Noname manuscript No.
(will be inserted by the editor)

Mean Field Portfolio Games

Guanxing Fu · Chao Zhou

Abstract We study mean field portfolio games with random parameters, where each player is concerned with not only her own wealth but also relative performance to her competitors. We use the martingale optimality principle approach to characterize the unique Nash equilibrium in terms of a mean field FBSDE with quadratic growth, which is solvable under a weak interaction assumption. Motivated by the weak interaction assumption, we establish an asymptotic expansion result in powers of the competition parameter. When the market parameters do not depend on the Brownian paths, we obtain the Nash equilibrium in closed form.

G. Fu
Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
E-mail: guanxing.fu@polyu.edu.hk
C. Zhou
Department of Mathematics and Risk Management Institute, National University of Singapore
E-mail: matzc@nus.edu.sg

1 Introduction

Mean field games (MFGs) are a powerful tool to study large population games, where each individual player has negligible influence on the outcome of the game. Introduced independently by Huang et al. [19] and Lasry and Lions [23], MFGs have received considerable attention in the probability and financial mathematics literature. In this paper, we study a classs of mean field portfolio games with random market parameters by the martingale optimality principle (MOP) approach.

Assume that there are N risky assets in the market, with price dynamics of asset $i \in \{1, \ldots, N\}$ following

$$dS_t^i = S_t^i \Big(h_t^i \, dt + \sigma_t^i \, W_t^i + \sigma_t^{i0} \, dW_t^0 \Big), \qquad (1.1)$$

where the return rate h^i and the volatility (σ^i, σ^{i0}) are assumed to be bounded and progressively measurable stochastic processes; W^i is a Brownian motion describing the idiosyncratic noise to the asset i; and W^0 is a Brownian motion that is independent of W^i, describing common noise to all risky assets. The interest rate of the risk-free asset is assumed to be zero for simplicity. Let X^i be the wealth process of player i, who trades asset i, and $\overline{X}^{-i}$ be the performance index of player i. Each player solves a utility maximization problem and she is concerned with not only her own wealth X^i, but also the "difference" between her wealth and the performance index.

We further assume that the risk preference of players is characterized by power utility functions, i.e., player $i \in \{1, 2, \ldots, N\}$ chooses the fraction of her wealth invested in the risky asset i to maximize the objective function:

$$\max_{\pi^i} \quad \mathbb{E}\left[\frac{1}{\gamma^i} \left(X_T^i (\overline{X}_T^{-i})^{-\theta^i} \right)^{\gamma^i} \right], \qquad (1.2)$$

where the wealth process X^i follows

$$dX_t^i = \pi_t^i X_t^i \Big(h_t^i \, dt + \sigma_t^i \, dW_t^i + \sigma_t^{i0} \, dW_t^0 \Big), \quad X_0^i = x^i, \qquad (1.3)$$

where $\overline{X}^{-i} = \big(\Pi_{j \neq i} X^j \big)^{\frac{1}{N-1}}$ is the geometric average of all players' wealth except for player i; $\gamma^i \in (-\infty, 1)/\{0\}$ is the degree of risk aversion; and $\theta^i \in [0,1]$ is the relative competition parameter: player i is more concerned with her own terminal wealth X_T^i if θ^i is closer to 0 and more concerned with the relative distance $\frac{X_T^i}{\overline{X}_T^{-i}}$ between her terminal wealth and the performance index if θ^i is closer to 1. The goal is to find a Nash equilibrium (NE) $(\pi^{1,*}, \ldots, \pi^{N,*})$ such that $\pi^{i,*}$ is the optimal strategy for player i and no one wants to change her strategy unilaterally.

By MOP, as in Hu et al. [18] and Rouge and El Karoui [25] for single player's utility maximization problems, the unique NE for the N-player game (1.2)-(1.3) can be characterized by a multidimensional FBSDE with quadratic growth; see Section 5. Although such FBSDE is solvable, the equations are

tedious. The analysis can be significantly simplified by studying the following associated MFG

$$
\begin{cases}
\text{1. Fix } \mu \text{ in some suitable space.} \\
\text{2. Solve the optimization problem} \\
\quad \mathbb{E}\left[\dfrac{1}{\gamma}(X_T \mu_T^{-\theta})^\gamma\right] \to \max \ \text{over } \pi \\
\quad \text{such that } dX_t = \pi_t X_t (h_t \, dt + \sigma_t \, dW_t + \sigma_t^0 \, dW_t^0), \ X_0 = x. \\
\text{3. Search for the fixed point } \mu_t = \exp\left(\mathbb{E}[\log X_t^* | \mathcal{F}_t^0]\right), \ t \in [0,T], \\
\quad X^* \text{ is the optimal wealth from 2} \\
\quad \text{and } \mathcal{F}^0 \text{ is the filtration generated by } W^0.
\end{cases}
\tag{1.4}
$$

In the MFG (1.4), by the approach introduced in [19,23], it is only necessary to consider a representative player's utility maximization problem with μ fixed, which in turn should be consistent with the aggregation of optimal wealth.

In the mathematical finance literature, the first result on portfolio games with relative performance concerns, including N-player games and MFGs, was obtained by Espinosa and Touzi [8]: in the context of a complete market, the unique NE was established for general utility functions; in the context of an incomplete market where each player has a heterogeneous portfolio constraint, by assuming that the drift and volatility of the log price are deterministic, using a BSDE approach, [8] obtained a unique NE for exponential utility functions, which was called *deterministic Nash equilibrium* in that paper. Moreover, the convergence from N-player games to MFGs was also studied in [8]. Later, Frei and dos Reis [9] studied similar portfolio games to [8] from a different perspective: they constructed counterexamples where no NE exists. In contrast to [8,9], where all players trade common stocks, Lacker and Zariphopoulou [22] investigated N-player and mean field portfolio games where the stock price follows (1.1), but with constant market parameters. Using a PDE approach, [22] established a *constant equilibrium* that was proven to be unique among all constant ones. In addition, portfolio games with mean field interaction have been examined in [6, 7, 17, 21], where in [6, 7] dos Reis and Platonov studied N-player games and MFGs with forward utilities; and in [21], Lacker and Soret extended the CRRA model in [22] to include consumption, by using PDE approaches. In [17], Hu and Zariphopoulou studied portfolio games in an Itô-diffusion environment.

In this paper, we study portfolio games with power utility functions. The games with exponential utility functions and log utility functions can be studied in the same manner; refer to Remark 2.4 and Remark 2.5. Our paper makes two main contributions. The first one is the wellposedness result of NE for the portfolio game (1.4). We first establish a one-to-one correspondence between the NE of (1.4) with random market parameters and some mean field FBSDE with quadratic growth. Such correspondence result is crucial to prove the uniqueness of the NE result. We then solve the FBSDE under a weak interaction assumption, i.e., the competition parameter θ is assumed to be small. Such

assumption is widely used in the game theory and financial mathematics literature; see [4, 5, 10, 11, 13, 15], among others. In order to achieve this, our idea is to first consider the difference between the FBSDE and the benchmark one when $\theta = 0$ to cancel out non-homogenous terms, and then transform the resulting FBSDE into a mean field quadratic BSDE. It is worth noting that although a transformation argument from FBSDE to BSDE was also used in [8], there is an essential difference from our transformation: the terminal condition of our resulting BSDE is bounded, which makes it convenient to apply the theory of quadratic BSDE. To solve the BSDE, our idea is to decompose the driver into two parts. Specifically, one part does not depend on θ and the other part depends on θ, and all mean field terms belong to the second part so that they can be controlled by θ. To the best of our knowledge, both our existence and uniqueness results are new in the literature. In particular, the contribution of our uniqueness result is two-fold. On the one hand, our uniqueness result partially generalizes the uniqueness result in [8] in the sense that our utility functions can be beyond exponential ones, our stock price is driven by both idiosyncratic and common noise, and both the market parameters and the admissible strategies can be random; however, there is no trading constraint in our paper. On the other hand, when the market parameters are independent of the Brownian paths, we construct a unique NE in L^∞ in closed form, which is beyond constant strategies. This result completely generalizes the result in [22], where only constant NE was studied. Such generalization strongly relies on our FBSDE approach, which shares a similar idea to [3, 8, 9] and is more powerful than the PDE approach in [22].

Our second contribution is an asymptotic expansion result. Motivated by the weak interaction assumption, we provide an approximation in any order of the value function and the optimal investment for the model with competition in terms of the solutions to the benchmark model without competition when the investor is only concerned with her own wealth. These results enable us to obtain the value function and the optimal investment based only on the benchmark model in the case of a small competition parameter. In order to obtain the asymptotic expansion results, our idea is to start with the FBSDE charaterization of the NE, and establish the expansion of the solution to the FBSDE by studying the iterative system of FBSDEs with coefficients in the BMO space. Our analysis relies on the application of energy inequality and reverse Hölder's inequality for BMO martingales. Asymptotic expansion results in stochastic optimization and game setting were also studied in [16] and [4, 5] by PDE analysis. In [16], Horst et al. investigated a single-player optimal liquidation problem under ambiguity with respect to price impact parameters. They established a first-order approximation result of the robust model for small uncertainty factors, while our approximation is in any order. In [4, 5], Chan and Sircar analyzed continuous time Bertrand and Cournot competitions as MFGs. Instead of solving the forward-backward PDE characterizing the NE, they established a formal asymptotic expansion result in powers of the competition parameter; no rigorous proof of the asymptotic result was provided

there. It is worth noting that the market parameters in [4,5,16] are deterministic, while ours are random.

The remainder of this paper is organized as follows:
• After the introduction of notation, in Section 2, we establish an equivalent relationship between the existence of NE of the MFG (1.4) and the solvability of some mean field FBSDE.
• In Section 3, we study the MFG (1.4) in detail; in particular, Section 3 addresses the wellposedness of the MFG with general market parameters by solving the FBSDE introduced in Section 2. Moreover, when the market parameters do not depend on the Brownian paths, we find the NE in closed form.
• In Section 4, the asymptotic result is established. Specifically, the logarithm of value function and the optimal investment are expanded into any order in powers of θ.
• In Section 5, we comment on the result of N-player games.

Notation. In the probability space $(\Omega, \mathbb{P}, \mathbb{K} = (\mathcal{K}_t)_{0 \leq t \leq T})$, a two-dimensional Brownian motion $\overline{W} = (W, W^0)^\top$ is defined, where W is the idiosyncratic noise for the representative player, and W^0 is the common noise for all players. In addition, $\mathbb{G} = \{\mathcal{G}_t, t \in [0, T]\}$ is assumed to be the augmented natural filtration of $\overline{W}$. The augmented natural filtration of W^0 is denoted by $\mathbb{F}^0 = \{\mathcal{F}_t^0, t \in [0, T]\}$. Let $\mathcal{A}$ be a σ-algebra that is independent of $\mathbb{G}$, to describe heterogeneous types of players. Let $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ be the σ-algebra generated by $\mathcal{A}$ and $\mathbb{G}$.

For a random variable ξ, let $\|\xi\|$ be the essential supremum of its absolute value $|\xi|$, and let $\underline{\xi}$ be its essential infimum. For a sub σ-algebra $\mathbb{H}$ of $\mathbb{F}$, let $\mathrm{Prog}(\Omega \times [0, T]; \mathbb{H})$ be the space of all stochastic processes that are $\mathbb{H}$-progressively measurable. For each $\eta \in \mathrm{Prog}(\Omega \times [0, T]; \mathbb{H})$, define $\|\eta\|_\infty = \mathrm{ess\,sup}_{\omega \in \Omega, t \in [0,T]} |\eta_t(\omega)|$. Let $L_{\mathbb{H}}^\infty$ be the space of all stochastic processes that are essentially bounded, i.e.,

$$L_{\mathbb{H}}^\infty = \{\eta \in \mathrm{Prog}(\Omega \times [0, T]; \mathbb{H}) : \|\eta\|_\infty < \infty\}.$$

Let $S_{\mathbb{H}}^\infty$ be the subspace of $L_{\mathbb{H}}^\infty$, where the trajectories of all processes are continuous. Furthermore, for a probability measure $\mathbb{Q}$ and for $p > 1$, define

$$S_{\mathbb{Q},\mathbb{H}}^p = \left\{ \eta \in \mathrm{Prog}(\Omega \times [0, T]; \mathbb{H}) : \eta \text{ has continuous trajectory} \right.$$
$$\left. \text{and } \|\eta\|_{S_{\mathbb{Q},\mathbb{H}}^p} := \left(\mathbb{E}^{\mathbb{Q}} \left[\sup_{0 \leq t \leq T} |\eta_t|^p \right] \right)^{1/p} < \infty \right\}$$

and

$$M_{\mathbb{Q},\mathbb{H}}^p = \left\{ \eta \in \mathrm{Prog}(\Omega \times [0, T]; \mathbb{H}) : \|\eta\|_{M_{\mathbb{Q},\mathbb{H}}^p}^p := \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T |\eta_t|^2 \, dt \right)^{\frac{p}{2}} \right] < \infty \right\}.$$

Define the BMO space under $\mathbb{Q}$ as

$$H^2_{BMO,\mathbb{Q},\mathbb{H}} = \left\{ \eta \in \mathrm{Prog}(\Omega \times [0,T];\mathbb{H}) : \right.$$

$$\left. \|\eta\|^2_{BMO,\mathbb{Q},\mathbb{H}} := \sup_{\tau:\mathbb{H}-\mathrm{stopping\ time}} \left\| \mathbb{E}^{\mathbb{Q}} \left[\int_\tau^T |\eta_t|^2\, dt \middle| \mathcal{F}_\tau \right] \right\|_\infty < \infty \right\}.$$

In particular, if $\mathbb{Q} = \mathbb{P}$, which is the physical measure, and/or $\mathbb{H} = \mathbb{F}$, we drop the dependence on $\mathbb{Q}$ and/or $\mathbb{H}$ in the definition of the above spaces.

Assumption 1.1 The initial wealth x, the competition parameter θ, and the risk aversion parameter γ of the population are assumed to be bounded $\mathcal{A}$-measurable random variables. In addition, x is valued in $(0,\infty)$, θ is valued in $[0,1]$, and γ is valued in $(-\infty,1)/\{0\}$.

Assume the return rate $h \in L^\infty$ and the volatility $\widetilde{\sigma} := (\sigma,\sigma^0)^\top \in L^\infty \times L^\infty$. Furthermore, $|\gamma|$, $|\sigma| + |\sigma^0|$ are bounded away from 0, i.e., $|\gamma| \geq \underline{\gamma} > 0$ a.s. and $\mathrm{ess\,inf}_{\omega \in \Omega} \inf_{t \in [0,T]} \widetilde{\sigma}^\top \widetilde{\sigma} > 0$.

Moreover, we assume that the space of admissible strategies is H^2_{BMO}.

Remark 1.2 (1) In Assumption 1.1, the assumption that (x, γ, θ) is $\mathcal{A}$-measurable is consistent with the formulation in [8, Remark 5.10] and the *random type* introduced in [22];

(2) The space of admissible strategies is consistent with [9] and [24];

2 MFGs and Mean Field FBSDEs Are Equivalent

In this section, we prove that the solvability of some mean field FBSDE is sufficient and necessary for the solvability of the MFG (1.4). This equivalent result is key to establish the uniqueness result of NE. The sufficient part is proven by MOP in [18], and the necessary part is proven by the dynamic programming principle in [8, Theorem 4.7] and [9, Lemma 3.2], where the N-player game with exponential utility functions and trading constraint, but without idiosyncratic noise, was studied.

Before the main result of this section, we introduce the definition of NE.

Definition 2.1 We say that the pair (μ^*, π^*) is an NE of (1.4), if $\pi^* \in H^2_{BMO}$, $\mu_t^* = \exp\left(\mathbb{E}[\log X_t^* | \mathcal{F}_t^0]\right)$, for $t \in [0,T]$, and $\mathbb{E}\left[\frac{1}{\gamma}\left(X_T^{\pi^*}(\mu_T^*)^{-\theta}\right)^\gamma\right] \geq \mathbb{E}\left[\frac{1}{\gamma}\left(X_T^\pi(\mu_T^*)^{-\theta}\right)^\gamma\right]$ for each admissible strategy π. Specifically, $(\mu_t^*)_{0 \leq t \leq T}$ is called a solution to (1.4).

