde{\mathbf{U}}, \widetilde{\mathbf{H}}) \in \mathcal{X} \cap H^2_{loc}(\Omega), \nabla P \in L^2(\Omega)$ *that satisfies the estimate*

$$\|(\widetilde{\mathbf{U}}, \widetilde{\mathbf{H}})\|_{\mathcal{X}} \leq C \varepsilon^{-\frac14} |\log \varepsilon|^{\frac{3+\eta}{2}} \left(\|(\mathbf{f_U}, \mathbf{f_H})\|_{L^2(\Omega)} + \varepsilon^{-\frac14} \|Z^{\frac12} (\mathbf{f_U}, \mathbf{f_H})\|_{L^2(\Omega)} \right), \tag{1.12}$$

where C *is independent of* ε.

Remark 1.2. Comparing our work with the result in [8] for steady Navier–Stokes equations, there are three main differences.

(a) The shear flow in [8] is monotonic near the boundary and remains positive for all $Y > 0$. These assumptions are crucial for the stability since they prevent the reverse flow and boundary layer separation. While there is neither a monotonicity nor a positivity assumption on the velocity field background in our result, instead the only structural condition we need is (1.7), which is a natural one for stability of shear flow for the MHD system [41]. It reflects the stabilization effect of a tangential magnetic field on the boundary layer. It is also emphasized that we do not make any assumptions on the strength of γ_0 in (1.7).

(b) Another essential requirement for the stability result in [8] is the smallness condition on the periodicity ϱ of the tangential variable, which means that the stability result is only local in space. However, such smallness for ϱ is not necessary in our result. In some sense we have proven the almost global stability for $(\mathbf{U}_s, \mathbf{H}_s)$. In fact, one can recover from (1.11) that the periodicity ϱ can be arbitrarily large, provided that $\bar{M}$, which measures the perturbations of the profile (U_s, H_s) around its far field (U_E, H_E), is suitably small.

(c) Our analysis is quite different from [8]. As the authors of [8] mentioned in their paper that they are not able to get direct estimates of the perturbation $\widetilde{\mathbf{U}}$, instead they construct the solution to the problem of $\widetilde{\mathbf{U}}$ via a complicated iteration process. However, by using a key transform inspired by [33] we can establish the estimates of $\widetilde{\mathbf{U}}$ through a direct energy method.

Remark 1.3. We stress that the result in the present paper is a generalization of [33] to the steady case. Unlike the unsteady case, for the steady case it is difficult to establish the L^2 estimates for the equations (1.1), which are degenerate on the boundary because of the no-slip condition for velocity. It leads to some essential difficulties in the mathematical analysis.

In what follows, we briefly point out the difficulties and explain the main ingredients in our proof.

(a) *Good unknown functions.* First, to prove Theorem 1.1, the key step is to analyze the linear system (3.1). Similar to the Navier–Stokes equations [8], one of the difficulties in the analysis of (3.1) comes from the large stretching terms $v\partial_y U_s - g\partial_y H_s$ and $v\partial_y H_s - g\partial_y U_s$ which behave like $O(\varepsilon^{-\frac{1}{2}})(v, g)$. As in [33], our strategy to overcome this difficulty is to introduce new unknowns $(\widehat{\mathbf{U}}, \widehat{\mathbf{H}}) = (\hat{u}, \hat{v}, \hat{h}, \hat{g})$ that are defined in section 3.2, in which the nondegeneracy of tangential magnetic field (1.6) plays an important role. Notice that the transformation performed in the present paper is slightly different from that in [33], since it keeps the divergence-free condition for both velocity field and magnetic field, which is important for proof in this paper. By reformulating (3.1) into a system for these new unknowns, the previously mentioned stretching terms are directly canceled; see (3.8). We would like to mention that in the work by Masmoudi and Wong [36], a different set of good unknowns has been used in the study of the Prandtl system, where the monotonicity condition on velocity field plays an important role.

(b) *L^2-coercivity.* The good unknown functions provide an advantage to obtain uniform-in-ε estimates of $\varepsilon^{\frac{1}{2}}\|\nabla(\widehat{\mathbf{U}}, \widehat{\mathbf{H}})\|_{L^2}$ via $\|(\widehat{\mathbf{U}}, \widehat{\mathbf{H}})\|_{L^2}$; see Lemma 3.5. Then it remains to establish the estimate of $\|(\widehat{\mathbf{U}}, \widehat{\mathbf{H}})\|_{L^2}$ to make the process self-contained. However, in contrast to the previous work [33] for the unsteady case, it is hard to obtain the L^2 estimate directly. Moreover, there is a difficulty from the degeneracy due to the no-slip boundary condition. Therefore, another key ingredient in the proof is to establish an L^2-coercivity estimate of linearized steady MHD operator around the boundary layer profile. To illustrate the main idea, let us consider the main part of (3.12) for the nth Fourier mode of the good unknown function $(\widehat{\mathbf{U}}, \widehat{\mathbf{H}})$:

$$-i\tilde{n}G_s\left(\frac{y}{\sqrt{\varepsilon}}\right)\widehat{\mathbf{H}}_n + (i\tilde{n}p_n, \partial_y p_n) - \varepsilon\mu(\partial_y^2 - \tilde{n}^2)\widehat{\mathbf{U}}_n = \cdots,$$
$$- i\tilde{n}\widehat{\mathbf{U}}_n - \varepsilon\kappa(\partial_y^2 - \tilde{n}^2)\widehat{\mathbf{H}}_n = \cdots,$$

where $G_s(Y) = H_s^2(Y) - U_s^2(Y)$. Thanks to the nondegeneracy assumption (1.7), G_s has a strictly positive lower bound. A natural multiplier is $(\widehat{\mathbf{H}}_n, \widehat{\mathbf{U}}_n)$ to obtain the estimate of $|\tilde{n}|^{\frac{1}{2}}\|(\widehat{\mathbf{U}}_n, \widehat{\mathbf{H}}_n)\|_{L^2}$. However, such a multiplier is not compatible with the diffusion terms of $\widehat{\mathbf{U}}_n$ because the boundary term $\hat{h}_n\partial_y\hat{u}_n|_{y=0}$ appears due to the mixed boundary condition (1.2). And this boundary term is clearly hard to control for this degenerate system.

For this, we will establish a weighted estimate of the solution with an appropriate weight function $Z^{\frac{1}{2}}(y)$ which vanishes on the boundary; see Lemma 3.7. Then the interpolation inequality (2.8) allows us to obtain the estimate of $\|(\widehat{\mathbf{U}}, \widehat{\mathbf{H}})\|_{L^2}$. In this process, since the unweighted estimates and the weighted estimates are strongly coupled, we must keep track of the dependence of the constants on the frequency n, the length of torus ϱ, and $\bar{M}$ in each step. The smallness assumption in (1.11) is crucial for closing the estimate in $\mathcal{X}$.

(c) *Choice of weight function.* The key issue in Lemma 3.5 is to obtain a gain of $\varepsilon^{\frac{1}{4}}$ in the weighted estimate of the magnetic field, which is crucial to recover the unweighted L^2 estimate via the interpolation inequality (2.8). Therefore, the hypothesis of $Z(y) \sim y$ near the boundary $\{y = 0\}$ is natural, since multiplying terms involving Y-derivatives of the boundary layer profile by the weight $Z^{\frac{1}{2}}(y)$ yields a gain of $\varepsilon^{\frac{1}{4}}$. The main reason for the tricky construction of $Z(y)$ in section 2 is as follows. We consider the vorticity formulation (3.21) (to avoid the commutator $[Z, \partial_y P_n]$). Thanks to the divergence-free condition for the good unknown function $(\hat{h}, \hat{g})$, denote by $\hat{\psi}_n$ the nth Fourier coefficient of the stream function $\hat{\psi}$ of $(\hat{h}, \hat{g})$. Applying the multiplier $Z\hat{\psi}_n$ to the vorticity equation produces the following good terms:

$$\begin{aligned}
&\mathrm{Im}\langle -i\tilde{n}[G_s(Y)\omega_{h,n} + \partial_y G_s(Y)\hat{h}_n], Z\hat{\psi}_n\rangle \\
&= \int_0^\infty \tilde{n}G_s(Y)Z(y)|\widehat{\mathbf{H}}_n|^2 \mathrm{d}y + \mathrm{Im}\int_0^\infty i\tilde{n}\partial_y Z\, G_s(Y)\partial_y\hat{\psi}_n\overline{\hat{\psi}_n}\mathrm{d}y \\
&= \underbrace{\tilde{n}\int_0^\infty G_s(Y)Z(y)|\widehat{\mathbf{H}}_n|^2\mathrm{d}y}_{\mathcal{J}_1} + \underbrace{\frac{\tilde{n}}{2}\int_0^\infty \frac{-\partial_y[G_s(Y)\partial_y Z]}{2}|\hat{\psi}_n|^2\mathrm{d}y}_{\mathcal{J}_2}.
\end{aligned}$$

Here $Y = \frac{y}{\sqrt{\varepsilon}}$, and $\mathcal{J}_1$ gives the desired weighted boundedness on $\widehat{\mathbf{H}}_n$. So the function $Z(y)$ is designed so that the most singular part in the lower-order term $\mathcal{J}_2$ is canceled. See Lemma 2.1 for necessary details. Notice that such a process is not appropriate for the vorticity equation of the magnetic field in (3.21), simply because it is not strictly concave near the boundary. Fortunately, due to the boundary condition $\widetilde{\mathbf{U}}|_{y=0} = \mathbf{0}$, the unweighted norm

$\|\widehat{\mathbf{U}}_n\|_{L^2}$ can be obtained directly by applying the natural multiplier $\widehat{\mathbf{U}}_n$ to the second equality in (3.12); see Lemma 3.6 below.

(d) *Commutator estimates.* Note that the commutator $[\partial_y, Z] = \partial_y Z$ is not a boundary layer term; then the gain of $\varepsilon^{\frac{1}{4}}$ does not apply to it. Therefore, another key point in our weighted estimate is to control the lower-order terms involving this commutator in a suitable way. To this end, we take as an example an inner product term $\langle R_n, \partial_y Z \partial_y^{-1} \hat{h}_n \rangle$, where R_n is an inhomogeneous source. We observe that $\partial_y Z$ is supported on $[0,2]$, and the integral operator ∂_y^{-1} gives an extra $Z(y)$ near the boundary; then it implies a trivial bound of this term by virtue of the Hardy inequality as

$$|\langle R_n, \partial_y Z \partial_y^{-1} \hat{h}_n \rangle| = \left| \left\langle y R_n, \partial_y Z \frac{\partial_y^{-1} \hat{h}_n}{y} \right\rangle \right| \le C \|Z R_n\|_{L^2} \|\hat{h}_n\|_{L^2}$$
$$\le C\big(\varepsilon^{-\frac{1}{4}} \|Z^{\frac{1}{2}} R_n\|_{L^2}\big) \cdot \big(\varepsilon^{\frac{1}{4}} \|h_n\|_{L^2}\big),$$

which will lead to a growth of $\varepsilon^{-\frac{1}{4}}$ in our linear estimate (3.5). If so, an extra $\varepsilon^{\frac{1}{2}}$ on the perturbation of external force $(\mathbf{f_U}, \mathbf{f_H})$ is required to compensate such a growth in the nonlinear analysis. In order to minimize the negative power of ε, our main idea is as follows. First, we use the weighted Hardy inequality, instead of the classical one, in the above treatment:

$$|\langle R_n, \partial_y Z \partial_y^{-1} \hat{h}_n \rangle| = \left| \left\langle \sqrt{y} R_n, \partial_y Z \frac{\partial_y^{-1} \hat{h}_n}{\sqrt{y}} \right\rangle \right|$$
$$\le C \big\|Z^{\frac{1}{2}} R_n\big\|_{L^2} \big\|Z^{\frac{1}{2}} |\log Z|^{1+} \hat{h}_n\big\|_{L^2}.$$

We emphasize that the logarithmic-type weight in $\big\|Z^{\frac{1}{2}} |\log Z|^{1+} \hat{h}_n\big\|_{L^2}$ is necessary since it is the critical case for Hardy inequality; see Lemma 2.3 for details. Second, we establish the control of $\big\|Z^{\frac{1}{2}} |\log Z|^{1+} \hat{h}_n\big\|_{L^2}$ via $\big\|Z^{\frac{1}{2}} \hat{h}_n\big\|_{L^2}$ and $\varepsilon^{\frac{1}{4}} \big\|\hat{h}_n\big\|_{L^2}$ but with the price of a logarithmic singularity $|\log \varepsilon|^{1+}$; see Lemma 2.4. Such singularity will cause a growth of $|\log \varepsilon|^{\frac{3}{2}+}$ in the linear estimate (3.5), and that is why we need the logarithmic coefficients in the main result. The above process is also applied to treat commutators in the weighted estimate of the vorticity. We refer to Lemma 3.9 for details.

The rest of paper is organized as follows. In section 2 we will introduce the function $Z(y)$ and establish some related interpolation inequalities. In section 3, we will show the linear stability which is the key step of the proof. The nonlinear stability and the proof of Theorem 1.1 will be given in section 4.

Notation. Throughout this paper, the positive constants which are independent of ε are denoted by C and c. It may vary from line to line. The constants $C_a, C_b, \dots$ represent the generic positive constants depending on a, $b, \dots$, respectively. We say $A \sim B$ if there exist two positive constants C_1 and C_2 such that $C_1 A \le B \le C_2 A$, and $A \sim_\eta B$ if the constants C_1 and C_2 depend on η. $A \lesssim B$ means that there exists a positive constant C such that $A \le CB$, and $A \lesssim_\eta B$ means that the constant C depends on η. For any complex number a, we denote by $\bar{a}$ its complex conjugate. For any two complex value functions f and g which depend on y, the notation $\langle\ ,\ \rangle$ represents the standard $L^2(\mathbb{R}_+)$ inner product, i.e., $\langle f, g \rangle = \int_0^\infty f \bar{g} dy$. Finally, we denote $\|\cdot\|_{L^p}$ as the standard $L^p(\mathbb{R}_+)$-norm and $\|\cdot\|_{L^p(\Omega)}$ as the $L^p(\Omega)$-norm.

2. Weight function. In this section, we specify the weight function $Z^{\frac{1}{2}}(y)$ through the construction of $Z(y)$ and establish some related interpolation inequalities. Recall $G_s(Y) = H_s^2(Y) - U_s^2(Y)$, and it is easy to obtain from the assumptions (1.5)–(1.7) that

$$\gamma_0 \le G_s(Y) \le \bar{\gamma}^2, \qquad \sup_{Y\ge 0}(1+Y)^3|G'_s(Y)| \lesssim \bar{M}.$$

We construct a C^1-function $\widetilde{G}(y)$, $y \in \mathbb{R}_+$, satisfying

$$\widetilde{G}(y) := \begin{cases} \dfrac{1}{G_s(y/\sqrt{\varepsilon})}, & 0 \le y \le 1, \\ 0, & y \ge 2, \end{cases} \tag{2.1}$$

and

$$\frac{1}{2\bar{\gamma}^2} \le \widetilde{G}(y) \le \frac{2}{\gamma_0}, \quad \left|\widetilde{G}'(y)\right| \lesssim \bar{M}\varepsilon \text{ for } y \in \left[1, \frac{3}{2}\right]; \qquad \widetilde{G}'(y) \le 0 \text{ for } y \in \left[\frac{3}{2}, 2\right]. \tag{2.2}$$

It is not difficult to know such function $\widetilde{G}(y)$ exists due to the fact

$$\left(\frac{1}{G_s(y/\sqrt{\varepsilon})}\right)'\Bigg|_{y=1} = \varepsilon \cdot \left(-\frac{Y^3 G'_s(Y)}{G_s^2(Y)}\right)\Bigg|_{Y=\frac{1}{\sqrt{\varepsilon}}} \lesssim \bar{M}\varepsilon.$$

Then we define the function

$$Z(y) := \int_0^y \widetilde{G}(y')dy'. \tag{2.3}$$

One can see that $Z \in C^2(\mathbb{R}_+)$. In the following lemma, we give some basic properties of $Z(y)$, which will be frequently used later.

LEMMA 2.1. *There exists a positive constant C_0 independent of ε and $\bar{M}$ such that the following estimates hold for $Z(y)$:*

(1) $0 \le Z(y) \le C_0$, $Z'(y) \ge 0$, *and*

$$C_0^{-1} y \le Z(y) \le C_0 y \quad \text{for } y \in [0,2], \qquad Z(y) \equiv \int_0^2 \widetilde{G}(y')dy' \triangleq \bar{Z} \quad \text{for } y \ge 2. \tag{2.4}$$

(2) *Estimates near the boundary:*

$$G_s\left(\frac{y}{\sqrt{\varepsilon}}\right) Z'(y) \equiv 1 \text{ for } y \in [0,1], \quad |y^k Z''(y)| \le C_0 \bar{M} \varepsilon^{\frac{k-1}{2}} \text{ for } y \in \left[0, \frac{3}{2}\right],\ 0 \le k \le 3, \tag{2.5}$$

and estimates away from the boundary

$$-\left(G_s(\frac{y}{\sqrt{\varepsilon}})Z'(y)\right)' \ge -C_0\bar{M}\varepsilon \quad \text{for } y \ge 1; \qquad Z''(y) \le 0, \quad \text{for } y \ge \frac{3}{2}. \tag{2.6}$$

(3) *Global-in-y weighted estimates:*

$$|(1+y)Z'(y)| \le C_0, \quad |yZ''(y)| \le C_0(\bar{M}+1) \quad \text{for } y \in \mathbb{R}_+. \tag{2.7}$$

Proof. It suffices to show (2.6) and (2.7), since the other estimates are straightforward. As $Z'(y)=\widetilde{G}(y), Z''(y)=\widetilde{G}'(y)$, from (2.2) for $\widetilde{G}$ it is easy to get $Z''(y)\le 0$ for $y\ge \frac{3}{2}$. Then, we have that for $y\ge 1$,

$$-\left(G_s\left(\frac{y}{\sqrt{\varepsilon}}\right)Z'(y)\right)'=-G_s\left(\frac{y}{\sqrt{\varepsilon}}\right)\widetilde{G}'(y)-\varepsilon^{-\frac{1}{2}}G'_s\left(\frac{y}{\sqrt{\varepsilon}}\right)\widetilde{G}(y).$$

By using (2.2), it implies

$$\left|G_s\left(\frac{y}{\sqrt{\varepsilon}}\right)\widetilde{G}'(y)\right|\le C_0\varepsilon \quad \text{for } 1\le y\le \frac{3}{2}; \qquad G_s\left(\frac{y}{\sqrt{\varepsilon}}\right)\widetilde{G}'(y)\le 0 \quad \text{for } y\ge \frac{3}{2},$$

and

$$\left|G'_s\left(\frac{y}{\sqrt{\varepsilon}}\right)\widetilde{G}(y)\right|\le C_0\varepsilon^{\frac{3}{2}}\frac{|Y^3G'_s(Y)|}{y^3}\le C_0\bar{M}\varepsilon^{\frac{3}{2}}y^{-3}$$

with $Y=y/\sqrt{\varepsilon}$. Therefore, (2.6) follows immediately. The proof of (2.7) is similar and we omit it for brevity. This completes the proof of the lemma. ◻

Next we establish an interpolation inequality which is analogous to Proposition 2.4 in [8].

LEMMA 2.2. *Let $Z(y)$ be the weight function defined in* (2.3) *and C_0 be the positive constant given in Lemma* 2.1. *It holds that for any $g\in H^1(\mathbb{R}_+)$,*

$$\|g\|_{L^2}\le 2\sqrt{2C_0}\|Z^{\frac{1}{2}}g\|_{L^2}^{\frac{2}{3}}\|\partial_y g\|_{L^2}^{\frac{1}{3}}+C_0\|Z^{\frac{1}{2}}g\|_{L^2}. \tag{2.8}$$

Proof. Since $Z(y)\sim y$ for $y\in[0,2]$ and $Z(y)\equiv \bar{Z}$ for $y\ge 2$, the inequality (2.8) follows from an argument similar to that in [6, 8]. Nevertheless, we give its proof for completeness. Let $0<\eta\le 2$ be a constant which will be chosen later. Then from (2.4) one has

$$\begin{aligned}\|g\|_{L^2}^2&=\int_0^\eta|g(y)|^2dy+\int_\eta^\infty|g(y)|^2dy\le\eta\|g\|_{L^\infty}^2+\int_\eta^\infty\frac{1}{Z(y)}Z(y)|g(y)|^2dy\\&\le 2\eta\|g\|_{L^2}\|\partial_y g\|_{L^2}+C_0\eta^{-1}\|Z^{\frac{1}{2}}g\|_{L^2}^2,\end{aligned}\tag{2.9}$$

where we have used the fact

$$Z(y)\ge Z(\eta)\ge C_0^{-1}\eta \quad \text{for } y\ge\eta,$$

and the classical interpolation inequality $\|g\|_{L^\infty}^2\le 2\|g\|_{L^2}\|\partial_y g\|_{L^2}$. Then we optimize the right-hand side of (2.9) with respect to $\eta\in(0,2]$. On one hand, when $\frac{\sqrt{C_0}\|Z^{\frac{1}{2}}g\|_{L^2}}{\sqrt{2\|g\|_{L^2}\|\partial_y g\|_{L^2}}}\le 2$, we choose $\eta=\frac{\sqrt{C_0}\|Z^{\frac{1}{2}}g\|_{L^2}}{\sqrt{2\|g\|_{L^2}\|\partial_y g\|_{L^2}}}$ in (2.9) to obtain

$$\|g\|_{L^2}^2\le 2\sqrt{2C_0}\|Z^{\frac{1}{2}}g\|_{L^2}\sqrt{\|g\|_{L^2}\|\partial_y g\|_{L^2}},$$

which implies

$$\|g\|_{L^2}\le 2\sqrt{2C_0}\|Z^{\frac{1}{2}}g\|_{L^2}^{\frac{2}{3}}\|\partial_y g\|_{L^2}^{\frac{1}{3}}. \tag{2.10}$$

On the other hand, when $\frac{\sqrt{C_0}\|Z^{\frac{1}{2}}g\|_{L^2}}{\sqrt{2\|g\|_{L^2}\|\partial_y g\|_{L^2}}}>2$, it implies $\|g\|_{L^2}\|\partial_y g\|_{L^2}<\frac{C_0}{8}\|Z^{\frac{1}{2}}g\|_{L^2}^2$. We apply this inequality to (2.9) and let $\eta=2$ to get

$$\|g\|_{L^2}^2<C_0\|Z^{\frac{1}{2}}g\|_{L^2}^2. \tag{2.11}$$

Combining (2.10) with (2.11) yields the desired estimate (2.8). ◻

The following two lemmas are crucial for controlling some lower-order terms involving the commutator $[Z,\partial_y]$.

