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A Rearrangement Minimization Problem Related to a Nonlinear Parametric Boundary Value Problem [1]

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Abstract: This paper deals with a rearrangement minimization problem, which is associated with a nonlinear parametric boundary value problem. When the parameter is positive and less than the principal eigenvalue of the p-Laplacian type operator, we obtain that the nonlinear parametric boundary value problem has a unique solution. We then establish the solvability of the rearrangement minimization problem. Finally, based on Stampacchia truncation method, we establish the regularity property of the solution to the nonlinear boundary value problem, and then we investigate the symmetric property of the solution to the rearrangement minimization problem when the domain is a ball at the origin.

Keywords: Parameter, p-Laplacian type operator, Energy functional, Rearrangement functions, Optimization problem.

1. Introduction

Consider the classical Poisson's problem:

$$\begin{cases} -\Delta u = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Δ is the Laplace operator, Ω is a smooth bounded domain of $\mathbb{R}^N$ $(N \geq 2)$, $g : \Omega \to \mathbb{R}$ is a measurable function and u is a real-valued function. In the case of $N = 2$, problem (1) models the deformation of an isotropic elastic membrane that is hinged along the boundary under the vertical force $g(x)$.

The energy functional associated with (1) is defined by

$$\Psi(g) = \int_\Omega g u_g dx \tag{2}$$

where u_g denotes the unique solution of (1). Let O be a measurable subset of Ω, denote $|O| = \text{meas}(O)$ as the measure of O. Let e_0 be a measurable function on Ω. If the measurable function e on Ω satisfies

$$|\{x \in \Omega : e(x) \geq a\}| = |\{x \in \Omega : e_0(x) \geq a\}|, \quad \forall a \in \mathbb{R}, \tag{3}$$

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then e is called a rearrangement of e_0. Denote $\mathscr{R}(e_0)$ as the set of all rearrangement functions of e_0.

There are two rearrangement optimization problems related to (1):

$$(\text{Opt}_1) \qquad \sup_{e\in\mathscr{R}(g)} \Psi(e)$$

and

$$(\text{Opt}_2) \qquad \inf_{e\in\mathscr{R}(g)} \Psi(e),$$

where $\Psi(e)$ is defined by (2) by changing g to e.

Burton [5, 6] investigated the maximization problem (Opt_1) and the minimization problem (Opt_2) and proved that both problems (Opt_1) and (Opt_2) have solutions. Since then, the rearrangement optimization problems related to semilinear and quasilinear elliptic boundary value problems have been extensively investigated, see the references [9, 11, 18, 20, 21]. The rearrangement optimization problems related to the nonlocal operators have been investigated by [3, 4, 22]. Qiu et. al studied the minimization and maximization rearrangement optimization problems related to quasilinear elliptic boundary value problems in [21]. Liu and Emamizadeh investigated three rearrangement optimization problems where the energy functional is connected with the Dirichlet or Robin boundary value problems in [18]. Emamizadeh et.al obtained a comparison result for solutions to (p,q)-Laplace equation via Schwarz symmetrization in [10]. Besides rearrangement optimization problems, the other optimization problems which are related to boundary value problems or variational inequality problems involving p-Laplacian have been investigated by many authors, see, for example, [7, 19, 26, 27].

Recently, we have considered the rearrangement maximization and minimization problems related to the following boundary value problem in [23]:

$$(\mathscr{P}_0) \qquad \begin{cases} \text{div}A(-\nabla u) + w(x,u) = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the operator $\text{div}A(-\nabla u)$ is a p-Laplacian type operator $(p > N)$ introduced in [2] and defined in details in Section 2. The way to get the solvability of the optimization problems related to $(\mathscr{P}_0)$ in [23] is essentially based on truncated function method, the peculiarity of the p-Laplacian type operator, the property of the rearrangement function, together with the fact that the imbedding $\mathbf{X} \to C(\bar{\Omega})$ is compact in the case of $p > N$, where $\mathbf{X}$ is a Sobolev space which is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = (\int_\Omega |\nabla u|^p dx)^{1/p}$.

In the present paper, we will consider a rearrangement minimization problem related to the following nonlinear parametric boundary value problem:

$$(\mathscr{P}_\mu) \qquad \begin{cases} \text{div}A(-\nabla u) - \mu V(x)|u|^{p-2}u + w(x,u) = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ is a parameter, $w(x,t) : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function, $V \in L^\infty(\Omega)$, $g \in L^q(\Omega)$ with some $1 \leq q < \infty$ and $\text{div}A(-\nabla u)$ is a p-Laplacian type operator $(1 < p < N)$. It is easy to see that (1) and the boundary value problems involving the p-Laplacian are all special cases of the problem $(\mathscr{P}_\mu)$ with $\mu = 0$, $w = 0$ and $\mathrm{div}A(-\nabla u) = -\Delta u$, or $w \neq 0$ and $\mathrm{div}A(-\nabla u) = -\Delta_p u$.

For a fixed $g \in L^q(\Omega)$, when $0 < \mu < \mu_0$ (the principal eigenvalue related to the p-Laplacian type operator), we will prove that $(\mathscr{P}_\mu)$ has a unique solution which is denoted by u_g (cf. Theorem 3..1 in Section 3.).

We are interested in the following rearrangement minimization problem related to the nonlinear parametric boundary value problem $(\mathscr{P}_\mu)$:

$$(\mathscr{R}mp) \qquad \inf_{e \in \mathscr{R}(g)} \Phi(e),$$

where $\mathscr{R}(g)$ denotes a rearrangement class generated by g and $\Phi(e)$ is defined as

$$\Phi(e) = \frac{1}{p}\int_\Omega A(-\nabla u_e)(-\nabla u_e)dx - \frac{\mu}{p}\int_\Omega V(x)|u_e|^p dx + \int_\Omega W(x,u_e)dx - \int_\Omega e u_e dx, \tag{4}$$

where u_e is the corresponding solution of $(\mathscr{P}_\mu)$ when the data function g is changed to $e \in \mathscr{R}(g)$ and $W(x,u) = \int_0^u w(x,t)dt$.

We shall point out that the rearrangement minimization problem $(\mathscr{R}mp)$ related to the boundary value problem $(\mathscr{P}_\mu)$, has been well investigated in the case of $\mu = 0$, $\mathrm{div}A(-\nabla u) = -\Delta_p u$, and $w = 0$ or $w \neq 0$, cf. [5,6,9,20,21]. In the case of $\mu \neq 0$, $\mathrm{div}A(-\nabla u) = -\Delta_p u$, $w = 0$ and $g = 0$, $(\mathscr{P}_\mu)$ is actually an eigenvalue problem, the rearrangement optimization problems related to the principal eigenvalues of p-Laplace eigenvalue problems have been studied in [1,8]. However, so far the problem $(\mathscr{R}mp)$ related to the boundary value problem $(\mathscr{P}_\mu)$ has not been studied yet in the case of $\mu \neq 0$, $w \neq 0$ and $g \neq 0$.