The NE in Definition 2.1 is of open-loop type. We will make a remark on the closed-loop NE in Remark 5.3. Now we introduce the main result of this section, where we adapt the argument in [8] and [9] to our MFG (1.4) with power utility functions.

Proposition 2.2 (1) *If an NE (μ^*, π^*) of the MFG (1.4) exists, with $(\widehat{\mu}^*, \pi^*) \in S^2_{\mathbb{F}^0} \times H^2_{BMO}$ and*

$$\mathbb{E}\left[\frac{1}{\gamma}e^{\gamma(\widehat{X}^{\pi^*}_T - \theta\widehat{\mu}^*_T)}\Big|\mathcal{F}_\cdot\right] \quad \text{satisfying } R_p \text{ for some } p > 1, \qquad (2.1)$$

where $\widehat{X}^{\pi^*} = \log X^{\pi^*}$ is the log-wealth and $\widehat{\mu}^* = \log \mu^*$, and R_p is introduced in Appendix B, then the following mean field FBSDE admits a solution, such that $(Z, Z^0) \in H^2_{BMO} \times H^2_{BMO}$,

$$\begin{cases} d\widehat{X}_t = \dfrac{h_t + \sigma_t Z_t + \sigma^0_t Z^0_t}{(1-\gamma)(\sigma^2_t + (\sigma^0_t)^2)}\left\{\left(h_t - \dfrac{h_t + \sigma_t Z_t + \sigma^0_t Z^0_t}{2(1-\gamma)}\right) dt + \sigma_t\, dW_t + \sigma^0_t\, dW^0_t\right\}, \\[3mm] -dY_t = \left(\dfrac{Z^2_t + (Z^0_t)^2}{2} + \dfrac{\gamma}{2(1-\gamma)}\dfrac{(h_t + \sigma_t Z_t + \sigma^0_t Z^0_t)^2}{\sigma^2_t + (\sigma^0_t)^2}\right) dt - Z_t\, dW_t - Z^0_t\, dW^0_t, \\[3mm] \widehat{X}_0 = \log x, \ Y_T = -\gamma\theta\mathbb{E}[\widehat{X}_T|\mathcal{F}^0_T]. \end{cases}$$
$$(2.2)$$

(2) *If the FBSDE (2.2) admits a solution, such that $(Z, Z^0) \in H^2_{BMO} \times H^2_{BMO}$, then the MFG (1.4) admits an NE (μ^*, π^*), such that $(\widehat{\mu}^*, \pi^*) \in S^2_{\mathbb{F}^0} \times H^2_{BMO}$ and (2.1) holds.*

The relationship is given by $\pi^* = \frac{h + \sigma Z + \sigma^0 Z^0}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)}$.

Proof **(1)** Let (μ^*, π^*) be an NE of (1.4) such that (2.1) holds. Define

$$M^\pi_t = e^{\gamma\widehat{X}^\pi_t} \operatorname*{ess\,sup}_{\kappa \in H^2_{BMO}} \mathbb{E}\left[\frac{1}{\gamma}e^{\gamma(\widehat{X}^\kappa_T - \widehat{X}^\kappa_t - \theta\widehat{\mu}^*_T)}\Big|\mathcal{F}_t\right],$$

where $\widehat{X}^\kappa$ is the log-wealth associated with the strategy κ. Following the argument in [8, Theorem 4.7] and [9, Lemma 3.2], M^π has a continuous version which is a supermartingale for all π and a martingale for π^*, and there exists a $\breve{Z} \in H^2_{BMO} \times H^2_{BMO}$, such that

$$M^{\pi^*}_t = M^{\pi^*}_0 e^{\int_0^t \breve{Z}^\top_s\, d\overline{W}_s - \frac{1}{2}\int_0^t \breve{Z}^\top_s \breve{Z}_s\, ds}.$$

Straightforward calculation implies that

$$\begin{aligned} M^\pi_t &= e^{-\gamma(\widehat{X}^{\pi^*}_t - \widehat{X}^\pi_t)}M^{\pi^*}_t \\ &= M^{\pi^*}_0 \mathcal{E}\left(\int_0^t (-\gamma\pi^*_s\widetilde{\sigma}^\top_s + \gamma\pi_s\widetilde{\sigma}^\top_s + \breve{Z}^\top_s)\, d\overline{W}_s\right)\exp\left(\int_0^t \widetilde{f}_s\, ds\right), \end{aligned}$$

where

$$\begin{aligned} \widetilde{f} = &-\gamma\pi^* h + \frac{\gamma}{2}(\pi^*)^2\widetilde{\sigma}^\top\widetilde{\sigma} + \gamma\pi h - \frac{\gamma}{2}\pi^2\widetilde{\sigma}^\top\widetilde{\sigma} \\ &+ \frac{1}{2}\left(\gamma\pi^*\widetilde{\sigma}^\top - \gamma\pi\widetilde{\sigma}^\top\right)\left(\gamma\pi^*\widetilde{\sigma} - \gamma\pi\widetilde{\sigma}\right) + (-\gamma\pi^*\widetilde{\sigma}^\top + \gamma\pi\widetilde{\sigma}^\top)\breve{Z}. \end{aligned}$$

Let $\mathring{Z} = \breve{Z} - \gamma\pi^*\widetilde{\sigma}$. Then $\widetilde{f}$ can be rewritten as

$$\widetilde{f} = \frac{\gamma}{2(1-\gamma)\widetilde{\sigma}^\top\widetilde{\sigma}}\left|(1-\gamma)\widetilde{\sigma}^\top\widetilde{\sigma}\pi^* - h - \widetilde{\sigma}^\top\mathring{Z}\right|^2 - \frac{\gamma}{2(1-\gamma)\widetilde{\sigma}^\top\widetilde{\sigma}}\left|(1-\gamma)\widetilde{\sigma}^\top\widetilde{\sigma}\pi - h - \widetilde{\sigma}^\top\mathring{Z}\right|^2.$$

Since M^π is a supermartingale, $\exp\left(\int_0^\cdot \widetilde{f}_s\, ds\right)$ is nonincreasing if $\gamma > 0$ and nondecreasing if $\gamma < 0$. As a result, $\frac{\widetilde{f}}{\gamma}$ is nonpositive. Thus, $\pi^* = \frac{h + \widetilde{\sigma}^\top \mathring{Z}}{(1-\gamma)\widetilde{\sigma}^\top \widetilde{\sigma}}$. Define $Y = \log\left(M^{\pi^*}\exp(-\gamma \widehat{X}^{\pi^*})\right)$. Then $(\log X^{\pi^*}, Y, \mathring{Z})$ satisfies (2.2).

(2) For each strategy $\pi \in H^2_{BMO}$, define $R_t^\pi = \frac{1}{\gamma} X_t^\gamma e^{Y_t}$, where $dY_t = f_t(Z_t, Z_t^0)\, dt + Z_t\, dW_t + Z_t^0\, dW_t^0$, $Y_T = -\gamma\theta\mu_T^*$ with f to be determined, such that

- R_0^π is independent of π.
- R^π is a supermartingale for all π and a martingale for some π^*. $\qquad$ (2.3)
- $R_T^\pi = \dfrac{1}{\gamma}\left(X_T(\mu_T^*)^{-\theta}\right)^\gamma$.

The three points in (2.3) indicate that $\mathbb{E}[R_T^{\pi^*}] = \mathbb{E}[R_0^{\pi^*}] = \mathbb{E}[R_0^\pi] \geq \mathbb{E}[R_T^\pi]$ for all π.

Note that

$$
\begin{aligned}
R_t^\pi = {} & \frac{1}{\gamma} x^\gamma \exp(Y_0)\exp\left\{\int_0^t \left(\gamma\pi_s h_s - \frac{\gamma\pi_s^2}{2}(\sigma_s^2 + (\sigma_s^0)^2) + f_s(Z_s, Z_s^0)\right.\right. \\
& \left.\left. + \frac{1}{2}(\gamma\pi_s\sigma_s + Z_s)^2 + \frac{1}{2}(\gamma\pi_s\sigma_s^0 + Z_s^0)^2\right)ds\right\} \\
& \times \mathcal{E}\left(\int_0^t (\gamma\pi_s\sigma_s + Z_s)\,dW_s + \int_0^t(\gamma\pi_s\sigma_s^0 + Z_s^0)\,dW_s^0\right) \\
:= {} & \frac{1}{\gamma}x^\gamma \exp(Y_0)\exp\left(\int_0^t \widetilde{f}_s(\pi_s, Z_s, Z_s^0)\,ds\right) \\
& \times \mathcal{E}\left(\int_0^t (\gamma\pi_s\sigma_s + Z_s)\,dW_s + \int_0^t(\gamma\pi_s\sigma_s^0 + Z_s^0)\,dW_s^0\right).
\end{aligned}
$$

The stochastic exponential $\mathcal{E}\left(\int_0^t(\gamma\pi_s\sigma_s + Z_s)\,dW_s + \int_0^t(\gamma\pi_s\sigma_s^0 + Z_s^0)\,dW_s^0\right)$ is a martingale since $\pi \in H^2_{BMO}$. In order to make R^π satisfy the second point of (2.3), we choose f, such that $\widetilde{f}(\pi, Z, Z^0)$ is nonpositive for all π and zero for some π^*. By rearranging terms we have the following equation

$$
\begin{aligned}
\widetilde{f}(\pi, Z, Z^0) = {} & -\frac{\gamma - \gamma^2}{2}(\sigma^2 + (\sigma^0)^2)\pi^2 + (\gamma h + \sigma\gamma Z + \gamma\sigma^0 Z^0)\pi \\
& + \frac{Z^2 + (Z^0)^2}{2} + f(Z, Z^0).
\end{aligned}
$$

By choosing

$$
\pi^* = \frac{h + \sigma Z + \sigma^0 Z^0}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)}
$$

and

$$
f(Z, Z^0) = -\frac{Z^2 + (Z^0)^2}{2} - \frac{\gamma}{2(1-\gamma)}\frac{(h + \sigma Z + \sigma^0 Z^0)^2}{\sigma^2 + (\sigma^0)^2},
$$

it holds that $\frac{\widetilde{f}}{\gamma}$ is nonpositive for all π, and $\widetilde{f}(\pi^*, Z, Z^0) = 0$. Thus, by (2.3), an NE of (1.4) exists if the following FBSDE admits a solution with $(Z, Z^0) \in H^2_{BMO} \times H^2_{BMO}$

$$
\begin{cases}
dX_t = \dfrac{h_t + \sigma_t Z_t + \sigma_t^0 Z_t^0}{(1 - \gamma)(\sigma_t^2 + (\sigma_t^0)^2)} X_t(h_t\, dt + \sigma_t\, dW_t + \sigma_t^0\, dW_t^0) \\[4mm]
-dY_t = \left(\dfrac{Z_t^2 + (Z_t^0)^2}{2} + \dfrac{\gamma}{2(1 - \gamma)} \dfrac{(h_t + \sigma_t Z_t + \sigma_t^0 Z_t^0)^2}{\sigma_t^2 + (\sigma_t^0)^2} \right) dt \\[4mm]
\qquad\quad - Z_t\, dW_t - Z_t^0\, dW_t^0, \\[2mm]
X_0 = x, \ \ Y_T = -\gamma\theta\mathbb{E}[\log X_T | \mathcal{F}_T^0].
\end{cases}
\tag{2.4}
$$

Let $\widehat{X} = \log X$. (2.4) is equivalent to (2.2). $\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.3 The proof of **(2)** in Proposition 2.2 relies on MOP in [18]. The essential difference between our proof and [18] is the choice of strategies; we consider the fraction of the wealth invested in stock π as our strategy, while in [18] Hu et al. considered the scaled one $\widetilde{\pi} := \pi(\sigma, \sigma^0)$ as a strategy. We claim that the choice in [18] is not appropriate in the game-theoretic version of utility maximization problems. Indeed, the choice of $\widetilde{\pi}$ leads to a different problem. The intuitive reason is that σ and σ^0 are not interchangeable in the game; the former is the volatility of the idiosyncratic noise, while the latter is the volatility of the common noise. Thus, we should not expect that $\frac{\widetilde{\pi}_1}{\sigma} = \frac{\widetilde{\pi}_2}{\sigma^0} = \pi$, where $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ are the two components of $\widetilde{\pi}$. To illustrate the difference resulting from the choice of strategies, we assume that all coefficients are $\mathcal{A}$-measurable random variables. Following the argument in [18, Section 3], the optimal strategy is given by

$$
\widetilde{\pi}^* = \frac{1}{1 - \gamma}(\boldsymbol{Z} + \Theta),
$$

where $\Theta = \left(\frac{\sigma h}{\sigma^2 + (\sigma^0)^2}, \frac{\sigma^0 h}{\sigma^2 + (\sigma^0)^2} \right)$, and $\boldsymbol{Z} = (Z, Z^0)$ together with some (X, Y) satisfies the FBSDE

$$
\begin{cases}
dX_t = \widetilde{\pi}_t^* \left(\Theta^\top - \dfrac{1}{2}(\widetilde{\pi}_t^*)^\top \right) dt + \widetilde{\pi}_t^*\, d\overline{W}_t, \\[4mm]
-dY_t = \left(\dfrac{\gamma|\boldsymbol{Z}_t + \Theta|^2}{2(1 - \gamma)} + \dfrac{|\boldsymbol{Z}_t|^2}{2} \right) dt - \boldsymbol{Z}_t\, d\overline{W}_t, \\[4mm]
X_0 = \log x, \quad Y_T = -\gamma\theta\widehat{\mu}_T^*,
\end{cases}
\tag{2.5}
$$

with $\widehat{\mu}_T^* = \mathbb{E}[X_T | \mathcal{F}_T^0]$. One can verify directly that the Z-component of the solution to (2.5) is

$$
\boldsymbol{Z} = \left(0, -\frac{\theta\gamma\mathbb{E}\left[\frac{\sigma^0 h}{(1 - \gamma)(\sigma^2 + (\sigma^0)^2)} \right]}{1 + \mathbb{E}\left[\frac{\theta\gamma}{1 - \gamma} \right]} \right).
$$

Thus, the optimal strategy is

$$\widetilde{\pi}^* = \frac{1}{1-\gamma}\left(\frac{\sigma h}{\sigma^2+(\sigma^0)^2},\ \frac{\sigma^0 h}{\sigma^2+(\sigma^0)^2} - \frac{\theta\gamma\mathbb{E}\left[\frac{\sigma^0 h}{(1-\gamma)(\sigma^2+(\sigma^0)^2)}\right]}{1+\mathbb{E}\left[\frac{\theta\gamma}{1-\gamma}\right]}\right) := \pi^*(\sigma,\sigma^0). \tag{2.6}$$

Multiplying $(\sigma,\sigma^0)^\top$ on both sides of (2.6), we get

$$\pi^* = \frac{1}{(1-\gamma)(\sigma^2+(\sigma^0)^2)}\left\{h - \frac{\theta\gamma\sigma^0\mathbb{E}\left[\frac{\sigma^0 h}{(1-\gamma)(\sigma^2+(\sigma^0)^2)}\right]}{1+\mathbb{E}\left[\frac{\theta\gamma}{1-\gamma}\right]}\right\},$$

which is surprisingly different from our Theorem 3.13 and Corollary 3.14, unless in the following two special cases.
• $\sigma = 0$: in this case, all players trade a common stock; this case was considered in [8,9].
• $\theta = 0$: in this case, there is no competition; this is the single-player utility maximization problem considered in [18].

Given the above argument, we provided the detailed proof in Proposition 2.2(**2**) for readers' convenience.

