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Evaluating Port Efficiency Dynamics: A Risk-based Approach

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This study proposes a new methodology to quantify the efficiency dynamics of a port over time. While efficiency evaluation has gained full attention in port management, researchers conducting related studies are challenged by temporal variations observed in the collected data. Existing approaches have almost exclusively relied on multivariate normal distributional assumptions of the input and output data, but empirical evidence from real data shows that the port operations data demonstrate long-tail distributions and violate the distributional assumptions. In addition, many existing models are intractable (non-convex) or lack interpretability. Motivated by these challenges, we develop an optimization-based approach for efficiency measurement under uncertainty that is compatible with the conventional non-parametric method. In particular, inspired by the coherent risk measure, we create a risk-based index to measure the efficiency of any operating unit by comparing its observations against a benchmark that is guaranteed to be production possible under a certain risk level. To facilitate the computation of the index, we develop a risk-based port efficiency evaluation (RPE) model, which can be reformulated as an exponential cone program (ECP) and solved efficiently by off-the-shelf solvers. We test our model for a multipurpose port on a real dataset of 3,394 observations showing the proposed approach's merits. We also provide evidence for the Chinese New Year effect from a port management perspective and draw managerial insights from the study.

Key words: port efficiency, stochastic efficiency, risk measure, conic programming

1. Introduction

As clearinghouses for nearly 80 percent of world merchandise in volume, seaports and their efficiency in processing cargo are essential. Port inefficiency, such as long waiting time at borders, inappropriate fees, and cumbersome administrative procedures, constitute obstacles to trade that are as serious as tariff barriers (UNCTAD 2014). Empirical evidence shows that an improvement in port efficiency from the 25th to the 75th percentile reduces transport charges by more than 10% (Clark et al. 2004).

Efficiency evaluations in port-related literature establish the relationship between inputs (mainly a port's physical facilities and labor force) and outputs (such as throughput or cargo movements), also known as the *technical efficiency* (see e.g., Suárez-Alemán et al. 2016). In this paper, we refer to the technical efficiency of ports as the *port efficiency*. Port efficiency evaluation is pioneered by Roll and Hayuth (1993), aiming to provide deeper insights into port performance to researchers and port managers. For example, Tongzon (2001), Jiang et al. (2012), Barros and Athanassiou (2015) and Wanke and Barros (2016) study the port efficiency in Australian, Asian, European, Brazilian ports (either seaports or river ports), respectively. For another example, Cullinane et al. (2004) compares the efficiency of the world's leading container ports over time with their annual cross-sectional data.

Understanding port efficiency brings far-reaching values. At the macro-level, it explains port productivity and how it interferes with regional economics. Suárez-Alemán et al. (2016) analyzes the evolution and drivers of productivity and efficiency changes across 70 developing regions from 2000 to 2010. The analysis shows that the participation of the private sector, the reduction of corruption in the public sector, improvements in liner connectivity and the existence of multimodal links increase the level of port efficiency in developing regions. Yuen et al. (2013) zooms into a context of container ports in China and estimates the operational efficiency of a set of sample ports. This study is followed by a regression study, revealing an interesting U-shaped relationship between Chinese ownership and container terminal efficiency. Having some Chinese ownership (but not as the major stakeholder) may enhance container terminal efficiency. At an operational level, efficiency evaluation provides an analytical tool for assessing and monitoring the relative efficiency of port operations over time or among different ports. It is also used to evaluate the consequence of port reform and the impact of regulation on port efficiencies (Woo et al. 2011).

Studies on port efficiency evaluation have long focused on applying and extending the Data Envelopment Analysis (DEA) and Stochastic Frontier Analysis (SFA) models to measure and compare port efficiencies among a set of ports to identify competitive factors, as summarized and reviewed by Panayides et al. (2009) and Lee and Song (2017). The rationale is to compare the performance of a decision-making unit (DMU) against an efficient frontier – a set of best practices identified from a collection of comparable DMUs. In the port context, the set of DMUs can be a set of ports or one port over different periods, corresponding to the cross-sectional data and time-series data, respectively, in a panel dataset. In this paper, we shall refer to the former as *port efficiency ranking* and the latter as *port efficiency dynamics*. A user studying port efficiency ranking zooms into a given evaluation period. The objective is to assess the relative performance of one particular port (the DMU for port efficiency ranking evaluation) against all comparable ports in generating multiple outputs from multiple inputs. For example, one can evaluate Shanghai's port efficiency by comparing it with the world's other top 20 busiest ports. Port efficiency ranking can be used to understand port competitiveness. Further, complemented by regression analysis, it helps to understand how a particular characteristic (such as ownership or business structure) leads to the efficiency differences (see, e.g., Suárez-Alemán et al. 2016 and Yuen et al. 2013). In contrast, port efficiency dynamics evaluation focuses on one port and has access to the day-to-day operational data of that port over a long time. The objective is to understand the relative performance in a particular period (the DMU for port efficiency dynamics evaluation) compared with all periods under assessment. For example, how Shanghai's port efficiency in March 2022 is affected by the COVID measures, as compared to the past months.

Four decades have passed since the introduction of the DEA model and the corresponding Charnes, Cooper, and Rhodes (CCR) ratio (Charnes et al. 1978). DEA and SFA have been predominate analytical tools in maritime transport, especially in port research. Woo et al. (2011) conducts a comprehensive literature review on seaport research over three decades (1980–2000s) from a methodology perspective. Among all identified papers relating to port policy, management, and operations, 10.2% of the data analysis papers use DEA as the primary technique; another 4.8% apply the SFA approach. Shi and Li (2017) further shows that among the journal articles in major outlets of maritime transport papers from 2000 to 2014, 10.32% of them apply a DEA or SFA method.

While we observe the growing trend of these analytical tools in port research, a fundamental question arises: *Are the existing approaches able to well address the port efficiency evaluation problem in real applications*? We answer this question by illustrating challenges from temporal variation of data in a port efficiency dynamics evaluation problem and identifying three pitfalls of existing (stochastic) efficiency evaluation approaches in Section 2.

Our contributions

We use the empirical evidence from a real dataset of an Asian multi-purpose port to explain the pitfalls associated with directly applying existing methodologies to a port management context (see §2.1). Motivated by this, we propose a general method[1] to measure efficiency from a risk perspective.

Our study contributes to the port efficiency evaluation literature as follows.

(a) We provide a new analytical tool to quantify port efficiency dynamics for future studies on port performance evaluation, which is useful for assessing and monitoring port productivity.

(b) Our method complements the existing literature, especially for applications when port activities' inputs and/or outputs are not normally distributed.

(c) We apply the proposed approach to address port efficiency evaluation challenges faced by an Asian multi-purpose port.

(d) Our application identifies some interesting phenomena, for example, the Chinese New Year (CNY) effect, from which we draw managerial insights.

This paper also contributes to the methodology of the efficiency evaluation under uncertainty in operations research literature as follows:

(a) We extend the classic axiom-based production possibility set (PPS) to a risk-based PPS, and propose a risk-based port efficiency index that preserves the desirable properties of the CCR ratio, which leads to an interpretable efficiency measure.

(b) We develop a risk-based port efficiency evaluation (RPE) model that evaluates a port's efficiency in a particular period with input and output uncertainty under a risk level. This model does not require parametric distributional assumptions on input and output data.

(c) We show that the RPE model is a tractable exponential cone program (ECP), which can be solved exactly and efficiently (in polynomial time) by off-the-shelf solvers.

The paper unfolds as follows. We introduce the port efficiency evaluation problem and identify the research gap in §2. In §3, we propose a risk-based efficiency index that preserves the desirable properties of the existing widely adopted efficiency index based on a newly proposed risk measure. We apply the theory to a port efficiency evaluation problem by

[1] A general method in the sense that a common assumption in the existing literature is relaxed.

developing an RPE model and computing the dynamics of a port's efficiency index. We show that the developed model is tractable as an ECP. Numerical results are presented in §4, and conclude the paper in §5.

2. Port efficiency evaluation and research gap

This paper is motivated by a practical challenge faced by a port operator of a large multi-purpose port, who aims to monitor the dynamics of the weekly workforce efficiency using panel data. The port handles mainly bulk cargo, with a small fraction of containers. The former is very labor-intensive, making monitoring workforce efficiency essential. The cargoes are unstandardized, so the conventional performance metric (e.g., TEU per hour) does not apply. We zoom into one particular port and evaluate the dynamics of the port's efficiency with a unique property: temporal variation. We next explain the data and illustrate this challenge.

We collected 3,394 observations of vessel throughput and cargo handling information, corresponding to all the stevedore operations of general cargo (GC) in the port over an evaluation horizon of 52 weeks during a time window from 1/4/2018 to 30/3/2019. This dataset contains information on the actual time of berthing (ATB), the actual time of un-berthing (ATU), first activity time (FAT), last activity time (LAT), raining hours within activity time, general cargo handled in tonnage, the number of containers handled in twenty-foot equivalent unit (TEU) if any, the deployment of foreman, winchman, and riggers. In this example, we consider a port's GC handling efficiency as its ability to generate output (i.e., deliver services) using limited resources (e.g., workforce and infrastructure). We define the week index of a stevedore operation according to the week its LAT falls into, and treat each week index as an individual DMU and each port visit as one observation to evaluate the port's weekly GC handling efficiency. We visualize the number of observations per DMU in Figure 1. We summarize the descriptive statistics of input and output variables of the multi-purpose port in Table 1. The inputs include:

(i) Total berthing time, which is calculated by ATU minus ATB;

(ii) Net activity time, which is calculated by LAT minus FAT and excludes the raining hours;

(iii) The number of foremen, signalman, or winchman (according to our interview with port manager, the ratio of foreman, signalman, and winchman is usually 1:1:1);

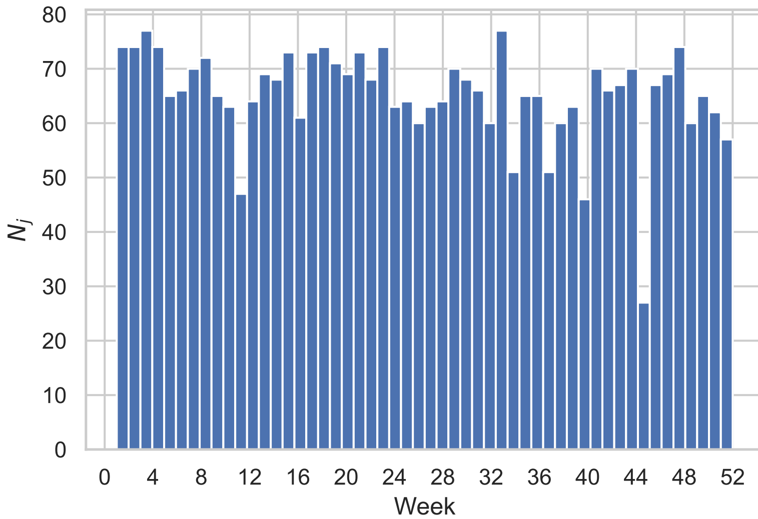


Figure 1 **The number of observations (i.e., port visits) in each DMU (i.e., a weekly evaluation period) in the GC handling data set**

(iv) The number of riggers deployed.

The outputs include general cargo handled in tonnage and the number of containers handled in TEU. We select the four input variables, consisting of the main ingredients of resources deployed in terms of a port's facilities and labor force. Such a choice of inputs is determined due to data availability and analogous to literature (e.g., Suárez-Alemán et al. 2016). Note that correlations among the input variables are natural and allowed.

Table 1 **Description for GC handling data**

Variables	Mean	Std	Min	Max
Vessel berthing time (in hours)	14.77	14.62	1.25	158.42
Net activity time (in hours)	11.41	12.83	1.00	98.43
Number of foremen/signalman/winchman	1.15	0.47	1.00	5.00
Number of riggers deployed	6.40	3.73	1.00	35.00
General cargo handled (in tonnes)	1,581.94	3539.67	8.60	55,896.88
The number of containers handled (in TEU)	18.31	32.94	0.00	503.00

This port efficiency evaluation problem demonstrates *temporal variations*: (i) there are multiple and random numbers of observations per week, and (ii) the input and output data within each week are not constants. The number of observations (N_j) per week depends on the number of port visits per evaluation period (see Figure 1) and demonstrates time-varying volatility. Further, consider the example mentioned above, among the observations within one particular week, many attributes (e.g., vessel berthing time and net activity time) change over time. For example, the berthing time per port visit within a week is not constant. The classic CCR ratio requires one value per input or output per DMU. If port managers wish to analyze how the port efficiency changes from week to week by the CCR ratio, they should identify the exact value per input or output per week.