LEMMA 2.3. *Let $R(y)$ be an L^2-function supported on $[0,2]$. Then for any $\eta>0$, there exists a positive constant C_η such that*

$$\big|\langle R,\partial_y^{-1}h\rangle\big|\le C_\eta\|Z^{\frac12}R\|_{L^2}\|Z^{\frac12}|\log Z|^{1+\frac{\eta}{3}}h\|_{L^2}. \tag{2.12}$$

Proof. By the Cauchy–Schwarz inequality,

$$\big|\langle R,\partial_y^{-1}h\rangle\big|\le\|Z^{\frac12}R\|_{L^2(0,2)}\|Z^{-\frac12}\partial_y^{-1}h\|_{L^2(0,2)}. \tag{2.13}$$

Recall the weighted Hardy inequality [29],

$$\Big\|u^{\frac1p}\partial_y^{-1}h\Big\|_{L^p}\lesssim\Big\|v^{\frac1q}h\Big\|_{L^q},\quad 1\le p\le q<\infty, \tag{2.14}$$

provided that the weight functions $u(y)$ and $v(y)$ satisfy

$$\sup_y\left(\int_{\{z\ge y\}}u(z)dz\right)^{\frac1q}\left(\int_{\{z\le y\}}v(z)^{1-p'}dz\right)^{\frac{1}{p'}}<\infty,\quad \frac1p+\frac1{p'}=1.$$

Then, it follows that by letting $p=q=2$, $u(y)=Z^{-1}(y)$, $v(y)=Z(y)|\log Z(y)|^{2+}$ in (2.14),

$$\|Z^{-\frac12}\partial_y^{-1}h\|_{L^2(0,2)}\lesssim\|Z^{\frac12}|\log Z|^{1+\frac{\eta}{3}}h\|_{L^2(0,2)}, \tag{2.15}$$

which, along with (2.13), implies (2.12) immediately.

For easy reference, we give an intuitive proof of (2.15) instead of using weighted Hardy inequality (2.14). Note that (2.15) holds automatically when y is away from zero, since $Z(y)$ is bounded from below by a positive constant. Hence we only need to focus on the case of y near zero. To this end, as $Z(y)\sim y$ with $y\in(0,1/2)$, for any $\eta>0$ we have the following pointwise estimate:

$$\begin{aligned}|\partial_y^{-1}h(y)|&\le C\int_0^y\xi^{-\frac12}|\log\xi|^{-(1+\frac{\eta}{3})}Z^{\frac12}|\log Z|^{1+\frac{\eta}{3}}|h(\xi)|d\xi\\&\le C\|Z^{\frac12}|\log Z|^{1+\frac{\eta}{3}}h\|_{L^2(0,2)}\cdot\left(\int_0^y\xi^{-1}|\log\xi|^{-2-\frac{2\eta}{3}}d\xi\right)^{\frac12}\\&\le C_\eta|\log y|^{-\frac12-\frac{\eta}{3}}\cdot\|Z^{\frac12}|\log Z|^{1+\frac{\eta}{3}}h\|_{L^2(0,2)}.\end{aligned}$$

Then,

$$\begin{aligned}\Big\|Z^{-\frac12}\partial_y^{-1}h\Big\|_{L^2(0,\frac12)}&\le C\left(\int_0^{\frac12}y^{-1}|\partial_y^{-1}h(y)|^2dy\right)^{\frac12}\\&\le C_\eta\|Z^{\frac12}|\log Z|^{1+\frac{\eta}{3}}h\|_{L^2(0,2)}\cdot\left(\int_0^2y^{-1}|\log y|^{-1-\frac{2\eta}{3}}dy\right)^{\frac12}\\&\le C_\eta\|Z^{\frac12}|\log Z|^{1+\frac{\eta}{3}}h\|_{L^2(0,2)}.\end{aligned}$$

Thus we obtain (2.15). ◻

LEMMA 2.4. *For any $\eta > 0$ and $\delta \geq 0$, there exists a positive constant $C_{\eta,\delta}$ independent of ε such that*

$$\|Z^{\frac{1}{2}}|\log Z|^{1+\frac{\eta}{3}}h\|_{L^2} \leq C_{\eta,\delta}|\log\varepsilon|^{1+\frac{\eta}{3}}\left(\|Z^{\frac{1}{2}}h\|_{L^2} + \varepsilon^{\frac{1}{4}+\delta}\|h\|_{L^2}\right). \tag{2.16}$$

Proof. We divide the integration interval into $[0,\varepsilon^{\frac{1}{2}+2\delta}]$ and $[\varepsilon^{\frac{1}{2}+2\delta},\infty)$. In the interval $[0,\varepsilon^{\frac{1}{2}+2\delta}]$, it holds that $Z(y)\sim y$. Let $\xi(y) := y|\log y|^{2+\frac{2\eta}{3}}$; then

$$\xi'(y) = |\log y|^{1+\frac{2\eta}{3}}\left[|\log y| - 2\left(1+\frac{\eta}{3}\right)\right] > 0.$$

Consequently, it holds that $|\xi(y)| \leq C\varepsilon^{\frac{1}{2}+2\delta}|\log\varepsilon|^{2+\frac{2\eta}{3}}$, which implies that

$$\int_0^{\varepsilon^{\frac{1}{2}+2\delta}} Z|\log Z|^{2+\frac{2\eta}{3}}|h|^2dy \leq C\int_0^{\varepsilon^{\frac{1}{2}+2\delta}} |\xi||h|^2dy \leq C\varepsilon^{\frac{1}{2}+2\delta}|\log\varepsilon|^{2+\frac{2\eta}{3}}\|h\|_{L^2}^2.$$

In the interval $[\varepsilon^{\frac{1}{2}+2\delta},\infty)$, since $Z(y)$ is bounded, it yields $|\log Z| \leq C|\log\varepsilon|$. Then it holds that

$$\int_{\varepsilon^{\frac{1}{2}+2\delta}}^{\infty} Z|\log Z|^{2+\frac{2\eta}{3}}|h|^2dy \leq C|\log\varepsilon|^{2+\frac{2\eta}{3}}\|Z^{\frac{1}{2}}h\|_{L^2}^2.$$

By combining these two inequalities, we obtain (2.16) and the proof of the lemma is completed. ∎

Combining (2.12) with (2.16) yields that for any $\eta > 0, \delta \geq 0$ and L^2-function R supported on $[0,2]$,

$$\left|\langle R, \partial_y^{-1}h\rangle\right| \leq C_{\eta,\delta}|\log\varepsilon|^{1+\frac{\eta}{3}}\|Z^{\frac{1}{2}}R\|_{L^2}\left(\|Z^{\frac{1}{2}}h\|_{L^2} + \varepsilon^{\frac{1}{4}+\delta}\|h\|_{L^2}\right). \tag{2.17}$$

We now conclude this section with the following lemma about the equivalence between the weighted estimates on the full gradient of a divergence-free vector field and the weighted estimates of its vorticity.

LEMMA 2.5. *Let $\mathbf{q} = (q_1, q_2)$ be a divergence-free vector field in Ω satisfying $q_2|_{y=0} = 0$. There exists a positive constant $C > 0$ such that*

$$\|Z^{\frac{1}{2}}\nabla\mathbf{q}\|_{L^2(\Omega)} \leq C\|Z^{\frac{1}{2}}\omega_{\mathbf{q}}\|_{L^2(\Omega)}, \tag{2.18}$$

where $\omega_{\mathbf{q}} = \partial_y q_1 - \partial_x q_2$ is the vorticity of $\mathbf{q}$.

Proof. Since $\partial_y q_1 = \omega_{\mathbf{q}} + \partial_x q_2$ and $\partial_y q_2 = -\partial_x q_1$, it suffices to prove

$$\|Z^{\frac{1}{2}}\partial_x\mathbf{q}\|_{L^2(\Omega)} \lesssim \|Z^{\frac{1}{2}}\omega_{\mathbf{q}}\|_{L^2(\Omega)}. \tag{2.19}$$

Let $\phi_{\mathbf{q}}$ be the stream function of $\mathbf{q}$ and $\phi_{\mathbf{q}}$ is determined by

$$\Delta\phi_{\mathbf{q}} = \omega_{\mathbf{q}}, \quad \text{in } \Omega; \qquad \phi_{\mathbf{q}}|_{y=0} = 0. \tag{2.20}$$

For convenience, we introduce $\tilde{Z}(y) := \frac{y}{1+y}$. According to (2.4), we can find two positive constants $\underline{c}$ and $\bar{c}$ such that

$$\underline{c}Z(y) \leq \tilde{Z}(y) \leq \bar{c}Z(y),$$

which implies the equivalence between the norms $\|Z^{\frac{1}{2}}f\|_{L^2(\Omega)}$ and $\|\tilde{Z}^{\frac{1}{2}}f\|_{L^2(\Omega)}$. Thus, we only need to show (2.19) for the weight $\tilde{Z}^{\frac{1}{2}}$. By taking the inner product of (2.20) with $\tilde{Z}\partial_x^2\phi_{\mathbf{q}}$, one has

$$\int_\Omega \tilde{Z}\partial_x^2\phi_{\mathbf{q}}\Delta\phi_{\mathbf{q}}dxdy = \int_\Omega \tilde{Z}\partial_x^2\phi_{\mathbf{q}}\omega_{\mathbf{q}}dxdy.$$

It follows from the Cauchy–Schwarz inequality that

$$\left|\int_\Omega \tilde{Z}\partial_x^2\phi_{\mathbf{q}}\omega_{\mathbf{q}}dxdy\right| \le \|\tilde{Z}^{\frac{1}{2}}\partial_x^2\phi_{\mathbf{q}}\|_{L^2(\Omega)}\|\tilde{Z}^{\frac{1}{2}}\omega_{\mathbf{q}}\|_{L^2(\Omega)} \le \|\tilde{Z}^{\frac{1}{2}}\partial_x q_2\|_{L^2(\Omega)}\|\tilde{Z}^{\frac{1}{2}}\omega_{\mathbf{q}}\|_{L^2(\Omega)}.$$

By integration by parts and using the fact that $\partial_y^2\tilde{Z}(y) = -\frac{2}{(1+y)^3} \le 0$, it yields

$$\begin{aligned}
\int_\Omega \tilde{Z}\partial_x^2\phi_{\mathbf{q}}\Delta\phi_{\mathbf{q}}dxdy &= \int_\Omega \tilde{Z}|\partial_x^2\phi_{\mathbf{q}}|^2dxdy - \int_\Omega \tilde{Z}\partial_x\phi_{\mathbf{q}}\partial_y^2(\partial_x\phi_{\mathbf{q}})dxdy \\
&= \int_\Omega \tilde{Z}\left[|\partial_x^2\phi_{\mathbf{q}}|^2 + |\partial_{xy}^2\phi_{\mathbf{q}}|^2\right]dxdy + \int_\Omega \partial_y\tilde{Z}\partial_{xy}^2\phi_{\mathbf{q}}\partial_x\phi_{\mathbf{q}}dxdy \\
&= \int_\Omega \tilde{Z}\left|\partial_x\nabla\phi_{\mathbf{q}}\right|^2dxdy - \frac{1}{2}\int_\Omega \partial_y^2\tilde{Z}|\partial_x\phi_{\mathbf{q}}|^2dxdy \\
&\ge \left\|\tilde{Z}^{\frac{1}{2}}\partial_x\mathbf{q}\right\|_{L^2(\Omega)}^2.
\end{aligned}$$

Combining the above two inequalities implies that $\|\tilde{Z}^{\frac{1}{2}}\partial_x\mathbf{q}\|_{L^2(\Omega)} \le C\|\tilde{Z}^{\frac{1}{2}}\omega_{\mathbf{q}}\|_{L^2(\Omega)}$, and (2.19) follows. Therefore, we obtain (2.18) and complete the proof of the lemma. □

3. Linear stability. To obtain the solution to nonlinear problem (1.8), we first consider the following linearized system:

$$\begin{cases}
U_s\partial_x\mathbf{U} + v\partial_yU_s\mathbf{e}_1 - H_s\partial_x\mathbf{H} - g\partial_yH_s\mathbf{e}_1 + \nabla P - \mu\varepsilon\Delta\mathbf{U} = \mathbf{f}, \\
U_s\partial_x\mathbf{H} + v\partial_yH_s\mathbf{e}_1 - H_s\partial_x\mathbf{U} - g\partial_yU_s\mathbf{e}_1 - \kappa\varepsilon\Delta\mathbf{H} = \mathbf{q}, \\
\nabla\cdot\mathbf{U} = \nabla\cdot\mathbf{H} = 0, \\
\mathbf{U}|_{y=0} = (\partial_yh, g)|_{y=0} = \mathbf{0},
\end{cases} \tag{3.1}$$

where $\mathbf{f} = (f_1, f_2)$ and $\mathbf{q} = (q_1, q_2)$ are given inhomogeneous source terms. Since U_s and H_s are independent of x, it is convenient to take the Fourier transform in x for (3.1) and study the following equivalent system:

(3.2)

$$\begin{cases}
i\tilde{n}U_s\mathbf{U}_n + v_n\partial_yU_s\mathbf{e}_1 - i\tilde{n}H_s\mathbf{H}_n - g_n\partial_yH_s\mathbf{e}_1 + (i\tilde{n}P_n, \partial_yP_n) - \mu\varepsilon(\partial_y^2 - \tilde{n}^2)\mathbf{U}_n = \mathbf{f}_n, \\
i\tilde{n}U_s\mathbf{H}_n + v_n\partial_yH_s\mathbf{e}_1 - i\tilde{n}H_s\mathbf{U}_n - g_n\partial_yU_s\mathbf{e}_1 - \kappa\varepsilon(\partial_y^2 - \tilde{n}^2)\mathbf{H}_n = \mathbf{q}_n, \\
i\tilde{n}u_n + \partial_yv_n = i\tilde{n}h_n + \partial_yg_n = 0, \\
(u_n, v_n, \partial_yh_n, g_n)|_{y=0} = \mathbf{0}.
\end{cases}$$

Here $n \in \mathbb{Z}$, $\tilde{n} = \frac{n}{\varrho}$, $\mathbf{U}_n = \mathbf{U}_n(y) = \big(u_n(y), v_n(y)\big)$ and $\mathbf{H}_n = \mathbf{H}_n(y) = (h_n(y), g_n(y))$ are nth Fourier coefficients of the velocity field $\mathbf{U}(x,y)$ and magnetic field $\mathbf{H}(x,y)$, respectively, and $\mathbf{f}_n = \mathbf{f}_n(y) = (f_{1,n}(y), f_{2,n}(y))$, $\mathbf{q}_n = \mathbf{q}(y) = (q_{1,n}(y), q_{2,n}(y))$ correspond to $\mathbf{f}(x,y)$ and $\mathbf{q}(x,y)$, respectively. Moreover, it is not difficult to check that the following compatibility condition for $\mathbf{q}$ is needed:

$$\nabla\cdot\mathbf{q} = 0,\ q_2|_{y=0} = 0; \tag{3.3}$$

then $q_{2,0} \equiv 0$ as a direct consequence of (3.3).

For simplicity of notation, we set $\mathbf{W} = (\mathbf{U}, \mathbf{H})$ and $\mathbf{W}_n = (\mathbf{U}_n, \mathbf{H}_n)$. Let $\mathcal{I}$ and ∂_y^{-1} be antiderivative operators defined by

$$\mathcal{I}f(y) = -\int_y^\infty f(y')dy', \qquad \partial_y^{-1} f(y) = \int_0^y f(y')dy',$$

respectively, for any $f \in L^1(\mathbb{R}_+)$. Recall the solution space $\mathcal{X}$ and its norm defined in (1.9) and (1.10), respectively. The solvability of the linear problem (3.1) is given by the following proposition.

PROPOSITION 3.1. *There exist positive constants* δ_1 *and* ε_1 *such that the following statement holds. If*

$$\varrho(\bar{M} + \bar{M}^4) \le \delta_1, \qquad \varepsilon \in (0, \varepsilon_1),$$

then for any $(\mathbf{f}, \mathbf{q})$ *satisfying* (3.3) *and*

$$(\mathcal{I}f_{1,0},\, \partial_y^{-1} q_{1,0}) \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+), \quad \mathcal{Q}_0(\mathbf{f}, \mathbf{q}) \in L^2(\Omega), \tag{3.4}$$

the linear problem (3.1) *admits a unique solution* $\mathbf{W} \in \mathcal{X}$ *that satisfies for any* $\eta > 0$

$$\begin{aligned} &\|\mathbf{W}\|_{\mathcal{X}} \\ &\le C\varepsilon^{-1} \left[\left\| (\mathcal{I}f_{1,0}, \partial_y^{-1} q_{1,0}) \right\|_{L^1} + \varepsilon^{\frac14} \left\| (\mathcal{I}f_{1,0}, \partial_y^{-1} q_{1,0}) \right\|_{L^2} + \left\| Z^{\frac12} (\mathcal{I}f_{1,0}, \partial_y^{-1} q_{1,0}) \right\|_{L^2} \right] \\ &+ C\varepsilon^{-\frac14} |\log \varepsilon|^{\frac{3+\eta}{2}} \left[\|\mathcal{Q}_0(\mathbf{f}, \mathbf{q})\|_{L^2(\Omega)} + \varepsilon^{-\frac14} \|Z^{\frac12} \mathcal{Q}_0(\mathbf{f}, \mathbf{q})\|_{L^2(\Omega)} \right]. \end{aligned} \tag{3.5}$$

Here the positive constant C *is independent of* ε.

The following three subsections are devoted to the proof of Proposition 3.1.

3.1. Estimate on zero mode. We first consider the zero mode $(\mathbf{U}_0, \mathbf{H}_0)$. When $n = 0$, the system (3.2) reduces to the following simple ODE system:

$$\begin{cases} v_0 \partial_y U_s - g_0 \partial_y H_s - \mu\varepsilon \partial_y^2 u_0 = f_{1,0}, \\ \partial_y p_0 - \mu\varepsilon \partial_y^2 v_0 = f_{2,0}, \\ v_0 \partial_y H_s - g_0 \partial_y U_s - \kappa\varepsilon \partial_y^2 h_0 = q_{1,0}, \\ -\kappa\varepsilon \partial_y^2 g_0 = 0, \\ \partial_y v_0 = \partial_y g_0 = 0, \\ (u_0, v_0, \partial_y h_0, g_0)|_{y=0} = \mathbf{0}. \end{cases} \tag{3.6}$$

We can explicitly solve (3.6) to have $v_0 = g_0 = 0$ and

$$\begin{aligned} u_0 &= \frac{1}{\mu\varepsilon} \int_0^y \int_{y'}^{+\infty} f_{1,0}(y'')dy''dy' = -\frac{1}{\mu\varepsilon} \int_0^y \mathcal{I}f_{1,0}(y')dy', \\ h_0 &= \frac{1}{\kappa\varepsilon} \int_y^\infty \int_0^{y'} q_{1,0}(y'')dy''dy' = \frac{1}{\kappa\varepsilon} \int_y^\infty \partial_y^{-1} q_{1,0}(y')dy'. \end{aligned}$$

As a direct consequence, one has the following lemma.