Remark 2.4 (MFGs with exponential utility functions) If each player uses an exponential utility criterion, then the MFG becomes

$$\begin{cases} \text{1. Fix } \mu \text{ in some suitable space.} \\ \text{2. Solve the optimization problem} \\ \quad \mathbb{E}\left[-e^{-\alpha(X_T-\theta\mu_T)}\right] \to \max \text{ over } \pi \\ \quad \text{such that } dX_t = \pi_t(h_t\,dt + \sigma_t\,dW_t + \sigma_t^0\,dW_t^0),\ X_0 = x_{exp}. \\ \text{3. Search for the fixed point } \mu_t = \mathbb{E}[X_t^*|\mathcal{F}_t^0],\ t \in [0,T], \\ \quad X^* \text{ is the optimal wealth from 2.} \end{cases} \tag{2.7}$$

By the same analysis as in Proposition 2.2, the existence of NE of the MFG (2.7) is equivalent to the solvability of the following FBSDE

$$\begin{cases} dX_t = \dfrac{\alpha\sigma_t Z_t + \alpha\sigma_t^0 Z_t^0 + h_t}{\alpha(\sigma_t^2+(\sigma_t^0)^2)}(h_t\,dt + \sigma_t\,dW_t + \sigma_t^0\,dW_t^0), \\ dY_t = \left(\dfrac{(\alpha\sigma_t Z_t + \alpha\sigma_t^0 Z_t^0 + h_t)^2}{2\alpha(\sigma_t^2+(\sigma_t^0)^2)} - \dfrac{\alpha}{2}(Z_t^2+(Z_t^0)^2)\right)dt + Z_t\,dW_t + Z_t^0\,dW_t^0, \\ X_0 = x_{exp},\ Y_T = \theta\mathbb{E}[X_T|\mathcal{F}_T^0]. \end{cases} \tag{2.8}$$

The relationship is given by $\pi^* = \frac{\alpha\sigma Z + \alpha\sigma Z^0 + h}{\alpha(\sigma^2+(\sigma^0)^2)}$. The FBSDE (2.8) can be solved in exactly the same manner as (2.2). Consequently, in this paper, we will only consider the game with power utility functions, and the analysis of the exponential case is available upon request.

Remark 2.5 (MFGs with log utility functions) If each player uses a log utility criterion, then the MFG becomes

$$
\begin{cases}
1.\ \text{Fix } \mu \text{ in some suitable space.} \\
2.\ \text{Solve the optimization problem} \\
\quad \mathbb{E}\left[\log\left(X_T \mu_T^{-\theta}\right)\right] \to \max \ \text{over } \pi \\
\quad \text{such that } dX_t = \pi_t X_t(h_t\, dt + \sigma_t\, dW_t + \sigma_t^0\, dW_t^0),\ X_0 = x_{log}. \\
3.\ \text{Search for the fixed point } \mu_t = \exp\left(\mathbb{E}[\log X_t^* | \mathcal{F}_t^0]\right),\ t \in [0,T], \\
\quad X^* \text{ is the optimal wealth from 2.}
\end{cases}
\tag{2.9}
$$

Note that $\arg\max_\pi \mathbb{E}\left[\log\left(X_T^\pi \mu_T^{-\theta}\right)\right] = \arg\max_\pi \mathbb{E}[\log X_T^\pi]$. Therefore, the MFG with log utility criterion is decoupled; each player makes her decision by disregarding her competitors. By [18], the NE of (2.9) is given by

$$
\mu^* = \exp\left(\mathbb{E}[\log X_T | \mathcal{F}_T^0]\right), \qquad \pi^* = \frac{h}{\sigma^2 + (\sigma^0)^2},
\tag{2.10}
$$

where X together with some (Y, Z, Z^0) is the unique solution to the (trivially solvable) FBSDE

$$
\begin{cases}
dX_t = \dfrac{h_t}{\sigma_t^2 + (\sigma_t^0)^2} X_t(h_t\, dt + \sigma_t\, dW_t + \sigma_t^0\, dW_t^0), \\[2mm]
-dY_t = \dfrac{h_t^2}{2(\sigma_t^2 + (\sigma_t^0)^2)}\, dt + Z_t\, dW_t + Z_t^0\, dW_t^0, \\[2mm]
X_0 = x_{log},\ Y_T = -\theta \mathbb{E}[\log X_T | \mathcal{F}_T^0].
\end{cases}
\tag{2.11}
$$

Let (μ', π') be any other NE of (2.9). Given μ', by MOP in [18], the optimal response is $\frac{h}{\sigma^2 + (\sigma^0)^2}$, which is unique since the log utility function is concave. Thus, $\pi' = \frac{h}{\sigma^2 + (\sigma^0)^2}$ and $\mu_t' = \exp\left(\mathbb{E}[\log X_t | \mathcal{F}_t^0]\right)$, $t \in [0,T]$ and X is the unique solution to (2.11). Therefore, $(\mu^*, \pi^*) = (\mu', \pi')$ and the NE of (2.9) is unique.

In Section 3, we will study the MFG (1.4) by examining the FBSDE (2.2). In particular, we will consider the difference between the FBSDE (2.2) and the benchmark one when the competition parameter $\theta = 0$, and then transform the resulting FBSDE into some BSDE.

3 Wellposedness of the FBSDE (2.2) and the MFG (1.4)

3.1 The Adjusted Mean Field FBSDE

By Proposition 2.2, to solve the MFG (1.4), it is equivalent to solve the mean field FBSDE (2.2). In order to solve (2.2), we compare (2.2) with the

benchmark FBSDE associated with the single player's utility maximization problem, i.e., the utility game with $\theta = 0$. When $\theta = 0$, (2.2) is decoupled into

$$
\begin{cases}
d\widehat{X}_t = \dfrac{h_t + \sigma_t Z_t + \sigma_t^0 Z_t^0}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \left\{ \left(h_t - \dfrac{h_t + \sigma_t Z_t + \sigma_t^0 Z_t^0}{2(1-\gamma)} \right) dt + \sigma_t \, dW_t + \sigma_t^0 \, dW_t^0 \right\}, \\[4mm]
-dY_t = \left(\dfrac{Z_t^2 + (Z_t^0)^2}{2} + \dfrac{\gamma}{2(1-\gamma)} \dfrac{(h_t + \sigma_t Z_t + \sigma_t^0 Z_t^0)^2}{\sigma_t^2 + (\sigma_t^0)^2} \right) dt - Z_t \, dW_t - Z_t^0 \, dW_t^0, \\[4mm]
\widehat{X}_0 = \log x, \ Y_T = 0.
\end{cases}
$$
(3.1)

The solvability of the FBSDE (3.1) is summarized in the following proposition.

Proposition 3.1 *The FBSDE* (3.1) *has a unique solution in* $\left(\bigcap_{p>1} S^p \right) \times S^\infty \times H^2_{BMO} \times H^2_{BMO}$.

Proof Theorem 7 in [18] implies that there exists a unique $(Y, Z, Z^0) \in S^\infty \times H^2_{BMO} \times H^2_{BMO}$ satisfying the BSDE in (3.1). By the energy inequality ([20, P.26]), it holds that $(Z, Z^0) \in \left(\bigcap_{p>1} M^p \right) \times \left(\bigcap_{p>1} M^p \right)$, which implies that $\widehat{X} \in \bigcap_{p>1} S^p$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

From now on, we denote the unique solution to (3.1) by $(X^{\bar{o}}, Y^{\bar{o}}, Z^{\bar{o}}, Z^{0,\bar{o}})$. Let $(\widehat{X}, Y, Z, Z^0)$ be a solution to (2.2) and we consider the difference

$$
(\overline{X}, \overline{Y}, \overline{Z}, \overline{Z}^0) := (\widehat{X} - X^{\bar{o}}, Y - Y^{\bar{o}}, Z - Z^{\bar{o}}, Z^0 - Z^{0,\bar{o}}), \qquad (3.2)
$$

which satisfies

$$
\begin{cases}
d\overline{X}_t = \left\{ \dfrac{\sigma_t \overline{Z}_t + \sigma_t^0 \overline{Z}_t^0}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \left(h_t - \dfrac{1}{1-\gamma}(h_t + \sigma_t Z_t^{\bar{o}} + \sigma_t^0 Z_t^{0,\bar{o}}) \right) \right. \\[4mm]
\qquad\quad \left. - \dfrac{\left(\sigma_t \overline{Z}_t + \sigma_t^0 \overline{Z}_t^0 \right)^2}{2(1-\gamma)^2(\sigma_t^2 + (\sigma_t^0)^2)} \right\} dt + \dfrac{\sigma_t \overline{Z}_t + \sigma_t^0 \overline{Z}_t^0}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} (\sigma_t \, dW_t + \sigma_t^0 \, dW_t^0) \\[4mm]
-d\overline{Y}_t = \left\{ \dfrac{(2Z_t^{\bar{o}} + \overline{Z}_t)\overline{Z}_t + (2Z_t^{0,\bar{o}} + \overline{Z}_t^0)\overline{Z}_t^0}{2} \right. \\[4mm]
\qquad\quad \left. + \dfrac{\gamma(2h_t + 2\sigma_t Z_t^{\bar{o}} + 2\sigma_t^0 Z_t^{0,\bar{o}} + \sigma_t \overline{Z}_t + \sigma_t^0 \overline{Z}_t^0)(\sigma_t \overline{Z}_t + \sigma_t^0 \overline{Z}_t^0)}{2(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \right\} dt \\[4mm]
\qquad\quad - \overline{Z}_t \, dW_t - \overline{Z}_t^0 \, dW_t^0, \\[4mm]
\overline{X}_0 = 0, \ \overline{Y}_T = -\theta\gamma\mathbb{E}[\overline{X}_T | \mathcal{F}_T^0] - \theta\gamma\mathbb{E}[X_T^{\bar{o}} | \mathcal{F}_T^0].
\end{cases}
$$
(3.3)

Remark 3.2 There are two reasons why we consider the difference instead of the original FBSDE. First, we want to solve (2.2) under a weak interaction assumption. In order to avoid any unreasonable assumptions on $(h, \gamma, \sigma, \sigma^0)$, we need to drop the non-homogenous terms without θ. This can be done by considering the difference. Second, (3.3) is the starting point of the asymptotic expansion result in Section 4.

3.2 The Equivalent BSDE

In this section, we will show that the FBSDE (3.3) is equivalent to some
BSDE. First, we will prove a general result on the equivalence between a class
of FBSDEs and BSDEs. Subsequently, we will show that our FBSDE (3.3) is
a special case and thus can be transformed into some BSDE.

We consider the following FBSDE

$$
\begin{cases}
d\mathcal{X}_t = dr(t, \mathcal{Y}_t, \mathcal{Z}_t, \mathcal{Z}_t^0)\, dt + \ diff(t, \mathcal{Y}_t, \mathcal{Z}_t, \mathcal{Z}_t^0)dW_t + diff^0(t, \mathcal{Z}_t, \mathcal{Z}_t^0)\, dW_t^0, \\
-d\mathcal{Y}_t = Dr(t, \mathcal{Y}_t, \mathcal{Z}_t, \mathcal{Z}_t^0)\, dt - \mathcal{Z}_t\, dW_t - \mathcal{Z}_t^0\, dW_t^0, \\
\quad \mathcal{X}_0 = \mathsf{x}, \quad \mathcal{Y}_T = \nu\mathbb{E}[\mathcal{X}_T | \mathcal{F}_T^0],
\end{cases}
$$
(3.4)

where the coefficients are assumed to satisfy the following conditions.

– **AS-1.** For any given $\widetilde{\mathcal{Y}}$, $\mathcal{Z}$ and $\mathcal{Z}^0$, the following mean field SDE for $\mathcal{Y}$ has
   a unique solution

$$
\begin{aligned}
\widetilde{\mathcal{Y}}_t = \mathcal{Y}_t &- \int_0^t \nu\mathbb{E}\left[dr(s, \mathcal{Y}_s, \mathcal{Z}_s, \mathcal{Z}_s^0)|\mathcal{F}_s^0\right]\, ds \\
&- \int_0^t \nu\mathbb{E}\left[diff^0(s, \mathcal{Z}_s, \mathcal{Z}_s^0)|\mathcal{F}_s^0\right]\, dW_s^0.
\end{aligned}
$$
(3.5)

The unique solution is denoted by $\mathcal{Y}_t = g_1(t, \widetilde{\mathcal{Y}}_{\cdot \wedge t}, \mathcal{Z}_{\cdot \wedge t}, \mathcal{Z}_{\cdot \wedge t}^0)$.
– **AS-2.** For each given $\widetilde{\mathcal{Z}}^0$ and $\mathcal{Z}$, the following equation for $\mathcal{Z}^0$ is uniquely
   solvable

$$
\widetilde{\mathcal{Z}}_t^0 = \mathcal{Z}_t^0 - \nu\mathbb{E}[diff^0(t, \mathcal{Z}_t, \mathcal{Z}_t^0)|\mathcal{F}_t^0].
$$
(3.6)

The unique solution is denoted by $\mathcal{Z}_t^0 = g_2(t, \mathcal{Z}_t, \widetilde{\mathcal{Z}}_t^0)$.

The next proposition shows that the solution to the FBSDE (3.4) satisfying
the above conditions has a one-to-one correspondence with the solution to
some BSDE. Such correspondence relies on the fact that $\mathcal{Y}$ depends on $\mathcal{X}$ in
a linear way and only through the terminal condition, as well as the unique
solvability of (3.5) and (3.6). The linear dependence allows us to rewrite and
split the terminal value into an integral on $[0, t]$ and an integral on $[t, T]$, where
the former integral can be merged into the solution to a new BSDE and the
latter integral can be merged into the coefficient of the new BSDE. The unique
solvability of (3.5) and (3.6) yields the one-to-one correspondence.

Proposition 3.3 *Under* **AS-1** *and* **AS-2**, *there is a one-to-one correspondence*
between the solution to the FBSDE (3.4) *and the solution to the following*

BSDE

$$\widetilde{\mathcal{Y}}_t$$

$$= \nu\mathbb{E}[\varkappa] - \int_t^T \widetilde{\mathcal{Z}}_s \, dW_s - \int_t^T \widetilde{\mathcal{Z}}_s^0 \, dW_s^0$$

$$+ \int_t^T \left\{ \nu\mathbb{E}\left[dr\left(s, g_1(s, \widetilde{\mathcal{Y}}_{\cdot \wedge s}, \widetilde{\mathcal{Z}}_{\cdot \wedge s}, g_2(\cdot \wedge s, \widetilde{\mathcal{Z}}_{\cdot \wedge s}, \widetilde{\mathcal{Z}}_{\cdot \wedge s}^0)), \widetilde{\mathcal{Z}}_s, g_2(s, \widetilde{\mathcal{Z}}_s, \widetilde{\mathcal{Z}}_s^0) \right) \middle| \mathcal{F}_s^0 \right] \right.$$

$$\left. + Dr\left(s, g_1(s, \widetilde{\mathcal{Y}}_{\cdot \wedge s}, \widetilde{\mathcal{Z}}_{\cdot \wedge s}, g_2(\cdot \wedge s, \widetilde{\mathcal{Z}}_{\cdot \wedge s}, \widetilde{\mathcal{Z}}_{\cdot \wedge s}^0)), \widetilde{\mathcal{Z}}_s, g_2(s, \widetilde{\mathcal{Z}}_s, \widetilde{\mathcal{Z}}_s^0) \right) \right\} ds.$$

$$(3.7)$$

Let the solution to (3.4) and the solution to (3.7) be $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{Z}^0)$ and $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}, \widetilde{\mathcal{Z}}^0)$, respectively. The relationship is given by for each $t \in [0, T]$

$$\begin{cases} \widetilde{\mathcal{Y}}_t = \mathcal{Y}_t - \int_0^t \nu\mathbb{E}[dr(s, \mathcal{Y}_s, \mathcal{Z}_s, \mathcal{Z}_s^0)|\mathcal{F}_s^0] \, ds - \int_0^t \nu\mathbb{E}[diff^0(s, \mathcal{Z}_s, \mathcal{Z}_s^0)|\mathcal{F}_s^0] \, dW_s^0 \\ \widetilde{\mathcal{Z}}_t = \mathcal{Z}_t \\ \widetilde{\mathcal{Z}}_t^0 = \mathcal{Z}_t^0 - \nu\mathbb{E}[diff^0(t, \mathcal{Z}_t, \mathcal{Z}_t^0)|\mathcal{F}_t^0]. \end{cases}$$

$$(3.8)$$

Proof From the dynamics of (3.4), we have the following

$$\mathcal{Y}_t = \nu\mathbb{E}[\mathcal{X}_T|\mathcal{F}_T^0] + \int_t^T Dr(s, \mathcal{Y}_s, \mathcal{Z}_s, \mathcal{Z}_s^0) \, ds - \int_t^T \mathcal{Z}_s \, dW_s - \int_t^T \mathcal{Z}_s^0 \, dW_s^0$$

$$= \nu\mathbb{E}[\varkappa] + \int_0^T \nu\mathbb{E}[dr(s, \mathcal{Y}_s, \mathcal{Z}_s, \mathcal{Z}_s^0)|\mathcal{F}_s^0] \, ds + \int_0^T \nu\mathbb{E}[diff^0(s, \mathcal{Y}_s, \mathcal{Z}_s, \mathcal{Z}_s^0)|\mathcal{F}_s^0] \, dW_s^0$$

$$+ \int_t^T Dr(s, \mathcal{Y}_s, \mathcal{Z}_s, \mathcal{Z}_s^0) \, ds - \int_t^T \mathcal{Z}_s \, dW_s - \int_t^T \mathcal{Z}_s^0 \, dW_s^0$$

$$= \nu\mathbb{E}[\varkappa] + \int_0^t \nu\mathbb{E}[dr(s, \mathcal{Y}_s, \mathcal{Z}_s, \mathcal{Z}_s^0)|\mathcal{F}_s^0] \, ds + \int_0^t \nu\mathbb{E}[diff^0(s, \mathcal{Z}_s, \mathcal{Z}_s^0)|\mathcal{F}_s^0] \, dW_s^0$$

$$+ \int_t^T \left\{ \nu\mathbb{E}[dr(s, \mathcal{Y}_s, \mathcal{Z}_s, \mathcal{Z}_s^0)|\mathcal{F}_s^0] + Dr(s, \mathcal{Y}_s, \mathcal{Z}_s, \mathcal{Z}_s^0) \right\} ds$$

$$- \int_t^T \mathcal{Z}_s \, dW_s - \int_t^T \left\{ \mathcal{Z}_s^0 - \nu\mathbb{E}[diff^0(s, \mathcal{Z}_s, \mathcal{Z}_s^0)|\mathcal{F}_s^0] \right\} dW_s^0.$$