A straightforward idea that addresses the temporal variation is to focus on the average of the input and output variables and calculate the CCR ratio. However, the average of each random variable can be distorted by some extreme scenarios. For example, the port might be inefficient in most instances but highly efficient in a particular instance, then simply taking the average could run into the risk of overestimating its efficiency. Essentially, taking sample average results in a significant loss of information as it only accounts for the first-moment information. Consequently, many studies have developed approaches to account for stochasticities.

Depending on the modeling of uncertainty in input and output variables, stochastic efficiency evaluation is developed into two directions: the regression-based econometric approaches under the statistical framework and the mathematical-programming-based approaches under the *Management Science* (MS) framework. We refer to Olesen and Petersen (2016) for a comprehensive literature review on stochastic efficiency measurement.

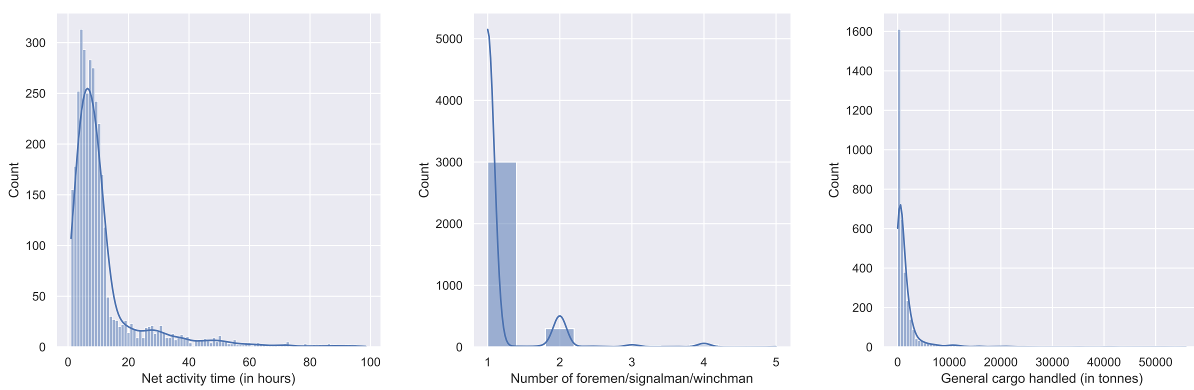
The regression-based models require an axiomatic approach to a data generating process (DGP). They usually consider a standard multiple-input, single-output, and cross-sectional model (e.g., Aigner et al. 1977, Kuosmanen and Johnson 2010). Under the statistical framework, consistency and asymptotic unbiasedness of the frontier estimator are the most critical properties of interest, which is beyond the scope of this study.

In contrast, the stochastic efficiency in an MS framework focuses on measuring the deviations of DMUs from the best practice frontier under data variations, without any reference to a specific statistical model involving the DGP. Initiated by Land et al. (1993), Olesen and Petersen (1995), Cooper et al. (1996), several models (e.g., Cooper et al. 1998, Wei et al. 2014) have been built under the chance-constrained principle proposed by Charnes and Cooper (1959). However, the computation of chance-constrained DEA requires transforming the probabilistic constraints into deterministic certainty equivalent constraints, which relies almost exclusively on a multivariate normal distribution assumption on the input and output data. This distributional assumption may not hold in many cases of practical importance, see Appendix A for a more detailed technical review.

2.1. Research gap

In the following, we elaborate on the three pitfalls in applying the existing approaches to port efficiency evaluation under uncertainty.

The first pitfall is the strong distributional assumption on data variation. Usually, the stochastic efficiency evaluation model is non-parametric, which does not assume any functional form on the efficient frontier. However, the computation of chance-constrained DEA requires a multivariate normal distribution assumption on the input and output data. As observed from the descriptive statistics (see Table 1), the parameters are clearly not normally distributed.



(a) Net activity hours (b) Number of foreman (c) General cargo handled

Figure 2 Histograms of three parameters in GC handling data, which demonstrate long-tail distributions

Figure 2 further illustrates the histogram of three parameters chosen as an example. They demonstrate a long-tail distribution. To further support this observation, we perform an omnibus normality test that combines skewness and kurtosis (d'Agostino 1971, d'Agostino and Pearson 1973) for all input and output variables. The test results reject all the null hypotheses under a significance level 0.05. Thus, the distributional assumption required to solve chance-constrained DEA models is violated. To make the paper self-contained, we present simulation studies in Appendix B to see how distribution mis-specification error translates to problematic efficiency evaluation.

The second pitfall is the lack of an interpretable index. The initiators of chance-constrained DEA phrased their comments in a review paper as follows: *"A related aspect of these types of extensions, found, e.g., in Olesen and Petersen (1995) and Cooper et al. (1998), is the fact that the authors are very silent on the matter of how to measure or estimate inefficiency."*(Olesen and Petersen 2016) The phrase "these types of extensions" refers to the chance-constrained efficiency evaluation model. The indices or evaluations under existing stochastic efficiency models are not as interpretable as the well-recognized CCR ratio established based on the axioms on the PPS. In other words, there is a missing link between the chance-constrained DEA model and index properties demonstrated in the classic axiom-based models.

The intractability of the existing models is another pitfall. As pointed out by Chen and Zhu (2019), the chance-constrained DEA is both merit in modeling and a burden in computation. Under normal distribution assumption, a few models can be reformulated as second-order cone programs (SOCPs), which are solvable to the global optimality by the off-the-shelf solvers. Nevertheless, many models are not. Many researchers have pointed out that solving models to optimality is challenging (e.g., Udhayakumar et al. 2011 and Wei et al. 2014).

To sum up, we need an interpretable and tractable optimization framework with minimal distributional assumptions under a stochastic context to evaluate port efficiency dynamics.

3. The risk-based port efficiency evaluation (RPE) model

This section develops the RPE model to estimate the efficiency index per evaluation period (e.g., per week or month) over multiple observations within the evaluation horizon. We first provide a formal definition of the problem, after which we develop a new theory to handle the uncertainties that appear in the problem. Leveraging the nice properties demonstrated in the new theory, we derive an RPE model to measure port efficiency under uncertainty.

Notations. To facilitate presentation, we adopt the following notations throughout the study. We use the calligraphic font to denote a set such as $\mathcal{T}$. We denote by $|\mathcal{N}|$ the cardinality of a set $\mathcal{N}$. We use tilde $(\tilde{\cdot})$ to denote uncertain parameters and use $\mathbb{P}$ to denote probability measure on sample space Ω. We denote by $\mathbb{E}_{\mathbb{P}}[\tilde{\boldsymbol{z}}]$ the expectation of $\tilde{\boldsymbol{z}}$ under probability distribution $\tilde{\boldsymbol{z}} \sim \mathbb{P}$. The inequality between two uncertain parameters $\tilde{t} \geq \tilde{v}$ describes state-wise dominance, i.e., $\tilde{t}(\omega) \geq \tilde{v}(\omega)$ for all $\omega \in \Omega$. The inequality between two vectors $\boldsymbol{x} \geq \boldsymbol{y}$ corresponds to the element-wise comparison.

3.1. Problem definition

As discussed in §1, the objective of a port efficiency evaluation model is to establish the relationship between inputs and outputs from panel data, either in terms of port efficiency ranking or dynamics. In this section, we shall use port efficiency dynamics for illustration. All models and results explained in this section can extend to port efficiency ranking without additional difficulty by changing the definition of a DMU from "a period of a port" to "a port in a period".

To mimic a real dataset (see, e.g., the GC handling data described in Table 1), we consider the panel data consisting of the input and output observations over a set of periods $\{(x_{j,m}^{\ell}, y_{j,s}^{\ell})\}_{j\in\mathcal{J},m\in\mathcal{M},s\in\mathcal{S},\ell=1,\ldots,N_j}$, as explained in Table 2.

Consider the operations of one particular port over a horizon that is partitioned into a set of periods $\mathcal{J}$, each indexed by j. In a period j, the port generates random outputs (denoted by a vector of random variable $\tilde{\boldsymbol{y}}_j$) using the inputs (denoted by a vector of random variable $\tilde{\boldsymbol{x}}_j$). A special case is the deterministic setting, where the port use resources $\boldsymbol{x}_j$ to handle cargo quantified as $\boldsymbol{y}_j$. Then, the problem will be simplified and thus handled by the classic DEA models (or the CCR ratio). However, we are unaware of the distribution of each input or output element, e.g., the distribution of general cargo handled (in tonnes), but we have the past day-to-day operational data. Within each period j, there are N_j port visits as observations indexed by ℓ, $\ell = 1, ..., N_j$, with the usage of inputs (vector $\boldsymbol{x}_j^{\ell}$) to

Table 2 Description for panel data

$\mathcal{J}$	a set of periods indexed by $j \in \mathcal{J}$ (each period corresponds to one DMU);
$\mathcal{M}$	a set of input parameters indexed by $m \in \mathcal{M}$;
$\mathcal{S}$	a set of output parameters indexed by $s \in \mathcal{S}$;
N_j	the number of port visits in period j;
ℓ	the index of port visit, $\ell = 1, ..., N_j$;
$x^{\ell}_{j,m}$	the quantity of input m during the ℓth port visit in period j, $x^{\ell}_{j,m} \geq 0$;
$y^{\ell}_{j,s}$	the quantity of output s during the ℓth port visit in period j, $y^{\ell}_{j,s} \geq 0$;
$\boldsymbol{x}^{\ell}_j$	a vector for all inputs during the ℓth port visit in period j, $\boldsymbol{x}^{\ell}_j \in \mathbb{R}^{\lvert\mathcal{M}\rvert}$;
$\boldsymbol{y}^{\ell}_j$	a vector of all outputs during the ℓth port visit in period j, $\boldsymbol{y}^{\ell}_j \in \mathbb{R}^{\lvert\mathcal{S}\rvert}$;
$\tilde{\boldsymbol{x}}_j$	a vector of random variables for inputs (i.e., infrastrature and labour) in period j;
$\tilde{\boldsymbol{y}}_j$	a vector of random variables for outputs (i.e., bulk and containerized cargo handled) in period j.

deliver outputs (vector $\boldsymbol{y}^{\ell}_j$). The objective is to evaluate the port efficiency dynamics, say $\rho(\tilde{\boldsymbol{x}}_j, \tilde{\boldsymbol{y}}_j)$, in its ability to generate outputs with inputs.

We approach the problem through the coherent risk measure theory. A coherent risk measure summarizes a random loss by a scalar with salient properties to account for its associated risk and hence provides a fundamental approach for optimization under uncertainty (see, e.g., Rockafellar 2007).

3.2. A risk-based efficiency index

In this subsection, we first present essential properties of the CCR ratio in deterministic efficiency evaluation and use them to define interpretable efficiency score. Then we propose a risk-based port efficiency index and finally show this index preserves those properties owing to some desirable properties of a newly defined risk measure, inspired by the theory of coherent risk measure (Artzner et al. 1999). We explain how the index translate the score into insights in port efficiency context.

An efficiency index is best understood and interpretable if (i) the index is explainable and provides insights to its users, and (ii) it is compatible with the essential properties possessed by the most widely adopted index, the CCR ratio. We start with a deterministic setting of the classic CCR ratio and review the axioms on the production possibility set (PPS) in literature.

3.2.1. Essential properties of the classic CCR ratio. Consider the input-output configuration $(\boldsymbol{x}_j, \boldsymbol{y}_j)$ observed for n DMUs indexed by $j = 1, ..., n$, where $\boldsymbol{x}_j$ is a vector of observed inputs and $\boldsymbol{y}_j$ is a vector of observed outputs of the DMU j. A production possibility set (PPS) is characterized to determine an "efficient" subset from observed data, based on the axioms of convexity, inefficiency postulate, ray unboundedness and minimum extrapolation (see Banker et al. 1984). The PPS can be formulated as

$$\mathcal{T}=\left\{(\boldsymbol{x},\boldsymbol{y})\mid \boldsymbol{x}\geq\sum_{j=1}^{n}\lambda_j\boldsymbol{x}_j;\boldsymbol{y}\leq\sum_{j=1}^{n}\lambda_j\boldsymbol{y}_j;\lambda_j\geq 0,\forall j=1,...,n\right\} \tag{1}$$

Based on the axioms of PPS, one can measure the efficiency of any DMU according to its input-output observation $(\boldsymbol{x},\boldsymbol{y})$ via the following linear program

[CCR]

$$\begin{aligned}
\textbf{Primal: } \mu^*(\boldsymbol{x},\boldsymbol{y}) = \min\ & \mu \\
\text{s.t. } & \sum_{j=1}^{n}\lambda_j\boldsymbol{x}_j-\mu\boldsymbol{x}\leq\boldsymbol{0} \\
& \boldsymbol{y}-\sum_{j=1}^{n}\lambda_j\boldsymbol{y}_j\leq\boldsymbol{0} \\
& \lambda_j\geq 0,\quad j=1,...,n
\end{aligned}
\qquad
\begin{aligned}
\textbf{Dual: } \max\ & \boldsymbol{u}^\top\boldsymbol{y} \\
\text{s.t. } & \boldsymbol{v}^\top\boldsymbol{x}=1 \\
& \boldsymbol{u}^\top\boldsymbol{y}_j-\boldsymbol{v}^\top\boldsymbol{x}_j\leq 0,\quad j=1,...,n \\
& \boldsymbol{u},\boldsymbol{v}\geq\boldsymbol{0}.
\end{aligned} \tag{2}$$

For a particular DMU under evaluation, indexed by o and denoted as DMU_o, the $\mu^*(\boldsymbol{x}_o,\boldsymbol{y}_o)$ obtained from above DEA model is called the CCR ratio (Charnes et al. 1978), which evaluates the technical efficiency of DMU_o in delivering multiple outputs from multiple inputs by comparing it against the PPS constructed. All DMUs under consideration can be ranked or compared according to their CCR ratios.