LEMMA 3.2. *From* (3.4) *it holds that*

$$\begin{aligned} \|(u_0, h_0)\|_{L^\infty} &\le C\varepsilon^{-1} \left\| (\mathcal{I}f_{1,0}, \partial_y^{-1} q_{1,0}) \right\|_{L^1}, \\ \|(\partial_y u_0, \partial_y h_0)\|_{L^2} &\le C\varepsilon^{-1} \left\| (\mathcal{I}f_{1,0}, \partial_y^{-1} q_{1,0}) \right\|_{L^2}, \\ \left\| Z^{\frac12} (\partial_y u_0, \partial_y h_0) \right\|_{L^2} &\le C\varepsilon^{-1} \left\| Z^{\frac12} (\mathcal{I}f_{1,0}, \partial_y^{-1} q_{1,0}) \right\|_{L^2}. \end{aligned}$$

3.2. Estimate on nonzero mode. Next we consider nonzero mode ($\mathbf{U}_n$, $\mathbf{H}_n$), $n \neq 0$. Since $\mathbf{H} = (h, g)$ is divergence-free, there exists a stream function $\psi(x, y)$ such that

$$h = \partial_y \psi, \quad g = -\partial_x \psi, \quad \psi|_{y=0} = 0,$$

and the equation of ψ is given by

$$U_s \partial_x \psi + H_s v - \kappa\varepsilon\Delta\psi = \partial_y^{-1} q_1.$$

Inspired by [33], we denote by

$$a_p(Y) = \frac{U_s(Y)}{H_s(Y)}, \qquad b_p(Y) = \frac{\partial_Y H_s(Y)}{H_s(Y)},$$

and introduce new "good unknown function" $\widehat{\mathbf{W}} = (\widehat{\mathbf{U}}, \widehat{\mathbf{H}}) = (\hat{u}, \hat{v}, \hat{h}, \hat{g})$:

$$\begin{cases} \hat{u}(x,y) = u(x,y) - \partial_y\big(a_p(Y)\psi(x,y)\big), \\ \hat{v}(x,y) = v(x,y) + \partial_x\big(a_p(Y)\psi(x,y)\big), \\ \hat{h}(x,y) = \partial_y\left(\frac{\psi(x,y)}{H_s(Y)}\right) = \frac{1}{H_s(Y)}\left(h(x,y) - \varepsilon^{-\frac{1}{2}} b_p(Y)\psi(x,y)\right), \\ \hat{g}(x,y) = -\partial_x\left(\frac{\psi(x,y)}{H_s(Y)}\right) = \frac{g(x,y)}{H_s(Y)} \end{cases} \tag{3.7}$$

with $Y = \frac{y}{\sqrt{\varepsilon}}$. Also, denote by

$$\hat{\psi}(x,y) = \frac{\psi(x,y)}{H_s(Y)},$$

and it is easy to check that $\hat{\psi}$ is the stream function of $\widehat{\mathbf{H}}$. Then by this transformation and some tedious calculations we can rewrite (3.1) into the following problem for $\widehat{\mathbf{W}}$:

(3.8)
$$\begin{cases} (1 + \dfrac{\mu}{\kappa})U_s\partial_x\widehat{\mathbf{U}} - G_s\partial_x\widehat{\mathbf{H}} + \varepsilon^{\frac{1}{2}}\left(\boldsymbol{A}_{\mathbf{U}}\partial_x\widehat{\mathbf{H}} + \boldsymbol{B}_{\mathbf{U}}\partial_y\widehat{\mathbf{H}}\right) + \boldsymbol{C}_{\mathbf{U}}\widehat{\mathbf{H}} + \varepsilon^{-\frac{1}{2}}\hat{\psi}\mathbf{D}_{\mathbf{U}} + \nabla P \\ \quad = \mu\varepsilon\Delta\widehat{\mathbf{U}} + \mathbf{R}_{\mathbf{U}}, \\ -\partial_x\widehat{\mathbf{U}} - 2\kappa\varepsilon^{\frac{1}{2}} b_p\partial_y\widehat{\mathbf{H}} + \boldsymbol{C}_{\mathbf{H}}\widehat{\mathbf{H}} + \varepsilon^{-\frac{1}{2}}\hat{\psi}\mathbf{D}_{\mathbf{H}} = \kappa\varepsilon\Delta\widehat{\mathbf{H}} + \mathbf{R}_{\mathbf{H}}, \\ \nabla\cdot\widehat{\mathbf{U}} = \nabla\cdot\widehat{\mathbf{H}} = 0, \\ \widehat{\mathbf{U}}|_{y=0} = (\partial_y\hat{h}, \hat{g})|_{y=0} = \mathbf{0}. \end{cases}$$

Here $\boldsymbol{A}_{\mathbf{U}}, \boldsymbol{B}_{\mathbf{U}}, \boldsymbol{C}_{\mathbf{U}}, \boldsymbol{C}_{\mathbf{H}}$ are matrices and $\mathbf{D}_{\mathbf{U}}, \mathbf{D}_{\mathbf{H}}$ are vectors. These terms depend only on μ, κ, U_s, H_s, and they have the following forms:

(3.9)
$$\boldsymbol{A}_{\mathbf{U}} = \begin{pmatrix} 0 & (\mu-\kappa)\partial_Y U_s \\ 0 & 0 \end{pmatrix},$$
$$\boldsymbol{B}_{\mathbf{U}} = \begin{pmatrix} (\kappa - 3\mu)\partial_Y U_s + 2\mu a_p \partial_Y H_s & 0 \\ 0 & 2\mu(a_p\partial_Y H_s - \partial_Y U_s) \end{pmatrix},$$
$$\boldsymbol{C}_{\mathbf{U}} = \frac{1}{H_s}$$
$$\begin{pmatrix} 2\kappa\partial_Y H_s \partial_Y U_s - 2\mu a_p(\partial_Y H_s)^2 + 3\mu(U_s\partial_Y^2 H_s - H_s\partial_Y^2 U_s) & 0 \\ 0 & \mu(U_s\partial_Y^2 H_s - H_s\partial_Y^2 U_s) \end{pmatrix},$$
$$\boldsymbol{C}_{\mathbf{H}} = \frac{\kappa}{H_s^2}\begin{pmatrix} 2(\partial_Y H_s)^2 - 3H_s\partial_Y^2 H_s & 0 \\ 0 & -H_s\partial_Y^2 H_s \end{pmatrix},$$

and

$$\begin{aligned}\mathbf{D_U} &= \frac{1}{H_s}\left(\kappa\partial_Y U_s\partial_Y^2 H_s - \mu a_p\partial_Y H_s\partial_Y^2 H_s + \mu U_s\partial_Y^3 H_s - \mu H_s\partial_Y^3 U_s,\, 0\right)^T,\\ \mathbf{D_H} &= \frac{\kappa}{H_s^2}\left(\partial_Y H_s\partial_Y^2 H_s - H_s\partial_Y^3 H_s,\, 0\right)^T.\end{aligned} \tag{3.10}$$

The source term $\mathbf{R} \triangleq (\mathbf{R_U}, \mathbf{R_H}) = (R_u, R_v, R_h, R_g)$ is given by

$$\begin{aligned}\mathbf{R_U} &= (R_u, R_v) = \left(f_1 - \frac{\mu}{\kappa}a_p q_1 + \frac{\varepsilon^{-\frac12}}{H_s}\left(\frac{\mu}{\kappa}a_p\partial_Y H_s - \partial_Y U_s\right)\partial_y^{-1}q_1,\, f_2 - \frac{\mu}{\kappa}a_p q_2\right)^T,\\ \mathbf{R_H} &= (R_h, R_g) = \frac{1}{H_s}\left(q_1 - \varepsilon^{-\frac12}b_p\partial_y^{-1}q_1,\, q_2\right)^T,\end{aligned}$$

where the divergence-free condition $\nabla\cdot\mathbf{q} = 0$ has been used.

Now, let us turn to the Fourier mode. According to (3.7), the nth Fourier coefficients $\widehat{\mathbf{W}}_n = (\widehat{\mathbf{U}}_n, \widehat{\mathbf{H}}_n) \triangleq (\hat u_n, \hat v_n, \hat h_n, \hat g_n)$ of $\widehat{\mathbf{W}}$ are given by

$$\begin{cases}\hat u_n(y) = u_n(y) - \partial_y\big(a_p(Y)\psi_n(y)\big),\\ \hat v_n(y) = v_n(y) + i\tilde n a_p(Y)\psi_n(y),\\ \hat h_n(y) = \partial_y\hat\psi_n(y) = \frac{1}{H_s(Y)}\left(h_n(y) - \varepsilon^{-\frac12}b_p(Y)\psi_n(y)\right),\\ \hat g_n(y) = -i\tilde n\hat\psi_n(y) = \frac{g_n(y)}{H_s(Y)}.\end{cases} \tag{3.11}$$

Here $\hat\psi_n(y)$ and $\psi_n(y)$ are the nth Fourier coefficients of $\hat\psi(x,y)$ and $\psi(x,y)$, respectively, and it holds that $\hat\psi_n(y) = \frac{\psi_n(y)}{H_s(Y)}$. Then, we obtain by taking the Fourier transformation in the problem (3.8) that

$$\begin{cases}i\tilde n\left[\left(1+\frac{\mu}{\kappa}\right)U_s\widehat{\mathbf{U}}_n - G_s\widehat{\mathbf{H}}_n + \varepsilon^{\frac12}\boldsymbol{A}_{\mathbf{U}}\widehat{\mathbf{H}}_n\right] + \varepsilon^{\frac12}\boldsymbol{B}_{\mathbf{U}}\partial_y\widehat{\mathbf{H}}_n + \boldsymbol{C}_{\mathbf{U}}\widehat{\mathbf{H}}_n + \varepsilon^{-\frac12}\hat\psi_n\mathbf{D_U}\\ \qquad\qquad + (i\tilde n p_n, \partial_y p_n)^T - \mu\varepsilon(\partial_y^2 - \tilde n^2)\widehat{\mathbf{U}}_n = \mathbf{R}_{\mathbf{U},n},\\ -i\tilde n\widehat{\mathbf{U}}_n - 2\kappa\varepsilon^{\frac12}b_p\partial_y\widehat{\mathbf{H}}_n + \boldsymbol{C}_{\mathbf{H}}\widehat{\mathbf{H}}_n + \varepsilon^{-\frac12}\hat\psi_n\mathbf{D_H} - \kappa\varepsilon(\partial_y^2 - \tilde n^2)\widehat{\mathbf{H}}_n = \mathbf{R}_{\mathbf{H},n},\\ i\tilde n\hat u_n + \partial_y\hat v_n = i\tilde n\hat h_n + \partial_y\hat g_n = 0,\\ \widehat{\mathbf{U}}_n|_{y=0} = (\partial_y\hat h_n, \hat g_n)|_{y=0} = \mathbf{0},\end{cases} \tag{3.12}$$

with the source $\mathbf{R}_n \triangleq (\mathbf{R}_{\mathbf{U},n}, \mathbf{R}_{\mathbf{H},n}) = (R_{u,n}, R_{v,n}, R_{h,n}, R_{g,n})$:

$$\begin{aligned}\mathbf{R}_{\mathbf{U},n} &= (R_{u,n}, R_{v,n})\\ &= \left(f_{1,n} - \frac{\mu}{\kappa}a_p q_{1,n} + \frac{\varepsilon^{-\frac12}}{H_s}\left(\frac{\mu}{\kappa}a_p\partial_Y H_s - \partial_Y U_s\right)\partial_y^{-1}q_{1,n},\, f_{2,n} - \frac{\mu}{\kappa}a_p q_{2,n}\right)^T,\\ \mathbf{R}_{\mathbf{H},n} &= (R_{h,n}, R_{g,n}) = \frac{1}{H_s}\left(q_{1,n} - \varepsilon^{-\frac12}b_p\partial_y^{-1}q_{1,n},\, q_{2,n}\right)^T.\end{aligned} \tag{3.13}$$

Before we estimate $\widehat{\mathbf{W}}_n$ in the new system (3.12), let us explain why $\widehat{\mathbf{W}}$ defined by (3.7) is a "good unknown function." For this, we first show in the next lemma the equivalence between the original unknown $\mathbf{W}_n$ and the newly defined $\widehat{\mathbf{W}}_n$. The proof is similar to that in [33], and we give it in the appendix.

LEMMA 3.3. *For any $1 < p \le \infty$, it holds that*

$$\|\mathbf{W}_n\|_{L^p} \sim_{\bar M} \|\widehat{\mathbf{W}}_n\|_{L^p}. \tag{3.14}$$

Moreover, we have

$$\begin{aligned}
\|Z^{\frac12}\mathbf{W}_n\|_{L^2} + \varepsilon^{\frac14}\|\mathbf{W}_n\|_{L^2} &\sim_{\bar M} \|Z^{\frac12}\widehat{\mathbf{W}}_n\|_{L^2} + \varepsilon^{\frac14}\|\widehat{\mathbf{W}}_n\|_{L^2},\\
\|\mathbf{W}_n\|_{L^2} + \varepsilon^{\frac12}\|\partial_y\mathbf{W}_n\|_{L^2} &\sim_{\bar M} \|\widehat{\mathbf{W}}_n\|_{L^2} + \varepsilon^{\frac12}\|\partial_y\widehat{\mathbf{W}}_n\|_{L^2},\\
\|\mathbf{W}_n\|_{L^2} + \varepsilon^{\frac14}\|Z^{\frac12}(\partial_y\mathbf{W}_n, i\tilde n\mathbf{W}_n)\|_{L^2} &\sim_{\bar M} \|\widehat{\mathbf{W}}_n\|_{L^2} + \varepsilon^{\frac14}\|Z^{\frac12}(\partial_y\widehat{\mathbf{W}}_n, i\tilde n\widehat{\mathbf{W}}_n)\|_{L^2}.
\end{aligned}$$

Next, the following lemma states that the coefficient matrices and vectors in the system (3.12) are of $O(1)$.

LEMMA 3.4. *There exists a positive constant C independent of ε such that*

$$\begin{aligned}
&\|(1+Y)\boldsymbol{A}_{\mathbf{U}}\|_{L^\infty_Y} + \|(1+Y)\boldsymbol{B}_{\mathbf{U}}\|_{L^\infty_Y} \le C\bar M,\\
&\|(1+Y)\boldsymbol{C}_{\mathbf{U}}\|_{L^\infty_Y} + \|(1+Y)\boldsymbol{C}_{\mathbf{H}}\|_{L^\infty_Y} + \|(1+Y)^2\mathbf{D}_{\mathbf{U}}\|_{L^\infty_Y}\\
&\quad + \|(1+Y)^2\mathbf{D}_{\mathbf{H}}\|_{L^\infty_Y} \le C\bar M(1+\bar M),\\
&\|\mathbf{R}_n\|_{L^2} + \varepsilon^{-\frac14}\|Z^{\frac12}\mathbf{R}_n\|_{L^2} \le C(1+\bar M)\left(\|(\mathbf{f}_n,\mathbf{q}_n)\|_{L^2} + \varepsilon^{-\frac14}\|Z^{\frac12}(\mathbf{f}_n,\mathbf{q}_n)\|_{L^2}\right).
\end{aligned} \tag{3.15}$$

Proof. We sketch the proof by showing the estimate on $\mathbf{R}_n$ because other estimates follow directly from (1.5) and the formulation (3.9), (3.10). According to the expression in (3.13), we treat the term $\varepsilon^{-\frac12}b_p\partial_y^{-1}q_{1,n}$ as an example. First by (1.5) and the Hardy inequality, it holds that

$$\|\varepsilon^{-\frac12}b_p\partial_y^{-1}q_{1,n}\|_{L^2} \le \|Yb_p\|_{L^\infty_Y}\|y^{-1}\partial_y^{-1}q_{1,n}\|_{L^2} \le \bar M\|q_{1,n}\|_{L^2},$$

and by (2.4),

$$\begin{aligned}
\left\|Z^{\frac12}\left(\varepsilon^{-\frac12}b_p\partial_y^{-1}q_{1,n}\right)\right\|_{L^2} &\le \varepsilon^{\frac14}\left\|\sqrt{\frac{Z(y)}{y}}\right\|_{L^\infty}\|Y^{\frac32}b_p\|_{L^\infty_Y}\|y^{-1}\partial_y^{-1}q_{1,n}\|_{L^2}\\
&\le C\bar M\varepsilon^{\frac14}\|q_{1,n}\|_{L^2}.
\end{aligned}$$

Hence, we obtain the estimate on $\mathbf{R}_n$. ∎

We are now ready to establish the uniform-in-ε estimate on $\widehat{\mathbf{W}}_n$ through (3.12). As the first step, the following lemma gives the L^2-estimate on the full derivatives of $\widehat{\mathbf{W}}_n$.

LEMMA 3.5. *Let $\widehat{\mathbf{W}}_n$ be the H^1-solution of the linear problem* (3.12). *There exists a positive constant C_3 independent of ε, $\tilde n$, and $\bar M$ such that*

$$\sqrt{\varepsilon}\left(\|\partial_y\widehat{\mathbf{W}}_n\|_{L^2} + |\tilde n|\|\widehat{\mathbf{W}}_n\|_{L^2}\right) \le C_3\bar M^{\frac12}(1+\bar M^{\frac12})\|\widehat{\mathbf{W}}_n\|_{L^2} + C_3\|\mathbf{R}_n\|_{L^2}^{\frac12}\|\widehat{\mathbf{W}}_n\|_{L^2}^{\frac12}. \tag{3.16}$$

Proof. We take the inner product of the first equality for $\widehat{\mathbf{U}}_n$ in (3.12) with $\widehat{\mathbf{U}}_n$ and the second equality for $\widehat{\mathbf{H}}_n$ in (3.12) with $G_s(\frac{y}{\sqrt{\varepsilon}})\widehat{\mathbf{H}}_n$, respectively, and then take the summation of these two equations. The real part of the final equation gives

$$
\begin{aligned}
&\mathrm{Re}\left\langle i\tilde{n}\varepsilon^{\frac{1}{2}}\boldsymbol{A}_{\mathbf{U}}\widehat{\mathbf{H}}_n+\varepsilon^{\frac{1}{2}}\boldsymbol{B}_{\mathbf{U}}\partial_y\widehat{\mathbf{H}}_n+\boldsymbol{C}_{\mathbf{U}}\widehat{\mathbf{H}}_n+\varepsilon^{-\frac{1}{2}}\hat{\psi}_n\mathbf{D}_{\mathbf{U}},\,\widehat{\mathbf{U}}_n\right\rangle\\
&-\mathrm{Re}\left\langle\mu\varepsilon(\partial_y^2-\tilde{n}^2)\widehat{\mathbf{U}}_n,\,\widehat{\mathbf{U}}_n\right\rangle+\mathrm{Re}\left\langle-2\kappa\varepsilon^{-\frac{1}{2}}b_p\partial_y\widehat{\mathbf{H}}_n+\boldsymbol{C}_{\mathbf{H}}\widehat{\mathbf{H}}_n+\varepsilon^{-\frac{1}{2}}\hat{\psi}_n\mathbf{D}_{\mathbf{H}},\,G_s\widehat{\mathbf{H}}_n\right\rangle\\
&-\mathrm{Re}\left\langle\kappa\varepsilon(\partial_y^2-\tilde{n}^2)\widehat{\mathbf{H}}_n,\,G_s\widehat{\mathbf{H}}_n\right\rangle=\mathrm{Re}\left\langle\mathbf{R}_{\mathbf{U},n},\,\widehat{\mathbf{U}}_n\right\rangle+\mathrm{Re}\left\langle\mathbf{R}_{\mathbf{H},n},\,G_s\widehat{\mathbf{H}}_n\right\rangle.
\end{aligned}\tag{3.17}
$$

Here we have used the fact

$$\left\langle(i\tilde{n}p_n,\partial_yp_n)^T,\,\widehat{\mathbf{U}}_n\right\rangle=0,$$

which follows from the integration by parts, the divergence-free condition $i\tilde{n}\hat{u}_n+\partial_y\hat{v}_n=0$, and the boundary condition $\hat{v}_n|_{y=0}=0$.