Define for each $t \in [0, T]$

$$\widetilde{\mathcal{Y}}_t = \mathcal{Y}_t - \int_0^t \nu\mathbb{E}[dr(s, \mathcal{Y}_s, \mathcal{Z}_s, \mathcal{Z}_s^0)|\mathcal{F}_s^0] \, ds - \int_0^t \nu\mathbb{E}[diff^0(s, \mathcal{Z}_s, \mathcal{Z}_s^0)|\mathcal{F}_s^0] \, dW_s^0$$

$$(3.9)$$

and

$$\begin{cases} \widetilde{\mathcal{Z}}_t = \mathcal{Z}_t \\ \widetilde{\mathcal{Z}}_t^0 = \mathcal{Z}_t^0 - \nu\mathbb{E}[diff^0(t, \mathcal{Z}_t, \mathcal{Z}_t^0)|\mathcal{F}_t^0]. \end{cases}$$

$$(3.10)$$

From (3.9) and (3.10), conditions (3.5) and (3.6) imply that

$$\mathcal{Z}^0_t = g_2(t, \widetilde{\mathcal{Z}}_t, \widetilde{\mathcal{Z}}^0_t) \tag{3.11}$$

and

$$\mathcal{Y}_t = g_1(t, \widetilde{\mathcal{Y}}_{\cdot \wedge t}, \mathcal{Z}_{\cdot \wedge t}, \mathcal{Z}^0_{\cdot \wedge t}) = g_1(t, \widetilde{\mathcal{Y}}_{\cdot \wedge t}, \widetilde{\mathcal{Z}}_{\cdot \wedge t}, g_2(\cdot \wedge t, \widetilde{\mathcal{Z}}_{\cdot \wedge t}, \widetilde{\mathcal{Z}}^0_{\cdot \wedge t})). \tag{3.12}$$

Then, $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}, \widetilde{\mathcal{Z}}^0)$ satisfies the BSDE (3.7).

Moreover, if there exists some $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}, \widetilde{\mathcal{Z}}^0)$ satisfying (3.7), by equations (3.10), (3.11) and (3.12), we can construct $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{Z}^0)$ satisfying (3.4). □

We introduce the following BSDE

$$\begin{aligned}
\widetilde{Y}_t = \theta \mathbb{E}[\log x] + \int_t^T &\left\{ \mathcal{J}_1(s, \widetilde{Z}_s, \widetilde{Z}^0_s) + \mathcal{J}_2(s; \widetilde{Z}, \widetilde{Z}^0, \theta) \right\} ds \\
&- \int_t^T \widetilde{Z}_s \, dW_s - \int_t^T \widetilde{Z}^0_s \, dW^0_s,
\end{aligned} \tag{3.13}$$

where $\mathcal{J}_1(\cdot; \widetilde{Z}, \widetilde{Z}^0)$ are terms that do not depend on θ

$$\begin{aligned}
&\mathcal{J}_1(\cdot, \widetilde{Z}, \widetilde{Z}^0) \\
&= \frac{1}{2} \left\{ 1 + \frac{\gamma \sigma^2}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} \right\} \widetilde{Z}^2 + \frac{1}{2} \left\{ 1 + \frac{\gamma(\sigma^0)^2}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} \right\} (\widetilde{Z}^0)^2 \\
&\quad + \left\{ Z^{\bar{o}} + \frac{\gamma \sigma(h + \sigma Z^{\bar{o}} + \sigma^0 Z^{0,\bar{o}})}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} \right\} \widetilde{Z} + \left\{ Z^{0,\bar{o}} + \frac{\gamma \sigma^0(h + \sigma Z^{\bar{o}} + \sigma^0 Z^{0,\bar{o}})}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} \right\} \widetilde{Z}^0 \\
&\quad + \frac{\gamma \sigma \sigma^0}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} \widetilde{Z} \widetilde{Z}^0,
\end{aligned}$$

and $\mathcal{J}_2(\cdot; \widetilde{Z}, \widetilde{Z}^0, \theta)$ are terms that depend on θ. The expression of $\mathcal{J}_2$ is cumbersome and we summarize it in Appendix A due to the convenience of the statement in the main text.

As a corollary of Proposition 3.3, we can show that the FBSDE (3.3) and the BSDE (3.13) are equivalent.

Corollary 3.4 *The wellposedness of the FBSDE* (3.3) *is equivalent to the wellposedness of the BSDE* (3.13). *The relationship is given by for each* $t \in$

$[0, T]$

$$\widetilde{Y}_t$$
$$= \overline{Y}_t + \theta\gamma \int_0^t \mathbb{E}\left[\frac{\sigma_s \overline{Z}_s + \sigma_s^0 \overline{Z}_s^0}{(1-\gamma)(\sigma_s^2 + (\sigma_s^0)^2)} \right.$$
$$\left. \times \left(h_s - \frac{1}{1-\gamma}(h_s + \sigma_s Z_s^{\bar{o}} + \sigma_s^0 Z_s^{0,\bar{o}}) \right) \middle| \mathcal{F}_s^0 \right] ds$$
$$- \theta\gamma \int_0^t \mathbb{E}\left[\frac{\left(\sigma_s \overline{Z}_s + \sigma_s^0 \overline{Z}_s^0 \right)^2}{2(1-\gamma)^2(\sigma_s^2 + (\sigma_s^0)^2)} \middle| \mathcal{F}_s^0 \right] ds$$
$$+ \theta\gamma \int_0^t \mathbb{E}\left[\frac{h_s + \sigma_s Z_s^{\bar{o}} + \sigma_s^0 Z_s^{0,\bar{o}}}{(1-\gamma)(\sigma_s^2 + (\sigma_s^0)^2)} \left(h_s - \frac{h_s + \sigma_s Z_s^{\bar{o}} + \sigma_s^0 Z_s^{0,\bar{o}}}{2(1-\gamma)} \right) \middle| \mathcal{F}_s^0 \right] ds$$
$$+ \theta\gamma \int_0^t \left\{ \mathbb{E}\left[\frac{\sigma_s \overline{Z}_s + \sigma_s^0 \overline{Z}_s^0}{(1-\gamma)(\sigma_s^2 + (\sigma_s^0)^2)} \sigma_s^0 \middle| \mathcal{F}_s^0 \right] \right.$$
$$\left. + \mathbb{E}\left[\frac{h_s + \sigma_s^0 Z_s^{\bar{o}} + \sigma_s^0 Z_s^{0,\bar{o}}}{(1-\gamma)(\sigma_s^2 + (\sigma_s^0)^2)} \sigma_s^0 \middle| \mathcal{F}_s^0 \right] \right\} dW_s^0$$

and

$$\widetilde{Z}_t = \overline{Z}_t, \quad \widetilde{Z}_t^0 = \overline{Z}_t^0 + \theta\gamma \mathbb{E}\left[\frac{\sigma \overline{Z}_t + \sigma_t^0 \overline{Z}_t^0}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \sigma_t^0 \middle| \mathcal{F}_t^0 \right]$$
$$+ \theta\gamma \mathbb{E}\left[\frac{h_t + \sigma Z_t^{\bar{o}} + \sigma_t^0 Z_t^{0,\bar{o}}}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \sigma_t^0 \middle| \mathcal{F}_t^0 \right]. \qquad (3.14)$$

Proof Let $\nu = (-\theta\gamma, -\theta\gamma)$ and $\mathcal{X} = (\overline{X}, X^{\bar{o}})^\top$, where $\overline{X}$ and $X^{\bar{o}}$ satisfy the forward dynamics of (3.3) and (3.1), respectively. Because the drift of $\mathcal{X}$ does not depend on the $\mathcal{Y}$-component, **AS-1** trivially holds. To verify **AS-2**, it is sufficient to solve a unique $\overline{Z}^0$ from (3.14).

Multiplied by $\frac{(\sigma^0)^2}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)}$ on both sides of the second equality in (3.14), and taking conditional expectations $\mathbb{E}[\cdots | \mathcal{F}_t^0]$ on both sides, we obtain an equality for $\mathbb{E}\left[\frac{(\sigma_t^0)^2}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \widetilde{Z}_t^0 \middle| \mathcal{F}_t^0 \right]$ in terms of $\mathbb{E}\left[\frac{(\sigma_t^0)^2}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \overline{Z}_t^0 \middle| \mathcal{F}_t^0 \right]$, from which we get for each $t \in [0, T]$

$$\mathbb{E}\left[\frac{(\sigma_t^0)^2}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \overline{Z}_t^0 \middle| \mathcal{F}_t^0 \right]$$
$$= \frac{\mathbb{E}\left[\frac{(\sigma_t^0)^2}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \widetilde{Z}_t^0 \middle| \mathcal{F}_t^0 \right] - \mathbb{E}\left[\frac{\theta\gamma(\sigma_t^0)^2}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \middle| \mathcal{F}_t^0 \right] \mathbb{E}\left[\frac{\sigma_t \sigma_t^0}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \widetilde{Z}_t \middle| \mathcal{F}_t^0 \right]}{1 + \mathbb{E}\left[\frac{\theta\gamma(\sigma_t^0)^2}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \middle| \mathcal{F}_t^0 \right]}$$
$$- \frac{\mathbb{E}\left[\frac{\theta\gamma(\sigma_t^0)^2}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \middle| \mathcal{F}_t^0 \right] \mathbb{E}\left[\frac{h_t \sigma_t^0 + \sigma_t \sigma_t^0 Z_t^{\bar{o}} + (\sigma_t^0)^2 Z^{0,\bar{o}}}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \middle| \mathcal{F}_t^0 \right]}{1 + \mathbb{E}\left[\frac{\theta\gamma(\sigma_t^0)^2}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \middle| \mathcal{F}_t^0 \right]}.$$
$$\qquad (3.15)$$

Taking (3.15) back into (3.14) and rearranging terms, we obtain $\overline{Z}^0$ in terms of $\widetilde{Z}^0$ and $\widetilde{Z}$

$$
\begin{aligned}
\overline{Z}_t^0 = \widetilde{Z}_t^0 &- \frac{\theta\gamma\mathbb{E}\left[\frac{\sigma_t\sigma_t^0}{(1-\gamma)(\sigma_t^2+(\sigma_t^0)^2)}\widetilde{Z}_t\,\middle|\,\mathcal{F}_t^0\right] + \theta\gamma\mathbb{E}\left[\frac{(\sigma_t^0)^2}{(1-\gamma)(\sigma_t^2+(\sigma_t^0)^2)}\widetilde{Z}_t^0\,\middle|\,\mathcal{F}_t^0\right]}{1+\mathbb{E}\left[\frac{\theta\gamma(\sigma_t^0)^2}{(1-\gamma)(\sigma_t^2+(\sigma_t^0)^2)}\,\middle|\,\mathcal{F}_t^0\right]} \\
&- \frac{\theta\gamma\mathbb{E}\left[\frac{h_t\sigma_t^0+\sigma_t\sigma_t^0 Z_t^{\bar{o}}+(\sigma_t^0)^2 Z_t^{0,\bar{o}}}{(1-\gamma)(\sigma_t^2+(\sigma_t^0)^2)}\,\middle|\,\mathcal{F}_t^0\right]}{1+\mathbb{E}\left[\frac{\theta\gamma(\sigma_t^0)^2}{(1-\gamma)(\sigma_t^2+(\sigma_t^0)^2)}\,\middle|\,\mathcal{F}_t^0\right]}, \quad t\in[0,T].
\end{aligned}
$$

(3.16)

Thus, **AS-2** is verified, and Proposition 3.3 implies that the FBSDE (3.3) is equivalent to the BSDE (3.13). $\qquad\square$

3.3 Wellposedness of the BSDE (3.13) and the FBSDE (2.2)

The BSDE (3.13) is a quadratic one of conditional mean field type, and it does not satisfy the assumptions in [14]. In particular, the quadratic growth in (3.13) comes from both $(\widetilde{Z},\widetilde{Z}^0)$ and the conditional expectation of $(\widetilde{Z},\widetilde{Z}^0)$; refer to the expression for $\mathcal{J}_2$ in Appendix A. In addition, the feature of $\mathcal{J}_2$ is that it includes all mean field terms, and each term in $\mathcal{J}_2$ can be controlled by θ. This observation motivates us to solve (3.13) under a weak interaction assumption, where the competition parameter θ is assumed to be sufficiently small.