The aim of the present paper is to investigate the rearrangement minimization problem $(\mathscr{R}mp)$ in the case of $1 < p < N$ and $0 < \mu < \mu_0$ for some positive parameter μ_0 or $\mu = 0$. Noting that the imbedding $\mathbf{X} \to C(\bar{\Omega})$ is compact in the case of $p > N$(cf. [23]), which is not valid again in the case of $1 < p < N$ and the methods to obtain the uniqueness result of $(\mathscr{P}_\mu)$ mentioned above for $\mu = 0$ is not available for the case of $\mu \neq 0$. So that we have to overcome these difficulties by some new tricks. Firstly, by exploring some variational methods and techniques, together with the peculiarity of $\mu_0 > 0$, we show that the problem $(\mathscr{P}_\mu)$ has exact one solution. Then we obtain the solvability of the problem $(\mathscr{R}mp)$. Finally, based on Stampacchia truncation method, we establish the regularity property of the solution to the problem $(\mathscr{P}_0)$. Then by the symmetric property of the domain Ω, we prove that the solution of $(\mathscr{R}mp)$ has the Schwarz symmetric property.

This paper is organized as follows. In Section 2, we present some preliminaries, which will be used in sequel. In Section 3, for a fixed $g \in L^q(\Omega)$, when $0 < \mu < \mu_0$ (the principal eigenvalue related to the p-Laplacian type operator), we show that the nonlinear parametric boundary value problem $(\mathscr{P}_\mu)$ is solvable and has a unique solution. Section 4 is devoted to the solvability of the rearrangement minimization problem $(\mathscr{R}mp)$. In Section 5, by exploring Stampacchia truncation method, we establish the regularity property of the solution to the problem $(\mathscr{P}_0)$. We also show that, the solution of the minimization problem $(\mathscr{R}mp)$ related to $(\mathscr{P}_0)$ has the symmetric property if the domain has the symmetric property.

2. Preliminaries

Let e_0 be a measurable function defined in a smooth bounded domain Ω of $\mathbb{R}^N$ $(N \geq 2)$. We denote by $\mathscr{R}(e_0)$ the set of all the rearrangements of e_0, $\rightharpoonup$ and $\rightarrow$ for the weak convergence and strong convergence respectively. Let $1 < p < N$. The critical Sobolev exponent is given by $p^* = Np/(N-p)$. Throughout this paper we always suppose that $\alpha : \mathbb{R}^N \mapsto [0, \infty)$ is a continuous differentiable convex function satisfying

$$\alpha(t\xi) = t\alpha(\xi) \text{ for } t > 0 \text{ and } \xi \in \mathbb{R}^N. \tag{5}$$

Assume that $A(0) = 0$, $A(\xi) = \alpha^{p-1}(\xi)\nabla\alpha(\xi)$ for $\xi \in \mathbb{R}^N \backslash \{0\}$, and there exist positive constants Γ and γ such that

$$(A(\xi) - A(\eta)) \cdot (\xi - \eta) \geq \gamma(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \tag{6}$$

$$|A(\xi) - A(\eta)| \leq \Gamma(|\xi| + |\eta|)^{p-2}|\xi - \eta| \tag{7}$$

for all $\xi, \eta \in R^N$. It follows from (6) and (7) that

$$\gamma|\xi|^p \leq A(\xi) \cdot \xi \leq \Gamma|\xi|^p \tag{8}$$

for all $\xi \in R^N$.

Remark 2..1. *Note that* (8) *holds, we may throughout this paper consider that the Sobolev space* $\mathbf{X}$ *is the completion of* $C_0^\infty(\Omega)$ *with respect to the norm*

$$\|u\| := \Big(\int_\Omega A(-\nabla u) \cdot (-\nabla u)\, dx \Big)^{1/p}. \tag{9}$$

Remark 2..2. *Let* $\alpha(\xi) = (\sum_{i=1}^n c_i|\xi_i|^s)^{1/s}$ *where* $c_i > 0, s \geq 2$. *Then* α *satisfying* (5). *Moreover, if* $\alpha \in C^2(\mathbb{R}^N \backslash \{0\})$ *satisfies* (5) *and if there exists* $\sigma > 0$ *such that*

$$\sum_{i,j=1}^n \frac{\partial^2\alpha(\xi)}{\partial\xi_i\partial\xi_j}\eta_i\eta_j \geq \sigma|\eta| \quad \textit{whenever} \quad \alpha(\xi) = 1 \quad \textit{and} \quad \nabla\alpha(\xi)\cdot\eta = 0,$$

then A *satisfies* (6) *and* (7).

Definition 2..1. *We say that the problem* $(\mathscr{P}_\mu)$ *has a solution* $u \in \mathbf{X}$, *if*

$$\int_\Omega \big(A(-\nabla u)(-\nabla v) - \mu V(x)|u|^{p-2}uv + w(x,u)v - gv\big)\, dx = 0,\ \forall v \in \mathbf{X}.$$

Let $u \in \mathbf{X}$. Remember that the energy functional $J_g : \mathbf{X} \to \mathbb{R}$ corresponding to $(\mathscr{P}_\mu)$ is

$$J_g(u) = \frac{1}{p}\int_\Omega A(-\nabla u)(-\nabla u)dx - \frac{\mu}{p}\int_\Omega V(x)|u|^p dx + \int_\Omega W(x,u)dx - \int_\Omega gu dx, \tag{10}$$

where $W(x,u) = \int_0^u w(x,t)dt$. Then $J_g \in C^1(\mathbf{X},\ \mathbb{R})$ and

$$\langle J_g'(u), v\rangle = \int_\Omega \big(A(-\nabla u)(-\nabla v) - \mu V(x)|u|^{p-2}uv + w(x,u)v - gv\big)\, dx,\ \forall v \in \mathbf{X}.$$

Therefore, $u \in \mathbf{X}$ is a solution of the problem $(\mathscr{P}_\mu)$ if and only if $\langle J_g'(u), v\rangle = 0$, $\forall v \in \mathbf{X}$.

Definition 2..2. (Definition 16.5 in [17]) *Assume that g is a non-negative measurable function on* Ω *and* $\tau_g : \mathbb{R} \mapsto [0,\infty)$ *is given by*

$$\tau_g(s) = |\{x \in \Omega : g(x) > s\}|, \quad \forall s \in \mathbb{R}.$$

If the function $g^* : B(0,r) \mapsto [0,\infty)$ *satisfies*

$$g^*(x) = \inf\left\{s \in [0,\infty) : \tau_g(s) \le \omega_N |x|^N\right\}, \forall x \in B(0,r),$$

where ω_N *is the volume of the unit ball in* $\mathbb{R}^N$, $r := (|\Omega|/\omega_N)^{1/N}$. *Then* g^* *is called the Schwarz symmetric decreasing rearrangement of g.*

In the following, we assume that B is a ball centered at the origin in $\mathbb{R}^N$, e^* and g^* are the Schwarz symmetric decreasing rearrangements of e and g, respectively.