We can establish that the CCR ratio exhibits four essential properties that are natural for efficiency measurement listed in Proposition 1. All those properties are meaningful and well-established (Banker et al. 1984, Färe and Lovell 1978) except for the quasi-convexity. The detailed proof is presented in Appendix C.

Proposition 1. *The CCR ratio defined by the DEA model shown in formulation* (2) *has the following desirable properties:*

(a) Positive homogeneous of degree −1 in input vector $\boldsymbol{x}$*, i.e.,* $\mu^*(k\boldsymbol{x},\boldsymbol{y})=\mu^*(\boldsymbol{x},\boldsymbol{y})/k$ *for all* $k>0$*.*

(b) Positive homogeneous of degree 1 in output vector $\boldsymbol{y}$*, i.e.,* $\mu^*(\boldsymbol{x},k\boldsymbol{y})=k\mu^*(\boldsymbol{x},\boldsymbol{y})$ *for all* $k>0$*.*

(c) Monotonicity: A DMU with less inputs and more outputs should be more efficient, i.e., $\mu^*(\boldsymbol{x}',\boldsymbol{y}')\geq\mu^*(\boldsymbol{x},\boldsymbol{y})$ *if* $(-\boldsymbol{x}',\boldsymbol{y}')\geq(-\boldsymbol{x},\boldsymbol{y})$*.*

(d) *Quasi-convexity: The efficiency index of any convex combination* $(\boldsymbol{x},\boldsymbol{y})$ *and* $(\boldsymbol{x}',\boldsymbol{y}')$ *is no more than the higher efficiency index of them, i.e.,*

$$\mu^*(\kappa\boldsymbol{x}+(1-\kappa)\boldsymbol{x}',\kappa\boldsymbol{y}+(1-\kappa)\boldsymbol{y}')\le\max\{\mu^*(\boldsymbol{x},\boldsymbol{y}),\mu^*(\boldsymbol{x}',\boldsymbol{y}')\},\quad\forall\kappa\in(0,1).$$

Proposition 1 brings to light some meaningful interpretations of the CCR ratio. We explain them in a port efficiency evaluation context. If compared with period t, a particular port deploys k times of inputs to handle the same amount of cargo in period t', then the efficiency in period t' is $1/k$ of that in period t. Similarly, within a particular period, if port A uses k times of inputs to deliver the same output as port B, then B's efficiency is k times of A. A similar comparison holds regarding the discussion over the outputs. For example, if the resources or inputs used in period t are the same as in period t', but k times of cargoes are handled in period t, then the efficiency score in period t is k times that in period t'. The monotonicity implies that the efficiency score increases if the port deploys fewer resources but delivers more outputs. The quasi-convexity can be interpreted via a virtual port that combines part of port A's operations (e.g., several terminals within a port) with another part of port B's operations. The efficiency score associated with the virtual port is lower than that of the more efficient port between the two due to the effect of the less efficient port.

The four properties play an essential role for users adopting the CCR model to translate the score into meaningful explanations. We define these desirable properties as interpretability in our paper.

Definition 1 (Interpretable efficiency score). The efficiency score $\mu^*(\boldsymbol{x},\boldsymbol{y})$ for a DMU with input $\boldsymbol{x}$ and output $\boldsymbol{y}$ is interpretable under an efficiency evaluation model if it satisfies the desirable properties (a)-(d) in Proposition 1.

3.2.2. The risk-based efficiency index. We next develop an interpretable risk-based efficiency index. To this end, we first define a new risk measure for random vectors, called *Aggregated Entropic Value-at-Risk (AEVaR)*, and derive its nice properties. Then we use AEVaR to define the risk-based counterpart of the PPS in equation (1), based on which we propose a risk-based efficiency index and show it is interpretable owing to the nice properties of AEVaR.

We extend the coherent risk measure, *Entropic Value-at-Risk (EVaR)*, proposed by Ahmadi-Javid (2012) to random vectors and define the new risk measure AEVaR, which helps to define an interpretable and tractable risk-based efficiency index.

Definition 2. Let $\tilde{\boldsymbol{z}} \in \mathcal{Z}$ be a n-dimensional random vector, we define the AEVaR of $\tilde{\boldsymbol{z}}$ with confidence level $1-\beta$ as the functional $\eta_\beta : \mathcal{Z} \to \mathbb{R}$ with

$$\eta_\beta(\tilde{\boldsymbol{z}}) := \inf_{\theta>0} \theta \ln \left(\sum_{i=1}^{n} \mathbb{E}_{\mathbb{P}} \left[\exp\left(\frac{\tilde{z}_i}{\theta}\right)\right]\right) - \theta \ln \beta \tag{3}$$

for any $\beta \in (0,1)$.

The AEVaR defined above has the following properties.

Theorem 1. *The AEVaR defined in Equation* (3) *is non-increasing in* β *and satisfies the following properties for any* $\beta \in (0,1)$*:*

(a) Monotonicity: If $\tilde{\boldsymbol{z}} \geq \tilde{\boldsymbol{w}}$*, then* $\eta_\beta(\tilde{\boldsymbol{z}}) \geq \eta_\beta(\tilde{\boldsymbol{w}})$.

(b) Translation Invariance: $\eta_\beta(\tilde{\boldsymbol{z}} + t\mathbf{1}) = \eta_\beta(\tilde{\boldsymbol{z}}) + t$ *for all* $t \in \mathbb{R}$.

(c) Convexity: $\eta_\beta(\kappa\tilde{\boldsymbol{z}} + (1-\kappa)\tilde{\boldsymbol{w}}) \leq \kappa\eta_\beta(\tilde{\boldsymbol{z}}) + (1-\kappa)\eta(\tilde{\boldsymbol{w}})$ *for any* $\kappa \in (0,1)$.

(d) Positive Homogeneity: $\eta_\beta(k\tilde{\boldsymbol{z}}) = k\eta_\beta(\tilde{\boldsymbol{z}})$ *for any* $k \geq 0$.

Moreover, $\mathbb{P}[\tilde{\boldsymbol{z}} \leq \eta_\beta(\tilde{\boldsymbol{z}}) \cdot \mathbf{1}] \geq 1-\beta$*, which explains the reason of calling* $1-\beta$ *confidence level.*

The proof of Theorem 1 is in Appendix C. Note that when $n=1$, AEVaR reduces to the EVaR defined by Ahmadi-Javid (2012) and the properties $(a)-(d)$ coincide with the four properties of the well-known coherent risk measure called coherence in Artzner et al. (1999).

The properties in Theorem 1 are essential to establish the interpretability of the risk-based efficiency index. Specifically, the positive homogeneity, monotonicity, and convexity of AEVaR help establish the index's positive homogeneity, monotonicity, and quasi-convexity, respectively. We shall notice that the translation invariance property necessitates data scaling in the cases when the input and output parameters are of different magnitudes. The rest of the paper assumes that the input and output variables are scaled to the same magnitude.

Before introducing the risk-based efficiency index, we define the risk counterpart of the PPS using AEVaR.

Definition 3. We define a *risk-based PPS* with risk level β using AEVaR as

$$\mathcal{T}_\beta := \left\{ (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \mid \exists \lambda_j \geq 0, j=1,...,n: \ \eta_\beta \begin{pmatrix} \tilde{\boldsymbol{y}} - \sum_{j=1}^{n} \lambda_j \tilde{\boldsymbol{y}}_j \\ \sum_{j=1}^{n} \lambda_j \tilde{\boldsymbol{x}}_j - \tilde{\boldsymbol{x}} \end{pmatrix} \leq 0 \right\}. \tag{4}$$

The risk-based PPS $\mathcal{T}_\beta$ is a stochastic counterpart of $\mathcal{T}$ in (1). It is worth emphasizing that $\mathcal{T}_\beta$ is a set of random vectors while $\mathcal{T}$ is a set of deterministic ones, so $\mathcal{T}_\beta$ cannot be visualized as in Olesen and Petersen (1995). We point out its connection to the classic DEA model by the fact that

$$\eta_\beta \begin{pmatrix} \tilde{\boldsymbol{y}} - \sum_{j=1}^n \lambda_j \tilde{\boldsymbol{y}}_j \\ \sum_{j=1}^n \lambda_j \tilde{\boldsymbol{x}}_j - \tilde{\boldsymbol{x}} \end{pmatrix} \leq 0 \text{ implies } \mathbb{P} \begin{bmatrix} \tilde{\boldsymbol{y}} \leq \sum_{j=1}^n \lambda_j \tilde{\boldsymbol{y}}_j \\ \sum_{j=1}^n \lambda_j \tilde{\boldsymbol{x}}_j \leq \tilde{\boldsymbol{x}} \end{bmatrix} \geq 1 - \beta,$$

which is the chance-constrained counterpart of $\sum_{j=1}^n \lambda_j \boldsymbol{x}_j \leq \boldsymbol{x}$ and $\boldsymbol{y} \leq \sum_{j=1}^n \lambda_j \boldsymbol{y}_j$ with parameter β. This sufficient condition implies that for a port with inputs and outputs within the risk-based PPS $\mathcal{T}_\beta$, we are at least $1-\beta$ confident that the inputs can deliver the outputs by benchmarking the port operation with a set of past stochastic inputs and outputs $(\tilde{\boldsymbol{x}}_j, \tilde{\boldsymbol{y}}_j)$, $j = 1, ..., n$.

We next show $\mathcal{T}_\beta$ is a convex cone, which is essential for preserving the properties $(a), (b), (d)$ in Proposition 1.

Proposition 2. *The risk-based PPS $\mathcal{T}_\beta$ is a convex cone for any $\beta \in (0,1)$.*

Proof of Proposition 2. Given any $\beta \in (0,1)$, for any $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in \mathcal{T}_\beta$, $k > 0$, we have $(k\tilde{\boldsymbol{x}}, k\tilde{\boldsymbol{y}}) \in \mathcal{T}_\beta$ from positive homogeneity of η_β. And for any $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ and $(\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{y}}') \in \mathcal{T}_\beta$, $\kappa \in (0,1)$, we have $\kappa(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) + (1-\kappa)(\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{y}}') \in \mathcal{T}_\beta$ from convexity of η_β. Hence $\mathcal{T}_\beta$ is a convex cone. Q.E.D.

We define the risk-based efficiency index based on the risk-based PPS.

Definition 4. We define the *risk-based efficiency index* as

$$\rho_\beta(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) := \inf \left\{ \mu > 0 : (\mu \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in \mathcal{T}_\beta \right\}. \tag{5}$$

The main idea of the risk-based efficiency index defined in Equation (5) is to find the minimum positive scale factor of the inputs such that we are at least $1-\beta$ confident that the scaled inputs $(\mu\tilde{\boldsymbol{x}})$ can still deliver the outputs by benchmarking it to a collection of comparable DMUs.

Finally, we show in Theorem 2 the risk-based efficiency index is as interpretable as the CCR ratio by leveraging the nice properties of extended coherent risk measures.

Theorem 2. *The risk-based efficiency index $\rho_\beta(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is interpretable (according to Definition 1).*

In a port context, Theorem 2 allows us to interpret the risk-based efficiency index in the same way as how we explain the CCR ratio that is widely adopted in port literature; see the detailed explanations in the paragraph after Proposition 1. We also note that the stochastic PPS in Olesen and Petersen (1995) offers an extension of the classic PPS, but the resulting efficiency index is not as interpretable as the CCR ratio.