To make it convenient to apply the theory of quadratic BSDE, by a change of measure, we can transform $\mathcal{J}_1$ into a pure quadratic term. Indeed, define

$$
\begin{aligned}
\frac{d\mathbb{P}^{\bar{o}}}{d\mathbb{P}} = \mathcal{E}\bigg(&\int_0^{\cdot}\left\{Z_s^{\bar{o}}+\frac{\gamma\sigma_s(h_s+\sigma_s Z_s^{\bar{o}}+\sigma_s^0 Z_s^{0,\bar{o}})}{(1-\gamma)(\sigma_s^2+(\sigma_s^0)^2)}\right\}dW_s \\
&+ \int_0^{\cdot}\left\{Z_s^{0,\bar{o}}+\frac{\gamma\sigma_s^0(h_s+\sigma_s Z_s^{\bar{o}}+\sigma_s^0 Z_s^{0,\bar{o}})}{(1-\gamma)(\sigma_s^2+(\sigma_s^0)^2)}\right\}dW_s^0\bigg) \\
:= \mathcal{E}\bigg(&\int_0^{\cdot}\mathcal{M}_s\,d(W_s,W_s^0)^{\top}\bigg),
\end{aligned}
$$

(3.17)

where

$$
\mathcal{M} = \left(Z^{\bar{o}}+\frac{\gamma\sigma(h+\sigma Z^{\bar{o}}+\sigma^0 Z^{0,\bar{o}})}{(1-\gamma)(\sigma^2+(\sigma^0)^2)},\; Z^{0,\bar{o}}+\frac{\gamma\sigma^0(h+\sigma Z^{\bar{o}}+\sigma^0 Z^{0,\bar{o}})}{(1-\gamma)(\sigma^2+(\sigma^0)^2)}\right).
$$

(3.18)

By Proposition 3.1 and [20, Theorem 2.3], it holds that $\mathbb{P}^{\bar{o}}$ is a probability measure, and the Girsanov theorem yields that

$$
\begin{aligned}
\widetilde{Y}_t = &-\theta\gamma\mathbb{E}[\log x] + \int_t^T\left(\mathcal{J}_1^{\bar{o}}(s,\widetilde{Z}_s,\widetilde{Z}_s^0)+\mathcal{J}_2(s;\widetilde{Z},\widetilde{Z}^0,\theta)\right)ds \\
&-\int_t^T\widetilde{Z}_s\,d\widetilde{W}_s - \int_t^T\widetilde{Z}_s^0\,d\widetilde{W}_s^0,
\end{aligned}
$$

(3.19)

where

$$
\begin{aligned}
W_t^{\bar{o}} := \ & (\widetilde{W}_t, \widetilde{W}_t^0)^\top \\
= \ & \left(W_t - \int_0^t \left\{ Z_s^{\bar{o}} + \frac{\gamma \sigma_s (h_s + \sigma_s Z_s^{\bar{o}} + \sigma_s^0 Z_s^{0,\bar{o}})}{(1-\gamma)(\sigma_s^2 + (\sigma_s^0)^2)} \right\} ds, \right. \\
& \left. W_t^0 - \int_0^t \left\{ Z_s^{0,\bar{o}} + \frac{\gamma \sigma_s^0 (h_s + \sigma_s Z_s^{\bar{o}} + \sigma_s^0 Z_s^{0,\bar{o}})}{(1-\gamma)(\sigma_s^2 + (\sigma_s^0)^2)} \right\} ds \right)^\top
\end{aligned}
\tag{3.20}
$$

is a two-dimensional Brownian motion under $\mathbb{P}^{\bar{o}}$, and

$$
\begin{aligned}
\mathcal{J}_1^{\bar{o}}(\cdot, \widetilde{Z}, \widetilde{Z}^0) = \ & \frac{1}{2} \left\{ 1 + \frac{\gamma \sigma^2}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} \right\} \widetilde{Z}^2 \\
& + \frac{1}{2} \left\{ 1 + \frac{\gamma (\sigma^0)^2}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} \right\} (\widetilde{Z}^0)^2 \\
& + \frac{\gamma \sigma \sigma^0}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} \widetilde{Z} \widetilde{Z}^0 .
\end{aligned}
\tag{3.21}
$$

Therefore, solving (3.13) under $\mathbb{P}$ is equivalent to solving (3.19) under $\mathbb{P}^{\bar{o}}$. To do so, we will use a fixed point argument as in [26] to study a general BSDE with (3.19) as a special case.

Lemma 3.5 *Define a BSDE*

$$
-d\widehat{\mathcal{Y}}_t = \left\{ f_1(t, \widehat{\mathcal{Z}}_t) + f_2(t; \widehat{\mathcal{Z}}, \theta) \right\} dt - \widehat{\mathcal{Z}}_t \, dW_t^{\bar{o}}, \quad \mathcal{Y}_T = \theta \mathbf{x}.
\tag{3.22}
$$

The random coefficients f_1 and f_2 are assumed to satisfy the following conditions: there exists an $\mathcal{A}$-measurable, positive and bounded random variable c_1, an increasing positive locally bounded function C_1, and a positive constant C_2 such that for any $\widehat{z}, \widehat{z}' \in H^2_{BMO, \mathbb{P}^{\bar{o}}}$ it holds that:

ASS-1. $|f_1(t, \widehat{z})| \le c_1 |\widehat{z}|^2$ and $\left\| |2 c_1 f_2(\cdot; \widehat{z}, \theta)|^{\frac{1}{2}} \right\|^2_{BMO, \mathbb{P}^{\bar{o}}} \le \|\theta\| C_1(\|\widehat{z}\|_{BMO, \mathbb{P}^{\bar{o}}})$.

ASS-2. $|f_1(t, \widehat{z}) - f_1(t, \widehat{z}')| \le c_1 |\widehat{z} - \widehat{z}'| (|\widehat{z}| + |\widehat{z}'|)$, *and*

$$
\begin{aligned}
& \|f_2(\cdot; \widehat{z}, \theta) - f_2(\cdot; \widehat{z}', \theta)\|_{BMO, \mathbb{P}^{\bar{o}}} \\
& \le \|\theta\| C_2 \|\widehat{z} - \widehat{z}'\|_{BMO, \mathbb{P}^{\bar{o}}} \left(1 + \|\widehat{z}\|_{BMO, \mathbb{P}^{\bar{o}}} + \|\widehat{z}'\|_{BMO, \mathbb{P}^{\bar{o}}} \right).
\end{aligned}
$$

Then, for each fixed $0 < R \le \frac{1}{4\sqrt{2}\|c_1\|}$, there exists a constant θ^* that is small enough and only depends on R, $\mathbf{x}$, c_1, C_1 and C_2, such that for each $0 \le \|\theta\| \le \theta^*$ the BSDE (3.22) admits a unique solution $(\widehat{\mathcal{Y}}, \widehat{\mathcal{Z}})$ with $\widehat{\mathcal{Z}}$ located in the R-ball of $H^2_{BMO, \mathbb{P}^{\bar{o}}}$. Furthermore, the solution satisfies the following estimate

$$
\|\widehat{\mathcal{Y}}\|_\infty \le \|\theta \mathbf{x}\| - \frac{1}{2\underline{c_1}} \log \left\{ 1 - \left\| |2 c_1 f_2(\cdot; \widehat{\mathcal{Z}}, \theta)|^{\frac{1}{2}} \right\|^2_{BMO, \mathbb{P}^{\bar{o}}} \right\}
\tag{3.23}
$$

and

$$\|\widehat{\mathcal{Z}}\|_{BMO,\mathbb{P}^{\bar{o}}}^2 \leq \frac{\frac{1}{2\underline{c_1}}e^{2\|c_1\|\|\theta\mathbf{x}\|} - \frac{1}{2\underline{c_1}} - \|\theta\mathbf{x}\|}{\underline{c_1}} \\ + \frac{e^{2\|c_1\|\|\widehat{\mathcal{Y}}\|_\infty} - 1}{\underline{c_1}}\left\||f_2(\cdot;\widehat{\mathcal{Z}},\theta)|^{\frac{1}{2}}\right\|_{BMO,\mathbb{P}^{\bar{o}}}^2, \tag{3.24}$$

where we recall that $\underline{c_1}$ is the essential infimum of c_1.

Proof **Step 1.** In the first step, we prove that for each fixed $R > 0$ and $\widehat{z} \in H_{BMO,\mathbb{P}^{\bar{o}}}^2$ with $\|\widehat{z}\|_{BMO,\mathbb{P}^{\bar{o}}}^2 \leq R$, there exists a unique solution to

$$-d\widehat{\mathcal{Y}}_t = \left\{f_1(t,\widehat{\mathcal{Z}}_t) + f_2(t;\widehat{z},\theta)\right\} dt - \widehat{\mathcal{Z}}_t\, dW_t^{\bar{o}}, \quad \mathcal{Y}_T = \theta\mathbf{x}. \tag{3.25}$$

In addition, the solution satisfies the estimates (3.23) and (3.24) with $\widehat{\mathcal{Z}}$ on the right side replaced by $\widehat{z}$.

In order to prove the claim in **Step 1**, we choose θ_1^*, such that $\theta_1^* C_1(R) < 1$. Then for all $0 \leq \|\theta\| \leq \theta_1^*$, we have by **ASS-1**

$$\left\||2c_1 f_2(\cdot;\widehat{z},\theta)|^{\frac{1}{2}}\right\|_{BMO,\mathbb{P}^{\bar{o}}}^2 \leq C_1(R)\|\theta\| < 1,$$

which implies that by [20, Theorem 2.2]

$$\mathbb{E}^{\mathbb{P}^{\bar{o}}}\left[e^{2c_1 \int_t^T f_2(s;\widehat{z},\theta)\, ds}\,\Big|\, \mathcal{F}_t\right] \leq \frac{1}{1 - \left\||2c_1 f_2(\cdot;\widehat{z},\theta)|^{\frac{1}{2}}\right\|_{BMO,\mathbb{P}^{\bar{o}}}^2} < \infty.$$

Thus, all conditions in [2, Proposition 3] are satisfied. It yields a solution $(\widehat{\mathcal{Y}},\widehat{\mathcal{Z}})$ of (3.25), such that the estimate for $\widehat{\mathcal{Y}}$ holds. In order to obtain the estimate for $\widehat{\mathcal{Z}}$, we define $\mathcal{T}(y) = \frac{\frac{1}{2c}e^{2c_1|y|} - \frac{1}{2c_1} - |y|}{c_1}$. Itô's formula implies that

$$\mathcal{T}(\widehat{\mathcal{Y}}_t) = \mathcal{T}(\theta\mathbf{x}) + \int_t^T \mathcal{T}'(\widehat{\mathcal{Y}}_s)\left(f_1(s,\widehat{\mathcal{Z}}_s) + f_2(s;\widehat{z},\theta)\right) ds$$

$$- \frac{1}{2}\int_t^T \mathcal{T}''(\widehat{\mathcal{Y}}_s)|\widehat{\mathcal{Z}}_s|^2\, ds - \int_t^T \mathcal{T}'(\mathcal{Y}_s)\widehat{\mathcal{Z}}_s\, dW_s^{\bar{o}}$$

$$\leq \mathcal{T}(\theta\mathbf{x}) + \int_t^T \left(c_1|\mathcal{T}'(\widehat{\mathcal{Y}}_s)| - \frac{1}{2}\mathcal{T}''(\widehat{\mathcal{Y}}_s)\right)|\widehat{\mathcal{Z}}_s|^2\, ds$$

$$+ \int_t^T \mathcal{T}'(\widehat{\mathcal{Y}}_s)f_2(s;\widehat{z},\theta)\, ds - \int_t^T \mathcal{T}'(\widehat{\mathcal{Y}}_s)\widehat{\mathcal{Z}}_s\, dW_s^{\bar{o}}$$

$$= \mathcal{T}(\theta\mathbf{x}) - \int_t^T |\widehat{\mathcal{Z}}_s|^2\, ds + \int_t^T \mathcal{T}'(\widehat{\mathcal{Y}}_s)f_2(s;\widehat{z},\theta)\, ds - \int_t^T \mathcal{T}'(\widehat{\mathcal{Y}}_s)\widehat{\mathcal{Z}}_s\, dW_s^{\bar{o}},$$

where the first inequality is obtained by **ASS-1**. Note that $\mathcal{T}$ is increasing for $y > 0$. Therefore, for any stopping time τ, it holds that

$$\mathbb{E}^{\mathbb{P}^{\bar{o}}}\left[\int_\tau^T |\widehat{\mathcal{Z}}_s|^2\, ds \,\middle|\, \mathcal{F}_\tau\right]$$

$$\leq \mathcal{T}(\|\theta\mathbf{x}\|) + \int_\tau^T \mathcal{T}'(\widehat{\mathcal{Y}}_s) f_2(s; \widehat{z}, \theta)\, ds$$

$$\leq \frac{\frac{1}{2\underline{c_1}}e^{2\|c_1\|\|\theta\mathbf{x}\|} - \frac{1}{2\underline{c_1}} - \|\theta\mathbf{x}\|}{\underline{c_1}} + \frac{e^{2\|c_1\|\|\widehat{\mathcal{Y}}\|_\infty} - 1}{\underline{c_1}} \left\||f_2(\cdot; \widehat{z}, \theta)|^{\frac{1}{2}}\right\|_{BMO,\mathbb{P}^{\bar{o}}}^2,$$

which implies the estimate for $\widehat{\mathcal{Z}}$.

Moreover, for any two solutions $(\widehat{\mathcal{Y}}, \widehat{\mathcal{Z}})$ and $(\widehat{\mathcal{Y}}', \widehat{\mathcal{Z}}')$ in $S^\infty \times H^2_{BMO,\mathbb{P}^{\bar{o}}}$, $(\Delta\widehat{\mathcal{Y}}, \Delta\widehat{\mathcal{Z}}) := (\widehat{\mathcal{Y}} - \widehat{\mathcal{Y}}', \widehat{\mathcal{Z}} - \widehat{\mathcal{Z}}')$ follows for some stochastic process L

$$\Delta\widehat{\mathcal{Y}}_t = \int_t^T L_s \Delta\widehat{\mathcal{Z}}_s\, ds - \int_t^T \Delta\widehat{\mathcal{Z}}_s\, dW_s^{\bar{o}},$$

where $|L| \leq c_1(|\widehat{\mathcal{Z}}| + |\widehat{\mathcal{Z}}'|)$. Define $\mathbb{Q}'$ by $\frac{d\mathbb{Q}'}{d\mathbb{P}^{\bar{o}}} = \mathcal{E}\left(\int_0 L_s\, dW_s^{\bar{o}}\right)$. [20, Theorem 2.3] implies that $\mathbb{Q}'$ is a probability measure. Consequently, we can rewrite the above equation as $\Delta\widehat{\mathcal{Y}}_t = -\int_t^T \Delta\widehat{\mathcal{Z}}_s\, dW_s^{\mathbb{Q}'}$, where $W^{\mathbb{Q}'} = W^{\bar{o}} - \int_0 L_s\, ds$ is a Brownian motion under $\mathbb{Q}'$. Obviously it holds that $\Delta\widehat{\mathcal{Y}} = \Delta\widehat{\mathcal{Z}} = 0$ and the uniqueness result follows.

Step 2. For each fixed R and each $\widehat{z}$ in the R-ball of $H^2_{BMO,\mathbb{P}^{\bar{o}}}$, **Step 1** yields a unique $(\widehat{\mathcal{Y}}, \widehat{\mathcal{Z}})$, such that (3.25) holds. According to the estimate for $(\widehat{\mathcal{Y}}, \widehat{\mathcal{Z}})$ in **Step 1**, we can choose a small $\theta_2^* \leq \theta_1^*$ depending on R, $\mathbf{x}$, c_1, C_1 and C_2, such that for any $0 \leq \|\theta\| \leq \theta_2^*$, it holds that $\|\widehat{\mathcal{Z}}\|_{BMO,\mathbb{P}^{\bar{o}}}^2 \leq R$. Thus, the mapping $\widehat{z} \mapsto \widehat{\mathcal{Z}}$ from the R-ball of $H^2_{BMO,\mathbb{Q}}$ to itself is well-defined. It remains to prove that this mapping is a contraction.