Lemma 2..1. (Theorem 16.9 in [17]) Assume that $e, g : B \mapsto [0,\infty)$ are measurable functions, then the inequality

$$\int_B eg dx \le \int_B e^* g^* dx$$

holds.

By Definition 2..2, if $e \in \mathscr{R}(g)$, then $e^* = g^*$.

Lemma 2..2. (Theorem 16.10 in [17]) *Assume that g is a non-negative measurable function on B, then the inequality*

$$\int_B h \circ g^* dx \le \int_B h \circ g dx$$

holds for any Borel function $h : [0,\infty) \mapsto [0,\infty)$.

Lemma 2..3. (Lemma 2.7 in [23]) Assume that $1 < p < \infty$.

(i) If $g \in W_0^{1,p}(B)$ is non-negative, then $g^* \in W_0^{1,p}(B)$ and

$$\int_B \alpha^p(-\nabla g) dx \ge \int_B \alpha^p(-\nabla g^*) dx, \tag{11}$$

where $\alpha : \mathbb{R}^N \mapsto [0,\infty)$ is a convex function of class $C^1(\mathbb{R}^N - \{0\})$ satisfying (5) and there exists a positive constant a_0, such that $\alpha(\xi) = a_0$, for all $\xi \in \mathbb{R}^N$ and $|\xi| = 1$.

(ii) If (11) holds and $t \in [0, \text{ess sup}_{x\in B}\, g(x))$, then $g^{-1}(t,\infty)$ is a translate of $g^{*-1}(t,\infty)$. Moreover, if

$$\left|\left\{x \in B : \nabla g(x) = 0, 0 < g(x) < \operatorname*{ess\,sup}_{y\in B} g(y)\right\}\right| = 0,$$

then $g = g^*$.

Lemma 2..4. (Lemma B.1 in [15]) *Let $\phi : [t_0, +\infty) \to [0, +\infty)$ be nonincreasing such that*

$$\phi(t_2) \leq \frac{C_0}{(t_2 - t_1)^\sigma} |\phi(t_1)|^\beta, \quad t_2 > t_1 \geq t_0,$$

where C_0, σ, β are positive constants with $\beta > 1$. Then

$$\phi(t_0 + d) = 0,$$

where

$$d^\sigma = C_0 |\phi(t_0)|^{\beta - 1} 2^{\frac{\sigma\beta}{\beta - 1}}.$$

Defined by

$$\mu_0 = \inf \Big\{ \int_\Omega A(-\nabla v) \cdot (-\nabla v)\, dx : \int_\Omega |V(x)| |v|^p\, dx = 1,\ v \in \mathbf{X} \Big\}. \tag{12}$$

Lemma 2..5. *Let μ_0 be defined by* (12) *and $V(x) \in L^\infty(\Omega)$. Then the principal eigenvalue $\mu_0 > 0$.*

Proof. Assume that $\{v_n\}$ is the minimizing sequence for μ_0, i.e.,

$$\int_\Omega |V(x)| |v_n|^p\, dx = 1 \quad \text{and} \quad \int_\Omega A(-\nabla v_n) \cdot (-\nabla v_n)\, dx \to \mu_0, \tag{13}$$

as $n \to \infty$. (13), combined with Remark 2..1, yields that $\{v_n\}$ is a bounded sequence. Because $\mathbf{X}$ is reflexive, there exists a weak convergent subsequence of $\{v_n\}$. Without loss of generality, we may assume that $v_n \rightharpoonup \hat{v} \in \mathbf{X}$. Since the embeddings $\mathbf{X} \hookrightarrow L^p(\Omega)$ is compact, $v_n \to \hat{v}$ in $L^p(\Omega)$. This and $V(x) \in L^\infty(\Omega)$ imply

$$\int_\Omega |V(x)|\, \big| |v_n|^p - |\hat{v}|^p \big|\, dx \leq \|V\|_\infty \int_\Omega \big| |v_n|^p - |\hat{v}|^p \big|\, dx \to 0, \quad \text{as } n \to \infty.$$

Consequently,

$$\int_\Omega |V(x)|\, |v_n|^p\, dx \to \int_\Omega |V(x)|\, |\hat{v}|^p\, dx = 1, \quad \text{as } n \to \infty.$$

So $\hat{v} \not\equiv 0$. Note that the norm in $\mathbf{X}$ is weak lower semi-continuous. Thanks to (9) and (13) we obtain that

$$\begin{aligned} \mu_0 \leq \int_\Omega A(-\nabla \hat{v}) \cdot (-\nabla \hat{v}) dx &= \|\hat{v}\|^p \leq \liminf_{n \to \infty} \|v_n\|^p \\ &= \liminf_{n \to \infty} \int_\Omega A(-\nabla v_n) \cdot (-\nabla v_n) dx = \mu_0. \end{aligned}$$

Thus,

$$\mu_0 = \int_\Omega A(-\nabla \hat{v}) \cdot (-\nabla \hat{v})\, dx = \|\hat{v}\|^p.$$

So $\mu_0 > 0$ is the principal eigenvalue of the p-Laplacian type operator and we complete the proof. □

3. Existence and Uniqueness for the Solution of the Problem $(\mathscr{P}_\mu)$

In order to investigate the rearrangement minimization problem $(\mathscr{P}_\mu)$, we study the solvability of the problem $(\mathscr{P}_\mu)$ first. In the following, we assume that $1 < p < N$.

The assumptions on $w : \Omega \times \mathbb{R} \to \mathbb{R}$ are the following:

(w_1) $w(x,t)$ is a Carathéodory function, i.e., $x \mapsto w(x,t)$ is measurable for all $t \in \mathbb{R}$ and $t \mapsto w(x,t)$ is continuous for a.e. $x \in \Omega$.

(w_2) For some $0 < s < p-1$, $|w(x,t)| \leq C_1 + C_2|t|^s$, $\forall\ t \in \mathbb{R}$, a.e. $x \in \Omega$, where C_1, C_2 are constants.

Remark 3..1. *It is easy to check that if* $\frac{\partial w(x,t)}{\partial t} \geq 0$ *for all* $t \in \mathbb{R}$ *then* $w(x,\cdot)$ *will be non-decreasing and* $W(x,\cdot)$ *will be convex. Therefore,* (w_1) *holds.*

Next theorem is the main result of this section.

Theorem 3..1. *Let* $V(x) \in L^\infty(\Omega)$, $g \in L^q(\Omega)$, $q > N/p$. *We assume that* $(w_1), (w_2)$ *are satisfied and* $0 < \mu < \mu_0$, *where* μ_0 *is defined by* (12). *Then the problem* $(\mathscr{P}_\mu)$ *has a solution. Moreover, if* $W(x,\cdot)$ *is convex, a.e.* $x \in \Omega$, *where* $W(x,u) = \int_0^u w(x,t)dt$, *then there exists a unique solution* $u_g \in \mathbf{X}$ *of the problem* $(\mathscr{P}_\mu)$ *which satisfies that*

$$J_g(u_g) = \inf_{v \in \mathbf{X}} J_g(v),$$

where J_g *is defined by* (10).