3.3. A Risk-based Port Efficiency Evaluation (RPE) Model

This subsection is concerned with developing the RPE model to estimate the efficiency index per evaluation period (e.g., per week or month) over multiple observations within the evaluation horizon.

3.3.1. The β-RPE score We assume $(\tilde{\boldsymbol{x}}_j, \tilde{\boldsymbol{y}}_j)$ are mutually independent for $j \in \mathcal{J}$. This assumption of stochastic independence across the sample of DMUs is well known from the econometric literature (Olesen and Petersen 1995).

To calculate the risk-based efficiency index in equation (5) requires the true distributions (e.g., the distribution of vessel berthing time, general cargo handled per port visit, etc.), which is hardly known because we only have the panel data described in § 3.1. Therefore, we evaluate the expectation in AEVaR based on the *product of empirical marginal distribution* induced by the observations, i.e.,

$$\hat{\mathbb{P}} := \prod_{j \in \mathcal{J}} \left(\frac{1}{N_j} \sum_{\ell=1}^{N_j} \delta_{(\boldsymbol{x}_j^\ell, \boldsymbol{y}_j^\ell)}(\tilde{\boldsymbol{x}}_j, \tilde{\boldsymbol{y}}_j) \right),$$

where $\delta_{(\boldsymbol{x}_j^\ell, \boldsymbol{y}_j^\ell)}$ is a Dirac distribution concentrating unit mass at $(\boldsymbol{x}_j^\ell, \boldsymbol{y}_j^\ell)$. We denote by $\hat{\eta}_\beta(\tilde{\boldsymbol{z}})$ the empirical valuation of AEVaR, that is, $\hat{\eta}_\beta(\tilde{\boldsymbol{z}}) := \inf_{\theta > 0} \theta \ln \left(\sum_{i=1}^{n} \mathbb{E}_{\hat{\mathbb{P}}} \left[\exp\left(\frac{\tilde{z}_j}{\theta} \right) \right] \right) - \theta \ln \beta$. Next, we define an extended risk-based efficiency index, called the β-RPE score.

Definition 5. The *β-RPE score* of period o, where $\beta \in (0,1)$, is defined as the optimal value in the following model:

[RPE]

$$\rho_\beta^o := \inf \left\{ \mu > 0 : (\mu \tilde{\boldsymbol{x}}_o, \tilde{\boldsymbol{y}}_o) \in \mathcal{T}_\beta^o \right\} \tag{6}$$

where $\mathcal{T}_\beta^o := \left\{ (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \mid \exists \lambda_j \geq 0, j \in \mathcal{J} \backslash \{o\} : \ \hat{\eta}_\beta \begin{pmatrix} \tilde{\boldsymbol{y}} - \sum_{j \in \mathcal{J} \backslash \{o\}} \lambda_j \tilde{\boldsymbol{y}}_j \\ \sum_{j \in \mathcal{J} \backslash \{o\}} \lambda_j \tilde{\boldsymbol{x}}_j - \tilde{\boldsymbol{x}} \end{pmatrix} \leq 0 \right\}$.

The definition of β-RPE score, ρ^o_β, extends naturally from the risk-based efficiency index ρ_β, except for a modification on the corresponding risk-based PPS, $\mathcal{T}^o_\beta$, where the unit under evaluation is excluded from the reference set, that is, $j \in \mathcal{J}\backslash\{o\}$. This modification is inspired by Andersen and Petersen (1993). Note that a few evaluation periods may demonstrate the same risk-based efficiency index of one without such a modification, and they are all considered fully efficient. To see this, if we change all the $j \in \mathcal{J}\backslash\{o\}$ in the above definition into $j \in \mathcal{J}$, the efficiency index ρ^o_β for any period o, will always be less than or equal to one because it is always feasible to set $\lambda_j = 1$ for $j = o$ and $\lambda_j = 0$ otherwise. It is inevitably possible that a few evaluation periods may demonstrate the same risk-based efficiency index of one, so that distinction among them becomes impossible. This issue will be more pronounced when β is small. In that case, our risk-based PPS could be conservative, and the risk-based efficiency index tends to be over-estimated because more inputs are required to ensure production with a higher confidence level.

We next show that the exclusion of the unit under evaluation does not reduce the interpretability of the risk efficiency index.

THEOREM 3. *Given $(\tilde{\boldsymbol{x}}_j, \tilde{\boldsymbol{y}}_j)_{j\in\mathcal{J}}$, the β-RPE score ρ^o_β is interpretable (according to Definition 1) for any $\beta \in (0,1)$.*

The proof of Theorem 3 is presented in Appendix C. Theorem 3 implies that given historical data described in Table 2, we can use the β-RPE score to address the efficiency evaluation problem described in §3.1. Further, the resulting β-RPE score ρ^o_β is interpretable in the sense that it satisfies the desirable properties demonstrated by the commonly used CCR ratio.

3.3.2. Efficient computation via exponential cone programming (ECP) We next show the β-RPE score can be computed exactly and efficiently (in polynomial time). In particular, the computation of an β-RPE score turns out to be an ECP, a new class of conic program (Boyd and Vandenberghe 2004) that generalizes linear program to incorporate generalized inequalities defined by an exponential cone (a three-dimensional convex cone) involving exponentials and logarithms (see, e.g., Chares 2009). It can be computed exactly and efficiently by leveraging the recent advancements of solvers, such as MOSEK (MOSEK ApS 2020) and SCS (O'donoghue et al. 2016).

Proposition 3. *The β-RPE model can be reformulated as an ECP below:*

*[β-**RPE-ECP**]*

$$
\begin{aligned}
\rho_\beta^o = \quad & \inf_{\substack{\mu,\theta,\boldsymbol{\kappa},\boldsymbol{t},\boldsymbol{\nu},\boldsymbol{r},\\ \boldsymbol{q},\boldsymbol{\xi},\boldsymbol{\gamma},\boldsymbol{p},\boldsymbol{\lambda}}} \mu \\
& \text{s.t.} \sum_{j\in\mathcal{J}} t_{j,s} + \kappa_s \leq 0 && \forall s \in \mathcal{S} \\
& \theta \exp\left(\frac{-\kappa_s}{\theta}\right) \leq \nu_s && \forall s \in \mathcal{S} \\
& \sum_{\ell=1}^{N_o} r_{o,s}^\ell \leq N_o \theta && \forall s \in \mathcal{S} \\
& \theta \exp\left(\frac{y_{o,s}^\ell - t_{o,s}}{\theta}\right) \leq r_{o,s}^\ell && \forall s \in \mathcal{S}, \quad \forall \ell = 1,...,N_o \\
& \sum_{\ell=1}^{N_j} r_{j,s}^\ell \leq N_j \theta && \forall s \in \mathcal{S}, \quad \forall j \in \mathcal{J}\backslash\{o\} \\
& \theta \exp\left(\frac{-\lambda_j y_{j,s}^\ell - t_{j,s}}{\theta}\right) \leq r_{j,s}^\ell && \forall s \in \mathcal{S}, \quad \forall j \in \mathcal{J}\backslash\{o\}, \quad \forall \ell = 1,...,N_j \\
& \sum_{j\in\mathcal{J}} q_{j,m} + \xi_m \leq 0 && \forall m \in \mathcal{M} \\
& \theta \exp\left(\frac{-\xi_m}{\theta}\right) \leq \gamma_m && \forall m \in \mathcal{M} \\
& \sum_{\ell=1}^{N} p_{o,m}^\ell \leq N_o \theta && \forall m \in \mathcal{M} \\
& \theta \exp\left(\frac{-\mu x_{o,m}^\ell - q_{o,m}}{\theta}\right) \leq p_{o,m}^\ell && \forall m \in \mathcal{M}, \quad \forall \ell = 1,...,N_o \\
& \sum_{\ell=1}^{N_j} p_{j,m}^\ell \leq N_j \theta && \forall m \in \mathcal{M}, \quad \forall j \in \mathcal{J}\backslash\{o\} \\
& \theta \exp\left(\frac{\lambda_j x_{j,m}^\ell - q_{j,m}}{\theta}\right) \leq p_{j,m}^\ell && \forall m \in \mathcal{M}, \quad \forall j \in \mathcal{J}\backslash\{o\}, \quad \forall \ell = 1,...,N_j \\
& \sum_{s\in\mathcal{S}} \nu_s + \sum_{m\in\mathcal{M}} \gamma_m \leq \beta\theta \\
& \boldsymbol{\lambda} \geq \mathbf{0}, \theta > 0, \mu > 0
\end{aligned}
$$

The proof of above reformulation is in Appendix C. We note that, in the proof, the use of AEVaR in the definition of RPE model is essential for exploiting the stochastic independence among all evaluation periods in the reformulation.

The β-RPE-ECP model is an ECP since all the constraints involving exponentials can be characterized using exponential cones. For example, the constraints $\theta \exp\left(\frac{y^{\ell}_{o,s} - t_{o,s}}{\theta}\right) \leq r^{\ell}_{o,s}$ can be represented by $\left(r^{\ell}_{o,s}, \theta, y^{\ell}_{o,s} - t_{o,s}\right) \in \mathcal{K}_{\exp}$, where $\mathcal{K}_{\exp}$ is an exponential cone (see e.g., Chares 2009) defined as

$$\mathcal{K}_{\exp} := \left\{(z_1, z_2, z_3) : z_1 \geq z_2 \exp\left(\frac{z_3}{z_2}\right), z_2 > 0\right\} \bigcup \left\{(z_1, 0, z_3) : z_1 \geq 0, z_3 \leq 0\right\}.$$

The ECP can be solved to global optimality very efficiently by off-the-shelf solvers.

3.3.3. Parametric extensions of β-RPE score We have assumed no prior knowledge about the distribution of port resources, labor force, throughput and other inputs/outputs so far, except for the stochastic independence among the sample of evaluation periods. If certain parametric information on $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{y}}$ is available, the model may remain to be representable by ECP. To see this, we first illustrate the link between our model and the moment generation function (MGF) of a random variable.

PROPOSITION 4. *Assume $\tilde{y}_{o,s}$ follows a distribution with MGF $M_{\tilde{y}_{o,s}}(t)$, then the β-RPE model can be reformulated as follows:*

*[β-**RPE-MGF**]*

$$\begin{aligned}
\rho^o_\beta = \inf\ & \mu \\
\text{s.t. }\ & \theta \ln M_{\tilde{y}_{o,s}}(1/\theta) + \sum_{j \in \mathcal{J}\setminus\{o\}} \theta \ln M_{\tilde{y}_{j,s}}(-\lambda_j/\theta) - \theta \ln(\nu_s/\theta) \leq 0 \qquad \forall s \in \mathcal{S} \\
& \sum_{j \in \mathcal{J}\setminus\{o\}} \theta \ln M_{\tilde{x}_{j,m}}(\lambda_j/\theta) + \theta \ln M_{\tilde{x}_{o,m}}(-\mu/\theta) - \theta \ln(\gamma_m/\theta) \leq 0 \quad \forall m \in \mathcal{M} \\
& \left(\sum_{s \in \mathcal{S}} \nu_s + \sum_{m \in \mathcal{M}} \gamma_m\right) \leq \beta\theta \\
& \boldsymbol{\lambda} \geq \mathbf{0}, \theta > 0
\end{aligned} \tag{7}$$

We present the following examples to demonstrate how to represent the terms involving moments in model (7) such as $\theta \ln M_{\tilde{y}_{o,s}}(1/\theta)$, by convex cones. We consider these terms in their epigraph form, for example, $\theta \ln M_{\tilde{y}_{o,s}}(1/\theta) \leq t_{o,s}$.

EXAMPLE 1. For any random variable X that follows a binomial distribution $B(n,p)$, a normal distribution $N(\mu, \sigma^2)$, or a poisson distribution $Pois(\lambda)$, the epigraph of

$\theta \ln M_X(a/\theta)$ is conic representable (see e.g., Ben-Tal and Nemirovski 2001), as summarized in Table 3. (Note that, we adopt the commonly used notation for each distribution, so there are some conflicts of notations between the following example and the other part of this paper. We limit the usage of these notations to the statement and the results shown in Table 3.) The technical details are presented in Appendix D.