For any two $\widehat{z}$ and $\widehat{z}'$ in the R-ball of $H^2_{BMO,\mathbb{P}^{\bar{o}}}$, **Step 1** yields a unique solution $(\widehat{\mathcal{Y}}, \widehat{\mathcal{Z}})$ and $(\widehat{\mathcal{Y}}', \widehat{\mathcal{Z}}')$ of (3.25), corresponding to $\widehat{z}$ and $\widehat{z}'$, respectively. Applying Itô's formula to the BSDE of $(\Delta\widehat{\mathcal{Y}}, \Delta\widehat{\mathcal{Z}}) := (\widehat{\mathcal{Y}} - \widehat{\mathcal{Y}}', \widehat{\mathcal{Z}} - \widehat{\mathcal{Z}}')$, by **ASS-2** and Young's inequality we have

$$\|\Delta\widehat{\mathcal{Y}}\|_\infty^2 + \|\Delta\widehat{\mathcal{Z}}\|_{BMO,\mathbb{P}^{\bar{o}}}^2$$

$$\leq 2\|\Delta\widehat{\mathcal{Y}}\|_\infty \|c_1\| \|\Delta\widehat{\mathcal{Z}}\|_{BMO,\mathbb{P}^{\bar{o}}} \left(\|\widehat{\mathcal{Z}}\|_{BMO,\mathbb{P}^{\bar{o}}} + \|\widehat{\mathcal{Z}}'\|_{BMO,\mathbb{P}^{\bar{o}}}\right)$$

$$\quad + 2C_2\|\Delta\widehat{\mathcal{Y}}\|_\infty \|\theta\| \|\Delta\widehat{z}\|_{BMO,\mathbb{P}^{\bar{o}}} (1 + \|\widehat{z}\|_{BMO,\mathbb{P}^{\bar{o}}} + \|\widehat{z}'\|_{BMO,\mathbb{P}^{\bar{o}}})$$

$$\leq 4R\|c_1\| \|\Delta\widehat{\mathcal{Y}}\|_\infty \|\Delta\widehat{\mathcal{Z}}\|_{BMO,\mathbb{P}^{\bar{o}}} + 2C_2\|\theta\|(1 + 2R)\|\Delta\widehat{\mathcal{Y}}\|_\infty \|\Delta\widehat{z}\|_{BMO,\mathbb{P}^{\bar{o}}}$$

$$\leq \frac{1}{2}\|\Delta\widehat{\mathcal{Y}}\|_\infty^2 + 8R^2\|c_1\|^2 \|\Delta\widehat{\mathcal{Z}}\|_{BMO,\mathbb{P}^{\bar{o}}}^2 + \frac{1}{2}\|\Delta\widehat{\mathcal{Y}}\|_\infty^2 + 2C_2^2\|\theta\|^2(1 + 2R)^2\|\Delta\widehat{z}\|_{BMO,\mathbb{P}^{\bar{o}}}^2.$$

First, choose R, such that $8R^2\|c_1\|^2 \leq \frac{1}{4}$. Second, choose $\theta^* \leq \theta_2^*$, such that $2C_2^2(\theta^*)^2(1+2R) \leq \frac{1}{4}$. Then, for all $0 \leq \|\theta\| \leq \theta^*$, it holds that $\|\Delta\widehat{\mathcal{Z}}\|_{BMO,\mathbb{P}^{\bar{o}}} \leq \frac{1}{2}\|\Delta\widehat{z}\|_{BMO,\mathbb{P}^{\bar{o}}}^2$, which implies a contraction. $\square$

As a corollary of Lemma 3.5, we get the wellposedness result of (3.19).

Theorem 3.6 *Let Assumption 1.1 hold. Let $c_1 = \frac{1}{1-\gamma}$ and choose R, such that $R \leq \frac{1}{4\sqrt{2}\|c_1\|}$. Then, there exists a positive constant θ^* only depending on R, T, γ, σ, σ^0 and h, such that for all $0 \leq \|\theta\| \leq \theta^*$, there exists a unique $(\widetilde{Y}, \widetilde{Z}, \widetilde{Z}^0) \in S^\infty \times H^2_{BMO,\mathbb{P}^{\bar{o}}} \times H^2_{BMO,\mathbb{P}^{\bar{o}}}$ satisfying (3.19), and the estimates (3.23) and (3.24).*

Proof It is sufficient to verify that the driver of (3.19) satisfies **ASS-1** and **ASS-2** in Lemma 3.5. Let $\widehat{\mathcal{Z}} = (\widetilde{Z}, \widetilde{Z}^0)$. First, by the definition of $\mathcal{J}_1^{\bar{o}}$ in (3.21), it is straightforward to verify that

$$|\mathcal{J}_1^{\bar{o}}(t, \widehat{\mathcal{Z}})| \leq c_1|\widehat{\mathcal{Z}}|^2, \quad \text{and} \quad |\mathcal{J}_1^{\bar{o}}(t, \widehat{\mathcal{Z}}) - \mathcal{J}_1^{\bar{o}}(t, \widehat{\mathcal{Z}}')| \leq c_1|\widehat{\mathcal{Z}} - \widehat{\mathcal{Z}}'|(|\widehat{\mathcal{Z}}| + |\widehat{\mathcal{Z}}'|).$$

Furthermore, by the expression of $\mathcal{J}_2$ in Appendix A and [12, Lemma A.1], there exists a positive constant C_2 that only depends on γ, σ, σ^0, h, $\|Z^{\bar{o}}\|_{BMO,\mathbb{P}^{\bar{o}}}$ and $\|Z^{0,\bar{o}}\|_{BMO,\mathbb{P}^{\bar{o}}}$, such that

$$\|\mathcal{J}_2(t; \widehat{\mathcal{Z}}, \theta) - \mathcal{J}_2(t; \widehat{\mathcal{Z}}', \theta)\|_{BMO,\mathbb{P}^{\bar{o}}}$$
$$\leq \|\theta\|C_2\|\widehat{\mathcal{Z}} - \widehat{\mathcal{Z}}'\|_{BMO,\mathbb{P}^{\bar{o}}} \left(1 + \|\widehat{\mathcal{Z}}\|_{BMO,\mathbb{P}^{\bar{o}}} + \|\widehat{\mathcal{Z}}'\|_{BMO,\mathbb{P}^{\bar{o}}}\right).$$

Again using Appendix A and [12, Lemma A.1], there exists a positive locally bounded function C_1 depending on γ, σ, σ^0 and h, such that

$$\left\||2c_1\mathcal{J}_2(\cdot; \widehat{\mathcal{Z}})|^{\frac{1}{2}}\right\|^2_{BMO,\mathbb{P}^{\bar{o}}} \leq \|\theta\|C_1(\|\widehat{\mathcal{Z}}\|_{BMO,\mathbb{P}^{\bar{o}}}).$$

Therefore, **ASS-1** and **ASS-2** are satisfied. From Lemma 3.5, we obtain the desired results. □

The following corollary implies that the unique solution to (3.13) can be controlled by θ. In particular, the triple $\left(\|\widetilde{Y}\|_\infty, \|\widetilde{Z}\|_{BMO}, \|\widetilde{Z}^0\|_{BMO}\right)$ goes to 0 as $\|\theta\|$ goes to 0. This result is used to establish the convergence result in Corollary 3.8, which will be used in Section 4.

Corollary 3.7 *Let Assumption 1.1 hold and $(\widetilde{Y}, \widetilde{Z}, \widetilde{Z}^0)$ be the unique solution to (3.13). Then, it holds that*

$$\lim_{\|\theta\|\to 0} \left(\|\widetilde{Y}\|_\infty + \|\widetilde{Z}\|_{BMO} + \|\widetilde{Z}^0\|_{BMO}\right) = 0.$$

and

$$\lim_{\|\theta\|\to 0} \left(\|\widetilde{Z}\|_{M^p} + \|\widetilde{Z}^0\|_{M^p}\right) = 0.$$

Proof Note that $\widetilde{Z}$ and $\widetilde{Z}^0$ belong to the R-ball of $H^2_{BMO,\mathbb{P}^{\bar{o}}}$, where R is independent of θ by Theorem 3.6. Thus, by (3.23) and letting $\|\theta\| \to 0$, we get $\|\widetilde{Y}\|_\infty \to 0$. Taking this convergence into (3.24), we get the convergence of $\widehat{\mathcal{Z}} := (\widetilde{Z}, \widetilde{Z}^0)$ in $H^2_{BMO,\mathbb{P}^{\bar{o}}}$. The convergence of $\widehat{\mathcal{Z}}$ in H^2_{BMO} can be obtained by [13, Lemma A.1]. The convergence in M^p is obtained by the energy inequality; see [20, P.26]. □

Another corollary of Theorem 3.6 is that the FBSDE (3.3), and thus the FBSDE (2.2) are wellposed.

Corollary 3.8 *Let Assumption 1.1 hold and $0 \leq \|\theta\| \leq \theta^*$, where θ^* is the constant determined in Theorem 3.6. The FBSDE (3.3) has a unique solution $(\overline{X}, \overline{Y}, \overline{Z}, \overline{Z}^0) \in \left(\bigcap_{p>1} S^p\right) \times \left(\bigcap_{p>1} S^p\right) \times H^2_{BMO} \times H^2_{BMO}$ with the convergence*

$$\lim_{\|\theta\| \to 0} \left(\|\overline{Z}\|_{M^p} + \|\overline{Z}^0\|_{M^p} \right) = 0$$

and the FBSDE (2.2) is wellposed in $\left(\bigcap_{p>1} S^p\right) \times \left(\bigcap_{p>1} S^p\right) \times H^2_{BMO} \times H^2_{BMO}$.

Proof By Theorem 3.6 and Corollary 3.4, there exists a unique $(\overline{X}, \overline{Y}, \overline{Z}, \overline{Z}^0)$ satisfying (3.3). Theorem 3.6, Proposition 3.1, (3.14), (3.16), and [13, Lemma A.1] imply that $(\overline{Z}, \overline{Z}^0) \in H^2_{BMO} \times H^2_{BMO}$. In order to prove $\overline{X} \in \bigcap_{p>1} S^p$, we make a change of measure by defining

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\left(-\int_0^\cdot \widetilde{\sigma}^\top (\widetilde{\sigma}^\top \widetilde{\sigma})^{-1} \left\{ h_s - \frac{h_s + \sigma_s Z_s^{\bar{o}} + \sigma_s^0 Z_s^{0,\bar{o}}}{1-\gamma} - \frac{\sigma_s \overline{Z}_s + \sigma_s^0 \overline{Z}_s^0}{2(1-\gamma)} \right\} d\overline{W}_s \right)$$

$$:= \mathcal{E}\left(-\int_0^\cdot \widehat{\mathcal{M}}_s \, d\overline{W}_s \right),$$

where we recall that $\widetilde{\sigma} = (\sigma, \sigma^0)^\top$. Then, the dynamics of $\overline{X}$ can be rewritten as

$$d\overline{X}_t = \frac{\sigma_t \overline{Z}_t + \sigma_t^0 \overline{Z}_t^0}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \widetilde{\sigma}_t^\top \, d\widehat{W}_t,$$

where

$$\widehat{W}_t = \overline{W}_t + \int_0^t \widetilde{\sigma}_s(\widetilde{\sigma}_s^\top \widetilde{\sigma}_s)^{-1} \left\{ h_s - \frac{h_s + \sigma_s Z_s^{\bar{o}} + \sigma_s^0 Z_s^{0,\bar{o}}}{1-\gamma} - \frac{\sigma_s \overline{Z}_s + \sigma_s^0 \overline{Z}_s^0}{2(1-\gamma)} \right\} ds$$

is a $\mathbb{Q}$-Brownian motion. For each $p > 1$, it holds that

$$\mathbb{E}^{\mathbb{Q}}\left[\sup_{0 \leq t \leq T} |\overline{X}_t|^p \right] \leq C\mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T \left(\frac{\sigma_t \overline{Z}_t + \sigma_t^0 \overline{Z}_t^0}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \right)^2 \widetilde{\sigma}_t^\top \widetilde{\sigma}_t \, dt \right)^{\frac{p}{2}} \right]$$

$$\leq C\left(\|\overline{Z}\|_{BMO,\mathbb{Q}}^2 + \|\overline{Z}^0\|_{BMO,\mathbb{Q}}^2 \right)$$

$$< \infty,$$

where the above three inequalities are obtained by using BDG's inequality, energy inequality and [13, Lemma A.1], respectively.

By the definition of $\mathbb{Q}$, for any $p > 1$ and any $1 < q < p_{\widehat{\mathcal{M}}}$, where we recall $p_{\widehat{\mathcal{M}}}$ is defined in Appendix B, it holds that

$$
\mathbb{E}\left[\sup_{0 \leq t \leq T} |\overline{X}_t|^p\right]
$$

$$
= \mathbb{E}^{\mathbb{Q}}\left[\mathcal{E}\left(\int_0^T \widehat{\mathcal{M}}_t \, d\widehat{W}_t\right) \sup_{0 \leq t \leq T} |\overline{X}_t|^p\right]
$$

$$
\leq \mathbb{E}^{\mathbb{Q}}\left[\mathcal{E}\left(\int_0^T \widehat{\mathcal{M}}_t \, d\widehat{W}_t\right)^q\right]^{\frac{1}{q}} \mathbb{E}^{\mathbb{Q}}\left[\sup_{0 \leq t \leq T} |\overline{X}_t|^{pq^*}\right]^{\frac{1}{q^*}} \qquad \left(\frac{1}{q} + \frac{1}{q^*} = 1\right)
$$

$$
< \infty,
$$

where the last inequality is given by Lemma B.1. Therefore, $\overline{X} \in \bigcap_{p>1} S^p$. The same arguement implies that $\overline{Y} \in \bigcap_{p>1} S^p$. The convergence is obtained by Corollary 3.7, (3.14), and (3.16).

The wellposedness of (2.2) follows from (3.2), Proposition 3.1, and Theorem 3.6. $\qquad\qquad\square$

3.4 Wellposedness of the MFG (1.4)

The main result in Section 3 is the following wellposedness result of MFG (1.4). In particular, our NE is unique.

Theorem 3.9 *Let Assumption 1.1 hold and $0 \leq \|\theta\| \leq \theta^*$, where θ^* is the constant determined in Theorem 3.6. Let $(\widehat{X}^*, Y^*, Z^*, Z^{0,*})$ be the unique solution to (2.2), and $\mu_t^* = \exp\left(\mathbb{E}[\widehat{X}_t^* | \mathcal{F}_t^0]\right)$, $t \in [0, T]$. Under μ^*, the unique optimal response for the representative player is*

$$
\pi^* = \frac{h + \sigma Z^* + \sigma^0 Z^{0,*}}{(1 - \gamma)(\sigma^2 + (\sigma^0)^2)} \tag{3.26}
$$

and the value function given one realization of (θ, γ, x) follows

$$
V(\theta, \gamma, x, \mu^*) = \frac{1}{\gamma} x^\gamma e^{Y_0^*}.
$$

Moreover, (μ^*, π^*) is the unique NE of (1.4), where $\log \mu^* \in \bigcap_{p>1} S_{\mathbb{F}^0}^p$ and $\pi^* \in H_{BMO}^2$.

Proof By Proposition 2.2, there is a one-to-one correspondence between the FBSDE (2.2) and the NE of (1.4). Therefore, the existence and uniqueness result of NE can be obtained by Corollary 3.8. $\qquad\qquad\square$

Remark 3.10 By [18, Proposition 15], for each realization of the random variables (θ, γ, x), the value function satisfies the dynamic version

$$V(t, X_t^*; \theta, \gamma, x, \mu^*) = \frac{1}{\gamma}(X_t^*)^\gamma e^{Y_t^*}, \qquad (3.27)$$

where

$$V(t, X_t^*; \theta, \gamma, x, \mu^*) := \max_\pi \mathbb{E}\left[\frac{1}{\gamma}\left(X_T(\mu_T^*)^{-\theta}\right)^\gamma \middle| \mathcal{F}_t\right].$$

The dynamic version of value function (3.27) will be used in Section 4.

3.5 The Portfolio Game under $\mathcal{A}$-Measurable Market Parameters

In this section, we consider a special case, in which the market parameters are $\mathcal{A}$-measurable. We recall that $\mathcal{A}$-measurability corresponds to heterogeneous types. In addition to Assumption 1.1, we make the following assumption:

Assumption 3.11 For each $t \in [0, T]$, the return rate h_t and the volatility (σ_t, σ_t^0) are measurable w.r.t. $\mathcal{A}$.

Under Assumption 3.11, we will construct a unique solution to the FBSDE (2.2) in closed form. As a result, we get an NE of the MFG (1.4) in closed form, which is proven to be unique in L^∞. Furthermore, when all of the market parameters become time-independent, we revisit the model in [22], and we prove that the constant equilibrium obtained in [22] is the unique one in L^∞, not only in the space of constant equilibria, as shown in [22].

The following proposition shows the closed form solution to the FBSDE (2.2) under Assumption 1.1 and Assumption 3.11.