Proof. Assume that $v_n \rightharpoonup v$ in $\mathbf{X}$ as $n \to \infty$. Note that $1 < p < N$, and

$$1 < q' := q/(q-1) < \frac{N}{N-p} < p^*,$$

where $p^* := Np/(N-p)$ is the critical Sobolev exponent. By the Sobolev embedding theorem, the embedding $\mathbf{X} \hookrightarrow L^{q'}(\Omega)$ is continuous and compact. Therefore $v_n \to v$ in $L^{q'}(\Omega)$ as $n \to \infty$.

By the condition (w_2), $0 < s < p-1$. Similarly the embedding $\mathbf{X} \hookrightarrow L^{s+1}(\Omega)$ is continuous and compact, and then $v_n \to v$ in $L^{s+1}(\Omega)$ as $n \to \infty$. Since the Nemytskij operator $u \mapsto W(x,u)$ from $L^{s+1}(\Omega)$ to $L^1(\Omega)$ is continuous, $W(x,v_n) \to W(x,v)$ in $L^1(\Omega)$ as $n \to \infty$. Therefore,

$$\int_\Omega W(x,v_n)dx \to \int_\Omega W(x,v)dx \tag{14}$$

as $n \to \infty$. Consequently,

$$
\begin{aligned}
&\liminf_{n\to\infty} J_g(v_n) \\
&= \liminf_{n\to\infty} \left(\frac{1}{p}\int_\Omega A(-\nabla v_n)(-\nabla v_n) - \frac{\mu}{p}\int_\Omega V(x)|v_n|^p dx + W(x,v_n) - g v_n dx \right) \\
&\geq \frac{1}{p}\int_\Omega A(-\nabla v)(-\nabla v)dx - \frac{\mu}{p}\int_\Omega V(x)|v|^p dx + \int_\Omega W(x,v)dx - \limsup_{n\to\infty}\int_\Omega g v_n dx \\
&\geq J_g(v) - \limsup_{n\to\infty} \|g\|_{L^q(\Omega)} \|v_n - v\|_{L^{q'}(\Omega)} \\
&= J_g(v).
\end{aligned}
$$

That is, J_g is weakly lower semi-continuous.

By using (w_2) again,

$$
\left| \int_\Omega W(x,u)dx \right| \leq \int_\Omega \left| \int_0^u w(x,v)\,dv \right| dx \leq C\int_\Omega |u|^{s+1}dx \leq C\|u\|^{s+1}, \quad \forall u \in \mathbf{X}, \tag{15}
$$

and

$$
\left| \int_\Omega g u dx \right| \leq \|g\|_{L^q(\Omega)} \|u\|_{L^{q'}(\Omega)} \leq C\|u\|, \quad \forall u \in \mathbf{X}. \tag{16}
$$

When $0 < \mu < \mu_0$, from (9), (10), (15), (16), Lemma 2..5 and $0 < s < p-1$ that

$$
J_g(u) \geq \frac{1}{p}\Big(1 - \frac{\mu}{\mu_0}\Big)\|u\|^p - C\|u\|^{s+1} - C\|u\| \to \infty,
$$

as $\|u\| \to \infty$. Therefore J_g is coercive.

Thus, there exists $u_g \in \mathbf{X}$ such that

$$
J_g(u_g) = \inf_{v\in\mathbf{X}} J_g(v).
$$

By conditions (w_1), (w_2), and Lemma 2.16 in [25], we obtain that $J_g \in C^1(\mathbf{X},\mathbb{R})$. So that u_g must be a critical point of the functional J_g, i.e.,

$$
\langle J_g'(u_g), v\rangle = 0,\ \forall v \in \mathbf{X}.
$$

and then a solution of the problem $(\mathscr{P}_\mu)$.

In the following, we assert that J_g has a unique minimum point. Moreover, if $W(x,\cdot)$ is convex we assert that J_g has a unique minimum point in the following. Since $0 < \mu < \mu_0$,

$$
\begin{aligned}
\frac{\mu_0+\mu}{\mu_0}\int_\Omega A(-\nabla u)(-\nabla u)dx &\geq \int_\Omega A(-\nabla u)(-\nabla u)dx - \mu\int_\Omega V(x)|u|^p dx \\
&\geq \frac{\mu_0-\mu}{\mu_0}\int_\Omega A(-\nabla u)(-\nabla u)dx.
\end{aligned} \tag{17}
$$

Define by

$$
\|u\|_\mu := (\int_\Omega A(-\nabla u)(-\nabla u)dx - \mu\int_\Omega V(x)|u|^p dx)^{1/p}.
$$

It follows from (17) and (9) that $\|u\|_\mu$ is a new norm in $\mathbf{X}$ which is equivalent to the norm $\|u\|$. Thus, for any $t \in (0,1), u, v \in \mathbf{X}$, we have

$$\|tu+(1-t)v\|_\mu \leq t\|u\|_\mu + (1-t)\|v\|_\mu.$$

By the condition (w_1), $W(x,\cdot)$ is convex. Since $W(x,\cdot)$ is convex,

$$\begin{aligned}
&J_g(tu+(1-t)v)\\
&= \frac{1}{p}\|tu+(1-t)v\|_\mu^p + \int_\Omega W(tu+(1-t)v)dx - \int_\Omega g(x)(tu+(1-t)v)dx\\
&\leq \frac{1}{p}(t\|u\|_\mu + (1-t)\|v\|_\mu)^p + \int_\Omega W(tu+(1-t)v)dx - \int_\Omega g(x)(tu+(1-t)v)dx\\
&< \frac{t}{p}\|u\|_\mu^p + \frac{1-t}{p}\|v\|_\mu^p + t\int_\Omega W(u)dx + (1-t)\int_\Omega W(v)dx - t\int_\Omega gudx - (1-t)\int_\Omega gvdx\\
&= tJ_g(u) + (1-t)J_g(v).
\end{aligned}$$

Hence, J_g is a strict convex functional. Then J_g has a unique minimum point u_g which solves the problem $(\mathscr{P}_\mu)$. □

Remark 3..2. *It is easy to see that if $w(x,\cdot)$ is non-decreasing then $W(x,\cdot)$ will be convex, where $W(x,u) = \int_0^u w(x,t)dt$.*

4. Existence of a Solution to the Problem $(\mathscr{R}mp)$

Based on the work in Section 3, we turn our attention in this section to derive the solvability of the rearrangement minimization problem $(\mathscr{R}mp)$ and study more property of the solution to $(\mathscr{R}mp)$.