Next we estimate terms in (3.17). For the diffusion terms, by integration by parts and the boundary condition $\widehat{\mathbf{U}}_n|_{y=0}=(\partial_y\hat{h}_n,\hat{g}_n)|_{y=0}=\mathbf{0}$, we write

$$-\mathrm{Re}\left\langle\mu\varepsilon(\partial_y^2-\tilde{n}^2)\widehat{\mathbf{U}}_n,\,\widehat{\mathbf{U}}_n\right\rangle=\mu\varepsilon\left(\tilde{n}^2\|\widehat{\mathbf{U}}_n\|_{L^2}^2+\|\partial_y\widehat{\mathbf{U}}_n\|_{L^2}^2\right).$$

Then by using (1.5) and (1.7), we have

$$
\begin{aligned}
&-\mathrm{Re}\left\langle\kappa\varepsilon(\partial_y^2-\tilde{n}^2)\widehat{\mathbf{H}}_n,\,G_s\widehat{\mathbf{H}}_n\right\rangle\\
&\geq\gamma_0\kappa\varepsilon\left(\tilde{n}^2\|\widehat{\mathbf{H}}_n\|_{L^2}^2+\|\partial_y\widehat{\mathbf{H}}_n\|_{L^2}^2\right)+\kappa\varepsilon^{\frac{1}{2}}\mathrm{Re}\left\langle\partial_y\widehat{\mathbf{H}}_n,\,\partial_yG_s\widehat{\mathbf{H}}_n\right\rangle\\
&\geq\frac{\gamma_0\kappa}{2}\varepsilon\left(\tilde{n}^2\|\widehat{\mathbf{H}}_n\|_{L^2}^2+\|\partial_y\widehat{\mathbf{H}}_n\|_{L^2}^2\right)-C\bar{M}^2\|\widehat{\mathbf{H}}_n\|_{L^2}^2.
\end{aligned}
$$

By the Cauchy–Schwarz inequality and the bound given in Lemma 3.4, it holds that

$$
\begin{aligned}
&\left|\left\langle i\tilde{n}\varepsilon^{\frac{1}{2}}\boldsymbol{A}_{\mathbf{U}}\widehat{\mathbf{H}}_n+\varepsilon^{\frac{1}{2}}\boldsymbol{B}_{\mathbf{U}}\partial_y\widehat{\mathbf{H}}_n+\boldsymbol{C}_{\mathbf{U}}\widehat{\mathbf{H}}_n,\,\widehat{\mathbf{U}}_n\right\rangle\right|\\
&\leq\left[\varepsilon^{\frac{1}{2}}\left(|\tilde{n}|\|\boldsymbol{A}_{\mathbf{U}}\|_{L_Y^\infty}\|\widehat{\mathbf{H}}_n\|_{L^2}+\|\boldsymbol{B}_{\mathbf{U}}\|_{L_Y^\infty}\|\partial_y\widehat{\mathbf{H}}_n\|_{L^2}\right)+\|\boldsymbol{C}_{\mathbf{U}}\|_{L_Y^\infty}\|\widehat{\mathbf{H}}_n\|_{L^2}\right]\|\widehat{\mathbf{U}}_n\|_{L^2}\\
&\leq\frac{\gamma_0\kappa}{8}\varepsilon\left(\tilde{n}^2\|\widehat{\mathbf{H}}_n\|_{L^2}^2+\|\partial_y\widehat{\mathbf{H}}_n\|_{L^2}^2\right)+C\bar{M}(1+\bar{M})\|\widehat{\mathbf{W}}_n\|_{L^2}^2.
\end{aligned}
$$

By using $\hat{\psi}_n=\partial_y^{-1}\hat{h}_n$ and the Hardy inequality, one has $\|y^{-1}\hat{\psi}_n\|_{L^2}\lesssim\|\hat{h}_n\|_{L^2}$, and thus it follows from the bound on $\mathbf{D}_{\mathbf{U}}$ given in Lemma 3.4 that

$$\left|\left\langle\varepsilon^{-\frac{1}{2}}\hat{\psi}_n\mathbf{D}_{\mathbf{U}},\,\widehat{\mathbf{U}}_n\right\rangle\right|\leq\|Y\mathbf{D}_{\mathbf{U}}\|_{L_Y^\infty}\|y^{-1}\hat{\psi}_n\|_{L^2}\|\widehat{\mathbf{U}}_n\|_{L^2}\leq C\bar{M}(1+\bar{M})\|\widehat{\mathbf{W}}_n\|_{L^2}^2.$$

Similarly, one has

$$
\begin{aligned}
&\left|\left\langle-2\kappa\varepsilon^{-\frac{1}{2}}b_p\partial_y\widehat{\mathbf{H}}_n+\boldsymbol{C}_{\mathbf{H}}\widehat{\mathbf{H}}_n+\varepsilon^{-\frac{1}{2}}\hat{\psi}_n\mathbf{D}_{\mathbf{H}},\,G_s\widehat{\mathbf{H}}_n\right\rangle\right|\\
&\quad\leq\frac{\gamma_0\kappa}{8}\varepsilon\|\partial_y\widehat{\mathbf{H}}_n\|_{L^2}^2+C\bar{M}(1+\bar{M})\|\widehat{\mathbf{W}}_n\|_{L^2}^2.
\end{aligned}
$$

Also, it is easy to obtain

$$\left|\left\langle\mathbf{R}_{\mathbf{U},n},\,\widehat{\mathbf{U}}_n\right\rangle\right|+\left|\left\langle\mathbf{R}_{\mathbf{H},n},\,G_s\widehat{\mathbf{H}}_n\right\rangle\right|\leq C\|\mathbf{R}_n\|_{L^2}\|\widehat{\mathbf{W}}_n\|_{L^2}.$$

Plugging the above estimates into (3.17) yields

$$\varepsilon\left(\|\partial_y\widehat{\mathbf{W}}_n\|_{L^2}^2+|\tilde{n}|^2\|\widehat{\mathbf{W}}_n\|_{L^2}^2\right)\leq C\bar{M}(1+\bar{M})\|\widehat{\mathbf{W}}_n\|_{L^2}^2+\ C\|\mathbf{R}_n\|_{L^2}\|\|\widehat{\mathbf{W}}_n\|_{L^2},$$

which implies the estimate (3.16), and this completes the proof of the lemma. ∎

Next we establish a uniform-in-ε L^2-estimate on the velocity field $\widehat{\mathbf{U}}_n$.

LEMMA 3.6. *There exists a positive constant C_4 independent of ε, $\tilde{n}$, and $\bar{M}$ such that*

$$|\tilde{n}|^{\frac{1}{2}}\big\|\widehat{\mathbf{U}}_n\big\|_{L^2} \leq C_4\bar{M}^{\frac{1}{2}}(1+\bar{M}^{\frac{1}{2}})\big\|\widehat{\mathbf{W}}_n\big\|_{L^2} + C_4\|\mathbf{R}_n\|_{L^2}^{\frac{1}{2}}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^{\frac{1}{2}}. \tag{3.18}$$

Proof. From the second equation for $\widehat{\mathbf{H}}_n$ in (3.12), we write

$$-i\tilde{n}\widehat{\mathbf{U}}_n = \kappa\varepsilon(\partial_y^2-\tilde{n}^2)\widehat{\mathbf{H}}_n + 2\kappa\varepsilon^{\frac{1}{2}}b_p\partial_y\widehat{\mathbf{H}}_n - \boldsymbol{C}_{\mathbf{H}}\widehat{\mathbf{H}}_n - \varepsilon^{-\frac{1}{2}}\hat{\psi}_n\mathbf{D}_{\mathbf{H}} + \mathbf{R}_{\mathbf{H},n}.$$

Then, taking the inner product of the above equality with $-\widehat{\mathbf{U}}_n$ yields

$$\begin{aligned} i\tilde{n}\big\|\widehat{\mathbf{U}}_n\big\|_{L^2}^2 =& -\kappa\varepsilon\left\langle(\partial_y^2-\tilde{n}^2)\widehat{\mathbf{H}}_n,\, \widehat{\mathbf{U}}_n\right\rangle + \left\langle 2\kappa\varepsilon^{\frac{1}{2}}b_p\partial_y\widehat{\mathbf{H}}_n - \boldsymbol{C}_{\mathbf{H}}\widehat{\mathbf{H}}_n - \varepsilon^{-\frac{1}{2}}\hat{\psi}_n\mathbf{D}_{\mathbf{H}},\, \widehat{\mathbf{U}}_n\right\rangle \\ &+ \left\langle \mathbf{R}_{\mathbf{H},n},\, \widehat{\mathbf{U}}_n\right\rangle. \end{aligned} \tag{3.19}$$

We estimate the right-hand side of (3.19) term by term. First, by integration by parts and the boundary condition $\widehat{\mathbf{U}}_n|_{y=0}=\mathbf{0}$, it is easy to get

$$\kappa\varepsilon\left|\left\langle(\partial_y^2-\tilde{n}^2)\widehat{\mathbf{H}}_n,\, \widehat{\mathbf{U}}_n\right\rangle\right| \leq \kappa\varepsilon\left(\big\|\partial_y\widehat{\mathbf{W}}_n\big\|_{L^2}^2 + \tilde{n}^2\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2\right).$$

Second, it follows by the Cauchy–Schwarz inequality and the Hardy inequality that

$$\begin{aligned} &\left|\left\langle 2\kappa\varepsilon^{\frac{1}{2}}b_p\partial_y\widehat{\mathbf{H}}_n - \boldsymbol{C}_{\mathbf{H}}\widehat{\mathbf{H}}_n - \varepsilon^{-\frac{1}{2}}\hat{\psi}_n\mathbf{D}_{\mathbf{H}},\, \widehat{\mathbf{U}}_n\right\rangle\right| \\ &\leq \left(2\kappa\sqrt{\varepsilon}\|b_p\|_{L_Y^\infty}\|\partial_Y\widehat{\mathbf{H}}_n\|_{L^2} + \|\boldsymbol{C}_{\mathbf{H}}\|_{L_Y^\infty}\|\widehat{\mathbf{H}}_n\|_{L^2} + \|Y\mathbf{D}_{\mathbf{H}}\|_{L_Y^\infty}\|y^{-1}\hat{\psi}_n\|_{L^2}\right)\|\widehat{\mathbf{U}}_n\|_{L^2} \\ &\leq C\varepsilon\big\|\partial_Y\widehat{\mathbf{H}}_n\big\|_{L^2}^2 + C\bar{M}(1+\bar{M})\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2, \end{aligned}$$

where we have used (3.15) in the last inequality. Note that

$$\left|\left\langle \mathbf{R}_{\mathbf{H},n},\, \widehat{\mathbf{U}}_n\right\rangle\right| \leq \|\mathbf{R}_{\mathbf{H},n}\|_{L^2}\big\|\widehat{\mathbf{U}}_n\big\|_{L^2}.$$

Hence, we apply the above three inequalities to (3.19) and obtain

$$\begin{aligned} |\tilde{n}|\,\big\|\widehat{\mathbf{U}}_n\big\|_{L^2}^2 \leq& C\varepsilon\left(\big\|\partial_y\widehat{\mathbf{W}}_n\big\|_{L^2}^2 + \tilde{n}^2\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2\right) + C\bar{M}(1+\bar{M})\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2 \\ &+ \|\mathbf{R}_{\mathbf{H},n}\|_{L^2}\big\|\widehat{\mathbf{U}}_n\big\|_{L^2}, \end{aligned}$$

which, along with (3.16), gives the estimate (3.18) and this completes the proof of the lemma. ∎

Next we turn to the L^2-estimate of $\widehat{\mathbf{H}}_n$. We point out that if we estimate $\|\widehat{\mathbf{H}}_n\|_{L^2}$ in a similar way as Lemma 3.6, a boundary term $\hat{h}_n\partial_y\hat{u}_n|_{y=0}$ appears due to the mixed boundary condition (1.2). Clearly, it is impossible to control this term with the low regularity of the solution. In order to overcome this difficulty, in what follows we turn to establish a weighted estimate on $\widehat{\mathbf{H}}_n$ with the weight $Z^{\frac{1}{2}}(y)$. Notice that the function $Z(y)$ depends on the variable y. To avoid the commutator with pressure term P, we use the vorticity formulation. Let $\omega_u = \partial_y\hat{u} - \partial_x\hat{v}$ and $\omega_h = \partial_y\hat{h} - \partial_x\hat{g}$ be the vorticity of $(\hat{u},\hat{v})$ and $(\hat{h},\hat{g})$, respectively. We recall that $\hat{\psi}$ is the stream function of $\widehat{\mathbf{H}}$ and denote by $\hat{\phi}$ the stream function of $\widehat{\mathbf{U}}$, i.e.,

$$\hat{\phi}_y = \hat{u}, \quad -\hat{\phi}_x = \hat{v}, \quad \hat{\phi}|_{y=0} = 0.$$

We denote respectively by $\omega_{u,n}$ and $\omega_{h,n}$ the nth Fourier coefficients of ω_u and ω_h. Similarly, the nth Fourier coefficient of $\hat{\phi}$ is denoted by $\hat{\phi}_n$. That is,

$$\omega_{u,n} = curl\, \widehat{\mathbf{U}}_n = \partial_y \hat{u}_n - i\tilde{n}\hat{v}_n = (\partial_y^2 - \tilde{n}^2)\hat{\phi}_n,$$
$$\omega_{h,n} = curl\, \widehat{\mathbf{H}}_n = \partial_y \hat{h}_n - i\tilde{n}\hat{g}_n = (\partial_y^2 - \tilde{n}^2)\hat{\psi}_n.$$

From the system (3.12) for $\widehat{\mathbf{W}}_n$, we use the second equation in (3.12) to eliminate $\widehat{\mathbf{U}}_n$ in the first equation. Then it holds that

(3.20)
$$\begin{aligned}
&- i\tilde{n}G_s\widehat{\mathbf{H}}_n + (i\tilde{n}p_n, \partial_y p_n)^T - \mu\varepsilon(\partial_y^2 - \tilde{n}^2)\widehat{\mathbf{U}}_n - \varepsilon(\kappa+\mu)U_s(\partial_y^2 - \tilde{n}^2)\widehat{\mathbf{H}}_n \\
&= \mathbf{R}_{\mathbf{U},n} + \frac{\mu+\kappa}{\kappa}U_s\mathbf{R}_{\mathbf{H},n} - i\tilde{n}\sqrt{\varepsilon}\boldsymbol{A}_{\mathbf{U}}\widehat{\mathbf{H}}_n - \sqrt{\varepsilon}\Big(\boldsymbol{B}_{\mathbf{U}} - 2(\kappa+\mu)a_p\partial_Y H_s\boldsymbol{I}_2\Big)\partial_y\widehat{\mathbf{H}}_n \\
&\quad - \left(\boldsymbol{C}_{\mathbf{U}} + \frac{\mu+\kappa}{\kappa}U_s\boldsymbol{C}_{\mathbf{H}}\right)\widehat{\mathbf{H}}_n - \varepsilon^{-\frac{1}{2}}\hat{\psi}_n\left(\mathbf{D}_{\mathbf{U}} + \frac{\mu+\kappa}{\kappa}U_s\mathbf{D}_{\mathbf{H}}\right) \\
&\triangleq \widetilde{\mathbf{R}}_{\mathbf{U},n},
\end{aligned}$$

where $\boldsymbol{I}_2$ is the identity matrix of order 2. By taking curl on the above equation and the second equation in (3.12), respectively, we arrive at the following system for $\omega_{u,n}$ and $\omega_{h,n}$:

(3.21)
$$\begin{cases}
-i\tilde{n}\, curl\big(G_s\widehat{\mathbf{H}}_n\big) - \varepsilon\mu(\partial_y^2 - \tilde{n}^2)\omega_{u,n} - \varepsilon(\kappa+\mu)\, curl\left(U_s(\partial_y^2 - \tilde{n}^2)\widehat{\mathbf{H}}_n\right) = curl\, \widetilde{\mathbf{R}}_{\mathbf{U},n}, \\
\qquad -i\tilde{n}\omega_{u,n} - \varepsilon\kappa(\partial_y^2 - \tilde{n}^2)\omega_{h,n} = curl\, \widetilde{\mathbf{R}}_{\mathbf{H},n},
\end{cases}$$

where

$$\widetilde{\mathbf{R}}_{\mathbf{H},n} = \mathbf{R}_{\mathbf{H},n} + 2\kappa\sqrt{\varepsilon}\partial_y\widehat{\mathbf{H}}_n - \boldsymbol{C}_{\mathbf{H}}\widehat{\mathbf{H}}_n + \varepsilon^{-\frac{1}{2}}\hat{\psi}_n\mathbf{D}_{\mathbf{H}}.$$

The weighted estimte on $\widehat{\mathbf{W}}_n$ is given in the following lemma.

LEMMA 3.7. *For sufficiently small ε, there exists a positive constant C_5 independent of ε, $\tilde{n}$, and $\bar{M}$ such that for any $\eta > 0$ and $\delta \geq 0$, it holds that*

(3.22)
$$\begin{aligned}
|\tilde{n}|^{\frac{1}{2}}\|Z^{\frac{1}{2}}\widehat{\mathbf{W}}_n\|_{L^2} &\leq C_5|\tilde{n}|^{-\frac{1}{2}}|\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac{1}{2}}\mathbf{R}_n\big\|_{L^2} + C_5\varepsilon^{\frac{1}{4}}\bar{M}^{\frac{1}{2}}\big\|\mathbf{R}_n\big\|_{L^2}^{\frac{1}{2}}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^{\frac{1}{2}} \\
&\quad + C_5\varepsilon^{\frac{1}{4}}(1+\bar{M}^{\frac{1}{2}})\big\|\mathbf{R}_n\big\|_{L^2}^{\frac{1}{4}}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^{\frac{3}{4}} + C_5\varepsilon^{\frac{1}{4}}\bar{M}^{\frac{1}{4}}(1+\bar{M}^{\frac{5}{4}})\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}.
\end{aligned}$$

Proof. By taking the inner product of the first and second equations in (3.21) with $sgn(\tilde{n})\mu^{-1}Z\hat{\psi}_n$ and $sgn(\tilde{n})\kappa^{-1}Z\hat{\phi}_n$, respectively, and adding them together, then taking its imaginary part, we obtain

$$\sum_{i=1}^{4} I_i = 0, \tag{3.23}$$

where

$$\begin{aligned}
I_1 &= -|\tilde{n}|\,\mathrm{Re}\left(\big\langle curl\big(G_s\widehat{\mathbf{H}}_n\big),\, \mu^{-1}Z\hat{\psi}_n\big\rangle + \big\langle \omega_{u,n},\, \kappa^{-1}Z\hat{\phi}_n\big\rangle\right), \\
I_2 &= -\varepsilon sgn(\tilde{n})\,\mathrm{Im}\left(\big\langle(\partial_y^2 - \tilde{n}^2)\omega_{u,n},\, Z\hat{\psi}_n\big\rangle + \big\langle(\partial_y^2 - \tilde{n}^2)\omega_{h,n},\, Z\hat{\phi}_n\big\rangle\right), \\
I_3 &= -\varepsilon sgn(\tilde{n})\,\mathrm{Im}\big\langle(\mu+\kappa)curl\left(U_s(\partial_y^2 - \tilde{n}^2)\widehat{\mathbf{H}}_n\right),\, \mu^{-1}Z\hat{\psi}_n\big\rangle, \\
I_4 &= -\frac{sgn(\tilde{n})}{\mu}\mathrm{Im}\big\langle curl\, \widetilde{\mathbf{R}}_{\mathbf{U},n},\, Z\hat{\psi}_n\big\rangle - \frac{sgn(\tilde{n})}{\kappa}\mathrm{Im}\big\langle curl\, \widetilde{\mathbf{R}}_{\mathbf{H},n},\, Z\hat{\phi}_n\big\rangle.
\end{aligned}$$

We estimate I_1 to I_4 term by term. For I_1, by integration by parts and using the boundary condition $\hat{\psi}_n|_{y=0} = \hat{\phi}_n|_{y=0} = 0$, it holds that

$$\begin{aligned} I_1 =& |\tilde{n}| \left(\mu^{-1} \|\sqrt{G_s Z}\, \widehat{\mathbf{H}}_n\|_{L^2}^2 + \kappa^{-1} \|\sqrt{Z}\, \widehat{\mathbf{U}}_n\|_{L^2}^2\right) \\ &+ |\tilde{n}| \mathrm{Re} \left(\mu^{-1} \langle G_s \hat{h}_n,\, \partial_y Z\, \hat{\psi}_n\rangle + \kappa^{-1} \langle \hat{u}_n,\, \partial_y Z\, \hat{\phi}_n\rangle\right) \\ :=& I_{1,1} + I_{1,2}. \end{aligned}$$

From (1.7), it follows that

$$I_{1,1} \geq c_0 |\tilde{n}| \left\| Z^{\frac{1}{2}} \widehat{\mathbf{W}}_n \right\|_{L^2}^2$$

for some positive constants c_0 independent of ε and $\bar{M}$. For $I_{1,2}$, we notice that $\hat{h}_n = \partial_y \hat{\psi}_n, \hat{u}_n = \partial_y \hat{\phi}_n$, and $\partial_y Z \equiv 0,\ y \geq 2$. Then by integration by parts and using boundary condition $\hat{\psi}_n|_{y=0} = \hat{\phi}_n|_{y=0} = 0$, we can write it as

$$\begin{aligned} I_{1,2} &= \frac{|\tilde{n}|}{2} \int_0^2 \partial_y Z \left[\mu^{-1} G_s \partial_y (|\hat{\psi}_n|^2) + \kappa^{-1} \partial_y (|\hat{\phi}_n|^2)\right] dy \\ &= -\frac{|\tilde{n}|}{2\mu} \int_0^2 \partial_y \left(G_s \partial_y Z\right) |\hat{\psi}_n|^2 dy - \frac{|\tilde{n}|}{2\kappa} \int_0^2 \partial_y^2 Z |\hat{\phi}_n|^2 dy. \end{aligned}$$