Table 3 Conic representation of $\theta \ln M_X(a/\theta) \le c$ under different distribution

Distribution	MGF $M_X(t)$	$\theta \ln M_X(a/\theta) \le c$	Representable by
Binomial(n,p)	$(1-p+pe^t)^n$	$(1-p)\theta e^{-c/n\theta} + p\theta e^{a/\theta - c/n\theta} \le \theta$	exponential cone
Normal(μ,σ^2)	$e^{t\mu+\frac{1}{2}\sigma^2 t^2}$	$\sqrt{(\sqrt{2}a\sigma)^2 + (c-a\mu-\theta)^2} \le c - a\mu + \theta$	second-order cone
Poisson(λ)	$e^{\lambda(e^t-1)}$	$\lambda\theta e^{a/\theta} \le c + \lambda\theta$	exponential cone

4. A case study: The effect of major festivals and week days on port efficiency

In this section, we present two possible applications of β-RPE-ECP model. We study the effect of festivals and week days on port operation, aiming to answer the following two questions.

(Q1) How do the major festivals affect port efficiency?

(Q2) Which day in a week does a port demonstrate the highest/lowest efficiency?

We answer these two questions according to the β-RPE-ECP model and discuss how to translate these answers to actionable insights. We normalize the relative efficiency score by dividing it by the highest relative efficiency. Note that the normalization does not affect the interpretability.

4.1. Q1: Festivals versus port efficiency

We first solve the real port efficiency problem introduced in §2, which can hardly be addressed by existing approaches. Through this case study, we highlight the practical relevance of our new approach.

We plot the normalized efficiency score over time in Figure 3, which shows that the efficiency ranking is affected by the risk level β. For example, when β is 0.1 or 0.3, Week 24 is the most efficient, but when β is 0.5, Week 41 is the most efficient (the week immediate after New Year of 2019). This change in ranking is unavoidable, but the efficiency ranking is relatively robust to the β in this example.

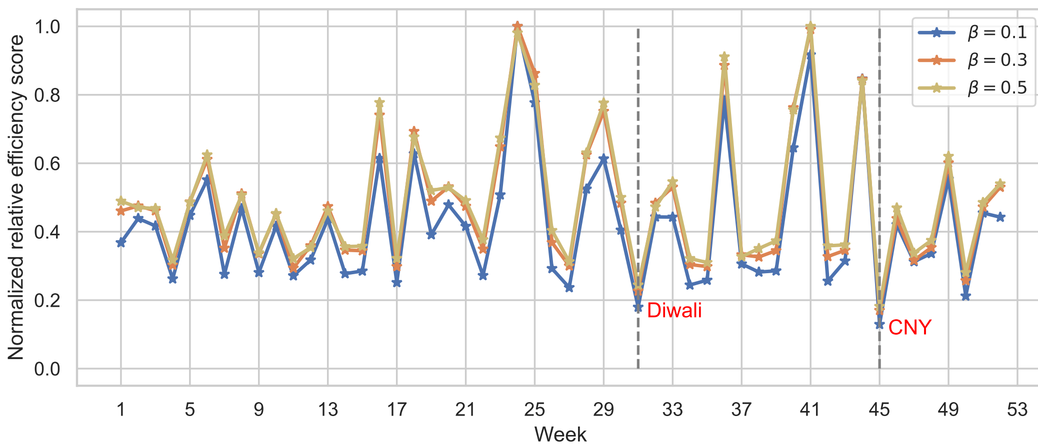


Figure 3 **GC handling efficiency over time under different** β

We observe the pronounced variability in efficiency over the 52 weeks in Figure 3. The dynamics of efficiency indices can be interpreted according to the results shown in Theorem 3. First, the efficiency score will be lowered if a particular evaluation period takes more infrastructure or workforce to deliver the same services. Further, if the k times of the resources are deployed to deliver the same amount of services, the risk-based efficiency score will be scaled by $1/k$. As a mirror effect, if k times of services are delivered using the same amount of resources, the efficiency index will be scaled by a multiple of k. Hence, if we zoom into the confidence level of 0.9 ($\beta = 0.1$). We can see that compared with the port's most efficient operations, the efficiency indices for most of the time are only 30% to 60% of its full efficiency. In other words, the port used $1.67 \sim 3.3$ times of resources to generate the same output or deployed the same amount of inputs but handled only $30\% \sim 60\%$ of the cargo that could have been done out of their peak ability.

Considering the additional dimension of randomness in the input and output data from a risk perspective can refine the efficiency ranking. This is evidenced by comparing the risk-based model with its deterministic counterpart that takes sample averages over all inputs and outputs per period. Figure 4 illustrates the result of a benchmark that averages all inputs and outputs per period and then computes the CCR ratio by excluding the unit under evaluation from the reference set, referred to as the *extended CCR ratio*. Clearly, compared with Figure 4, Figure 3 demonstrates a different ranking. For example, week 23 is the most efficient week according to the extended CCR ratio, but week 25 is more efficient than week 23 in the $\beta-$RPE model. Similar differences appear between weeks 49 and 52, weeks 6 and 7, among others. Such differences can be explained by the availability of information incorporated from data. Essentially, taking sample average aggregates all available data points in each period. It only accounts for the first-moment information of the empirical distribution of data points, resulting in a significant loss of information. In contrast, the $\beta-$RPE model tries to make the best use of data by constructing the interpretable risk-based PPS that incorporates all the moment information of data points via the moment generating function.

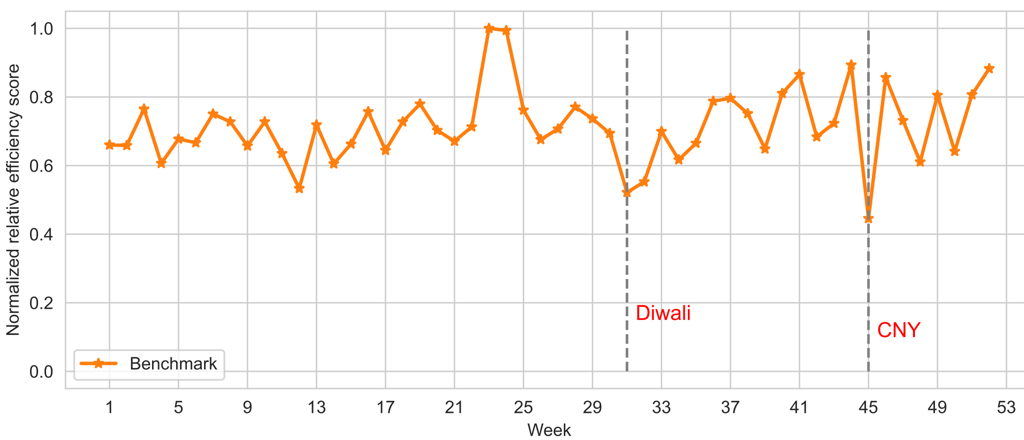


Figure 4 **Extended CCR ratio over time**

It is worth emphasizing that the efficiency scores and rankings are relative and model-specific. In particular, only the results calculated by the same model (either the $\beta-$RPE model or the extended CCR model) are comparable. For example, we can compare the $\beta-$RPE score in week 1 with that in week 5 to see which week shows higher efficiency under a confidence level of β. We can also compare the $\beta-$RPE score in the week of CNY under $\beta = 0.1$ with that when $\beta = 0.3$ to see how the relative efficiency varies with the confidence level. However, we cannot compare the score of CNY week calculated with the extended CCR model with that evaluated by the $\beta-$RPE score, because the underlying PPSs, based on which the scores are evaluated, are totally irrelevant.

The Chinese New Year effect in port management. Figure 3 illustrates an interesting phenomenon. Regardless of the β level chosen, Week 45, spanning 03/02/2019 to 10/02/2019, has the lowest GC handling efficiency. As a matter of fact, the Chinese New

Year falls in this Week 45. Based on our interview with the port manager, there are two possible explanations.

The internal reason arises from the crew management. Stevedore services in this port are carried out in groups. Each group consists of one foreman, one signalman, one winchman and six riggers, with the foreman being the group leader. Typically, every crew works with a fixed foreman in a fixed group of nine. During the CNY, the workforce scheduling may be affected by the public holiday, so many workers may not work in the group that they are familiar with. It is also possible that some vessels have to wait for groups to come upon berthing.

The external reason arises from the low cargo volume near CNY. Mainland China is a significant trading counterpart of the hinterland of this multipurpose port. The service demand is usually high ahead of the CNY and low during the CNY because many shippers shift their shipping operations ahead of the CNY to accommodate the surge in transportation-related cost and the manufacturing shutdown in China during CNY. With low general cargo handling volume, the handling efficiency is affected by the economies of scale adversely.

It is also worth noting that the second most inefficient week is Week 31, spanning 4/11/2018 to 11/11/2018, in which the Diwali falls. We can also see a relatively low efficiency around Christmas that falls in Week 38, spanning 23/12/2018 to 29/12/2018.

4.2. Q2: The temporal variation of port efficiency within a week

The dataset described in Table 1 also allows us to evaluate port efficiency according to the day of the week, providing a valuable indicator for port managers to understand the temporal variation in port operations.

We consider a set of 7 DMUs; each corresponds to one day of a week. Using the RPE model, we compare each DMU against the risk-based PPS constructed by the other six DMUs. The normalized efficiency score per day is summarized in Table 4 for various choices of confidence level β. The number of vessels serviced per work day is relatively balanced, slightly above or below 485 instances.

Figure 5 visualizes the efficiency scores under different confidence. Clearly, the choice of β does not affect the trend or comparison within the week in this study. However, it seems counterintuitive that we observe high efficiency levels on weekends, with the most efficient day of the entire week being Saturday. Tuesday is surprisingly the most productive

Table 4 Normalized β-RPE score ρ^o_β under different β (higher score implies higher efficiency)

Day of the week		Mon	Tue	Wed	Thur	Fri	Sat	Sun	Most efficient	Least efficient
Sample size (N_j)		480	476	488	494	477	471	508		
ρ^o_β under	$\beta = 0.1$	0.64	1.00	0.64	0.66	0.56	1.00	0.93	Tue & Sat	Fri
	$\beta = 0.2$	0.60	0.98	0.63	0.67	0.55	1.00	0.93	Sat	Fri
	$\beta = 0.3$	0.60	0.98	0.63	0.67	0.54	1.00	0.95	Sat	Fri
	$\beta = 0.4$	0.61	0.99	0.63	0.68	0.54	1.00	0.96	Sat	Fri
	$\beta = 0.5$	0.61	1.00	0.63	0.69	0.55	1.00	0.97	Tue & Sat	Fri
Extended CCR ratio		0.90	0.91	0.86	0.96	0.92	0.96	1.00	Sun	Wed

day among weekdays, very close to Saturday, with special cases happening for $\beta = 0.1$ or $\beta = 0.5$ when the efficiency scores on those two days are equal. Friday, on the contrary, demonstrates the lowest efficiency in handling cargoes with resources. Despite a different geographical setting, a survey in 2019 by Accountemps, a global human resource consulting firm, suggests similar findings through interviews of human resources (HR) managers in Canada[2]. It suggests that workplace productivity peaks on Tuesday, with 39% percent of respondents ranking Tuesday as the most productive day of the week. Thursday and Friday are the least productive days, with each receiving just 3% of the responses.

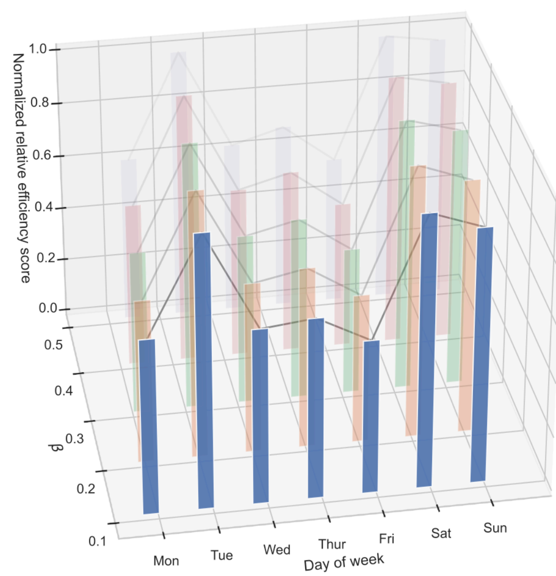


Figure 5 GC handling efficiency for day of week under different β

[2] Source: "Early Bird Gets The Worm: New Survey Reveals Productivity Peaks For Workers in Canada"; URL: `https://www.roberthalf.ca/en/early-bird-gets-the-worm-new-survey-reveals-productivity-peaks-for-workers-in-canada`. Accessed on May 2022.

We can cluster the seven days of a week into high-efficiency and low-efficiency days, with Monday, Wednesday, Thursday and Friday representing the low-efficiency cluster ($\rho_\beta^o \in [0.5, 0.7]$). The weekends, together with Tuesday, belong to the high-efficiency cluster ($\rho_\beta^o \in [0.9, 1]$). Compared with the high-efficiency cluster, the port deploys around 1.5 times of resources to handle the same amount of cargo in the low cluster – a significant difference for port managers to draw attention to.