If all the assumptions in Theorem 3..1 are satisfied, then when the data function g is changed to $e \in \mathscr{R}(g)$, the problem $(\mathscr{P}_\mu)$ has a unique solution u_e in $\mathbf{X}$. We have

Theorem 4..1. *Let $1 < p < N$, $V \in L^\infty(\Omega)$, $g \in L^q(\Omega)$, $q > N/p$. Assume that $(w_1),(w_2)$ are satisfied, $w(x,\cdot)$ is non-decreasing, a.e. $x \in \Omega$ and $0 < \mu < \mu_0$, where μ_0 is defined by (12). Then $(\mathscr{R}mp)$ has a solution $\hat{g} \in \mathscr{R}(g)$, i.e.,*

$$\Phi(\hat{g}) = J_{\hat{g}}(u_{\hat{g}}) = \inf_{e\in\mathscr{R}(g)} J_e(u_e) = \inf_{e\in\mathscr{R}(g)} \Phi(e),$$

where $\Phi(e)$ is defined by (4), $u_e \in \mathbf{X}$ is the solution of $(\mathscr{P}_\mu)$ and J_e is defined by (10) by changing the data function g to $e \in \mathscr{R}(g)$.

Proof. It follows from (10) and (12) that

$$\begin{aligned}
J_e(u_e) &= \frac{1}{p}\int_\Omega A(-\nabla u_e)(-\nabla u_e)dx - \frac{\mu}{p}\int_\Omega V(x)|u_e|^p dx + \int_\Omega (W(x,u_e) - eu_e)dx\\
&\geq \frac{1}{p}\left(1 - \frac{\mu}{\mu_0}\right)\|u_e\|^p - C(\|e\|_{L^q(\Omega)}\|u_e\| + \|u_e\|^{s+1}).
\end{aligned} \tag{18}$$

This implies that $\inf_{e\in\mathscr{R}(g)} J_e(u_e) > -\infty$.

Assume that $\{g_i\} \subset \mathscr{R}(g)$ is the minimizing sequence of functions, such that,

$$J_{g_i}(u_{g_i}) = \inf_{e\in\mathscr{R}(g)} J_e(u_e) + \varepsilon_i, \tag{19}$$

with $\varepsilon_i \to 0^+$ as $i \to \infty$. Denote by $u_i := u_{g_i}$ and $J_i(u_i) := J_{g_i}(u_{g_i})$. Then from (18) and (19) that $\{u_i\}$ is bounded in $\mathbf{X}$. Since $\mathbf{X}$ is reflexive, there exists a subsequence which is still denoted by $\{u_i\}$, such that $u_i \rightharpoonup u \in \mathbf{X}$, and then $u_i \to u \in L^{q'}(\Omega)$, $u_i \to u \in L^p(\Omega)$ as $i \to \infty$. It follows from the same arguments in the proof of Lemma 2..5 that

$$\lim_{i\to\infty} \int_\Omega V(x)\,|u_i|^p\,dx = \int_\Omega V(x)\,|u|^p\,dx. \tag{20}$$

Since $\|g_i\|_{L_q(\Omega)} \equiv \|g\|_{L_q(\Omega)}$, $\{g_i\}$ is also bounded in $L^q(\Omega)$. Similarly, there exists a subsequence which is still denoted by $\{g_i\}$, such that

$$g_i \rightharpoonup \bar{g} \in \overline{\mathscr{R}(g)^{q,w}},$$

as $i \to \infty$, where $\overline{\mathscr{R}(g)^{q,w}}$ denotes the weak closure of $\mathscr{R}(g)$ in $L^q(\Omega)$. Since $u \in L^{q'}(\Omega)$,

$$\lim_{i\to\infty} \left| \int_\Omega (g_i - \bar{g})u\,dx \right| = 0.$$

By the Hölder inequality, we have

$$\begin{aligned} \left| \int_\Omega (g_i u_i - \bar{g}u)dx \right| &\le \left| \int_\Omega g_i(u_i - u)dx \right| + \left| \int_\Omega (g_i - \bar{g})u\,dx \right| \\ &\le \|g_i\|_{L^q(\Omega)} \|u_i - u\|_{L^{q'}(\Omega)} + \left| \int_\Omega (g_i - \bar{g})u\,dx \right|. \end{aligned}$$

Thus

$$\lim_{i\to\infty} \left| \int_\Omega (g_i u_i - \bar{g}u)dx \right| = 0. \tag{21}$$

We deduce that

$$\begin{aligned} \inf_{e\in\mathscr{R}(g)} J_e(u_e) &= \lim_{i\to\infty} J_i(u_i) \\ &\ge \frac{1}{p}\int_\Omega A(-\nabla u)(-\nabla u)dx - \mu \int_\Omega V(x)|u|^p dx + \int_\Omega (W(x,u) - \bar{g}u)dx, \end{aligned} \tag{22}$$

where the first equality comes from (19), the last inequality follows from (14), (20), (21) and the weakly lower semi-continuity of norm.

By Lemma 2.3 in [14], the linear functional $l : \overline{\mathscr{R}(g)^{q,w}} \mapsto \mathbb{R}, h \to \int_\Omega hu\,dx$ has a maximizer $\hat{g} \in \mathscr{R}(g)$. Because $\bar{g} \in \overline{\mathscr{R}(g)^{q,w}}$,

$$\int_\Omega \bar{g}u\,dx \le \int_\Omega \hat{g}u\,dx. \tag{23}$$

From (22) and (23), we have

$$\inf_{e\in\mathscr{R}(g)} J_e(u_e) \geq \frac{1}{p}\int_\Omega A(-\nabla u)(-\nabla u)dx - \mu\int_\Omega V(x)|u|^p dx + \int_\Omega (W(x,u)-\hat{g}u)dx. \tag{24}$$

By Theorem 3..1 and (10),

$$\begin{aligned} J_{\hat{g}}(u_{\hat{g}}) &= \inf_{v\in\mathbf{X}} J_{\hat{g}}(v) \\ &= \inf_{v\in\mathbf{X}}\int_\Omega \left(\frac{1}{p}A(-\nabla v)(-\nabla v) - \mu\int_\Omega V(x)|v|^p dx + W(x,v) - \hat{g}v\right)dx \\ &\leq \frac{1}{p}\int_\Omega A(-\nabla u)(-\nabla u)dx - \mu\int_\Omega V(x)|u|^p dx + \int_\Omega (W(x,u)-\hat{g}u)dx. \end{aligned} \tag{25}$$

Combining with (24) and (25), we get that $J_{\hat{g}}(u_{\hat{g}}) \leq \inf_{e\in\mathscr{R}(g)} J_e(u_e)$.

On the other hand, since $\hat{g}\in\mathscr{R}(g)$, $\inf_{e\in\mathscr{R}(g)} J_e(u_e) \leq I_{\hat{g}}(u_{\hat{g}})$.