We estimate the two terms on the right-hand side of the above identity. For the first one, we use (2.5) and (2.6) to obtain

$$\begin{aligned} -\frac{|\tilde{n}|}{2\mu} \int_0^2 \partial_y \left(G_s \partial_y Z\right) |\hat{\psi}_n|^2 dy &= -\frac{|\tilde{n}|}{2\mu} \int_1^2 \partial_y \left(G_s \partial_y Z\right) |\hat{\psi}_n|^2 dy \\ &\geq -C\bar{M}\varepsilon |\tilde{n}| \int_1^2 y^{-2} |\hat{\psi}_n|^2 dy \geq -C\bar{M}\varepsilon |\tilde{n}|\, \|\hat{h}_n\|_{L^2}^2, \end{aligned}$$

where we have used the Hardy inequality in the last inequality. Similarly, the second term is bounded from below as

$$\begin{aligned} -\frac{|\tilde{n}|}{2\kappa} \int_0^2 \partial_y^2 Z |\hat{\phi}_n|^2 dy &= -\frac{|\tilde{n}|}{2\kappa} \left(\int_0^{3/2} \partial_y^2 Z\, |\hat{\phi}_n|^2 dy + \int_{3/2}^2 \partial_y^2 Z\, |\hat{\phi}_n|^2 dy\right) \\ &\geq -\frac{|\tilde{n}|}{2\kappa} \int_0^{3/2} |y^2 \partial_y^2 Z| \cdot \left|\frac{\hat{\phi}_n}{y}\right|^2 dy \geq -C\sqrt{\varepsilon} \bar{M} |\tilde{n}| \|\hat{u}_n\|_{L^2}^2. \end{aligned}$$

By combining the above estimates related to I_1, one has

$$I_1 \geq c_0 |\tilde{n}|\, \|Z^{\frac{1}{2}} \widehat{\mathbf{W}}_n\|_{L^2}^2 - C\bar{M} |\tilde{n}| \left(\varepsilon \|\hat{h}_n\|_{L^2}^2 + \sqrt{\varepsilon} \|\hat{u}_n\|_{L^2}^2\right). \tag{3.24}$$

Next we consider I_2. The boundary conditions $Z|_{y=0} = \hat{\phi}_n|_{y=0} = \hat{\psi}_n|_{y=0} = 0$ allow us to use integration by parts twice. That is, we have

$$\begin{aligned} I_2 =& -sgn(\tilde{n}) \varepsilon \mathrm{Im} \Big(\langle \omega_{u,n},\, Z \omega_{h,n}\rangle + \langle \omega_{h,n},\, Z \omega_{u,n}\rangle \Big) \\ &- 2 sgn(\tilde{n}) \varepsilon \mathrm{Im} \Big(\langle \omega_{u,n},\, \partial_y Z \hat{h}_n\rangle + \langle \omega_{h,n},\, \partial_y Z \hat{u}_n\rangle \Big) \\ &- sgn(\tilde{n}) \varepsilon \mathrm{Im} \Big(\langle \omega_{u,n},\, \partial_y^2 Z \hat{\psi}_n\rangle + \langle \omega_{h,n},\, \partial_y^2 Z \hat{\phi}_n\rangle \Big) \\ :=& I_{2,1} + I_{2,2} + I_{2,3}. \end{aligned}$$

It is straightforward to see that $I_{2,1}=0$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}|I_{2,2}| &\le 2\varepsilon\left|\langle\omega_{u,n},\,\partial_y Z\hat{h}_n\rangle+\langle\omega_{h,n},\partial_y Z\hat{u}_n\rangle\right|\\ &\le C\varepsilon\|\partial_y Z\|_{L^\infty}\left(\|\omega_{u,n}\|_{L^2}\|\hat{h}_n\|_{L^2}+\|\omega_{h,n}\|_{L^2}\|\hat{u}_n\|_{L^2}\right)\\ &\le C\varepsilon\left(\|\partial_y\widehat{\mathbf{W}}_n\|_{L^2}+|\tilde{n}|\|\widehat{\mathbf{W}}_n\|_{L^2}\right)\|\widehat{\mathbf{W}}_n\|_{L^2}.\end{aligned}$$

And by the Hardy inequality and (2.7),

$$\begin{aligned}|I_{2,3}| &\le \varepsilon\left|\langle\omega_{u,n},\,\partial_y^2 Z\hat{\psi}_n\rangle+\langle\omega_{h,n},\,\partial_y^2 Z\hat{\phi}_n\rangle\right|\\ &\le C\varepsilon\|y\partial_y^2 Z\|_{L^\infty}\left(\|\omega_{u,n}\|_{L^2}\|y^{-1}\hat{\psi}_n\|_{L^2}+\|\omega_{h,n}\|_{L^2}\|y^{-1}\hat{\phi}_n\|_{L^2}\right)\\ &\le C(1+\bar{M})\varepsilon\left(\|\partial_y\widehat{\mathbf{W}}_n\|_{L^2}+|\tilde{n}|\|\widehat{\mathbf{W}}_n\|_{L^2}\right)\|\widehat{\mathbf{W}}_n\|_{L^2}.\end{aligned}$$

Hence combining the above three estimates yields

$$|I_2|\le C(1+\bar{M})\varepsilon\left(\|\partial_y\widehat{\mathbf{W}}_n\|_{L^2}+|\tilde{n}|\|\widehat{\mathbf{W}}_n\|_{L^2}\right)\|\widehat{\mathbf{W}}_n\|_{L^2}. \tag{3.25}$$

The term I_3 can be treated in a similar way. In fact, by integration by parts and the boundary conditions $Z|_{y=0}=\hat{\psi}_n|_{y=0}=0$, we have

$$I_3=-\,\varepsilon sgn(\tilde{n})\,(1+\frac{\kappa}{\mu})\mathrm{Im}\left\langle\partial_y\hat{h}_n,(2\partial_y ZU_s+Z\partial_y U_s)\hat{h}_n+\partial_y(U_s\partial_y Z)\hat{\psi}_n\right\rangle.$$

Note that from (2.7), it holds that

$$|\partial_y ZU_s|\le\|\partial_y Z\|_{L^\infty}\|U_s\|_{L^\infty_Y}\le C,\quad |Z\partial_y U_s|\le\|y^{-1}Z\|_{L^\infty}\|Y\partial_Y U_s\|_{L^\infty_Y}\le C(1+\bar{M}), \tag{3.26}$$

and

$$\begin{aligned}|y\partial_y(U_s\partial_y Z)| &\le |y\partial_y^2 Z\,U_s|+\varepsilon^{-\frac12}|y\partial_y Z\partial_Y U_s|\\ &\le\|y\partial_y^2 Z\|_{L^\infty}\|U_s\|_{L^\infty_Y}+\|\partial_y Z\|_{L^\infty}\|Y\partial_Y U_s\|_{L^\infty_Y}\le C(1+\bar{M}).\end{aligned}$$

Thus one has

$$\begin{aligned}|I_3| &\le C\varepsilon\|\partial_y\hat{h}_n\|_{L^2}\\ &\quad\times\left((\|\partial_y ZU_s\|_{L^\infty}+\|Z\partial_y U_s\|_{L^\infty})\|\hat{h}_n\|_{L^2}+\|y\partial_y(U_s\partial_y Z)\|_{L^\infty}\|y^{-1}\hat{\psi}_n\|_{L^2}\right)\\ &\le C(1+\bar{M})\varepsilon\left(\left\|\partial_y\widehat{\mathbf{W}}_n\right\|_{L^2}+|\tilde{n}|\left\|\widehat{\mathbf{W}}_n\right\|_{L^2}\right)\left\|\widehat{\mathbf{H}}_n\right\|_{L^2}.\end{aligned} \tag{3.27}$$

For I_4, we first estimate $|\mathrm{Im}\langle curl\,\widetilde{\mathbf{R}}_{\mathbf{U},n},\,Z\hat{\psi}_n\rangle|$. By integration by parts,

$$\left\langle curl\,\widetilde{\mathbf{R}}_{\mathbf{U},n},\,Z\hat{\psi}_n\right\rangle=-\left\langle\widetilde{\mathbf{R}}_{\mathbf{U},n},\,Z\widehat{\mathbf{H}}_n\right\rangle-\left\langle\tilde{R}_{u,n},\,\partial_y Z\hat{\psi}_n\right\rangle, \tag{3.28}$$

where $\widetilde{\mathbf{R}}_{\mathbf{U},n}=(\tilde{R}_{u,n},\tilde{R}_{v,n})$. As from (3.20), it holds that

$$\begin{aligned}\widetilde{\mathbf{R}}_{\mathbf{U},n}=&\mathbf{R}_{\mathbf{U},n}+\frac{\mu+\kappa}{\kappa}U_s\mathbf{R}_{\mathbf{H},n}-i\tilde{n}\sqrt{\varepsilon}\boldsymbol{A}_{\mathbf{U}}\widehat{\mathbf{H}}_n-\sqrt{\varepsilon}\Big(\boldsymbol{B}_{\mathbf{U}}-2(\kappa+\mu)a_p\partial_Y H_s\boldsymbol{I}_2\Big)\partial_y\widehat{\mathbf{H}}_n\\ &-\left(\boldsymbol{C}_{\mathbf{U}}+\frac{\mu+\kappa}{\kappa}U_s\boldsymbol{C}_{\mathbf{H}}\right)\widehat{\mathbf{H}}_n-\varepsilon^{-\frac12}\hat{\psi}_n\left(\mathbf{D}_{\mathbf{U}}+\frac{\mu+\kappa}{\kappa}U_s\mathbf{D}_{\mathbf{H}}\right).\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\left|\left\langle \widetilde{\mathbf{R}}_{\mathbf{U},n},\, Z\widehat{\mathbf{H}}_n\right\rangle\right| \\
&\lesssim \left\|Z^{\frac12}\mathbf{R}_n\right\|_{L^2}\left\|Z^{\frac12}\widehat{\mathbf{H}}_n\right\|_{L^2} + \|y^{-1}Z\|_{L^\infty}\left\|\widehat{\mathbf{H}}_n\right\|_{L^2}\cdot\Big\{\varepsilon|\tilde n|\left\|Y\boldsymbol{A}_{\mathbf{U}}\right\|_{L^\infty_Y}\left\|\widehat{\mathbf{H}}_n\right\|_{L^2} \\
&\quad + \varepsilon\left(\left\|Y\boldsymbol{B}_{\mathbf{U}}\right\|_{L^\infty_Y} + \|Y\partial_Y H_s\|_{L^\infty_Y}\right)\left\|\partial_y\widehat{\mathbf{H}}_n\right\|_{L^2} \\
&\quad + \sqrt{\varepsilon}\left(\left\|Y\boldsymbol{C}_{\mathbf{U}}\right\|_{L^\infty_Y} + \left\|Y\boldsymbol{C}_{\mathbf{H}}\right\|_{L^\infty_Y}\right)\left\|\widehat{\mathbf{H}}_n\right\|_{L^2} \\
&\quad + \sqrt{\varepsilon}\left(\left\|Y^2\mathbf{D}_{\mathbf{U}}\right\|_{\mathbf{L}^\infty_{\mathbf{Y}}} + \left\|\mathbf{Y^2D_H}\right\|_{\mathbf{L}^\infty_{\mathbf{Y}}}\right)\left\|y^{-1}\hat\psi_n\right\|_{L^2}\Big\} \\
&\lesssim \left\|Z^{\frac12}\mathbf{R}_n\right\|_{L^2}\left\|Z^{\frac12}\widehat{\mathbf{H}}_n\right\|_{L^2} + \varepsilon\bar M\left(\left\|\partial_y\widehat{\mathbf{H}}_n\right\|_{L^2} + |\tilde n|\left\|\widehat{\mathbf{H}}_n\right\|_{L^2}\right)\left\|\widehat{\mathbf{H}}_n\right\|_{L^2} \\
&\qquad + \sqrt{\varepsilon}\bar M(1+\bar M)\left\|\widehat{\mathbf{H}}_n\right\|^2_{L^2}.
\end{aligned}\tag{3.29}$$

Similarly, by noting that $\hat\psi_n = \partial_y^{-1}\hat h_n$, it holds that for any $\eta > 0$ and $\delta \geq 0$,

$$\begin{aligned}
&\left|\left\langle \tilde R_{u,n},\, \partial_y Z\hat\psi_n\right\rangle\right| \\
&\lesssim |\log\varepsilon|^{1+\frac{\eta}{3}}\|\partial_y Z\|_{L^\infty}\left\|Z^{\frac12}\mathbf{R}_n\right\|_{L^2}\left(\left\|Z^{\frac12}\hat h_n\right\|_{L^2} + \varepsilon^{\frac14+\delta}\left\|\hat h_n\right\|_{L^2}\right) \\
&\quad + \|\partial_y Z\|_{L^\infty}\left\|y^{-1}\hat\psi_n\right\|_{L^2}\Big\{\varepsilon|\tilde n|\left\|Y\boldsymbol{A}_{\mathbf{U}}\right\|_{L^\infty_Y}\left\|\widehat{\mathbf{H}}_n\right\|_{L^2} \\
&\quad + \varepsilon\left(\left\|Y\boldsymbol{B}_{\mathbf{U}}\right\|_{L^\infty_Y} + \|Y\partial_Y H_s\|_{L^\infty_Y}\right)\left\|\partial_y\widehat{\mathbf{H}}_n\right\|_{L^2} \\
&\quad + \sqrt{\varepsilon}\left(\left\|Y\boldsymbol{C}_{\mathbf{U}}\right\|_{L^\infty_Y} + \left\|Y\boldsymbol{C}_{\mathbf{H}}\right\|_{L^\infty_Y}\right)\left\|\widehat{\mathbf{H}}_n\right\|_{L^2} \\
&\quad + \sqrt{\varepsilon}\left(\left\|Y^2\mathbf{D}_{\mathbf{U}}\right\|_{\mathbf{L}^\infty_{\mathbf{Y}}} + \left\|\mathbf{Y^2D_H}\right\|_{\mathbf{L}^\infty_{\mathbf{Y}}}\right)\left\|y^{-1}\hat\psi_n\right\|_{L^2}\Big\} \\
&\lesssim |\log\varepsilon|^{1+\frac{\eta}{3}}\left\|Z^{\frac12}\mathbf{R}_n\right\|_{L^2}\left(\left\|Z^{\frac12}\hat h_n\right\|_{L^2} + \varepsilon^{\frac14+\delta}\left\|\hat h_n\right\|_{L^2}\right) \\
&\quad + \varepsilon\bar M\left(\left\|\partial_y\widehat{\mathbf{H}}_n\right\|_{L^2} + |\tilde n|\left\|\widehat{\mathbf{H}}_n\right\|_{L^2}\right)\left\|\widehat{\mathbf{H}}_n\right\|_{L^2} + \sqrt{\varepsilon}\bar M(1+\bar M)\left\|\widehat{\mathbf{H}}_n\right\|^2_{L^2},
\end{aligned}\tag{3.30}$$

where we have used (2.17) to obtain the first term on the right-hand side of the first inequality. Applying (3.29) and (3.30) to (3.28) yields

$$\begin{aligned}
&\left|\left\langle curl\,\widetilde{\mathbf{R}}_{\mathbf{U},n},\, Z\hat\psi_n\right\rangle\right| \\
&\lesssim |\log\varepsilon|^{1+\frac{\eta}{3}}\left\|Z^{\frac12}\mathbf{R}_n\right\|_{L^2}\left(\left\|Z^{\frac12}\widehat{\mathbf{H}}_n\right\|_{L^2} + \varepsilon^{\frac14+\delta}\left\|\hat h_n\right\|_{L^2}\right) \\
&\quad + \varepsilon\bar M\left(\left\|\partial_y\widehat{\mathbf{H}}_n\right\|_{L^2} + |\tilde n|\left\|\widehat{\mathbf{H}}_n\right\|_{L^2}\right)\left\|\widehat{\mathbf{H}}_n\right\|_{L^2} + \sqrt{\varepsilon}\bar M(1+\bar M)\left\|\widehat{\mathbf{H}}_n\right\|^2_{L^2} \\
&\leq \frac{c_0|\tilde n|}{4}\left\|Z^{\frac12}\widehat{\mathbf{H}}_n\right\|^2_{L^2} + C|\tilde n|^{-1}|\log\varepsilon|^{2+\frac{2\eta}{3}}\left\|Z^{\frac12}\mathbf{R}_n\right\|^2_{L^2} \\
&\quad + C\varepsilon^{\frac14+\delta}|\log\varepsilon|^{1+\frac{\eta}{3}}\left\|Z^{\frac12}\mathbf{R}_n\right\|_{L^2}\|\hat h_n\|_{L^2} \\
&\quad + C\varepsilon\bar M\left(\left\|\partial_y\widehat{\mathbf{H}}_n\right\|_{L^2} + |\tilde n|\left\|\widehat{\mathbf{H}}_n\right\|_{L^2}\right)\left\|\widehat{\mathbf{H}}_n\right\|_{L^2} + C\sqrt{\varepsilon}\bar M(1+\bar M)\left\|\widehat{\mathbf{H}}_n\right\|^2_{L^2}.
\end{aligned}\tag{3.31}$$

Similarly, one can obtain

$$\begin{aligned}
&\left|\left\langle curl\,\widetilde{\mathbf{R}}_{\mathbf{H},n},\, Z\hat\phi_n\right\rangle\right| \\
&\leq \frac{c_0|\tilde n|}{4}\left\|Z^{\frac12}\widehat{\mathbf{U}}_n\right\|^2_{L^2} + C|\tilde n|^{-1}|\log\varepsilon|^{2+\frac{2\eta}{3}}\left\|Z^{\frac12}\mathbf{R}_n\right\|^2_{L^2} \\
&\quad + C\varepsilon^{\frac14+\delta}|\log\varepsilon|^{1+\frac{\eta}{3}}\left\|Z^{\frac12}\mathbf{R}_n\right\|_{L^2}\|\hat u_n\|_{L^2} \\
&\quad + C\varepsilon\bar M\left\|\partial_y\widehat{\mathbf{H}}_n\right\|_{L^2}\left\|\widehat{\mathbf{U}}_n\right\|_{L^2} + C\sqrt{\varepsilon}\bar M(1+\bar M)\left\|\widehat{\mathbf{H}}_n\right\|_{L^2}\left\|\widehat{\mathbf{U}}_n\right\|_{L^2}.
\end{aligned}$$

Then combining the above two estimates gives

$$
\begin{aligned}
|I_4| \le{}& \frac{c_0|\tilde{n}|}{2}\big\|Z^{\frac12}\widehat{\mathbf{W}}_n\big\|_{L^2}^2 \\
&+ C|\tilde{n}|^{-1}|\log\varepsilon|^{2+\frac{2\eta}{3}}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2}^2 + C\varepsilon^{\frac14+\delta}|\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2}\|\widehat{\mathbf{W}}_n\|_{L^2} \\
&+ C\varepsilon\bar{M}\left(\big\|\partial_y\widehat{\mathbf{H}}_n\big\|_{L^2} + |\tilde{n}|\big\|\widehat{\mathbf{H}}_n\big\|_{L^2}\right)\big\|\widehat{\mathbf{W}}_n\big\|_{L^2} + C\sqrt{\varepsilon}\bar{M}(1+\bar{M})\|\widehat{\mathbf{W}}_n\|_{L^2}^2.
\end{aligned}
\tag{3.32}
$$

Thus we complete the estimates of $I_1 - I_4$. By substituting (3.24), (3.25), (3.27), and (3.32) into (3.23), we obtain

$$
\begin{aligned}
|\tilde{n}|\,\|Z^{\frac12}\widehat{\mathbf{W}}_n\|_{L^2}^2 \lesssim{}& \sqrt{\varepsilon}\bar{M}|n|\|\hat{u}_n\|_{L^2}^2 + \varepsilon(1+\bar{M})\left(\|\partial_y\widehat{\mathbf{W}}_n\|_{L^2} + |\tilde{n}|\|\widehat{\mathbf{W}}_n\|_{L^2}\right)\|\widehat{\mathbf{W}}_n\|_{L^2} \\
&+ |\tilde{n}|^{-1}|\log\varepsilon|^{2+\frac{2\eta}{3}}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2}^2 + \varepsilon^{\frac14+\delta}|\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2}\|\widehat{\mathbf{W}}_n\|_{L^2} \\
&+ \sqrt{\varepsilon}\bar{M}(1+\bar{M})\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2.
\end{aligned}
\tag{3.33}
$$

From (3.16) and (3.18) one has

$$
\sqrt{\varepsilon}\left(\|\partial_y\widehat{\mathbf{W}}_n\|_{L^2} + |\tilde{n}|\|\widehat{\mathbf{W}}_n\|_{L^2}\right) \lesssim \bar{M}^{\frac12}(1+\bar{M}^{\frac12})\|\widehat{\mathbf{W}}_n\|_{L^2} + \|\mathbf{R}_n\|_{L^2}^{\frac12}\|\widehat{\mathbf{W}}_n\|_{L^2}^{\frac12}
$$

and

$$
|n|\|\hat{u}_n\|_{L^2}^2 \lesssim \bar{M}(1+\bar{M})\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2 + \|\mathbf{R}_n\|_{L^2}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}.
$$

Then substituting the above two inequalities into (3.33) implies

$$
\begin{aligned}
|\tilde{n}|\,\|Z^{\frac12}\widehat{\mathbf{W}}_n\|_{L^2}^2 \lesssim{}& \sqrt{\varepsilon}\bar{M}\left[\bar{M}(1+\bar{M})\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2 + \|\mathbf{R}_n\|_{L^2}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}\right] \\
&+ \sqrt{\varepsilon}(1+\bar{M})\|\widehat{\mathbf{W}}_n\|_{L^2}\left(\bar{M}^{\frac12}(1+\bar{M}^{\frac12})\|\widehat{\mathbf{W}}_n\|_{L^2} + \|\mathbf{R}_n\|_{L^2}^{\frac12}\|\widehat{\mathbf{W}}_n\|_{L^2}^{\frac12}\right) \\
&+ |\tilde{n}|^{-1}|\log\varepsilon|^{2+\frac{2\eta}{3}}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2}^2 + \varepsilon^{\frac14+\delta}|\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2}\|\widehat{\mathbf{W}}_n\|_{L^2} \\
&+ \sqrt{\varepsilon}\bar{M}(1+\bar{M})\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2 \\
\lesssim{}& |\tilde{n}|^{-1}|\log\varepsilon|^{2+\frac{2\eta}{3}}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2}^2 + \sqrt{\varepsilon}\bar{M}\big\|\mathbf{R}_n\big\|_{L^2}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2} \\
&+ \sqrt{\varepsilon}(1+\bar{M})\big\|\mathbf{R}_n\big\|_{L^2}^{\frac12}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^{\frac32} + \sqrt{\varepsilon}\bar{M}^{\frac12}(1+\bar{M}^{\frac52})\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2,
\end{aligned}
$$

provided ε small enough. Hence we obtain (3.22) and then complete the proof of the lemma. ∎

To recover the L^2-estimate of $\widehat{\mathbf{W}}_n$ by the interpolation inequality (2.8), we have the following lemma.