We compare the above results with the extended CCR ratio, presented in the last row of Table 4. Again, a completely different trend was observed. By taking the sample average over the input and output data and with only first-moment information incorporated, the efficiency of Tuesday and Wednesday was underrated (in terms of ranking), but Thursday was overrated. This comparison, together with the one explained in §4.1, suggests that incorporating more information from data by considering data uncertainties and variations does make a difference.

4.3. Discussions of managerial insights

Despite answering two different questions, the above two examples contribute to the same function in port management – assessing and monitoring the relative technical efficiency of a port. The approach developed in this study provides a general and effective analytical tool, which is the prerequisite for port performance management and control in establishing standards, measuring the actual performance, comparing the performance with the standards, and taking corrective action.

First, the efficiency score allows port operators to establish a baseline in stevedore operations. For example, the results visualized in Figure 3 identify the most efficient weeks. These are meaningful benchmarks for the port operators to understand the highest achievable capability currently in cargo handling functionality, resource usage, change detection, cost control and value assessment.

Second, well-designed and executed port efficiency monitoring and assessment procedures are critical tools for operators to capture the trend over time and better utilize their resources. For example, provided that historical data suggests poor efficiency during the week of CNY, special attention should be paid to the stevedore operations. The external reason for low cargo volume is unavoidable, but better crew management can improve unnecessary waste of waiting time through workforce planning and scheduling. In the second example, it is valuable for port managers to scrutinize why Thursday and Friday are the least efficient days of the week. The findings will help inform planning and prioritization. Such monitoring and assessments play a foundational role in continuous improvement in today's business environment when ports are facing strong pressures in order to meet modern demand characteristics.

Finally, although the discussion so far has been devoted to port efficiency dynamics that compare the relative performance in a particular period compared with all periods under assessment, the applications of our approach are not exhaustive. For example, it can be applied to study the relative efficiency performance of one group of workers compared with other groups. Accordingly, reward and punishment programs can be developed and executed. For another example, it can be used to study port efficiency ranking that assesses the relative performance of one port compared with comparable ports to understand port competitiveness profile and make continuous monitoring and improvement.

5. Conclusions

We have addressed a port efficiency evaluation problem under random variations of the observed input and output data. We identified significant gaps arising from the existing approaches' incompatibility with the port management context. First, the existing approaches have relied almost exclusively on the a-priori normal distributional assumption. We have provided a piece of empirical evidence from real data sets that many input and output variables of port operations demonstrate long-tail distributions so that the normal distribution requirement is violated. Second, the past studies on stochastic efficiency evaluation are often silent about the interpretation of the efficiency indices, unlike the fundamental and well-known CCR ratio. Finally, the intractability of the existing models creates a burden in computation, which hinders the consideration of data variation issues in applications.

We have developed a risk-based efficiency evaluation approach to close the gaps. We start by defining the interpretable efficiency score as an index with four essential properties of the classic CCR ratio for deterministic efficiency evaluation. Then we proposed a risk-based index (β-RPE score) for efficiency evaluation under uncertainty. The risk-based index is based on a risk-based PPS that consists of a set of random input-output vectors guaranteed to be production possible under a certain risk level associated with the AEVaR. The nice properties of AEVaR allow us to show that the proposed risk-based efficiency index is interpretable. We proposed a risk-based efficiency evaluation (β-RPE) model for port efficiency evaluation with panel data and derived its tractable ECP reformulation to facilitate the computation under both non-parametric and parametric distributional assumptions.

We have demonstrated two applications of our approach in a real example. We have identified two interesting phenomena. First, the CNY effect – the operational efficiency during the CNY period is significantly lower than during ordinary periods. Second, port efficiency fluctuates with the day of the week, with Saturday and Friday being the most and least efficient days, respectively. In the same vein, researchers and port managers can use our model to evaluate and quantify port efficiency, especially when their dataset demonstrates temporal variation. This can be followed by regression studies to investigate the causal effects between external factors (such as vessel cancellations and the COVID-19) and port efficiency.

Finally, although the β-RPE model is motivated from a challenge we faced in port efficiency evaluation, the proposed theory and models are widely applicable to other contexts of technical efficiency evaluation. Taking transportation researches for instance, possible applications include but not limited to airline and airport management (e.g., Merkert and Hensher 2011 and Assaf et al. 2014), bus transit and road transportation (e.g., Viton 1997, Dervaux et al. 1998, and Lederman and Wynter 2011), railway freight transportation (e.g., Hilmola 2007 and Kleinová 2016) and port management (see, e.g., Merk and Dang 2012, Suárez-Alemán et al. 2016).

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Appendix A: A technical review on stochastic efficiency evaluation

Several studies have developed approaches to account for the stochasticities in efficiency evaluation (e.g., Olesen and Petersen 1995, Wei et al. 2014).

Olesen and Petersen (1995) present the below model (referred to as OP model hereinafter) :

[OP]

$$\begin{aligned}\sup_{\boldsymbol{u},\boldsymbol{v}}\ & \boldsymbol{u}^\top \overline{\boldsymbol{y}}_o + \Phi^{-1}(1-\alpha_o)\sqrt{(\boldsymbol{v}^\top, -\boldsymbol{u}^\top)\boldsymbol{\Lambda}_o(\boldsymbol{v}^\top, -\boldsymbol{u}^\top)^\top} \\ \text{s.t. } & \boldsymbol{v}^\top \overline{\boldsymbol{x}}_o = 1 \\ & \boldsymbol{u}^\top \overline{\boldsymbol{y}}_j - \boldsymbol{v}^\top \overline{\boldsymbol{x}}_j + \Phi^{-1}(1-\alpha_j)\sqrt{(\boldsymbol{v}^\top, -\boldsymbol{u}^\top)\boldsymbol{\Lambda}_j(\boldsymbol{v}^\top, -\boldsymbol{u}^\top)^\top} \le 0, \quad j = 1, ..., n \\ & \boldsymbol{u}, \boldsymbol{v} \ge \boldsymbol{0}, \end{aligned} \tag{8}$$

where the observations are assumed to be mutually independent random vectors $(\tilde{\boldsymbol{x}}_j, \tilde{\boldsymbol{y}}_j)$ following multivariate normal distribution with mean $(\bar{\boldsymbol{x}}_j, \bar{\boldsymbol{y}}_j)$ and covariance matrix $\boldsymbol{\Lambda}_j$ among DMUs. The notation $\Phi^{-1}(\cdot)$ denotes the inverse function of the cumulative distribution function of the standard normal distribution and $\alpha_j \le 0.5$ captures the probability violation tolerance level for $j = 1, ..., n$. Observe that the second set of constraints can be considered as a chance constraint under a given tolerance level α_j so that $\mathbb{P}\left[\boldsymbol{u}^\top \boldsymbol{y}_j - \boldsymbol{v}^\top \boldsymbol{x}_j \le 0\right] \ge 1 - \alpha_j$ for all $j = 1, ..., n$. For $\alpha_o < 0.5$, we take supreme of the convex objective function and the problem becomes non-convex. In addition, the objective function remains unexplained in the paper. To understand this objective function, we consider its hypo-graph form $\boldsymbol{u}^\top \overline{\boldsymbol{y}}_o + \Phi^{-1}(1-\alpha_o)\sqrt{(\boldsymbol{v}^\top, -\boldsymbol{u}^\top)\boldsymbol{\Lambda}_o(\boldsymbol{v}^\top, -\boldsymbol{u}^\top)^\top} \ge t$. Observe that its chance-constrained equivalence is $\mathbb{P}\left[\dfrac{\boldsymbol{u}^\top \tilde{\boldsymbol{y}}_o}{\boldsymbol{v}^\top \tilde{\boldsymbol{x}}_o} \le 1 + \dfrac{t-1}{\boldsymbol{v}^\top \tilde{\boldsymbol{x}}_o}\right] \le 1 - \alpha_o$. In other words, the model (8) tries to find the greatest t such that the DMU_o has an efficiency level of less than or equal to $1 + \dfrac{t-1}{\boldsymbol{v}^\top \tilde{\boldsymbol{x}}_o}$ with a confidence of at most $1 - \alpha_o$. Recall that α_o is the tolerable violation probability with $\alpha_o \le 0.5$, so "at most $1 - \alpha_o$" turns out to be inexplicable.

Wei et al. (2014) propose another chance-constrained DEA model (referred to as WCW model hereinafter), which is more interpretable:

[WCW]

$$\begin{aligned}\sup_{\boldsymbol{u},\boldsymbol{v},\mu}\ & \mu \\ \text{s.t. } & \mathbb{P}\left[\frac{\boldsymbol{u}^\top \tilde{\boldsymbol{y}}_o}{\boldsymbol{v}^\top \tilde{\boldsymbol{x}}_o} \ge \mu\right] \ge 1 - \alpha_o \\ & \mathbb{P}\left[\frac{\boldsymbol{u}^\top \tilde{\boldsymbol{y}}_j}{\boldsymbol{v}^\top \tilde{\boldsymbol{x}}_j} \le 1\right] \ge 1 - \alpha_j \quad j = 1, ..., n \\ & \boldsymbol{u}, \boldsymbol{v} \ge \boldsymbol{0}. \end{aligned} \tag{9}$$

The computation of this model still relies on the assumption of the multivariate normal distribution, under which the WCW model can be solved by evaluating a sequence of second-order cone program (SOCP) bisectionally[3].

To the best of our knowledge, there are two recent studies, Chen and Zhu (2019) and Beraldi and Bruni (2020) that relax the normal distribution assumption. Chen and Zhu (2019) construct an ambiguity set with mean and covariance information, and model the problem as a distributionally robust optimization (DRO) problem (refer to Delage and Ye 2010). The DRO optimizes the objective function over the worst-case distribution in the ambiguity set. Despite the merits of the model, the ambiguity set used in Chen and

[3] Note that algorithm proposed in Wei et al. (2014) is more complicated than solving a sequence of SOCP feasibility problem bisectionally.

Zhu (2019) that only pins down the mean and covariance implies that the worst-case distribution can be restricted to support on at most three points (see Popescu 2007), which seems unnatural in the applications. In addition, such ambiguity set may lead to conservative or even infeasible models (see Theorem 6 in Chen and Zhu 2019). Beraldi and Bruni (2020) consider a special case of stochastic DEA with deterministic inputs and stochastic outputs under discrete distribution. They transform the chance-constrained problem with a risk-based objective as a mixed-integer linear program using big-M formulations, which is not polynomial time solvable (unless P=NP).

Appendix B: A simple motivating example for misspecification risk

Consider the following example that compares the technical efficiency of two ports, indexed by $j = 1, 2$. Each port uses a one-dimensional input, observed as $\tilde{x}_j$, to produce a one-dimensional output, $\tilde{y}_j$, and all observations $(\tilde{x}_j, \tilde{y}_j)$'s are independent. We assume the following DGP: $\tilde{y}_j = 1$ almost surely for $j = 1, 2$; and the DGP of $\tilde{x}_j$ follows one of the following three settings:

(a) $\tilde{x}_j \sim \text{Normal}(5/2 + j/10, 1/4)$;

(b) $\tilde{x}_j$ is supported on two points, $2 + j/10$ and $3 + j/10$, with equal probability;

(c) $\tilde{x}_j \sim \text{Uniform}[5/2 + j/10 - \sqrt{3}/2, 5/2 + j/10 + \sqrt{3}/2]$.

By the example setting, the efficiency of port 1 should be greater than that of port 2, since $-\tilde{x}_2$ is stochastically dominated by $-\tilde{x}_1$. Note also $\tilde{x}_j$ have the same mean and variance under the three DGPs for $j \in \{1, 2\}$.

We simulate 500 datasets under each DGP setting and each dataset contains the same number of observations for each port, namely $N_1 = N_2$. The efficiency of each port is evaluated using: (i) the OP model, (ii) the WCW model under joint normal distributional assumption on $(\tilde{x}_j, \tilde{y}_j)$ and (iii) the RPE model that does not assume normality of data. The performances of the three models are compared by counting the fraction of datasets under which port 2 has higher score than port 1, referred to as *violation frequency*. Figure 6 shows that the violation frequency is not sensitive to the tolerance level α in all the models under the three DGP settings (We set $\beta = (|\mathcal{M}| + |\mathcal{S}|)\alpha$ for comparison, as β represents the risk level of a $(|\mathcal{M}| + |\mathcal{S}|)-$dimensional random vector (see the definition of $\mathcal{T}_\beta$).) In the rest of Appendix B we set $\alpha = 0.2, \beta = 0.4$.