Proposition 3.12 *Under Assumption 1.1 and Assumption 3.11, there exists a unique tuple $(\widehat{X}, Y, Z, Z^0) \in S^2 \times S^2 \times L^\infty \times L^\infty$ satisfying (2.2). In particular, the Z-component of the solution has the following closed form expression*

$$Z = 0, \qquad Z^0 = -\frac{\theta\gamma\mathbb{E}\left[\frac{\sigma^0 h}{(1-\gamma)(\sigma^2+(\sigma^0)^2)}\right]}{1+\mathbb{E}\left[\frac{\theta\gamma(\sigma^0)^2}{(1-\gamma)(\sigma^2+(\sigma^0)^2)}\right]}. \qquad (3.28)$$

Proof We first verify that (3.28), together with some $\widehat{X}$ and Y, satisfies (2.2) by construction. Our goal is to construct (Z, Z^0), such that (Z_t, Z_t^0) is $\mathcal{A}$-measurable for each $t \in [0, T]$. Assuming that (Z_t, Z_t^0) is $\mathcal{A}$-measurable for each $t \in [0, T]$ and taking the forward dynamics in (2.2) into the backward one in (2.2), we have the following

$$\begin{aligned}
Y_t = &-\theta\gamma\mathbb{E}[\log x] + \int_t^T \left\{\frac{Z_s^2 + (Z_s^0)^2}{2} + \frac{\gamma(h_s + \sigma_s Z_s + \sigma_s^0 Z_s^0)^2}{2(1-\gamma)(\sigma_s^2 + (\sigma_s^0)^2)}\right\} ds \\
&- \theta\gamma\int_0^T \mathbb{E}\left[\frac{h_s + \sigma_s Z_s + \sigma_s^0 Z_s^0}{(1-\gamma)(\sigma_s^2 + (\sigma_s^0)^2)}\left\{h_s - \frac{h_s + \sigma_s Z_s + \sigma_s^0 Z_s^0}{2(1-\gamma)}\right\}\right] ds \\
&- \theta\gamma\int_0^T \mathbb{E}\left[\frac{h_s + \sigma_s Z_s + \sigma_s^0 Z_s^0}{(1-\gamma)(\sigma_s^2 + (\sigma_s^0)^2)}\sigma_s^0\right] dW_s^0 - \int_t^T Z_s^0 \, dW_s^0 - \int_t^T Z_s \, dW_s,
\end{aligned}$$

where we use the assumption that (Z_t, Z_t^0) is $\mathcal{A}$-measurable, and that $\mathcal{A}$ and W^0 are independent. In order to make Y adapted, we let

$$Z \equiv 0, \qquad -\theta\gamma\mathbb{E}\left[\frac{h + \sigma Z + \sigma^0 Z^0}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)}\sigma^0\right] - Z^0 = 0,$$

which implies (3.28). Note that (3.28) together with the constructed Y is an adapted solution to (2.2).

Under Assumption 3.11, it holds that $Z^{\bar{\sigma}} = Z^{0,\bar{\sigma}} = 0$. To prove the uniqueness result, by Corollary 3.4, it is sufficient to show that the solution to (3.13) is unique in $S^2 \times L^\infty \times L^\infty$. Let $(\widetilde{Y}, \widetilde{Z}, \widetilde{Z}^0) \in S^2 \times L^\infty \times L^\infty$ and $(\widetilde{Y}', \widetilde{Z}', \widetilde{Z}^{0\prime}) \in S^2 \times L^\infty \times L^\infty$ be two solutions to (3.13). Denote by $(\Delta Y, \Delta Z, \Delta Z^0)$ the difference of these two solutions. Then, by the dynamics of (3.13) and by noting that $(\widetilde{Z}, \widetilde{Z}^0) \in L^\infty \times L^\infty$ and $(\widetilde{Z}', \widetilde{Z}^{0\prime}) \in L^\infty \times L^\infty$, the tuple $(\Delta Y, \Delta Z, \Delta Z^0)$ satisfies a conditional mean field BSDE with Lipschitz coefficients. Standard arguments imply that the unique solution is $(\Delta Y, \Delta Z, \Delta Z^0) = (0, 0, 0)$. $\qquad\square$

With the explicit solution in Proposition 3.12, we can construct an optimal strategy in closed form for the representative player, which is unique in L^∞.

Theorem 3.13 *Let Assumption 1.1 and Assumption 3.11 hold. Then, in L^∞ the unique optimal response of the respresentative player is given by*

$$\pi_t^* = \frac{h_t}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)}$$
$$-\frac{\theta\gamma\sigma_t^0}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)}\frac{\mathbb{E}\left[\frac{h_t\sigma_t^0}{(1-\gamma)(\sigma_t^2+(\sigma_t^0)^2)}\right]}{1 + \mathbb{E}\left[\frac{\theta\gamma(\sigma_t^0)^2}{(1-\gamma)(\sigma_t^2+(\sigma_t^0)^2)}\right]}, \quad t \in [0, T]. \tag{3.29}$$

Proof Taking (3.28) into (3.26), we get (3.29), which is unique in L^∞ by Proposition 3.12. $\qquad\square$

As a corollary, when all coefficients become time-independent, we revisit the MFG in [22].

Corollary 3.14 (Lacker & Zariphopoulou's MFGs revisited) *In addition to Assumption 1.1 and Assumption 3.11, assume that the return rate h and the volatility (σ, σ^0) are time-independent. Then, the unique optimal response is*

$$\pi^* = \frac{h}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} - \frac{\theta\gamma\sigma^0}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)}\frac{\mathbb{E}\left[\frac{h\sigma^0}{(1-\gamma)(\sigma^2+(\sigma^0)^2)}\right]}{1 + \mathbb{E}\left[\frac{\theta\gamma(\sigma^0)^2}{(1-\gamma)(\sigma^2+(\sigma^0)^2)}\right]}. \tag{3.30}$$

The constant equilibrium (3.30) is identical to [22, Theorem 19], which is unique in L^∞.

Remark 3.15 If $0 \leq \|\theta\| \leq \theta^*$ holds, then (3.28) is also unique in $H^2_{BMO} \times H^2_{BMO}$ by Corollary 3.8. Therefore, the constant equilibrium (3.30) is the unique one in H^2_{BMO}, and nonconstant equilibria do not exist in the subspace of H^2_{BMO} with (2.1) being true.

4 Asymptotic Expansion in Terms of θ

Motivated by the weak interaction assumption, we develop an asymptotic expansion result of the value function and the optimal investment. Specifically, we provide an approximation in any order of the value function and optimal investment for the model with competition in terms of the solution to the benchmark model without competition when the investor is only concerned with her own wealth. Our idea is to start with the FBSDE characterization of the NE, and translate the expansion of the value function and the optimal investment to that of the solution to the FBSDE.

Let V^θ and π^θ be the value function and the optimal investment of the MFG (1.4), and $V^{\bar{o}}$ and $\pi^{\bar{o}}$ be the value function and the optimal investment of the benchmark utility maximization problem (i.e. when $\theta = 0$ in (1.4)). Then by Theorem 3.9 and Remark 3.10, we have

$$V_t^\theta = \frac{1}{\gamma} e^{\gamma \widehat{X}_t^\theta + Y_t^\theta}, \qquad \pi^\theta = \frac{h + \sigma Z^\theta + \sigma^0 Z^{0,\theta}}{(1 - \gamma)(\sigma^2 + (\sigma^0)^2)},$$

$$V_t^{\bar{o}} = \frac{1}{\gamma} e^{\gamma X_t^{\bar{o}} + Y_t^{\bar{o}}}, \qquad \pi^{\bar{o}} = \frac{h + \sigma Z^{\bar{o}} + \sigma^0 Z^{0,\bar{o}}}{(1 - \gamma)(\sigma^2 + (\sigma^0)^2)},$$

where $(\widehat{X}^\theta, Y^\theta, Z^\theta, Z^{0,\theta})$ and $(X^{\bar{o}}, Y^{\bar{o}}, Z^{\bar{o}}, Z^{0,\bar{o}})$ are the unique solutions to (2.2) with θ and $\theta = 0$, respectively. As a result, it holds that

$$\log \frac{V^\theta}{V_t^{\bar{o}}} = \gamma(\widehat{X}_t^\theta - X_t^{\bar{o}}) + (Y_t^\theta - Y_t^{\bar{o}}), \ \pi^\theta - \pi^{\bar{o}} = \frac{\sigma(Z^\theta - Z^{\bar{o}}) + \sigma^0(Z^{0,\theta} - Z^{0,\bar{o}})}{(1 - \gamma)(\sigma^2 + (\sigma^0)^2)}.$$

Let $(\overline{X}^\theta, \overline{Y}^\theta, \overline{Z}^\theta, \overline{Z}^{0,\theta}) := (\widehat{X}^\theta - X^{\bar{o}}, Y^\theta - Y^{\bar{o}}, Z^\theta - Z^{\bar{o}}, Z^{0,\theta} - Z^{0,\bar{o}})$ be the unique solution to (3.3).

Our goal is to prove that there exist $(\widehat{X}^{(i)}, Y^{(i)}, Z^{(i)}, Z^{0,(i)})_{i=1,\dots,n}$, for any $n \geq 1$, such that it holds that

$$\widehat{X}^\theta - X^{\bar{o}} = \overline{X}^\theta = \sum_{i=1}^{n} \theta^i \widehat{X}^{(i)} + o(\theta^n), \ Y^\theta - Y^{\bar{o}} = \overline{Y}^\theta = \sum_{i=1}^{n} \theta^i Y^{(i)} + o(\theta^n),$$

$$Z^\theta - Z^{\bar{o}} = \overline{Z}^\theta = \sum_{i=1}^{n} \theta^i Z^{(i)} + o(\theta^n), \ Z^{0,\theta} - Z^{0,\bar{o}} = \overline{Z}^{0,\theta} = \sum_{i=1}^{n} \theta^i Z^{0,(i)} + o(\theta^n),$$

where we should note that in Section 4, θ^i means θ raised to the power of i. It should not be confused with player i's competition parameter θ^i in the N-player game, e.g., (1.2).

For $\varphi = \widehat{X}, Y, Z, Z^0$, define

$$\varphi^{\theta,(1)} = \frac{\varphi^\theta - \varphi^{\bar{o}}}{\theta} = \frac{\overline{\varphi}^\theta}{\theta},$$

which implies

$$
\begin{cases}
d\widehat{X}_t^{\theta,(1)} = \left\{ \dfrac{\sigma_t Z_t^{\theta,(1)} + \sigma_t^0 Z_t^{0,\theta,(1)}}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \left(h_t - \dfrac{1}{1-\gamma}(h_t + \sigma_t Z_t^{\bar{o}} + \sigma_t^0 Z_t^{0,\bar{o}}) \right) \right. \\
\qquad\qquad \left. - \dfrac{\left(\sigma_t \overline{Z}_t^{\theta} + \sigma_t^0 \overline{Z}_t^{0,\theta}\right)\left(\sigma_t Z_t^{\theta,(1)} + \sigma_t^0 Z_t^{0,\theta,(1)}\right)}{2(1-\gamma)^2(\sigma_t^2 + (\sigma_t^0)^2)} \right\} dt \\
\qquad\qquad + \dfrac{\sigma_t Z_t^{\theta,(1)} + \sigma_t^0 Z_t^{0,\theta,(1)}}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} (\sigma_t\, dW_t + \sigma_t^0\, dW_t^0) \\
-dY_t^{\theta,(1)} = \left\{ \dfrac{(2Z_t^{\bar{o}} + \overline{Z}_t^{\theta})Z_t^{\theta,(1)}}{2} + \dfrac{(2Z_t^{0,\bar{o}} + \overline{Z}_t^{0,\theta})Z_t^{0,\theta,(1)}}{2} \right. \\
\qquad\qquad + \dfrac{\gamma(2h_t + 2\sigma_t Z_t^{\bar{o}} + 2\sigma_t^0 Z_t^{0,\bar{o}} + \sigma_t \overline{Z}_t^{\theta} + \sigma_t^0 \overline{Z}_t^{0,\theta})}{2(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \\
\qquad\qquad \left. \times (\sigma_t Z_t^{\theta,(1)} + \sigma_t^0 Z_t^{0,\theta,(1)}) \right\} dt \\
\qquad\qquad - Z_t^{\theta,(1)}\, dW_t - Z_t^{0,\theta,(1)}\, dW_t^0, \\
\widehat{X}_0^{\theta,(1)} = 0, \quad Y_T^{\theta,(1)} = -\gamma\mathbb{E}[\overline{X}_T^{\theta}|\mathcal{F}_T^0] - \gamma\mathbb{E}[X_T^{\bar{o}}|\mathcal{F}_T^0].
\end{cases}
\tag{4.1}
$$

We now establish the convergence of $(\widehat{X}^{\theta,(1)}, Y^{\theta,(1)}, Z^{\theta,(1)}, Z^{0,\theta,(1)})$ in a suitable sense as $\theta \to 0$. Let the candidate limit of (4.1) satisfy

$$
\begin{cases}
d\widehat{X}_t^{(1)} = \dfrac{\sigma_t Z_t^{(1)} + \sigma_t^0 Z_t^{0,(1)}}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \left(h_t - \dfrac{1}{1-\gamma}(h_t + \sigma_t Z_t^{\bar{o}} + \sigma_t^0 Z_t^{0,\bar{o}}) \right) dt \\
\qquad\qquad + \dfrac{\sigma_t Z_t^{(1)} + \sigma_t^0 Z_t^{0,(1)}}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)}(\sigma_t\, dW_t + \sigma_t^0\, dW_t^0) \\
-dY_t^{(1)} = \left\{ Z_t^{\bar{o}} Z_t^{(1)} + Z_t^{0,\bar{o}} Z_t^{0,(1)} \right. \\
\qquad\qquad \left. + \dfrac{\gamma(h_t + \sigma_t Z_t^{\bar{o}} + \sigma_t^0 Z_t^{0,\bar{o}})(\sigma_t Z_t^{(1)} + \sigma_t^0 Z_t^{0,(1)})}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2}) \right\} dt \\
\qquad\qquad - Z_t^{(1)}\, dW_t - Z_t^{0,(1)}\, dW_t^0, \\
\widehat{X}_0^{(1)} = 0, \quad Y_T^{(1)} = -\gamma\mathbb{E}[X_T^{\bar{o}}|\mathcal{F}_T^0].
\end{cases}
\tag{4.2}
$$

The following lemma establishes the wellposedness result of (4.2) and the convergence from (4.1) to (4.2). In particular, the first-order approximation of the FBSDE is obtained.

Lemma 4.1 *Let Assumption 1.1 hold.*

(1) There exists a unique $(\widehat{X}^{(1)}, Y^{(1)}, Z^{(1)}, Z^{0,(1)}) \in \left(\bigcap_{p>1} S^p\right) \times \left(\bigcap_{p>1} S^p\right) \times \left(\bigcap_{p>1} M^p\right) \times \left(\bigcap_{p>1} M^p\right)$ satisfying (4.2);

(2) For each $p > 1$, the following convergence holds

$$\lim_{\theta \to 0} \left(\|\widehat{X}^{\theta,(1)} - \widehat{X}^{(1)}\|_{S^p} + \|Y^{\theta,(1)} - Y^{(1)}\|_{S^p} \right.$$
$$\left. + \|Z^{\theta,(1)} - Z^{(1)}\|_{M^p} + \|Z^{0,\theta} - Z^{0,(1)}\|_{M^p} \right) = 0. \tag{4.3}$$

Proof (1). Recall $\mathbb{P}^{\bar{o}}$ defined in (3.17). Then the backward dynamics in (4.2) can be rewritten as

$$dY_t^{(1)} = Z_t^{(1)} \, d\widetilde{W}_t + Z_t^{0,(1)} \, d\widetilde{W}_t^0, \qquad Y_T^{(1)} = -\gamma \mathbb{E}[X_T^{\bar{o}}|\mathcal{F}_T^0],$$

where $W^{\bar{o}} = (\widetilde{W}, \widetilde{W}^0)^{\top}$ is defined in (3.20). In order to apply [1, Theorem 3.5], it suffices to prove that for each $p > 1$

$$\mathbb{E}^{\mathbb{P}^{\bar{o}}}[(\mathbb{E}[X_T^{\bar{o}}|\mathcal{F}_T^0])^p] < \infty.$$

In fact, by the notation $\mathcal{M}$ in (3.18) and $p_{\mathcal{M}}$ in Appendix B, for any $p > 1$ and any $q \in (1, p_{\mathcal{M}})$

$$\mathbb{E}^{\mathbb{P}^{\bar{o}}}[(\mathbb{E}[|X_T^{\bar{o}}||\mathcal{F}_T^0])^p] = \mathbb{E}\left[\mathcal{E}_T(\mathcal{M})(\mathbb{E}[|X_T^{\bar{o}}||\mathcal{F}_T^0])^p\right]$$
$$\leq (\mathbb{E}[|\mathcal{E}_T(\mathcal{M})|^q])^{\frac{1}{q}} \left(\mathbb{E}\left[\left(\mathbb{E}[|X_T^{\bar{o}}||\mathcal{F}_T^0]\right)^{pq^*}\right]\right)^{\frac{1}{q^*}}$$
$$\leq K^{\frac{1}{q}}(q, \|\mathcal{M}\|_{BMO}) \left(\mathbb{E}\left[\left(\mathbb{E}[|X_T^{\bar{o}}||\mathcal{F}_T^0]\right)^{pq^*}\right]\right)^{\frac{1}{q^*}}$$
$$< \infty, \tag{4.4}$$

where the first inequality is given by Hölder's inequality and $\frac{1}{q} + \frac{1}{q^*} = 1$, the second inequality is given by Lemma B.1, and the last inequality is given by Proposition 3.1.