LEMMA 3.8. *There exist positive constants δ_1 and ε_1 such that if*

$$
\varrho(\bar{M} + \bar{M}^4) \le \delta_1, \qquad \varepsilon \in (0, \varepsilon_1),
$$

then

$$
\begin{aligned}
&\sqrt{\varepsilon}|\tilde{n}|^{\frac13}\left(\|\partial_y\widehat{\mathbf{W}}_n\|_{L^2} + |\tilde{n}|\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}\right) + |\tilde{n}|^{\frac23}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2} + \varepsilon^{-\frac14}|\tilde{n}|^{\frac56}\big\|Z^{\frac12}\widehat{\mathbf{W}}_n\big\|_{L^2} \\
&\quad \le C_6|\log\varepsilon|^{1+\frac{\eta}{3}}\left(\big\|\mathbf{R}_n\big\|_{L^2} + \varepsilon^{-\frac14}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2}\right),
\end{aligned}
\tag{3.34}
$$

where the positive constant C_6 is independent of ε and n.

Proof. We apply the estimates (3.16) and (3.22) to the interpolation inequality (2.8) for $\widehat{\mathbf{W}}_n$ and obtain

(3.35)
$$\begin{aligned}
&\big\|\widehat{\mathbf{W}}_n\big\|_{L^2} \le 2\sqrt{2C_0}\|Z^{\frac12}\widehat{\mathbf{W}}_n\|_{L^2}^{\frac23}\|\partial_y\widehat{\mathbf{W}}_n\|_{L^2}^{\frac13} + C_0\|Z^{\frac12}\widehat{\mathbf{W}}_n\|_{L^2}\\
&\le 2\sqrt{2C_0}|\tilde n|^{-\frac13}C_5^{\frac23}\Big\{|\tilde n|^{-\frac12}|\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2} + \varepsilon^{\frac14}\bar M^{\frac12}\big\|\mathbf{R}_n\big\|_{L^2}^{\frac12}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^{\frac12}\\
&\quad +\varepsilon^{\frac14}(1+\bar M^{\frac12})\big\|\mathbf{R}_n\big\|_{L^2}^{\frac14}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^{\frac34} + \varepsilon^{\frac14}\bar M^{\frac14}(1+\bar M^{\frac54})\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}\Big\}^{\frac23}\\
&\quad \cdot\varepsilon^{-\frac16}C_3^{\frac13}\Big\{\bar M^{\frac12}(1+\bar M^{\frac12})\|\widehat{\mathbf{W}}_n\|_{L^2} + \|\mathbf{R}_n\|_{L^2}^{\frac12}\|\widehat{\mathbf{W}}_n\|_{L^2}^{\frac12}\Big\}^{\frac13}\\
&\quad + C_0|\tilde n|^{-\frac12}C_5\Big\{|\tilde n|^{-\frac12}|\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2} + \varepsilon^{\frac14}\bar M^{\frac12}\big\|\mathbf{R}_n\big\|_{L^2}^{\frac12}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^{\frac12}\\
&\quad +\varepsilon^{\frac14}(1+\bar M^{\frac12})\big\|\mathbf{R}_n\big\|_{L^2}^{\frac14}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^{\frac34} + \varepsilon^{\frac14}\bar M^{\frac14}(1+\bar M^{\frac54})\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}\Big\}\\
&\le \Big[\frac14 + 2\sqrt{2C_0}C_3^{\frac13}C_5^{\frac23}|\tilde n|^{-\frac13}\bar M^{\frac13}(1+\bar M) + C_0C_5|\tilde n|^{-\frac12}\bar M^{\frac14}(1+\bar M^{\frac54})\varepsilon^{\frac14}\Big]\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}\\
&\quad + C|\tilde n|^{-\frac23}\big(1+|\tilde n|^{-\frac43}\big)|\log\varepsilon|^{1+\frac{\eta}{3}}\Big(\varepsilon^{-\frac14}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2} + \big\|\mathbf{R}_n\big\|_{L^2}\Big),
\end{aligned}$$

where the constant $C>0$ is independent of ε and $\tilde n$. Thus if we choose ϱ and $\bar M$ such that

$$2\sqrt{2C_0}C_3^{\frac13}C_5^{\frac23}\varrho^{\frac13}\bar M^{\frac13}(1+\bar M) \le \frac14, \tag{3.36}$$

then by the fact $|\tilde n|^{-1}\le\varrho$ for $n\neq 0$, (3.35) implies that for sufficiently small ε,

$$\big\|\widehat{\mathbf{W}}_n\big\|_{L^2} \lesssim |\tilde n|^{-\frac23}|\log\varepsilon|^{1+\frac{\eta}{3}}\Big(\varepsilon^{-\frac14}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2} + \big\|\mathbf{R}_n\big\|_{L^2}\Big). \tag{3.37}$$

Substituting (3.37) into (3.16) and (3.22), respectively, yields

(3.38)
$$\sqrt{\varepsilon}\Big(\|\partial_y\widehat{\mathbf{W}}_n\|_{L^2} + |\tilde n|\|\widehat{\mathbf{W}}_n\|_{L^2}\Big) \lesssim |\tilde n|^{-\frac13}|\log\varepsilon|^{1+\frac{\eta}{3}}\Big(\varepsilon^{-\frac14}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2} + \big\|\mathbf{R}_n\big\|_{L^2}\Big)$$

and

$$\|Z^{\frac12}\widehat{\mathbf{W}}_n\|_{L^2} \lesssim |\tilde n|^{-\frac56}|\log\varepsilon|^{1+\frac{\eta}{3}}\Big(\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2} + \varepsilon^{\frac14}\big\|\mathbf{R}_n\big\|_{L^2}\Big). \tag{3.39}$$

Combining (3.37)–(3.39) yields (3.34) and δ_1 is determined by (3.36). And this completes the proof of the lemma. ∎

Finally, in order to prove Proposition 3.1 we need to obtain the weighted estimate $\big\|Z^{\frac12}\partial_y\widehat{\mathbf{W}}_n\big\|_{L^2}$. Similarly, to avoid the commutator of the weight function and the pressure p, we take curl on the first equality of (3.12) to obtain

(3.40)
$$\begin{aligned}
&i\tilde n\, curl\Big[\Big(1+\frac{\mu}{\kappa}\Big)U_s\widehat{\mathbf{U}}_n - G_s\widehat{\mathbf{H}}_n\Big] - \mu\varepsilon(\partial_y^2-\tilde n^2)\omega_{u,n}\\
&= curl\Big(\mathbf{R}_{\mathbf{U},n} - i\tilde n\varepsilon^{\frac12}\boldsymbol{A}_{\mathbf{U}}\widehat{\mathbf{H}}_n - \varepsilon^{\frac12}\boldsymbol{B}_{\mathbf{U}}\partial_y\widehat{\mathbf{H}}_n - \boldsymbol{C}_{\mathbf{U}}\widehat{\mathbf{H}}_n - \varepsilon^{-\frac12}\hat\psi_n\mathbf{D}_{\mathbf{U}}\Big) \triangleq curl\,\widehat{\mathbf{R}}_{\mathbf{U},n}.
\end{aligned}$$

LEMMA 3.9. *For sufficiently small ε and any $\eta > 0$, there exists positive constant C, independent of ε and n, such that*

$$\begin{aligned}&\sqrt{\varepsilon}\left(\left\|Z^{\frac12}\partial_y\widehat{\mathbf{W}}_n\right\|_{L^2}+|\tilde n|\left\|Z^{\frac12}\widehat{\mathbf{W}}_n\right\|_{L^2}\right)\\ &\le C|\log\varepsilon|^{\frac12+\frac{\eta}{6}}\left(|\tilde n|^{-\frac12}\left\|Z^{\frac12}\mathbf{R}_n\right\|_{L^2}+|\tilde n|^{\frac12}\left\|Z^{\frac12}\widehat{\mathbf{W}}_n\right\|_{L^2}\right)\\ &\quad+C\varepsilon^{\frac14}\left(\left\|\widehat{\mathbf{W}}_n\right\|_{L^2}+\left\|\widehat{\mathbf{W}}_n\right\|_{L^2}^{\frac12}\left\|\mathbf{R}_n\right\|_{L^2}^{\frac12}\right).\end{aligned}\tag{3.41}$$

Proof. We take the inner product of (3.40) with $-Z\hat\phi_n$, and the second equation for $\widehat{\mathbf{H}}$ in (3.12) with $G_s(\frac{y}{\sqrt{\varepsilon}})Z\widehat{\mathbf{H}}_n$, respectively, then take the real part of its summation to obtain

$$\sum_{i=1}^{3}J_i=0,\tag{3.42}$$

where

$$\begin{aligned}J_1&=\tilde n\,\mathrm{Im}\left(\left\langle curl\left[\left(1+\frac{\mu}{\kappa}\right)U_s\widehat{\mathbf{U}}_n-G_s\widehat{\mathbf{H}}_n\right],\,Z\hat\phi_n\right\rangle+\left\langle\widehat{\mathbf{U}}_n,\,G_sZ\widehat{\mathbf{H}}_n\right\rangle\right),\\ J_2&=\varepsilon\mathrm{Re}\left(\left\langle\mu(\partial_y^2-\tilde n^2)\omega_{u,n},\,Z\hat\phi_n\right\rangle-\left\langle\kappa(\partial_y^2-\tilde n^2)\widehat{\mathbf{H}}_n,\,G_sZ\widehat{\mathbf{H}}_n\right\rangle\right),\\ J_3&=\mathrm{Re}\left(\left\langle curl\,\widehat{\mathbf{R}}_{\mathbf{U},n},\,Z\hat\phi_n\right\rangle\right.\\ &\quad\left.-\left\langle\mathbf{R}_{\mathbf{H},n}+2\kappa\varepsilon^{\frac12}b_p\partial_y\widehat{\mathbf{H}}_n-\boldsymbol{C}_{\mathbf{H}}\widehat{\mathbf{H}}_n-\varepsilon^{-\frac12}\hat\psi_n\mathbf{D}_{\mathbf{H}},\,G_sZ\widehat{\mathbf{H}}_n\right\rangle\right).\end{aligned}$$

We estimate $J_i, i=1,2,3$, term by term. First, for J_1 one has by integration by parts and the boundary condition $\hat\phi_n|_{y=0}=0$ that

$$\begin{aligned}&\left\langle curl\left[\left(1+\frac{\mu}{\kappa}\right)U_s\widehat{\mathbf{U}}_n-G_s\widehat{\mathbf{H}}_n\right],\,Z\hat\phi_n\right\rangle\\ &=-\left\langle\left(1+\frac{\mu}{\kappa}\right)U_s\widehat{\mathbf{U}}_n-G_s\widehat{\mathbf{H}}_n,\,Z\widehat{\mathbf{U}}_n\right\rangle-\left\langle\left(1+\frac{\mu}{\kappa}\right)U_s\hat u_n-G_s\hat h_n,\,Z_y\hat\phi_n\right\rangle.\end{aligned}$$

We apply the above equality to J_1 and get

$$J_1=-\tilde n\,\mathrm{Im}\left\langle\left(1+\frac{\mu}{\kappa}\right)U_s\hat u_n-G_s\hat h_n,\,Z_y\hat\phi_n\right\rangle,$$

and by virtue of (2.17) it implies that for sufficiently small ε,

$$\begin{aligned}|J_1|&\lesssim|\tilde n||\log\varepsilon|^{1+\frac{\eta}{3}}\left(\|Z^{\frac12}\hat u_n\|_{L^2}+\|Z^{\frac12}\hat h_n\|_{L^2}\right)\left(\|Z^{\frac12}\hat u_n\|_{L^2}+\varepsilon^{\frac14+\delta}\|\hat u_n\|_{L^2}\right)\\ &\lesssim|\tilde n||\log\varepsilon|^{1+\frac{\eta}{3}}\left\|Z^{\frac12}\widehat{\mathbf{W}}_n\right\|_{L^2}^2+\sqrt{\varepsilon}|\tilde n|\|\hat u_n\|_{L^2}^2.\end{aligned}\tag{3.43}$$

Second, as $\omega_{u,n}=(\partial_y^2-\tilde n^2)\hat\phi_n$, by integration by parts and the boundary conditions $Z|_{y=0}=\hat\phi_n|_{y=0}=0$, we can write J_2 as

$$\begin{aligned}J_2&=\varepsilon\mathrm{Re}\left(\mu\left\langle\omega_{u,n},\,Z\omega_{u,n}\right\rangle+\kappa\left\langle\partial_y\widehat{\mathbf{H}}_n,\,G_sZ\partial_y\widehat{\mathbf{H}}_n\right\rangle+\kappa\tilde n^2\left\langle\widehat{\mathbf{H}}_n,\,G_sZ\widehat{\mathbf{H}}_n\right\rangle\right)\\ &\quad+\varepsilon\mathrm{Re}\left(\mu\left\langle\omega_{u,n},\,2Z_y\hat u_n+Z_{yy}\hat\phi_n\right\rangle+\kappa\left\langle\partial_y\hat h_n,\,\partial_y(G_sZ)\hat h_n\right\rangle\right)\\ &\triangleq J_{2,1}+J_{2,2}.\end{aligned}$$

From (1.7), it is easy to get

$$J_{2,1} \geq \mu\varepsilon \big\|Z^{\frac{1}{2}}\omega_{u,n}\big\|_{L^2}^2 + \kappa\gamma_0\varepsilon\left(\big\|Z^{\frac{1}{2}}\partial_y\widehat{\mathbf{H}}_n\big\|_{L^2}^2 + \tilde{n}^2\big\|Z^{\frac{1}{2}}\widehat{\mathbf{H}}_n\big\|_{L^2}^2\right).$$

Similar to (3.26), it follows from $|\partial_y(G_sZ)| \leq C(1+\bar{M})$, (2.7), and the Hardy inequality that

$$\begin{aligned}|J_{2,2}| \lesssim{}& \varepsilon\|\omega_{u,n}\|_{L^2}\left(\|Z_y\|_{L^\infty}\|\hat{u}_n\|_{L^2} + \|yZ_{yy}\|_{L^\infty}\|y^{-1}\hat{\phi}_n\|_{L^2}\right)\\ &+ \varepsilon\big\|\partial_y\widehat{\mathbf{H}}_n\big\|_{L^2}\big\|\partial_y(G_sZ)\big\|_{L^\infty}\big\|\widehat{\mathbf{H}}_n\big\|_{L^2}\\ \lesssim{}& \varepsilon\left(\big\|\partial_y\widehat{\mathbf{W}}_n\big\|_{L^2} + |\tilde{n}|\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}\right)\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}.\end{aligned}$$

Combining the above three estimates yields

$$\begin{aligned}J_2 \geq{}& \mu\varepsilon\big\|Z^{\frac{1}{2}}\omega_{u,n}\big\|_{L^2}^2 + \kappa\gamma_0\varepsilon\left(\big\|Z^{\frac{1}{2}}\partial_y\widehat{\mathbf{H}}_n\big\|_{L^2}^2 + \tilde{n}^2\big\|Z^{\frac{1}{2}}\widehat{\mathbf{H}}_n\big\|_{L^2}^2\right)\\ &- C\varepsilon\left(\big\|\partial_y\widehat{\mathbf{W}}_n\big\|_{L^2} + |\tilde{n}|\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}\right)\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}.\end{aligned} \tag{3.44}$$

Next, for J_3, similar to (3.31), we obtain

$$\begin{aligned}\left|\left\langle curl\,\widehat{\mathbf{R}}_{\mathbf{U},n},\, Z\hat{\phi}_n\right\rangle\right| \lesssim{}& |\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac{1}{2}}\mathbf{R}_n\big\|_{L^2}\left(\big\|Z^{\frac{1}{2}}\widehat{\mathbf{U}}_n\big\|_{L^2} + \varepsilon^{\frac{1}{4}+\delta}\big\|\hat{u}_n\big\|_{L^2}\right)\\ &+ \varepsilon\left(\big\|\partial_y\widehat{\mathbf{H}}_n\big\|_{L^2} + |\tilde{n}|\big\|\widehat{\mathbf{H}}_n\big\|_{L^2}\right)\big\|\widehat{\mathbf{U}}_n\big\|_{L^2} + \sqrt{\varepsilon}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2.\end{aligned}$$

As for (3.29), one has

$$\begin{aligned}&\left|\left\langle \mathbf{R}_{\mathbf{H},n} + 2\kappa\varepsilon^{\frac{1}{2}}b_p\partial_y\widehat{\mathbf{H}}_n - \boldsymbol{C}_{\mathbf{H}}\widehat{\mathbf{H}}_n - \varepsilon^{-\frac{1}{2}}\hat{\psi}_n\mathbf{D}_{\mathbf{H}},\, G_sZ\widehat{\mathbf{H}}_n\right\rangle\right|\\ &\lesssim \big\|Z^{\frac{1}{2}}\mathbf{R}_n\big\|_{L^2}\big\|Z^{\frac{1}{2}}\widehat{\mathbf{H}}_n\big\|_{L^2} + \varepsilon\big\|\partial_y\widehat{\mathbf{H}}_n\big\|_{L^2}\big\|\widehat{\mathbf{H}}_n\big\|_{L^2} + \sqrt{\varepsilon}\big\|\widehat{\mathbf{H}}_n\big\|_{L^2}^2.\end{aligned}$$

Combining the above two inequalities yields

$$\begin{aligned}|J_3| \lesssim{}& |\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac{1}{2}}\mathbf{R}_n\big\|_{L^2}\left(\big\|Z^{\frac{1}{2}}\widehat{\mathbf{W}}_n\big\|_{L^2} + \varepsilon^{\frac{1}{4}+\delta}\big\|\hat{u}_n\big\|_{L^2}\right)\\ &+ \varepsilon\left(\big\|\partial_y\widehat{\mathbf{H}}_n\big\|_{L^2} + |\tilde{n}|\big\|\widehat{\mathbf{H}}_n\big\|_{L^2}\right)\big\|\widehat{\mathbf{W}}_n\big\|_{L^2} + \sqrt{\varepsilon}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2\\ \lesssim{}& |\tilde{n}||\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac{1}{2}}\widehat{\mathbf{W}}_n\big\|_{L^2}^2 + \sqrt{\varepsilon}|\tilde{n}|\|\hat{u}_n\|_{L^2}^2 + |\tilde{n}|^{-1}|\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac{1}{2}}\mathbf{R}_n\big\|_{L^2}^2\\ &+ \varepsilon\left(\big\|\partial_y\widehat{\mathbf{H}}_n\big\|_{L^2} + |\tilde{n}|\big\|\widehat{\mathbf{H}}_n\big\|_{L^2}\right)\big\|\widehat{\mathbf{W}}_n\big\|_{L^2} + \sqrt{\varepsilon}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2,\end{aligned} \tag{3.45}$$

provided ε small enough.