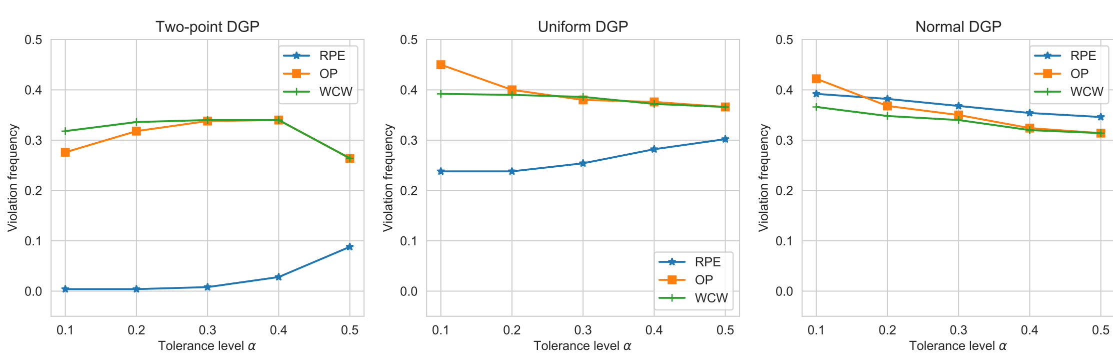

Figure 6 **Violation frequency versus tolerance or risk level** $(N_1 = N_2 = 10)$

Figure 7 visualizes the performances of the three models versus the numbers of observations, $N_1 = N_2 \in \{5, 10, 20, 50\}$, under each DGP setting. As illustrated in this figure, under the two-point DGP, the RPE model almost correctly identifies the efficiency ranking in all of the 500 datasets, even when the number of observations is very small. In contrast, the violation frequency of the OP model or WCW model is higher than that of the RPE model, especially when the number of observations is small. Similarly, under the uniform DGP setting, the RPE model outperforms the other two models. However, under the normal DGP setting, the OP and WCW models that assume the distribution of DGP correctly perform better than the non-parametric RPE model.

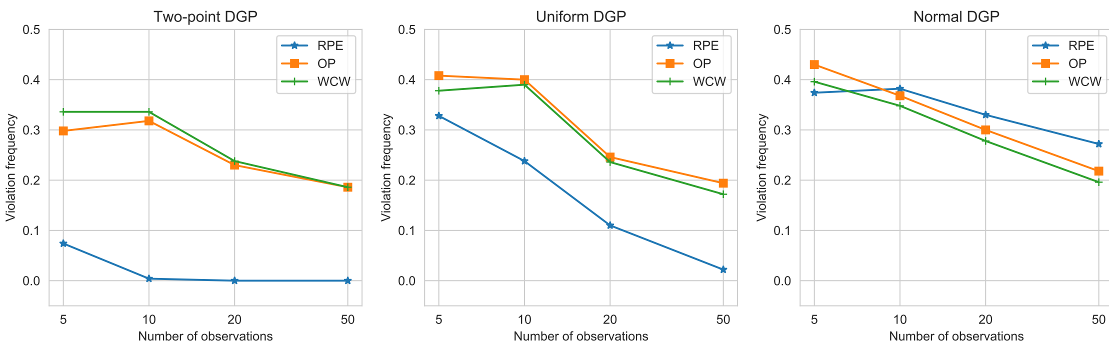

Figure 7 Violation frequency versus number of observations ($\alpha = 0.2, \beta = 0.4$)

The RPE model is robust to data uncertainty in highly ambiguous environment. As shown in Figure 7, RPE model outperforms the other two models when the number of observations is very small ($N_j = 5$). As the number of observations increases, the mean and variance of input data observed converges to those of the $\tilde{x}_1$ and $\tilde{x}_2$, so the performance of all the models improves. The violation frequency decreases at different speed due to the various convergence rate of each DGP.

In many real problems, researchers can hardly guess the distribution of parameters. This simulation study shows that, by simply assuming a normal distribution, efficiency evaluations are vulnerable to misspecification error, especially when the sample size is limited.

Appendix C: Proof of Theorems and Propositions

Proposition 1 *Proof.* The proofs for the first three properties are straightforward. We only show the forth property. For any $\kappa \in (0,1)$, $(\boldsymbol{x}, \boldsymbol{y})$ and $(\boldsymbol{x}', \boldsymbol{y}')$, suppose $\mu^*(\boldsymbol{x}, \boldsymbol{y}) \leq \mu$ and $\mu^*(\boldsymbol{x}', \boldsymbol{y}') \leq \mu$, then $(\mu\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{T}$ and $(\mu\boldsymbol{x}', \boldsymbol{y}') \in \mathcal{T}$ since $\mu^*(\boldsymbol{x}, \boldsymbol{y}) = \min\{\mu \mid (\mu\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{T}\}$. By definition, set $\mathcal{T}$ is convex, $(\kappa\mu\boldsymbol{x} + (1-\kappa)\mu\boldsymbol{x}', \kappa\boldsymbol{y} + (1-\kappa)\boldsymbol{y}') \in \mathcal{T}$. Hence, μ is feasible for $(\kappa\boldsymbol{x} + (1-\kappa)\boldsymbol{x}', \kappa\boldsymbol{y} + (1-\kappa)\boldsymbol{y}')$ and $\mu^*(\kappa\boldsymbol{x} + (1-\kappa)\boldsymbol{x}', \kappa\boldsymbol{y} + (1-\kappa)\boldsymbol{y}') \leq \mu$. Q.E.D.

Theorem 1 *Proof.* We check that $\eta_\beta(\tilde{\boldsymbol{z}})$ satisfies properties $(a)-(d)$:

(a) Monotone: For any $\tilde{\boldsymbol{z}} \geq \tilde{\boldsymbol{z}}'$ and any $\theta > 0$, clearly $\mathbb{E}_{\mathbb{P}}\left[e^{\tilde{z}_i/\theta}\right] \geq \mathbb{E}_{\mathbb{P}}\left[e^{\tilde{z}'_i/\theta}\right]$ for any $i = 1, ..., n$. Hence, $\theta \ln\left(\sum_{i=1}^{n} \mathbb{E}_{\mathbb{P}}\left[e^{\tilde{z}_i/\theta}\right]\right) - \theta\ln\beta \geq \theta\ln\left(\sum_{i=1}^{n}\mathbb{E}_{\mathbb{P}}\left[e^{\tilde{z}'_i/\theta}\right]\right) - \theta\ln\beta$ for any $\theta > 0$. It follows that $\eta_\beta(\tilde{\boldsymbol{z}}) \geq \eta_\beta(\tilde{\boldsymbol{z}}')$ after taking the infimum over $\theta > 0$.

(b) Translation invariant:

$$\begin{aligned}\eta_\beta(\tilde{\boldsymbol{z}}+t\mathbf{1}) &= \inf_{\theta>0}\theta\ln\left(\sum_{i=1}^n \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\tilde{z}_i+t}{\theta}\right)\right]\right)-\theta\ln\beta \\ &= \inf_{\theta>0}\theta\ln\left(\sum_{i=1}^n \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\tilde{z}_i}{\theta}\right)\right]\right)+t-\theta\ln\beta=\eta_\beta(\tilde{\boldsymbol{z}})+t\end{aligned}$$

(c) The convexity is straightforward as $\theta\ln\left(\sum_{i=1}^n \mathbb{E}_{\mathbb{P}}\left[\exp\left(\tilde{z}_i/\theta\right)\right]\right)-\theta\ln\beta$ is jointly convex in $\tilde{\boldsymbol{z}}$ and θ.

(d) Positive homogeneous: For any $k>0$, we have

$$\begin{aligned}\eta_\beta(k\tilde{\boldsymbol{z}}) &= \inf_{\theta>0}\theta\ln\left(\sum_{i=1}^n \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{k\tilde{z}_i}{\theta}\right)\right]\right)-\theta\ln\beta \\ &= k\inf_{\theta>0}\theta\ln\left(\sum_{i=1}^n \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\tilde{z}_i}{\theta}\right)\right]\right)-\theta\ln\beta \\ &= k\eta_\beta(\tilde{\boldsymbol{z}})\end{aligned}$$

Note also $\eta_\beta(\mathbf{0})=\inf_{\theta>0}\theta\ln(n/\beta)=0$ for any $\beta\in(0,1)$.

(e) For any random vector $\tilde{\boldsymbol{z}}$ of dimension n, any $t\in\mathbb{R}$, and $\theta>0$, we have

$$\begin{aligned}\mathbb{P}\left[\tilde{\boldsymbol{z}}\not\leq t\cdot\mathbf{1}\right]=\mathbb{P}\left[\bigcup_{i=1}^n\{\tilde{z}_i>t\}\right] &\leq \sum_{i=1}^n\mathbb{P}\left[\tilde{z}_i>t\right]=\sum_{i=1}^n\mathbb{P}\left[e^{\tilde{z}_i/\theta}>e^{t/\theta}\right] \\ &\leq \sum_{i=1}^n\mathbb{E}_{\mathbb{P}}\left[e^{\tilde{z}_i/\theta}\right]e^{-t/\theta}=e^{-t/\theta}\sum_{i=1}^n\mathbb{E}_{\mathbb{P}}\left[e^{\tilde{z}_i/\theta}\right].\end{aligned}$$

Hence $\inf_{\theta>0}e^{-t/\theta}\sum_{i=1}^n\mathbb{E}_{\mathbb{P}}\left[e^{\tilde{z}_i/\theta}\right]\leq\beta$ implies $\mathbb{P}\left[\tilde{\boldsymbol{z}}\leq t\cdot\mathbf{1}\right]>1-\beta$, so we have $\mathbb{P}\left[\tilde{\boldsymbol{z}}\leq\eta_\beta(\tilde{\boldsymbol{z}})\cdot\mathbf{1}\right]\geq 1-\beta$ by definition of $\eta_\beta(\tilde{\boldsymbol{z}})$. Q.E.D.

Theorem 2 *Proof.* The proof is straightforward and similar to that of Proposition 1. The positive homogeneity, monotonicity and quasi-convexity of the risk-based efficiency index ρ_β are derived from positive homogeneity, monotonicity and convexity of AEVaR η_β, respectively. Q.E.D.

Theorem 3 *Proof.* The properties follow from Theorem 2 except monotonicity. Suppose $(-\tilde{\boldsymbol{x}}_{j_1},\tilde{\boldsymbol{y}}_{j_1})\geq(-\tilde{\boldsymbol{x}}_{j_2},\tilde{\boldsymbol{y}}_{j_2})$, note $(\rho_\beta^{j_1}\tilde{\boldsymbol{x}}_{j_1},\tilde{\boldsymbol{y}}_{j_1})\in\mathcal{T}_\beta^{j_1}$ by definition, hence by monotonicity of $\hat{\eta}_\beta(\cdot)$ and $(-\tilde{\boldsymbol{x}}_{j_1},\tilde{\boldsymbol{y}}_{j_1})\geq(-\tilde{\boldsymbol{x}}_{j_2},\tilde{\boldsymbol{y}}_{j_2})$ we have $(\rho_\beta^{j_1}\tilde{\boldsymbol{x}}_{j_2},\tilde{\boldsymbol{y}}_{j_2})\in\mathcal{T}_\beta^{j_1}$. We claim $\mathcal{T}_\beta^{j_1}\subseteq\mathcal{T}_\beta^{j_2}$, then it follows that $(\rho_\beta^{j_1}\tilde{\boldsymbol{x}}_{j_2},\tilde{\boldsymbol{y}}_{j_2})\in\mathcal{T}_\beta^{j_2}$, hence $\rho_\beta^{j_1}\geq\rho_\beta^{j_2}$ by definition. To show $\mathcal{T}_\beta^{j_1}\subseteq\mathcal{T}_\beta^{j_2}$, take any $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})\in\mathcal{T}_\beta^{j_1}$, then there exists $\lambda_j\geq 0$ for $j\in\mathcal{J}\backslash\{j_1\}$ such that $\hat{\eta}_\beta\begin{pmatrix}\tilde{\boldsymbol{y}}-\sum_{j\in\mathcal{J}\backslash\{j_1\}}\lambda_j\tilde{\boldsymbol{y}}_j\\ \sum_{j\in\mathcal{J}\backslash\{j_1\}}\lambda_j\tilde{\boldsymbol{x}}_j-\tilde{\boldsymbol{x}}\end{pmatrix}\leq 0$. Since $(-\tilde{\boldsymbol{x}}_{j_1},\tilde{\boldsymbol{y}}_{j_1})\geq(-\tilde{\boldsymbol{x}}_{j_2},\tilde{\boldsymbol{y}}_{j_2})$, hence by monotonicity of $\hat{\eta}_\beta$, we have $\hat{\eta}_\beta\begin{pmatrix}\tilde{\boldsymbol{y}}-\sum_{j\in\mathcal{J}\backslash\{j_2\}}\lambda_j\tilde{\boldsymbol{y}}_j\\ \sum_{j\in\mathcal{J}\backslash\{j_2\}}\lambda_j\tilde{\boldsymbol{x}}_j-\tilde{\boldsymbol{x}}\end{pmatrix}\leq 0$ where $\lambda_{j_1}=\lambda_{j_2}\geq 0$. Hence $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})\in\mathcal{T}_\beta^{j_2}$. Q.E.D.