Therefore,

$$\inf_{e\in\mathscr{R}(g)} J_e(u_e) = I_{\hat{g}}(u_{\hat{g}}).$$

The proof is complete. □

Remark 4..1. *It is worth to note that the way to get the solvability of the problem* $(\mathscr{P}_0)$ *in* [23] *is essentially based on some variational methods and techniques, the truncated function method, the rearrangement optimization theory, and the fact that the imbedding* $\mathbf{X}\to C(\bar{\Omega})$ *is compact in the case of* $p>N$*. But in this paper, under the case of* $1<p<N$*, it is not able to embed* $\mathbf{X}$ *to* $C(\bar{\Omega})$ *again. Moreover, the boundary value problem* $(\mathscr{P}_\mu)$ *includes the parameter* $\mu>0$*. So the method used in* [23] *is not suitable for studying the minimization problem* $(\mathscr{R}mp)$*. In order to obtain the solvability of the problem* $(\mathscr{R}mp)$ *in Theorem* 4..1*, we need to use the different imbedding theorem, different approach and the peculiarity of* $\mu_0>0$*, where* μ_0 *is defined by* (12)*.*

5. Symmetric Property of a Solution to the Problem $(\mathscr{R}mp)$

Now we show that if the domain Ω is a ball and $\mu=0$, then the solution to $(\mathscr{R}mp)$ has some symmetric property. In order to obtain the symmetric property of the solution to $(\mathscr{R}mp)$, we establish the regularity property of the solution to the problem $(\mathscr{P}_0)$.

Theorem 5..1. *Let* $1<p<N$, $g\in L^q(\Omega)$, $q>N/p$*. We assume that* $(w_1),(w_2)$ *are satisfied,* $w(x,\cdot)$ *is non-decreasing, a.e.* $x\in\Omega$*. Then the problem* $(\mathscr{P}_0)$ *has a unique solution* $u_g\in\mathbf{X}\cap L^\infty(\Omega)$*.*

Proof. By Theorem 3..1, the problem $(\mathscr{P}_\mu)$ has a unique solution $u_g\in\mathbf{X}$. That is,

$$\int_\Omega (A(-\nabla u_g)(-\nabla v) + w(x,u_g)v - gv)\,dx = 0, \quad \forall v\in\mathbf{X}. \tag{26}$$

In the following, we will prove that $u_g\in L^\infty(\Omega)$.

In fact, for any $k > 0$, define by

$$u_{g,k} = \begin{cases} u_g - k & \text{for } u_g \geq k \\ 0 & \text{for } |u_g| \leq k \\ u_g + k & \text{for } u_g \leq -k, \end{cases} \tag{27}$$

and

$$\Omega_k = \{x \in \Omega : |u_g| \geq k\}.$$

By letting $v = u_{g,k}$ in (26), we obtain that

$$\int_\Omega \big(A(-\nabla u_g)(-\nabla u_{g,k}) + w(x,u_g)u_{g,k} - g u_{g,k}\big)\, dx = 0. \tag{28}$$

From (w_1), $w(x,\cdot)$ is non-decreasing, a.e. $x \in \Omega$, we have

$$\begin{aligned} \int_\Omega w(x,u_g)u_{g,k}dx &= \int_\Omega (w(x,u_g) - w(x,0))u_{g,k}dx + \int_\Omega w(x,0)u_{g,k}dx \\ &\geq \int_\Omega w(x,0)u_{g,k}dx. \end{aligned} \tag{29}$$

From (w_2), $|w(x,0)| \leq C_1$, a.e. $x \in \Omega$. Note that $g \in L^q(\Omega)$. So that

$$\begin{aligned} \gamma \|u_{g,k}\|^p &\leq \int_\Omega A(-\nabla u_{g,k})(-\nabla u_{g,k})dx \\ &= \int_\Omega A(-\nabla u_g)(-\nabla u_{g,k})dx \\ &= -\int_\Omega w(x,u_g)u_{g,k}dx + \int_\Omega g u_{g,k}dx \\ &\leq \int_{\Omega_k} (g - w(x,0))u_{g,k}dx \\ &\leq \Big(\int_{\Omega_k} |g - w(x,0)|^q dx\Big)^{\frac{1}{q}} \Big(\int_{\Omega_k} |u_{g,k}|^{p^*} dx\Big)^{\frac{1}{p^*}} \Big(\int_{\Omega_k} dx\Big)^{1-\frac{1}{q}-\frac{1}{p^*}} \\ &= \|g - w(x,0)\|_{L^q(\Omega)} \|u_{g,k}\|_{L^{p^*}(\Omega)} |\Omega_k|^{1-\frac{1}{q}-\frac{1}{p^*}}, \end{aligned} \tag{30}$$

where the first inequality follows from (6), the first equality comes from (27), the second equality holds from (28), the second inequality comes from (29), the third inequality is due to Hölder's inequality. Therefore, (30), combined with the Sobolev embedding theorem, gives

$$\|u_{g,k}\|^{p-1}_{L^{p^*}(\Omega)} \leq C_3 \|g - w(x,0)\|_{L^q(\Omega)} |\Omega_k|^{1-\frac{1}{q}-\frac{1}{p^*}}, \tag{31}$$

where C_3 is a positive constant. For any $l > k > 0$, we have $\Omega_l \subset \Omega_k$ and

$$\begin{aligned} (l-k)|\Omega_l|^{\frac{1}{p^*}} &\leq \left(\int_{\Omega_l} (|u_g| - k)^{p^*} dx\right)^{\frac{1}{p^*}} \\ &\leq \left(\int_{\Omega_k} (|u_g| - k)^{p^*} dx\right)^{\frac{1}{p^*}} \\ &= \|u_{g,k}\|_{L^{p^*}(\Omega)}. \end{aligned}$$

This, combined with (31), yields

$$|\Omega_l| \le \frac{C_4\|g-w(x,0)\|_{L^q(\Omega)}^{\frac{p^*}{p-1}}}{|l-k|^{p^*}}|\Omega_k|^{\beta},$$

where C_4 is a positive constant and

$$\beta = \frac{p^*}{p-1}(1-\frac{1}{q}-\frac{1}{p^*}) > \frac{Np}{(N-p)(p-1)}(1-\frac{p}{N}-\frac{N-p}{Np}) = 1.$$

Denote $\phi(t)=|\Omega_t|, \forall t>0$. Hence, from Lemma 2..4, there exists $d_0>0$ such that

$$\phi(d_0)=|\Omega_{d_0}|=0.$$

Actually, $d_0 = C_5\|g-w(x,0)\|_{L^q(\Omega)}^{(\beta-1)/(p-1)}$, where C_5 is a positive constant. Thus

$$|u_g(x)| \le C_5\|g-w(x,0)\|_{L^q(\Omega)}^{(\beta-1)/(p-1)}, \quad \text{a.e. } x\in\Omega.$$