Thus, we substitute (3.43), (3.44), and (3.45) into (3.42) to obtain

$$\begin{aligned}&\varepsilon\left(\big\|Z^{\frac{1}{2}}\omega_{u,n}\big\|_{L^2}^2 + \big\|Z^{\frac{1}{2}}\partial_y\widehat{\mathbf{H}}_n\big\|_{L^2}^2 + \tilde{n}^2\big\|Z^{\frac{1}{2}}\widehat{\mathbf{H}}_n\big\|_{L^2}^2\right)\\ &\lesssim |\tilde{n}||\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac{1}{2}}\widehat{\mathbf{W}}_n\big\|_{L^2}^2 + \sqrt{\varepsilon}|\tilde{n}|\|\hat{u}_n\|_{L^2}^2 + |\tilde{n}|^{-1}|\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac{1}{2}}\mathbf{R}_n\big\|_{L^2}^2\\ &\quad+ \varepsilon\left(\big\|\partial_y\widehat{\mathbf{H}}_n\big\|_{L^2} + |\tilde{n}|\big\|\widehat{\mathbf{H}}_n\big\|_{L^2}\right)\big\|\widehat{\mathbf{W}}_n\big\|_{L^2} + \sqrt{\varepsilon}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2.\end{aligned}$$

Then, applying (3.16) and (3.18) to the above inequality yields

$$\begin{aligned}&\varepsilon\left(\big\|Z^{\frac{1}{2}}\omega_{u,n}\big\|_{L^2}^2 + \big\|Z^{\frac{1}{2}}\partial_y\widehat{\mathbf{H}}_n\big\|_{L^2}^2 + \tilde{n}^2\big\|Z^{\frac{1}{2}}\widehat{\mathbf{H}}_n\big\|_{L^2}^2\right)\\ &\lesssim |\tilde{n}|^{-1}|\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac{1}{2}}\mathbf{R}_n\big\|_{L^2}^2 + |\tilde{n}|\,|\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac{1}{2}}\widehat{\mathbf{W}}_n\big\|_{L^2}^2\\ &\quad+ \sqrt{\varepsilon}\left(\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2 + \big\|\widehat{\mathbf{W}}_n\big\|_{L^2}\big\|\mathbf{R}_n\big\|_{L^2} + \big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^{\frac{3}{2}}\big\|\mathbf{R}_n\big\|_{L^2}^{\frac{1}{2}}\right)\\ &\lesssim |\tilde{n}|^{-1}|\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac{1}{2}}\mathbf{R}_n\big\|_{L^2}^2 + |\tilde{n}|\,|\log\varepsilon|^{1+\frac{\eta}{3}}\big\|Z^{\frac{1}{2}}\widehat{\mathbf{W}}_n\big\|_{L^2}^2\\ &\quad+ \sqrt{\varepsilon}\left(\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}^2 + \big\|\widehat{\mathbf{W}}_n\big\|_{L^2}\big\|\mathbf{R}_n\big\|_{L^2}\right).\end{aligned}$$

∎

This and Lemma 2.5 give (3.41). And the proof of the lemma is completed.

3.3. Final estimates. Now we are ready to give the proof of Proposition 3.1.

Proof of Proposition 3.1. Similar to [8], the existence of solution $(\mathbf{U},\mathbf{H})$ to the problem (3.1) follows from a standard procedure: we can replace $-\mu\varepsilon\Delta\mathbf{U}$ and $-\kappa\varepsilon\Delta\mathbf{H}$ by $-\mu\varepsilon\Delta\mathbf{U}+s\mathbf{U}$ and $-\mu\varepsilon\Delta\mathbf{H}+s\mathbf{H}$, respectively, with $s>0$. It is straightforward to show the existence for sufficiently large s. One can check that a priori estimate (3.5) is uniform in s. Therefore, the existence part follows from a standard continuity argument. We omit the details for brevity. In what follows, we focus on the a priori estimate (3.5). The proof is divided into two steps.

Step 1*: L^2-estimate*. By (3.34) and (3.41), we obtain

$$\sqrt{\varepsilon}\left(\|Z^{\frac12}\partial_y\widehat{\mathbf{W}}_n\|_{L^2}+|\tilde n|\|Z^{\frac12}\widehat{\mathbf{W}}_n\|_{L^2}\right)\lesssim|\tilde n|^{-\frac13}|\log\varepsilon|^{\frac{3+\eta}{2}}\left(\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2}+\varepsilon^{\frac14}\big\|\mathbf{R}_n\big\|_{L^2}\right).\tag{3.46}$$

Combining (3.34) with (3.46) yields

$$\begin{aligned}&|\tilde n|^{\frac23}\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}+\varepsilon^{-\frac14}|\tilde n|^{\frac56}\big\|Z^{\frac12}\widehat{\mathbf{W}}_n\big\|_{L^2}+\sqrt{\varepsilon}|\tilde n|^{\frac13}\left(\|\partial_y\widehat{\mathbf{W}}_n\|_{L^2}+|\tilde n|\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}\right)\\&\quad+\varepsilon^{\frac14}|\tilde n|^{\frac13}\left(\|Z^{\frac12}\partial_y\widehat{\mathbf{W}}_n\|_{L^2}+|\tilde n|\big\|Z^{\frac12}\widehat{\mathbf{W}}_n\big\|_{L^2}\right)\\&\lesssim|\log\varepsilon|^{\frac{3+\eta}{2}}\left(\big\|\mathbf{R}_n\big\|_{L^2}+\varepsilon^{-\frac14}\big\|Z^{\frac12}\mathbf{R}_n\big\|_{L^2}\right).\end{aligned}$$

Then by Lemmas 3.3 and 3.4 and the fact that $|\tilde n|^{-1}\le\varrho, n\neq0$, one has

$$\begin{aligned}&\big\|\mathbf{W}_n\big\|_{L^2}+\varepsilon^{-\frac14}\big\|Z^{\frac12}\mathbf{W}_n\big\|_{L^2}+\sqrt{\varepsilon}\left(\|\partial_y\mathbf{W}_n\|_{L^2}+|\tilde n|\big\|\mathbf{W}_n\big\|_{L^2}\right)\\&\quad+\varepsilon^{\frac14}\left(\|Z^{\frac12}\partial_y\mathbf{W}_n\|_{L^2}+|\tilde n|\big\|Z^{\frac12}\mathbf{W}_n\big\|_{L^2}\right)\\&\lesssim\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}+\varepsilon^{-\frac14}\big\|Z^{\frac12}\widehat{\mathbf{W}}_n\big\|_{L^2}+\sqrt{\varepsilon}\left(\|\partial_y\widehat{\mathbf{W}}_n\|_{L^2}+|\tilde n|\big\|\widehat{\mathbf{W}}_n\big\|_{L^2}\right)\\&\quad+\varepsilon^{\frac14}\left(\|Z^{\frac12}\partial_y\widehat{\mathbf{W}}_n\|_{L^2}+|\tilde n|\big\|Z^{\frac12}\widehat{\mathbf{W}}_n\big\|_{L^2}\right)\\&\le C|\log\varepsilon|^{\frac{3+\eta}{2}}\left(\|(\mathbf{f}_n,\mathbf{q}_n)\|_{L^2}+\varepsilon^{-\frac14}\|Z^{\frac12}(\mathbf{f}_n,\mathbf{q}_n)\|_{L^2}\right),\end{aligned}$$

where the positive constant C is independent of ε and n. Therefore, by the Parseval equality one has

$$\begin{aligned}&\varepsilon^{-\frac14}\|\mathcal{Q}_0\mathbf{W}\|_{L^2(\Omega)}+\varepsilon^{-\frac12}\|Z^{\frac12}\mathcal{Q}_0\mathbf{W}\|_{L^2(\Omega)}+\varepsilon^{\frac14}\|\nabla\mathcal{Q}_0\mathbf{W}\|_{L^2(\Omega)}+\|Z^{\frac12}\nabla\mathcal{Q}_0\mathbf{W}\|_{L^2(\Omega)}\\&=\varepsilon^{-\frac14}\left\|\Big\{\|\mathbf{W}_n\|_{L^2}\Big\}_{n\neq0}\right\|_{l^2}+\varepsilon^{-\frac12}\left\|\Big\{\|Z^{\frac12}\mathbf{W}_n\|_{L^2}\Big\}_{n\neq0}\right\|_{l^2}\\&\quad+\varepsilon^{\frac14}\left\|\Big\{\|(\partial_y\mathbf{W}_n,i\tilde n\mathbf{W}_n)\|_{L^2}\Big\}_{n\neq0}\right\|_{l^2}+\left\|\Big\{\|Z^{\frac12}(\partial_y\mathbf{W}_n,i\tilde n\mathbf{W}_n)\|_{L^2}\Big\}_{n\neq0}\right\|_{l^2}\\&\le C\varepsilon^{-\frac14}|\log\varepsilon|^{\frac{3+\eta}{2}}\left(\left\|\Big\{\|(\mathbf{f}_n,\mathbf{q}_n)\|_{L^2}\Big\}_{n\neq0}\right\|_{l^2}+\varepsilon^{-\frac14}\left\|\Big\{\|Z^{\frac12}(\mathbf{f}_n,\mathbf{q}_n)\|_{L^2}\Big\}_{n\neq0}\right\|_{l^2}\right)\\&=C\varepsilon^{-\frac14}|\log\varepsilon|^{\frac{3+\eta}{2}}\left(\|\mathcal{Q}_0(\mathbf{f},\mathbf{q})\|_{L^2(\Omega)}+\varepsilon^{-\frac14}\|Z^{\frac12}\mathcal{Q}_0(\mathbf{f},\mathbf{q})\|_{L^2(\Omega)}\right).\end{aligned}\tag{3.47}$$

Step 2*: L^∞-estimate*. By using (3.14) with $p=\infty$, the standard interpolation $\|f\|_{L^\infty}\le\sqrt{2}\|\partial_yf\|_{L^2}^{\frac12}\|f\|_{L^2}^{\frac12}$, and the estimate (3.34), we divide the estimation on $\|\mathbf{W}_n\|_{L^\infty}$ into two parts. First, for $1\le|n|\le\varepsilon^{-1}$,

$$\|\mathbf{W}_n\|_{L^\infty} \lesssim \|\widehat{\mathbf{W}}_n\|_{L^\infty} \le \sqrt{2}|\tilde{n}|^{-\frac{1}{2}}\varepsilon^{-\frac{1}{4}} \left(|\tilde{n}|^{\frac{1}{3}}\varepsilon^{\frac{1}{2}}\|\partial_y \widehat{\mathbf{W}}_n\|_{L^2}\right)^{\frac{1}{2}} \left(|\tilde{n}|^{\frac{2}{3}}\|\widehat{\mathbf{W}}_n\|_{L^2}\right)^{\frac{1}{2}}$$
$$\lesssim |n|^{-\frac{1}{2}}\varepsilon^{-\frac{1}{4}}|\log\varepsilon|^{1+\frac{\eta}{3}} \left(\|\mathbf{R}_n\|_{L^2} + \varepsilon^{-\frac{1}{4}}\|Z^{\frac{1}{2}}\mathbf{R}_n\|_{L^2}\right).$$

Second, for $|n| > \varepsilon^{-1}$,

$$\|\mathbf{W}_n\|_{L^\infty} \lesssim \|\widehat{\mathbf{W}}_n\|_{L^\infty} \le \sqrt{2}|\tilde{n}|^{-\frac{5}{6}}\varepsilon^{-\frac{1}{2}} \left(|\tilde{n}|^{\frac{1}{3}}\varepsilon^{\frac{1}{2}}\|\partial_y \widehat{\mathbf{W}}_n\|_{L^2}\right)^{\frac{1}{2}} \left(|\tilde{n}|^{\frac{4}{3}}\varepsilon^{\frac{1}{2}}\|\widehat{\mathbf{W}}_n\|_{L^2}\right)^{\frac{1}{2}}$$
$$\lesssim |n|^{-\frac{7}{12}}\varepsilon^{-\frac{1}{4}}|\log\varepsilon|^{1+\frac{\eta}{3}} \left(\|\mathbf{R}_n\|_{L^2} + \varepsilon^{-\frac{1}{4}}\|Z^{\frac{1}{2}}\mathbf{R}_n\|_{L^2}\right).$$

Thus, from the above two inequalities, it follows that by the Cauchy–Schwarz inequality,

(3.48)
$$\begin{aligned}\sum_{n\neq 0}\|\mathbf{W}_n\|_{L^\infty} \lesssim& \varepsilon^{-\frac{1}{4}}|\log\varepsilon|^{1+\frac{\eta}{3}} \sum_{1\le|n|\le\varepsilon^{-1}} |n|^{-\frac{1}{2}}\left(\|\mathbf{R}_n\|_{L^2} + \varepsilon^{-\frac{1}{4}}\|Z^{\frac{1}{2}}\mathbf{R}_n\|_{L^2}\right)\\
&+ \varepsilon^{-\frac{1}{4}}|\log\varepsilon|^{1+\frac{\eta}{3}} \sum_{|n|>\varepsilon^{-1}} |n|^{-\frac{7}{12}}\left(\|\mathbf{R}_n\|_{L^2} + \varepsilon^{-\frac{1}{4}}\|Z^{\frac{1}{2}}\mathbf{R}_n\|_{L^2}\right)\\
\lesssim& \varepsilon^{-\frac{1}{4}}|\log\varepsilon|^{1+\frac{\eta}{3}}\left(\left\|\left\{\|\mathbf{R}_n\|_{L^2}\right\}_{n\neq 0}\right\|_{l^2} + \varepsilon^{-\frac{1}{4}}\left\|\left\{\|Z^{\frac{1}{2}}\mathbf{R}_n\|_{L^2}\right\}_{n\neq 0}\right\|_{l^2}\right)\\
&\cdot\left[\left(\sum_{1\le|n|\le\varepsilon^{-1}}|n|^{-1}\right)^{\frac{1}{2}} + \left(\sum_{|n|>\varepsilon^{-1}}|n|^{-\frac{7}{6}}\right)^{\frac{1}{2}}\right]\\
\le& C\varepsilon^{-\frac{1}{4}}|\log\varepsilon|^{\frac{3}{2}+\frac{\eta}{3}}\left(\|\mathcal{Q}_0(\mathbf{f},\mathbf{q})\|_{L^2(\Omega)} + \varepsilon^{-\frac{1}{4}}\|Z^{\frac{1}{2}}\mathcal{Q}_0(\mathbf{f},\mathbf{q})\|_{L^2(\Omega)}\right),\end{aligned}$$

where we have used the third inequality in (3.15) and the Parseval equality in the last inequality. Finally, it is easy to obtain the desired estimate (3.5) from Lemma 3.2, (3.47), and (3.48). The proof of Proposition 3.1 is completed. ◻

4. Nonlinear stability. Recall the solution space $\mathcal{X}$ defined in (1.9). For any $(\mathbf{q},\mathbf{r}) \in \mathcal{X}$, we define the nonlinear map $\Phi(\mathbf{q},\mathbf{r}) = (\widetilde{\mathbf{U}},\widetilde{\mathbf{H}})$ as the solution to the following linear problem:

(4.1)
$$\begin{cases} U_s\partial_x\widetilde{\mathbf{U}} + \tilde{v}\partial_y U_s\mathbf{e}_1 - H_s\partial_x\widetilde{\mathbf{H}} - \tilde{g}\partial_y H_s\mathbf{e}_1 + \nabla P - \mu\varepsilon\Delta\widetilde{\mathbf{U}} = -\mathbf{q}\cdot\nabla\mathbf{q} + \mathbf{r}\cdot\nabla\mathbf{r} + \mathbf{f}_{\mathbf{U}},\\ U_s\partial_x\widetilde{\mathbf{H}} + \tilde{v}\partial_y H_s\mathbf{e}_1 - H_s\partial_x\widetilde{\mathbf{U}} - \tilde{g}\partial_y U_s\mathbf{e}_1 - \kappa\varepsilon\Delta\widetilde{\mathbf{H}} = -\mathbf{q}\cdot\nabla\mathbf{r} + \mathbf{r}\cdot\nabla\mathbf{q} + \mathbf{f}_{\mathbf{H}},\\ \nabla\cdot\widetilde{\mathbf{U}} = \nabla\cdot\widetilde{\mathbf{H}} = 0,\\ \widetilde{\mathbf{U}}|_{y=0} = (\partial_y\tilde{h},\tilde{g})|_{y=0} = \mathbf{0}.\end{cases}$$

The existence of solution operator Φ in $\mathcal{X}$ is guaranteed by Proposition 3.1 provided that the source term

$$(\tilde{\mathbf{f}}_{\mathbf{U}},\tilde{\mathbf{f}}_{\mathbf{H}}) \triangleq (-\mathbf{q}\cdot\nabla\mathbf{q} + \mathbf{r}\cdot\nabla\mathbf{r} + \mathbf{f}_{\mathbf{U}},\ -\mathbf{q}\cdot\nabla\mathbf{r} + \mathbf{r}\cdot\nabla\mathbf{q} + \mathbf{f}_{\mathbf{H}})$$

satisfies the compatibility conditions (3.3) and (3.4). Then the proof of the main result follows from showing the contractiveness of Φ in a suitable domain of $\mathcal{X}$ provided that the external force $(\mathbf{f}_{\mathbf{U}},\mathbf{f}_{\mathbf{H}})$ is suitably small. Consequently, it remains to verify (3.3) and (3.4) for $(\tilde{\mathbf{f}}_{\mathbf{U}},\tilde{\mathbf{f}}_{\mathbf{H}})$ and to prove Φ is a contraction map.

For $(\mathbf{q},\mathbf{r}) \in \mathcal{X}$, direct calculation shows that $(\tilde{\mathbf{f}}_{\mathbf{U}},\tilde{\mathbf{f}}_{\mathbf{H}})$ satisfies (3.3). To show (3.4) for $(\tilde{\mathbf{f}}_{\mathbf{U}},\tilde{\mathbf{f}}_{\mathbf{H}})$, let us recall the projections on the zeroth Fourier mode $\mathcal{P}_0$ and on nonzero Fourier mode $\mathcal{Q}_0$. Let $\mathbf{s}=(s_1,s_2)$ and $\mathbf{t}=(t_1,t_2)$ be any two divergence-free vectors satisfying boundary condition $s_2|_{y=0}=t_2|_{y=0}=0$. Then $\mathcal{P}_0\mathbf{s}$ and $\mathcal{P}_0\mathbf{t}$ depend only on y and $\mathcal{P}_0 s_2=\mathcal{P}_0 t_2=0$, which implies that

$$\begin{aligned}\mathbf{s}\cdot\nabla\mathbf{t}&=\mathcal{P}_0\mathbf{s}\cdot\nabla\mathcal{P}_0\mathbf{t}+\mathcal{P}_0\mathbf{s}\cdot\nabla\mathcal{Q}_0\mathbf{t}+\mathcal{Q}_0\mathbf{s}\cdot\nabla\mathcal{P}_0\mathbf{t}+\mathcal{Q}_0\mathbf{s}\cdot\nabla\mathcal{Q}_0\mathbf{t}\\&=\mathcal{P}_0 s_1\partial_x\mathcal{Q}_0\mathbf{t}+\mathcal{Q}_0 s_2(\partial_y\mathcal{P}_0 t_1)\mathbf{e}_1+\mathcal{Q}_0\mathbf{s}\cdot\nabla\mathcal{Q}_0\mathbf{t}.\end{aligned}\tag{4.2}$$

Observe that

$$\begin{aligned}\mathcal{P}_0(\mathbf{s}\cdot\nabla\mathbf{t})&=\mathcal{P}_0\left(\mathcal{Q}_0\mathbf{s}\cdot\nabla\mathcal{Q}_0\mathbf{t}\right)=\partial_y\mathcal{P}_0\left(\mathcal{Q}_0 s_2\mathcal{Q}_0\mathbf{t}\right)+\mathcal{P}_0\left(\mathcal{Q}_0 s_1\partial_x\mathcal{Q}_0\mathbf{t}-\partial_y\mathcal{Q}_0 s_2\mathcal{Q}_0\mathbf{t}\right)\\&=\partial_y\mathcal{P}_0\left(\mathcal{Q}_0 s_2\mathcal{Q}_0\mathbf{t}\right)+\mathcal{P}_0\left(\sum_{n\neq0,m\neq0}e^{i(\tilde n+\tilde m)x}\left(s_{1,n}i\tilde m-\partial_y s_{2,n}\right)\mathbf{t}_m\right)\\&=\partial_y\mathcal{P}_0\left(\mathcal{Q}_0 s_2\mathcal{Q}_0\mathbf{t}\right)-\sum_{n\neq0}(i\tilde n s_{1,n}+\partial_y s_{2,n})\mathbf{t}_{-n}=\partial_y\mathcal{P}_0\left(\mathcal{Q}_0 s_2\mathcal{Q}_0\mathbf{t}\right).\end{aligned}$$

Applying the above equality to $(\tilde{\mathbf{f}}_{\mathbf{U}},\tilde{\mathbf{f}}_{\mathbf{H}})$ yields

$$\left(\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{U}},\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{H}}\right)=\left(\partial_y\mathcal{P}_0\big(-\mathcal{Q}_0q_2\mathcal{Q}_0\mathbf{q}+\mathcal{Q}_0r_2\mathcal{Q}_0\mathbf{r}\big),\,\partial_y\mathcal{P}_0\big(-\mathcal{Q}_0q_2\mathcal{Q}_0\mathbf{r}+\mathcal{Q}_0r_2\mathcal{Q}_0\mathbf{q}\big)\right),$$

and then

$$\left(\mathcal{I}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{U}},\,\partial_y^{-1}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{H}}\right)=\left(\mathcal{P}_0\big(-\mathcal{Q}_0q_2\mathcal{Q}_0\mathbf{q}+\mathcal{Q}_0r_2\mathcal{Q}_0\mathbf{r}\big),\,\mathcal{P}_0\big(-\mathcal{Q}_0q_2\mathcal{Q}_0\mathbf{r}+\mathcal{Q}_0r_2\mathcal{Q}_0\mathbf{q}\big)\right).$$