Proposition 3 *Proof.* By definition of ρ_β^o and $\mathcal{T}_\beta^o$, we have

$$\begin{aligned}
\rho_\beta^o = \ & \inf \ \mu \\
& \text{s.t. } \theta \ln \mathbb{E}_{\hat{\mathbb{P}}}\left[\exp\left(\frac{\tilde{y}_{o,s} - \sum_{j\in\mathcal{J}\setminus\{o\}} \lambda_j \tilde{y}_{j,s}}{\theta}\right)\right] - \theta\ln(\nu_s/\theta) \le 0 && \forall s \in \mathcal{S} \\
& \quad \theta \ln \mathbb{E}_{\hat{\mathbb{P}}}\left[\exp\left(\frac{\sum_{j\in\mathcal{J}\setminus\{o\}} \lambda_j \tilde{x}_{j,m} - \mu \tilde{x}_{o,m}}{\theta}\right)\right] - \theta\ln(\gamma_m/\theta) \le 0 && \forall m \in \mathcal{M} \\
& \quad \left(\sum_{s\in\mathcal{S}} \nu_s + \sum_{m\in\mathcal{M}} \gamma_m\right) \le \beta\theta \\
& \quad \boldsymbol{\lambda} \ge \mathbf{0}, \theta > 0
\end{aligned} \tag{10}$$

For constraint $\theta \ln \mathbb{E}_{\hat{\mathbb{P}}}\left[\exp\left(\frac{\tilde{y}_{o,s} - \sum_{j\in\mathcal{J}\setminus\{o\}} \lambda_j \tilde{y}_{j,s}}{\theta}\right)\right] - \theta\ln(\nu_s/\theta) \le 0$, its left-hand side can be re-written by

$$\begin{aligned}
& \theta \ln\left(\mathbb{E}_{\hat{\mathbb{P}}}\left[\exp\left(\frac{\tilde{y}_{o,s}}{\theta}\right)\right] \prod_{j\in\mathcal{J}\setminus\{o\}} \mathbb{E}_{\hat{\mathbb{P}}}\left[\exp\left(\frac{-\lambda_j \tilde{y}_{j,s}}{\theta}\right)\right]\right) - \theta\ln(\nu_s/\theta) \\
= \ & \theta \ln \mathbb{E}_{\hat{\mathbb{P}}}\left[\exp\left(\frac{\tilde{y}_{o,s}}{\theta}\right)\right] + \sum_{j\in\mathcal{J}\setminus\{o\}} \theta \ln \mathbb{E}_{\hat{\mathbb{P}}}\left[\exp\left(\frac{-\lambda_j \tilde{y}_{j,s}}{\theta}\right)\right] - \theta\ln(\nu_s/\theta)
\end{aligned}$$

We look at the term $\theta \ln \mathbb{E}_{\hat{\mathbb{P}}}\left[\exp\left(\frac{\tilde{y}_{o,s}}{\theta}\right)\right]$, which has epigraph form

$$\begin{aligned}
\theta \ln \mathbb{E}_{\hat{\mathbb{P}}}\left[\exp\left(\frac{\tilde{y}_{o,s}}{\theta}\right)\right] \le t_{o,s} & \Longleftrightarrow \sum_{\ell=1}^{N_o} \theta \exp\left(\frac{y_{o,s}^\ell - t_{o,s}}{\theta}\right) \le N_o\theta \\
& \Longleftrightarrow \begin{cases} \sum_{\ell=1}^{N_o} r_{o,s}^\ell \le N_o\theta \\ \theta \exp\left(\frac{y_{o,s}^\ell - t_{o,s}}{\theta}\right) \le r_{o,s}^\ell \quad \forall \ell = 1, ..., N_o \end{cases}
\end{aligned}$$

where we introduce auxiliary variables $r_{o,s}^\ell$ in the last equivalence transformation, which holds because

$$\begin{aligned}
& \left\{(\theta, t_{o,s}) : \sum_{\ell=1}^{N_o} \theta \exp\left(\frac{y_{o,s}^\ell - t_{o,s}}{\theta}\right) \le N_o\theta\right\} \\
= & \left\{(\theta, t_{o,s}) : \exists r_{o,s}^l \text{ for all } \ell = 1, ..., N_o, \text{ such that } \sum_{\ell=1}^{N_o} r_{o,s}^\ell \le N_o\theta \text{ and } \theta \exp\left(\frac{y_{o,s}^\ell - t_{o,s}}{\theta}\right) \le r_{o,s}^\ell\right\}.
\end{aligned}$$

Similarly, $\theta \ln \mathbb{E}_{\hat{\mathbb{P}}}\left[\exp\left(\frac{-\lambda_j \tilde{y}_{j,s}}{\theta}\right)\right] \le t_{j,s}$ is equivalent to $\begin{cases} \sum_{\ell=1}^{N_j} r_{j,s}^\ell \le N_j\theta \\ \theta \exp\left(\frac{-\lambda_j y_{j,s}^\ell - t_{j,s}}{\theta}\right) \le r_{j,s}^\ell \quad \forall \ell = 1, ..., N_j \end{cases}$ for all $j \in \mathcal{J}\setminus\{o\}$. And $-\theta\ln(\nu_s/\theta) \le \kappa_s$ is equivalent to $\nu_s \ge \exp\left(-\frac{\kappa_s}{\theta}\right)$. Hence the constraint

$\theta \ln \mathbb{E}_{\hat{\mathbb{P}}}\left[\exp\left(\frac{\tilde{y}_{o,s} - \sum_{j\in\mathcal{J}} \lambda_j \tilde{y}_{j,s}}{\theta}\right)\right] - \theta \ln(\nu_s/\theta) \le 0$ is equivalent to

$$\begin{cases} \sum_{j\in\mathcal{J}} t_{j,s} + \kappa_s \le 0 \\ \sum_{\ell=1}^{N_o} r^{\ell}_{o,s} \le N_o \theta \\ \theta \exp\left(\frac{y^{\ell}_{o,s} - t_{o,s}}{\theta}\right) \le r^{\ell}_{o,s} & \forall \ell = 1, ..., N_o \\ \sum_{\ell=1}^{N_j} r^{\ell}_{j,s} \le N_j \theta & \forall j \in \mathcal{J}\backslash\{o\} \\ \theta \exp\left(\frac{-\lambda_j y^{\ell}_{j,s} - t_{j,s}}{\theta}\right) \le r^{\ell}_{j,s} & \forall j \in \mathcal{J}\backslash\{o\}, \quad \forall \ell = 1, ..., N_j \\ \exp\left(-\frac{\kappa_s}{\theta}\right) \le \nu_s \end{cases}$$

In the same way, we can transform all the constraints involving exponential and logarithm in formulation (10) and get the β-RPE-ECP formulation. Q.E.D.

Proposition 4 *Proof.* Consider model (10) with the empirical distribution $\hat{\mathbb{P}}$ replaced by the true distribution of $\mathbb{P}$. For the constraint $\theta \ln \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\tilde{y}_{o,s} - \sum_{j\in\mathcal{J}\backslash\{o\}} \lambda_j \tilde{y}_{j,s}}{\theta}\right)\right] - \theta \ln(\nu_s/\theta) \le 0$, we know the left-hand side is

$$\begin{aligned} &\theta \ln\left(\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\tilde{y}_{o,s}}{\theta}\right)\right] \prod_{j\in\mathcal{J}\backslash\{o\}} \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{-\lambda_j \tilde{y}_{j,s}}{\theta}\right)\right]\right) - \theta \ln(\nu_s/\theta) \\ &= \theta \ln \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\tilde{y}_{o,s}}{\theta}\right)\right] + \sum_{j\in\mathcal{J}\backslash\{o\}} \theta \ln \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{-\lambda_j \tilde{y}_{j,s}}{\theta}\right)\right] - \theta \ln(\nu_s/\theta) \\ &= \theta \ln M_{\tilde{y}_{o,s}}\left(\frac{1}{\theta}\right) + \sum_{j\in\mathcal{J}\backslash\{o\}} \theta \ln M_{\tilde{y}_{j,s}}\left(-\frac{\lambda_j}{\theta}\right) - \theta \ln(\nu_s/\theta) \end{aligned}$$

Similarly, $\theta \ln \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\sum_{j\in\mathcal{J}\backslash\{o\}} \lambda_j \tilde{x}_{j,m} - \mu \tilde{x}_{o,m}}{\theta}\right)\right] - \theta \ln(\gamma_m/\theta) \le 0$ is equivalent to $\sum_{j\in\mathcal{J}\backslash\{o\}} \theta \ln M_{\tilde{x}_{j,m}}(\lambda_j/\theta) + \theta \ln M_{\tilde{x}_{o,m}}(-\mu/\theta) - \theta \ln(\gamma_m/\theta) \le 0.$ Q.E.D.

Appendix D: Proof of results shown in Table 3

Proof. We derive the equivalent reformulation of $\theta \ln M_X(a/\theta) \le c$ under different distribution as follows.

Binomial: $\theta \ln(1-p+pe^{a/\theta})^n \le c \Longleftrightarrow (1-p)\theta e^{-c/n\theta} + p\theta e^{a/\theta - c/n\theta} \le \theta \begin{cases} (1-p)b_1 + pb_2 \le \theta \\ \theta e^{-c/n\theta} \le b_1 \\ \theta e^{(a-\frac{c}{n})/\theta} \le b_2 \end{cases}$ where we introduce auxiliary b_1 and b_2 in the last equivalence transformation. It holds as

$$\begin{aligned} &\left\{(\theta, a, c) : (1-p)\theta e^{-c/n\theta} + p\theta e^{a/\theta - c/n\theta} \le \theta\right\} \\ = &\left\{(\theta, a, c) : \exists b_1, b_2 \text{ such that } (1-p)b_1 + pb_2 \le \theta, \theta e^{-c/n\theta} \le b_1 \text{ and } \theta e^{(a-\frac{c}{n})/\theta} \le b_2\right\}. \end{aligned}$$

The last two constraints $\theta e^{-c/n\theta} \le b_1$ and $\theta e^{(a-\frac{c}{n})/\theta} \le b_2$, respectively, correspond to $(b_1, \theta, -c/n) \in \mathcal{K}_{\exp}$ and $\left(b_2, \theta, a - \frac{c}{n}\right) \in \mathcal{K}_{\exp}$.

Normal:

$$\begin{aligned} \theta \ln e^{\frac{a}{\theta}\mu + \frac{1}{2}\sigma^2(\frac{a}{\theta})^2} \le c \Longleftrightarrow (a\sigma)^2 \le 2\theta(c - a\mu) &\Longleftrightarrow \begin{cases} 2(a\sigma)^2 \le (c - a\mu + \theta)^2 - (c - a\mu - \theta)^2 \\ c - a\mu + \theta \ge 0 \end{cases}, \\ &\Longleftrightarrow \sqrt{(\sqrt{2}a\sigma)^2 + (c - a\mu - \theta)^2} \le c - a\mu + \theta \end{aligned}$$

which is a second-order cone representable (see e.g., Boyd and Vandenberghe 2004).

Poisson:

$$\theta \ln e^{\lambda(e^{a/\theta}-1)} \le c \Longleftrightarrow \lambda\theta e^{a/\theta} \le c + \lambda\theta \Longleftrightarrow \begin{cases} \theta e^{a/\theta} \le b \\ \lambda b \le c + \theta\lambda \end{cases}$$

where we introduce auxiliary variable b in the last equivalence transformation. It holds as $\left\{(\theta, a, c) : \lambda\theta e^{a/\theta} \le c + \lambda\theta\right\} = \left\{(\theta, a, c) : \exists b \text{ such that } \theta e^{a/\theta} \le b \text{ and } \lambda b \le c + \theta\lambda\right\}$. And $\theta e^{a/\theta} \le b$ corresponds to $(b, \theta, a) \in \mathcal{K}_{\text{exp}}$. Q.E.D.

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