Thus, $u_g \in L^\infty(\Omega)$. The proof is complete. □

Theorem 5..2. *Suppose that all the assumptions in Theorem* 5..1 *are satisfied, and* Ω *is a ball centered at the origin in* $\mathbb{R}^N$*,* $g(x)>0$ *and* $w(x,t)=w(t)\le 0$*, a.e.* $x\in\Omega$*,* $\forall\ t\in\mathbb{R}$*. Assume that* $\alpha:\mathbb{R}^N\mapsto[0,\infty)$ *is a convex function of class* $C^1(\mathbb{R}^N-\{0\})$ *satisfying* (5) *and there exists a positive constant* a_0*, such that* $\alpha(\xi)=a_0$*, for all* $\xi\in\mathbb{R}^N$ *and* $|\xi|=1$*. Then* g^* *will be the unique solution of the problem* $(\mathscr{R}mp)$*, where* g^* *is the Schwarz symmetric decreasing rearrangement of* g*.*

Proof. For $g\in L^q(\Omega)$, the problem $(\mathscr{P}_\mu)$ has a unique solution $u_g\in\mathbf{X}\cap L^\infty(\Omega)$ by Theorem 5..1. Thus u_g has the Hölder regularity follows as usual, see, for instance, Theorem 8.29 in [16]. It is easy to derive from $g(x)>0$ and $w(x,t)=w(t)\le 0$ that $J_g(|u_g|)\le J_g(u_g)$. By the uniqueness of the minimum point of J_g, we must have $u_g=|u_g|\ge 0$. Thanks to $g(x)>0$ and $w(x,t)=w(t)\le 0$ again, we have

$$\text{div}A(-\nabla u_g)(x) = g(x)-w(x,u_g(x))>0.$$

This, combined with Theorem 5 in [24], gives that $u_g(x)>0$, a.e. $x\in\Omega$.

By Theorem 4..1, the problem $(\mathscr{R}mp)$ has a solution. Let $\hat{g}\in\mathscr{R}(g)$ be the solution of the problem $(\mathscr{R}mp)$. By $g(x)>0$ and (3),

$$|\{x\in\Omega: g(x)\ge 0\}| = |\{x\in\Omega:\hat{g}(x)\ge 0\}|,$$

and then $\hat{g}(x)>0$, a.e. $x\in\Omega$. So that

$$\text{div}A(-\nabla u_{\hat{g}})(x) = \hat{g}(x)-w(u_{\hat{g}}(x))>0, \quad \text{a.e. } x\in\Omega.$$

By Lemma 7.7 in [13],

$$|\{x\in\Omega: u_{\hat{g}}(x)=c\}|=0, \quad \forall c\in\mathbb{R}.$$

By Lemma 2.4 and Lemma 2.9 in [6], the functional $L : h \mapsto \int_\Omega h u_{\hat{g}}, h \in \overline{\mathscr{R}(g)^{q,w}}$ has a unique maximizer $\varphi \circ u_{\hat{g}} \in \mathscr{R}(g)$ where φ is an increasing function.

For each $e \in \overline{\mathscr{R}(g)^{q,w}}$, if $e \in \mathscr{R}(g)$ then

$$\begin{aligned}
&\frac{1}{p}\int_\Omega A(-\nabla u_{\hat{g}})(-\nabla u_{\hat{g}})dx + \int_\Omega (W(u_{\hat{g}}) - \hat{g}u_{\hat{g}})dx \\
&= J_{\hat{g}}(u_{\hat{g}}) \le J_e(u_e) \\
&= \frac{1}{p}\int_\Omega A(-\nabla u_e)(-\nabla u_e)dx + \int_\Omega (W(u_e) - e u_e)dx \\
&\le \frac{1}{p}\int_\Omega A(-\nabla u_{\hat{g}})(-\nabla u_{\hat{g}})dx + \int_\Omega (W(u_{\hat{g}}) - e u_{\hat{g}})dx,
\end{aligned}$$

which implies that

$$\int_\Omega \hat{g}u_{\hat{g}}dx \ge \int_\Omega e u_{\hat{g}}dx. \tag{32}$$

If $e \in \overline{\mathscr{R}(g)^{q,w}}$ then there exists a sequence $\{g_n\} \subset \mathscr{R}(g),\ g_n \rightharpoonup e \in L^q(\Omega)$. From (32), it holds that

$$\int_\Omega \hat{g}u_{\hat{g}}dx \ge \int_\Omega g_n u_{\hat{g}}dx \to \int_\Omega e u_{\hat{g}}dx$$

as $n \to \infty$. Therefore, we have

$$\int_\Omega \hat{g}u_{\hat{g}}dx \ge \int_\Omega e u_{\hat{g}}dx, \forall e \in \overline{\mathscr{R}(g)^{q,w}}.$$

Since the maximizer of the linear functional $L : h \mapsto \int_\Omega h u_{\hat{g}}, h \in \overline{\mathscr{R}(g)^{q,w}}$ is unique,

$$\hat{g} = \varphi \circ u_{\hat{g}} \in \mathscr{R}(g).$$

Next, we are going to show that

$$\int_\Omega \alpha^p(-\nabla u^*_{\hat{g}})dx = \int_\Omega \alpha^p(-\nabla u_{\hat{g}})dx. \tag{33}$$

Since $J_{\hat{g}}(u_{\hat{g}}) \le J_{g^*}(u_{g^*}) \le J_{g^*}(u^*_{\hat{g}})$,

$$\begin{aligned}
&\frac{1}{p}\int_\Omega \alpha^p(-\nabla u_{\hat{g}})dx + \int_\Omega (W(u_{\hat{g}}) - \hat{g}u_{\hat{g}})dx \\
&= \frac{1}{p}\int_\Omega A(-\nabla u_{\hat{g}})(-\nabla u_{\hat{g}})dx + \int_\Omega (W(u_{\hat{g}}) - \hat{g}u_{\hat{g}})dx \\
&\le \frac{1}{p}\int_\Omega A(-\nabla u_{g^*})(-\nabla u_{g^*})dx + \int_\Omega (W(u_{g^*}) - g^*u_{g^*})dx \\
&= \frac{1}{p}\int_\Omega \alpha^p(-\nabla u_{g^*})dx + \int_\Omega (W(u_{g^*}) - g^*u_{g^*})dx \\
&\le \frac{1}{p}\int_\Omega \alpha^p(-\nabla u^*_{\hat{g}})dx + \int_\Omega (W(u^*_{\hat{g}}) - g^*u^*_{\hat{g}})dx,
\end{aligned}$$

where the first and second equalities follow from that $A(-\nabla u)(-\nabla u) = \alpha^p(-\nabla u)$.

From Lemma 2..1 and Lemma 2..2, we have

$$\frac{1}{p}\int_\Omega(\alpha^p(-\nabla u^*_{\hat g})-\alpha^p(-\nabla u_{\hat g}))dx$$
$$\geq \int_\Omega(W(u_{\hat g})-W(u^*_{\hat g})+g^*u^*_{\hat g}-\hat g u_{\hat g})dx\geq 0.$$

Combining with (11), we can see that (33) holds.