Since $|\mathcal{P}_0 f|\lesssim\|f\|_{L^1(\mathbb{T}_\varrho)}$, by the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}&\left\|\left(\mathcal{I}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{U}},\,\partial_y^{-1}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{H}}\right)\right\|_{L^1(\mathbb{R}_+)}\\&\quad\lesssim\|-\mathcal{Q}_0q_2\mathcal{Q}_0\mathbf{q}+\mathcal{Q}_0r_2\mathcal{Q}_0\mathbf{r}\|_{L^1(\Omega)}+\|-\mathcal{Q}_0q_2\mathcal{Q}_0\mathbf{r}+\mathcal{Q}_0r_2\mathcal{Q}_0\mathbf{q}\|_{L^1(\Omega)}\\&\quad\lesssim\|\mathcal{Q}_0(\mathbf{q},\mathbf{r})\|^2_{L^2(\Omega)}\lesssim\varepsilon^{\frac12}\|(\mathbf{q},\mathbf{r})\|^2_{\mathcal{X}},\end{aligned}\tag{4.3}$$

and by the Parseval equality,

$$\begin{aligned}&\left\|\left(\mathcal{I}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{U}},\,\partial_y^{-1}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{H}}\right)\right\|_{L^2(\mathbb{R}_+)}\\&\quad\le\|-\mathcal{Q}_0q_2\mathcal{Q}_0\mathbf{q}+\mathcal{Q}_0r_2\mathcal{Q}_0\mathbf{r}\|_{L^2(\Omega)}+\|-\mathcal{Q}_0q_2\mathcal{Q}_0\mathbf{r}+\mathcal{Q}_0r_2\mathcal{Q}_0\mathbf{q}\|_{L^2(\Omega)}\\&\quad\lesssim\|\mathcal{Q}_0(\mathbf{q},\mathbf{r})\|_{L^\infty(\Omega)}\|\mathcal{Q}_0(\mathbf{q},\mathbf{r})\|_{L^2(\Omega)}\lesssim\varepsilon^{\frac14}\|(\mathbf{q},\mathbf{r})\|^2_{\mathcal{X}},\end{aligned}\tag{4.4}$$

where we have used the fact that $\|\mathcal{Q}_0 f\|_{L^\infty(\Omega)}\le C\sum_{n\neq0}\|f_n\|_{L^\infty(\mathbb{R}_+)}$. Thus we verify the first part of (3.4). Moreover, by the commutativity of the weight $Z^{\frac12}$ and the projection operators $\mathcal{P}_0,\mathcal{Q}_0$, similar to (4.4), it holds that

$$\left\|Z^{\frac12}\left(\mathcal{I}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{U}},\,\partial_y^{-1}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{H}}\right)\right\|_{L^2(\mathbb{R}_+)}\lesssim\|\mathcal{Q}_0(\mathbf{q},\mathbf{r})\|_{L^\infty(\Omega)}\left\|Z^{\frac12}\mathcal{Q}_0(\mathbf{q},\mathbf{r})\right\|_{L^2(\Omega)}\lesssim\varepsilon^{\frac12}\|(\mathbf{q},\mathbf{r})\|^2_{\mathcal{X}}.\tag{4.5}$$

For the second part of (3.4), we use (4.2) to have

$$\begin{aligned}\|\mathcal{Q}_0(\mathbf{s}\cdot\nabla\mathbf{t})\|_{L^2(\Omega)}&\le\|\mathbf{s}\cdot\nabla\mathbf{t}\|_{L^2(\Omega)}\\&\lesssim\left(\|\mathcal{P}_0 s_1\|_{L^\infty(\mathbb{R}_+)}+\|\mathcal{Q}_0\mathbf{s}\|_{L^\infty(\Omega)}\right)\|\nabla\mathcal{Q}_0\mathbf{t}\|_{L^2(\Omega)}\\&\quad+\|\mathcal{Q}_0 s_2\|_{L^\infty(\Omega)}\|\partial_y\mathcal{P}_0 t_1\|_{L^2(\mathbb{R}_+)}.\end{aligned}$$

Apply the above inequality to $\mathcal{Q}_0(\tilde{\mathbf{f}}_{\mathbf{U}},\tilde{\mathbf{f}}_{\mathbf{H}})$ and obtain

(4.6)
$$\begin{aligned}\left\|\mathcal{Q}_0(\tilde{\mathbf{f}}_{\mathbf{U}},\tilde{\mathbf{f}}_{\mathbf{H}})\right\|_{L^2(\Omega)} &\lesssim \Big(\|\mathcal{P}_0(\mathbf{q},\mathbf{r})\|_{L^\infty(\mathbb{R}_+)}+\|\mathcal{Q}_0(\mathbf{q},\mathbf{r})\|_{L^\infty(\Omega)}\Big)\|\nabla\mathcal{Q}_0(\mathbf{q},\mathbf{r})\|_{L^2(\Omega)}\\ &\quad+\|\mathcal{Q}_0(\mathbf{q},\mathbf{r})\|_{L^\infty(\Omega)}\|\partial_y\mathcal{P}_0(\mathbf{q},\mathbf{r})\|_{L^2(\mathbb{R}_+)}+\|\mathcal{Q}_0(\mathbf{f}_{\mathbf{U}},\mathbf{f}_{\mathbf{H}})\|_{L^2(\Omega)}\\ &\lesssim \varepsilon^{-\frac14}\|(\mathbf{q},\mathbf{r})\|_{\mathcal{X}}^2+\|(\mathbf{f}_{\mathbf{U}},\mathbf{f}_{\mathbf{H}})\|_{L^2(\Omega)},\end{aligned}$$

which implies the second part of (3.4). Moreover, we can show that

(4.7)
$$\begin{aligned}\left\|Z^{\frac12}\mathcal{Q}_0(\tilde{\mathbf{f}}_{\mathbf{U}},\tilde{\mathbf{f}}_{\mathbf{H}})\right\|_{L^2(\Omega)} &\lesssim \Big(\|\mathcal{P}_0(\mathbf{q},\mathbf{r})\|_{L^\infty(\mathbb{R}_+)}+\|\mathcal{Q}_0(\mathbf{q},\mathbf{r})\|_{L^\infty(\Omega)}\Big)\|Z^{\frac12}\nabla\mathcal{Q}_0(\mathbf{q},\mathbf{r})\|_{L^2(\Omega)}\\ &\quad+\|\mathcal{Q}_0(\mathbf{q},\mathbf{r})\|_{L^\infty(\Omega)}\|Z^{\frac12}\partial_y\mathcal{P}_0(\mathbf{q},\mathbf{r})\|_{L^2(\mathbb{R}_+)}+\left\|Z^{\frac12}\mathcal{Q}_0(\mathbf{f}_{\mathbf{U}},\mathbf{f}_{\mathbf{H}})\right\|_{L^2(\Omega)}\\ &\lesssim \|(\mathbf{q},\mathbf{r})\|_{\mathcal{X}}^2+\left\|Z^{\frac12}(\mathbf{f}_{\mathbf{U}},\mathbf{f}_{\mathbf{H}})\right\|_{L^2(\Omega)}.\end{aligned}$$

Next, we apply Proposition 3.1 to the problem (4.1) and obtain

$$\begin{aligned}&\|\Phi(\mathbf{q},\mathbf{r})\|_{\mathcal{X}}=\|(\widetilde{\mathbf{U}},\widetilde{\mathbf{H}})\|_{\mathcal{X}}\\ &\lesssim\varepsilon^{-1}\Big(\left\|\Big(\mathcal{I}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{U}},\,\partial_y^{-1}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{H}}\Big)\right\|_{L^1}+\varepsilon^{\frac14}\left\|\Big(\mathcal{I}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{U}},\,\partial_y^{-1}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{H}}\Big)\right\|_{L^2}\\ &\quad+\left\|Z^{\frac12}\Big(\mathcal{I}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{U}},\,\partial_y^{-1}\mathcal{P}_0\tilde{\mathbf{f}}_{\mathbf{H}}\Big)\right\|_{L^2}\Big)\\ &\quad+\varepsilon^{-\frac14}|\log\varepsilon|^{\frac{3+\eta}{2}}\Big(\|\mathcal{Q}_0(\tilde{\mathbf{f}}_{\mathbf{U}},\tilde{\mathbf{f}}_{\mathbf{H}})\|_{L^2(\Omega)}+\varepsilon^{-\frac14}\|Z^{\frac12}\mathcal{Q}_0(\tilde{\mathbf{f}}_{\mathbf{U}},\tilde{\mathbf{f}}_{\mathbf{H}})\|_{L^2(\Omega)}\Big).\end{aligned}$$

Combining the above inequality and the estimates (4.3)–(4.7) yields

$$\begin{aligned}\|\Phi(\mathbf{q},\mathbf{r})\|_{\mathcal{X}} &\lesssim \varepsilon^{-\frac12}\|(\mathbf{q},\mathbf{r})\|_{\mathcal{X}}^2+\varepsilon^{-\frac14}|\log\varepsilon|^{\frac{3+\eta}{2}}\\ &\quad\times\left(\varepsilon^{-\frac14}\|(\mathbf{q},\mathbf{r})\|_{\mathcal{X}}^2+\|(\mathbf{f}_{\mathbf{U}},\mathbf{f}_{\mathbf{H}})\|_{L^2(\Omega)}+\varepsilon^{-\frac14}\left\|Z^{\frac12}(\mathbf{f}_{\mathbf{U}},\mathbf{f}_{\mathbf{H}})\right\|_{L^2(\Omega)}\right)\\ &\le C\varepsilon^{-\frac12}|\log\varepsilon|^{\frac{3+\eta}{2}}\|(\mathbf{q},\mathbf{r})\|_{\mathcal{X}}^2+C\varepsilon^{-\frac14}|\log\varepsilon|^{\frac{3+\eta}{2}}\\ &\quad\times\left(\|(\mathbf{f}_{\mathbf{U}},\mathbf{f}_{\mathbf{H}})\|_{L^2(\Omega)}+\varepsilon^{-\frac14}\left\|Z^{\frac12}(\mathbf{f}_{\mathbf{U}},\mathbf{f}_{\mathbf{H}})\right\|_{L^2(\Omega)}\right),\end{aligned}\tag{4.8}$$

where the constant $C>0$ is independent of ε. Therefore, (4.8) shows that the map Φ is well-defined from $\mathcal{X}$ to $\mathcal{X}$. Moreover, by a similar argument as above we can show that for any two vectors $(\mathbf{q}_1,\mathbf{r}_1),(\mathbf{q}_2,\mathbf{r}_2)\in\mathcal{X}$, it holds that

$$\|\Phi(\mathbf{q}_1-\mathbf{q}_2,\mathbf{r}_1-\mathbf{r}_2)\|_{\mathcal{X}}\le C\varepsilon^{-\frac12}|\log\varepsilon|^{\frac{3+\eta}{2}}\big(\|(\mathbf{q}_1,\mathbf{r}_1)\|_{\mathcal{X}}+\|(\mathbf{q}_2,\mathbf{r}_2)\|_{\mathcal{X}}\big)\big\|(\mathbf{q}_1-\mathbf{q}_2,\mathbf{r}_1-\mathbf{r}_2)\big\|_{\mathcal{X}}.$$

Now, we are able to choose suitable $(\mathbf{f}_{\mathbf{U}},\mathbf{f}_{\mathbf{H}})$ to establish the contractiveness of map Φ in a suitable domain of $\mathcal{X}$. Indeed, for any fixed $0<\alpha<1$, let

$$\|(\mathbf{f}_{\mathbf{U}},\mathbf{f}_{\mathbf{H}})\|_{L^2}+\varepsilon^{-\frac14}\|Z^{\frac12}(\mathbf{f}_{\mathbf{U}},\mathbf{f}_{\mathbf{H}})\|_{L^2(\Omega)}\le\delta_2\varepsilon^{\frac34}|\log\varepsilon|^{-(3+\eta)}\quad\text{with}\quad\delta_2=\frac{\alpha(2-\alpha)}{4C^2},$$

and we consider the domain of $\mathcal{X}$:

$$\mathcal{D}:=\left\{(\widetilde{\mathbf{U}},\widetilde{\mathbf{H}})\in\mathcal{X}\;\middle|\;\big\|(\widetilde{\mathbf{U}},\widetilde{\mathbf{H}})\big\|_{\mathcal{X}}\le\frac{\alpha\varepsilon^{\frac12}}{2C|\log\varepsilon|^{\frac{3+\eta}{2}}}\right\}.$$

It is straightforward to check that Φ is a contraction map from $\mathcal{D}$ to $\mathcal{D}$. Therefore, the existence and uniqueness of the solution to (1.8) follow from the fixed point theorem. In addition, the solution $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{H}})$ satisfies

$$\|(\widetilde{\mathbf{U}}, \widetilde{\mathbf{H}})\|_{\mathcal{X}} \leq \frac{2C}{2-\alpha}\varepsilon^{-\frac{1}{4}}|\log\varepsilon|^{\frac{3+\eta}{2}}\left(\|(\mathbf{f_U}, \mathbf{f_H})\|_{L^2(\Omega)} + \varepsilon^{-\frac{1}{4}}\|Z^{\frac{1}{2}}(\mathbf{f_U}, \mathbf{f_H})\|_{L^2(\Omega)}\right).$$

That is, we obtain (1.12). Finally, it is easy to see that $-\mu\varepsilon\Delta\widetilde{\mathbf{U}} + \nabla P \in L^2(\Omega)$ and $-\kappa\varepsilon\Delta\widetilde{\mathbf{H}} \in L^2(\Omega)$. Then by ellipticity of Stokes operators and Laplacian operators, we have $\nabla^2\widetilde{\mathbf{U}}, \nabla^2\widetilde{\mathbf{H}}, \nabla P \in L^2(\Omega)$. Therefore, the proof of Theorem 1.1 is completed.

5. Appendix.

We give the detailed proof of Lemma 3.3 as follows.

Proof of Lemma 3.3. We focus on the estimates on $\hat{h}_n$ since other components can be treated in a similar way. Recall that in (3.11)

$$\hat{h}_n = \partial_y\left(\frac{\psi_n}{H_s}\right) = \frac{1}{H_s}\left(h_n - \varepsilon^{-\frac{1}{2}}b_p\psi_n\right). \tag{5.1}$$

It follows by the Hardy inequality that

$$\|\hat{h}_n\|_{L^p} \leq C\|h_n\|_{L^p} + C\|Yb_p\|_{L^\infty_Y}\|y^{-1}\psi_n\|_{L^p} \leq C(1+\bar{M})\|h_n\|_{L^p}. \tag{5.2}$$

For the weighted L^2-norm, one has by using (2.4) that

$$\begin{aligned}\|Z^{\frac{1}{2}}\hat{h}_n\|_{L^2} &\leq C\|Z^{\frac{1}{2}}h_n\|_{L^2} + C\varepsilon^{\frac{1}{4}}\left\|\sqrt{\frac{Z(y)}{y}}\right\|_{L^\infty}\|Y^{\frac{3}{2}}b_p\|_{L^\infty_Y}\|y^{-1}\psi_n\|_{L^2}\\ &\leq C\|Z^{\frac{1}{2}}h_n\|_{L^2} + C\bar{M}\varepsilon^{\frac{1}{4}}\|h_n\|_{L^2},\end{aligned} \tag{5.3}$$

and by virtue of $g_n = -i\tilde{n}\psi_n$,

$$\begin{aligned}\varepsilon^{\frac{1}{4}}|\tilde{n}|\|Z^{\frac{1}{2}}\hat{h}_n\|_{L^2} &\leq C\varepsilon^{\frac{1}{4}}|\tilde{n}|\|Z^{\frac{1}{2}}h_n\|_{L^2} + C\left\|\sqrt{\frac{Z(y)}{y}}\right\|_{L^\infty}\|Y^{\frac{1}{2}}b_p\|_{L^\infty_Y}\|g_n\|_{L^2}\\ &\leq C\varepsilon^{\frac{1}{4}}|\tilde{n}|\|Z^{\frac{1}{2}}h_n\|_{L^2} + C\bar{M}\|g_n\|_{L^2}.\end{aligned} \tag{5.4}$$

Then, taking the y-derivative in (5.1) gives

$$\partial_y\hat{h}_n = \frac{1}{H_s}\left(\partial_y h_n - 2\varepsilon^{-\frac{1}{2}}b_p h_n + \varepsilon^{-1}(b_p^2 - \partial_Y b_p)\psi_n\right). \tag{5.5}$$

It follows that

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|\partial_y\hat{h}_n\|_{L^2} &\leq C\varepsilon^{\frac{1}{2}}\|\partial_y h_n\|_{L^2} + C\|b_p\|_{L^\infty_Y}\|h_n\|_{L^2} + C\left\|Y(b_p^2 - \partial_Y b_p)\right\|_{L^\infty_Y}\|y^{-1}\psi_n\|_{L^2}\\ &\leq C\varepsilon^{\frac{1}{2}}\|\partial_y h_n\|_{L^2} + C(1+\bar{M}^2)\|h_n\|_{L^2}\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}&\varepsilon^{\frac{1}{4}}\|Z^{\frac{1}{2}}\partial_y\hat{h}_n\|_{L^2}\\ &\leq C\varepsilon^{\frac{1}{4}}\|Z^{\frac{1}{2}}\partial_y h_n\|_{L^2}\\ &\quad + C\left\|\sqrt{\frac{Z(y)}{y}}\right\|_{L^\infty}\left[\|Y^{\frac{1}{2}}b_p\|_{L^\infty_Y}\|h_n\|_{L^2} + \left\|Y^{\frac{3}{2}}(b_p^2 - \partial_Y b_p)\right\|_{L^\infty_Y}\|y^{-1}\psi_n\|_{L^2}\right]\\ &\leq C\varepsilon^{\frac{1}{4}}\|Z^{\frac{1}{2}}\partial_y h_n\|_{L^2} + C(1+\bar{M}^2)\|h_n\|_{L^2}.\end{aligned} \tag{5.7}$$

In conclusion, we combine (5.2), (5.3), (5.4), (5.6), and (5.7) to get

(5.8)
$$\begin{aligned}
&\|\hat{h}_n\|_{L^p} \lesssim_{\bar{M}} \|h_n\|_{L^p}, \quad 1 < p \le \infty,\\
&\|Z^{\frac12}\hat{h}_n\|_{L^2} + \varepsilon^{\frac14}\|\hat{h}_n\|_{L^2} \lesssim_{\bar{M}} \left(\|Z^{\frac12}h_n\|_{L^2} + \varepsilon^{\frac14}\|h_n\|_{L^2}\right),\\
&\|\hat{h}_n\|_{L^2} + \varepsilon^{\frac12}\|\partial_y\hat{h}_n\|_{L^2} \lesssim_{\bar{M}} \left(\|h_n\|_{L^2} + \varepsilon^{\frac12}\|\partial_y h_n\|_{L^2}\right),\\
&\|\hat{h}_n\|_{L^2} + \varepsilon^{\frac14}\|Z^{\frac12}(\partial_y\hat{h}_n, \tilde{n}\hat{h}_n)\|_{L^2} \lesssim_{\bar{M}} \left(\|(h_n, g_n)\|_{L^2} + \varepsilon^{\frac14}\|Z^{\frac12}(\partial_y h_n, \tilde{n}h_n)\|_{L^2}\right).
\end{aligned}$$

On the other hand, we can express h_n in terms of $\hat{h}_n$. Indeed, from (5.1) one has $\psi_n = H_s\partial_y^{-1}\hat{h}_n$, and then

(5.9)
$$h_n = H_s\left(\hat{h}_n + \varepsilon^{-\frac12}b_p\partial_y^{-1}\hat{h}_n\right), \quad \partial_y h_n = H_s\left(\partial_y h_n + 2\varepsilon^{-\frac12}b_p h_n + \varepsilon^{-1}(b_p^2 + \partial_Y b_p)\psi_n\right).$$

Comparing (5.1), (5.5) with (5.9) and noting $\psi_n = \partial_y^{-1}h_n$, we use a similar argument as above to obtain

(5.10)
$$\begin{aligned}
&\|h_n\|_{L^p} \lesssim_{\bar{M}} \|\hat{h}_n\|_{L^p}, \quad 1 < p \le \infty,\\
&\|Z^{\frac12}h_n\|_{L^2} + \varepsilon^{\frac14}\|h_n\|_{L^2} \lesssim_{\bar{M}} \left(\|Z^{\frac12}\hat{h}_n\|_{L^2} + \varepsilon^{\frac14}\|\hat{h}_n\|_{L^2}\right),\\
&\|h_n\|_{L^2} + \varepsilon^{\frac12}\|\partial_y h_n\|_{L^2} \lesssim_{\bar{M}} \left(\|\hat{h}_n\|_{L^2} + \varepsilon^{\frac12}\|\partial_y\hat{h}_n\|_{L^2}\right),\\
&\|h_n\|_{L^2} + \varepsilon^{\frac14}\|Z^{\frac12}(\partial_y h_n, \tilde{n}h_n)\|_{L^2} \lesssim_{\bar{M}} \left(\|(\hat{h}_n, \hat{g}_n)\|_{L^2} + \varepsilon^{\frac14}\|Z^{\frac12}(\partial_y\hat{h}_n, \tilde{n}\hat{h}_n)\|_{L^2}\right).
\end{aligned}$$

Combining (5.8) with (5.10), we complete the proof of the lemma. ∎

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