At last, we show that

$$\left|\left\{x\in\Omega:\nabla u_{\hat g}=0,\ 0<u_{\hat g}(x)<\operatorname{ess\,sup}_{y\in\Omega}u_{\hat g}(y)\right\}\right|=0. \tag{34}$$

In fact, if $0<u_{\hat g}(x_0)<\operatorname{ess\,sup}_{x\in\Omega}u_{\hat g}(x)$ for some $x_0\in\Omega$, then define by

$$S=\{x\in\Omega:u_{\hat g}(x)\geq u_{\hat g}(x_0)\},$$

and

$$\bar u(x)=u_{\hat g}(x)-u_{\hat g}(x_0).$$

Then

$$divA(-\nabla\bar u(x))=divA(-\nabla u_{\hat g}(x))>0,\quad \text{a.e. } x\in\Omega.$$

By Theorem 5 in [24], we obtain that $\bar u(x)>0$ in the interior $\overset{\circ}{S}$ of S. Therefore, $u_{\hat g}(x)>u_{\hat g}(x_0)$ for all $x\in\overset{\circ}{S}$ which implies that x_0 must belong to the boundary of S. By the Hopf boundary lemma, we can see that $\frac{\partial\bar u}{\partial\nu}(x_0)=\frac{\partial u_{\hat g}}{\partial\nu}(x_0)\neq 0$, where ν is the outward unit normal to ∂S at x_0. So that

$$\left\{x\in\Omega:\nabla u_{\hat g}=0,\ 0<u_{\hat g}(x)<\operatorname{ess\,sup}_{y\in\Omega}u_{\hat g}(y)\right\}=\emptyset,$$

which justifies the validity of (34).

By Lemma 2..3, (33) and (34), it holds that $u_{\hat g}=u^*_{\hat g}$. Therefore, $\hat g=\varphi\circ u^*_{\hat g}$ will be a spherically symmetric decreasing function which coincides its Schwarz rearrangement, i.e., $\hat g=\hat g^*$. Since $\hat g\in\mathscr{R}(g)$, we must have $\hat g=g^*$. The proof is complete. □

Remark 5..1. *By exploring the Stampacchia truncation method, we establish the regularity property of the solution to the problem* $(\mathscr{P}_0)$ *first, see Theorem* 5..1*, then we obtain the solution of the problem* $(\mathscr{R}mp)$ *has the Schwarz symmetric property in Theorem* 5..2*.*

References

[1] C. Anedda, F. Cuccu, Steiner symmetry in the minimization of the first eigenvalue in problems involving the p-Laplacian. Proc. Amer. Math. 144, 3431-3440 (2016).

[2] J. S. Baek, Properties of solutions to a class of quasilinear elliptic problems, Ph.D thesis, Seoul National University in Seoul, 1992.

[3] J. F. Bonder, A. Ritorto, A. M. Salort, Shape optimization problems for nonlocal operators, Adv. Calc. Var. 11, 373-386 (2018).

[4] J. F. Bonder, J. D. Rossi, J. F. Spedaletti, Optimal design problems for the first p-fractional eigenvalue with mixed boundary conditions, Adv. Nonlinear Stud. 18, 323-335 (2018).

[5] G.R. Burton, Rearrangements of functions, maximization of convex functionals and vortex rings, Math. Ann. 276, 225-253 (1987).

[6] G.R. Burton, Variational problems on classes of rearrangements and multiple configurations for steady vortices, Ann. Inst. Henri Poincaré. 6, 295-319 (1989).

[7] Q.H. Chen, D. Chu, Roger C.E. Tan, Optimal control of obstacle for quasi-linear elliptic variational bilateral problems, SIAM J. Control Optim. 44, 1067-1080 (2005).

[8] C. Cosner, F. Cuccu, G. Porru, Optimization of the first eigenvalue of equations with indefinite weights. Adv. Nonlinear Stud. 13, 79-95 (2013).

[9] F. Cuccu, G. Porru, S. Sakaguchi, Optimization problems on general classes of rearrangements, Nonlinear Anal. 74 , 5554-5565 (2011).

[10] B. Emamizadeh, Y.C. Liu , G. Porru, Extensions of a result by G. Talenti to (p,q)-Laplace equations, Nonlinear Anal. 180 , 41-51 (2019).

[11] B. Emamizadeh, M. Marras, Rearrangement optimization problems with free boundary. Numer. Funct. Anal. Optim. 35, 404-422 (2014).

[12] B. Emamizadeh, J.V. Prajapat, Symmetry in rearrangement optimization problems, Electron. J. Differential Equations. 2009, 1-10 (2009).

[13] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Second edition, Springer, Berlin, 1998.

[14] B. Emamizadeh, J.V. Prajapat, Symmetry in rearrangement optimization problems, Electron. J. Differential Equations. 2009, 1-10 (2009).

[15] D. Kinderlehrer, G. Stampacchia, An introduction to variational inequalities and their applications, Pure and Applied Mathematics, 88. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.

[16] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 1998.

[17] G. Leoni, A First Course in Sobolev Spaces, Graduate Studies in Mathematics 105, American Mathematical Society, Providence, Rhode Island, 2009.

[18] Y.C. Liu, B. Emamizadeh, Converse symmetry and intermediate energy values in rearrangement optimization problems, SIAM J. Control Optim. 55, 2088-2107 (2017).

[19] G. Meng, Gang, P. Yan, M.R. Zhang, Meirong minimization of eigenvalues of one-dimensional p-Laplacian with integrable potentials, J. Optim. Theory Appl. 156, 294-319 (2013) .

[20] L.M. Del Pezzo, J.F. Bonder, Some optimization problems for p-Laplacian type equations, Appl. Math. Optim. 59, 365-381 (2009).

[21] C. Qiu, Y.S. Huang, Y.Y. Zhou, A class of rearrangement optimization problems involving the p-Laplacian, Nonlinear Anal., 112, 30-42 (2015).

[22] C. Qiu, Y.S. Huang, Y.Y. Zhou, Optimization problems involving the fractional laplacian, Electron. J. Differential Equations, 2016, 1-15 (2016).

[23] C. Qiu, X.Q. Yang, Y.Y. Zhou, Solvable optimization problems involving a p-Laplacian type operator, Appl. Anal. 2020, Online.

[24] J.L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12, 191-202 (1984).

[25] M. Willem, Minimax Theorems, Birkhauser, Basel, 1996.

[26] K. Zhang, K.L. Teo, A penalty-based method from reconstructing smooth local volatility surface from American options, J. Ind. Manag. Optim. 11, 631-644 (2015).

[27] Y.Y. Zhou, X.Q. Yang, K.L. Teo, Optimal control problems governed by a variational inequality via nonlinear Lagrangian methods, Optimization 55, 187-203 (2006).