Therefore, by [1, Theorem 3.5], there exists a unique $(Y^{(1)}, Z^{(1)}, Z^{0,(1)}) \in S_{\mathbb{P}^{\bar{o}}}^2 \times \left(\bigcap_{p>1} M_{\mathbb{P}^{\bar{o}}}^p\right) \times \left(\bigcap_{p>1} M_{\mathbb{P}^{\bar{o}}}^p\right)$, which implies that $(Y^{(1)}, Z^{(1)}, Z^{0,(1)}) \in S^2 \times \left(\bigcap_{p>1} M^p\right) \times \left(\bigcap_{p>1} M^p\right)$. Indeed, by the definition of $\mathbb{P}^{\bar{o}}$ in (3.17), it holds that

$$\frac{d\mathbb{P}}{d\mathbb{P}^{\bar{o}}} = \mathcal{E}\left(-\int_0^{\cdot} \mathcal{M}_s \, dW_s^{\bar{o}}\right).$$

Thus, by the same argument as (4.4), we have for any $p > 1$

$$\mathbb{E}\left[\left(\int_0^T (Z_s^{(1)})^2 \, ds\right)^{\frac{p}{2}}\right] = \mathbb{E}^{\mathbb{P}^{\bar{o}}}\left[\mathcal{E}_T\left(-\int_0^{\cdot} \mathcal{M}_s \, dW_s^{\bar{o}}\right)\left(\int_0^T (Z_s^{(1)})^2 \, ds\right)^{\frac{p}{2}}\right] < \infty.$$

The same result holds for $Z^{0,(1)}$. By the dynamics of $Y^{(1)}$ and standard estimate, we have $Y^{(1)} \in \bigcap_{p>1} S^p$.

(2). Let $\Delta\varphi^\theta = \varphi^{\theta,(1)} - \varphi^{(1)}$ for $\varphi = \widehat{X}, Y, Z, Z^0$. Then, $(\Delta Y^\theta, \Delta Z^\theta, \Delta Z^{0,\theta})$ satisfies

$$
\begin{cases}
-d\Delta Y_t^\theta = \left\{ \dfrac{1}{2}\overline{Z}_t^\theta Z_t^{\theta,(1)} + \dfrac{1}{2}\overline{Z}_t^{0,\theta} Z_t^{0,\theta,(1)} \right. \\[2mm]
\qquad\qquad + \dfrac{\gamma\left(\sigma_t Z_t^{\theta,(1)} + \sigma_t^0 Z_t^{0,\theta,(1)}\right)\left(\sigma_t \overline{Z}_t^\theta + \sigma_t^0 \overline{Z}_t^{0,\theta}\right)}{2(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \\[4mm]
\qquad\qquad + \left(Z_t^{\bar{o}} + \dfrac{\gamma\sigma_t(h_t + \sigma_t Z_t^{\bar{o}} + \sigma_t^0 Z_t^{0,\bar{o}})}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \right)\Delta Z_t^\theta \\[4mm]
\qquad\qquad \left. + \left(Z^{0,\bar{o}} + \dfrac{\gamma\sigma_t^0(h_t + \sigma_t Z_t^{\bar{o}} + \sigma_t^0 Z_t^{0,\bar{o}})}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \right)\Delta Z_t^{0,\theta} \right\} dt \\[4mm]
\qquad\qquad - \Delta Z_t^\theta\, dW_t - \Delta Z^{0,\theta}\, dW_t^0, \\[3mm]
\Delta Y_T^\theta = -\gamma\mathbb{E}[\overline{X}_T^\theta | \mathcal{F}_T^0].
\end{cases}
$$

The conditions in [1, Corollary 3.4] are satisfied. Indeed, by Corollary 3.8 and the energy inequality (see [20, P. 26]), it holds that for each $p > 1$, $\mathbb{E}\left[\left(\mathbb{E}[\overline{X}_T^\theta | \mathcal{F}_T^0]\right)^p\right] < \infty$. The same reason, together with the result in (1), implies that for each $p > 1$ the non-homogenous term in the driver of ΔY^θ is in M^p. Thus, [1, Assumption A.3] is satisfied. [1, Assumption A.2] holds due to Corollary 3.8 and Proposition 3.1. By [1, Corollary 3.4], it holds that for each $p > 1$ there exists $p' > 1$ and $p'' > 1$, such that

$$
\|\Delta Y^\theta\|_{S^p} + \|\Delta Z^\theta\|_{M^p} + \|\Delta Z^{0,\theta}\|_{M^p}
$$
$$
\leq C\left\{ \mathbb{E}\left[|\overline{X}_T^\theta|^{p'}\right] + (\|Z^{\theta,(1)}\|_{M^{p'}} + \|Z^{0,\theta,(1)}\|_{M^{p'}})(\|\overline{Z}^\theta\|_{M^{p'}} + \|\overline{Z}^{0,\theta}\|_{M^{p'}}) \right\}
$$
$$
\times \left\{ 1 + \left\| Z^{0,\bar{o}} + \dfrac{\gamma\sigma^0(h + \sigma Z^{\bar{o}} + \sigma^0 Z^{0,\bar{o}})}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} \right\|_{M^{p''}} \right.
$$
$$
\left. + \left\| Z^{\bar{o}} + \dfrac{\gamma\sigma(h + \sigma Z^{\bar{o}} + \sigma^0 Z^{0,\bar{o}})}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} \right\|_{M^{p''}} \right\}
$$
$$
\to 0,
$$

where C does not depend on θ, and the convergence is obtained by Corollary 3.8. $\qquad\square$

In order to establish higher order approximation, we define

$$
K^{(\theta,n-1),(\theta,1)} := \frac{\left(\sigma Z^{\theta,(n-1)} + \sigma^0 Z^{0,\theta,(n-1)}\right)\left(\sigma Z^{\theta,(1)} + \sigma^0 Z^{0,\theta,(1)}\right)}{2(\sigma^2 + (\sigma^0)^2)},
$$
$$
K^{(i),(\theta,n-i)} := \frac{\left(\sigma Z^{(i)} + \sigma^0 Z^{0,(i)}\right)\left(\sigma Z^{\theta,(n-i)} + \sigma^0 Z^{0,\theta,(n-i)}\right)}{2(\sigma^2 + (\sigma^0)^2)},
$$
$$
K^{(i),(n-i)} := \frac{\left(\sigma Z^{(i)} + \sigma^0 Z^{0,(i)}\right)\left(\sigma Z^{(n-i)} + \sigma^0 Z^{0,(n-i)}\right)}{2(\sigma^2 + (\sigma^0)^2)}.
$$

We now introduce the candidate $(\widehat{X}^{(n)}, Y^{(n)}, Z^{(n)}, Z^{0,(n)})$, $n \geq 2$, based on the above notation. Intuitively, $(\widehat{X}^{(n+1)}, Y^{(n+1)}, Z^{(n+1)}, Z^{0,(n+1)})$ is the limit of $(\widehat{X}^{\theta,(n+1)}, Y^{\theta,(n+1)}, Z^{\theta,(n+1)}, Z^{0,\theta,(n+1)})$, where $\varphi^{\theta,(n+1)} := \frac{\varphi^{\theta,(n)} - \varphi^{(n)}}{\theta}$, $\varphi = \widehat{X}, Y, Z, Z^0$. Thus, for each $n \geq 2$, we introduce two (decoupled) FBSDE systems iteratively

$$
\begin{cases}
d\widehat{X}_t^{\theta,(n)} = \left\{ \dfrac{\sigma_t Z_t^{\theta,(n)} + \sigma_t^0 Z_t^{0,\theta,(n)}}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \left\{ h_t - \dfrac{h_t + \sigma_t Z_t^{\bar{o}} + \sigma_t^0 Z_t^{0,\bar{o}}}{1-\gamma} \right\} - \dfrac{K_t^{(\theta,n-1),(\theta,1)}}{(1-\gamma)^2} \right. \\
\qquad\qquad \left. - \sum_{j=1}^{n-2} \dfrac{K_t^{(j),(\theta,n-j)}}{(1-\gamma)^2} \right\} dt + \dfrac{\sigma_t Z_t^{\theta,(n)} + \sigma_t^0 Z_t^{0,\theta,(n)}}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} (\sigma_t \, dW_t + \sigma_t^0 \, dW_t^0) \\[4pt]
-dY_t^{\theta,(n)} = \left\{ \dfrac{\gamma \left(\sigma_t Z_t^{\theta,(n)} + \sigma_t^0 Z_t^{0,\theta,(n)} \right)}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \left(\sigma_t Z_t^{\bar{o}} + \sigma_t^0 Z_t^{0,\bar{o}} + h_t \right) \right. \\
\qquad\qquad + \dfrac{\gamma K_t^{(\theta,n-1),(\theta,1)}}{1-\gamma} + \sum_{j=1}^{n-2} \dfrac{\gamma K_t^{(j),(\theta,n-j)}}{1-\gamma} + Z_t^{\bar{o}} Z_t^{\theta,(n)} + Z_t^{0,\bar{o}} Z_t^{0,\theta,(n)} \\
\qquad\qquad + \dfrac{1}{2} Z_t^{\theta,(n-1)} Z_t^{\theta,(1)} + \dfrac{1}{2} Z_t^{0,\theta,(n-1)} Z_t^{0,\theta,(1)} \\
\qquad\qquad \left. + \dfrac{1}{2} \sum_{j=1}^{n-2} Z_t^{(j)} Z_t^{\theta,(n-j)} + \dfrac{1}{2} \sum_{j=1}^{n-2} Z_t^{0,(j)} Z_t^{0,\theta,(n-j)} \right\} dt \\
\qquad\qquad - Z_t^{\theta,(n)} \, dW_t - Z_t^{0,\theta,(n)} \, dW_t^0 \\
\widehat{X}_0^{\theta,(n)} = 0, \ \ Y_T^{\theta,(n)} = -\gamma \mathbb{E}[\widehat{X}_T^{\theta,(n-1)} | \mathcal{F}_T^0]
\end{cases}
\tag{4.5}
$$

and

$$
\begin{cases}
d\widehat{X}_t^{(n)} = \left(\dfrac{\sigma_t Z_t^{(n)} + \sigma_t^0 Z_t^{0,(n)}}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \left\{ h_t - \dfrac{1}{1-\gamma}(h_t + \sigma_t Z_t^{\bar{o}} + \sigma_t^0 Z_t^{0,\bar{o}}) \right\} \right. \\
\qquad\qquad \left. - \sum_{j=1}^{n-1} \dfrac{K_t^{(j),(n-j)}}{(1-\gamma)^2} \right) dt + \dfrac{\left(\sigma_t Z_t^{(n)} + \sigma_t^0 Z_t^{0,(n)} \right)}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \left(\sigma_t \, dW_t + \sigma_t^0 \, dW_t^0 \right) \\[4pt]
-dY_t^{(n)} = \left\{ \dfrac{\gamma \left(\sigma_t Z_t^{(n)} + \sigma_t^0 Z_t^{0,(n)} \right)}{(1-\gamma)(\sigma_t^2 + (\sigma_t^0)^2)} \left(\sigma_t Z_t^{\bar{o}} + \sigma_t^0 Z_t^{0,\bar{o}} + h_t \right) + \sum_{j=1}^{n-1} \dfrac{\gamma K_t^{(j),(n-j)}}{1-\gamma} \right. \\
\qquad\qquad \left. + Z_t^{\bar{o}} Z_t^{(n)} + Z_t^{0,\bar{o}} Z_t^{0,(n)} + \dfrac{1}{2} \sum_{j=1}^{n-1} Z_t^{(j)} Z_t^{(n-j)} + \dfrac{1}{2} \sum_{j=1}^{n-1} Z_t^{0,(j)} Z_t^{0,(n-j)} \right\} dt \\
\qquad\qquad - Z_t^{(n)} \, dW_t - Z_t^{0,(n)} \, dW_t^0 \\
\widehat{X}_0^{(n)} = 0, \ \ Y_T^{(n)} = -\gamma \mathbb{E}[\widehat{X}_T^{(n-1)} | \mathcal{F}_T^0].
\end{cases}
\tag{4.6}
$$

In the FBSDEs (4.5) and (4.6), we use the convention that the sum vanishes whenever the lower bound of the index is larger than the upper bound of the index.

The next theorem is our main result in this section, which establishes the expansion result of the FBSDE, and thus the expansion result of the value function and the optimal investment.

Theorem 4.2 *Let Assumption 1.1 hold. We have for any $n \geq 1$*

$$
\widehat{X}^\theta = X^{\bar{o}} + \sum_{i=1}^n \theta^i \widehat{X}^{(i)} + o(\theta^n), \qquad Y^\theta = Y^{\bar{o}} + \sum_{i=1}^n \theta^i Y^{(i)} + o(\theta^n),
$$

$$
Z^\theta = Z^{\bar{o}} + \sum_{i=1}^n \theta^i Z^{(i)} + o(\theta^n), \qquad Z^{0,\theta} = Z^{0,\bar{o}} + \sum_{i=1}^n \theta^i Z^{0,(i)} + o(\theta^n),
$$

$$(4.7)$$

where $(\widehat{X}^{(n)}, Y^{(n)}, Z^{(n)}, Z^{0,(n)})$ is the unique solution to (4.6). In particular,

$$
\begin{cases}
\log \dfrac{V^\theta}{V^{\bar{o}}} = \displaystyle\sum_{i=1}^n \theta^i \left(\gamma \widehat{X}^{(i)} + Y^{(i)} \right) + o(\theta^n), \\[3mm]
\pi^\theta - \pi^{\bar{o}} = \dfrac{\sigma \sum_{i=1}^n \theta^i Z^{(i)} + \sigma^0 \sum_{i=1}^n \theta^i Z^{0,(i)}}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} + o(\theta^n).
\end{cases}
$$

Proof The proof is done by induction. Lemma 4.1 implies that

$$
\begin{cases}
\widehat{X}^\theta = X^{\bar{o}} + \theta \widehat{X}^{(1)} + o(\theta), \quad Y^\theta = Y^{\bar{o}} + \theta Y^{(1)} + o(\theta), \\[2mm]
Z^\theta = Z^{\bar{o}} + \theta Z^{(1)} + o(\theta), \quad Z^{0,\theta} = Z^{0,\bar{o}} + \theta Z^{0,(1)} + o(\theta),
\end{cases}
$$

which verifies (4.7) for $n = 1$. Furthermore, by definition, it holds that $\varphi^{\theta,(1)} = \frac{\varphi^\theta - \varphi^{\bar{o}}}{\theta}$, $\varphi = \widehat{X}, Y, Z, Z^0$.

Now assume that the result holds for $n \geq 2$, i.e., there exist unique tuples $(\widehat{X}^{\theta,(n)}, Y^{\theta,(n)}, Z^{\theta,(n)}, Z^{0,\theta,(n)})$ and $(\widehat{X}^{(n)}, Y^{(n)}, Z^{(n)}, Z^{0,(n)})$ in $\left(\bigcap_{p>1} S^p \right) \times \left(\bigcap_{p>1} S^p \right) \times \left(\bigcap_{p>1} M^p \right) \times \left(\bigcap_{p>1} M^p \right)$ satisfying (4.5) and (4.6), respectively, and for each $p > 1$

$$
\lim_{\theta \to 0} \left(\| \widehat{X}^{\theta,(n)} - \widehat{X}^{(n)} \|_{S^p} + \| Y^{\theta,(n)} - Y^{(n)} \|_{S^p} \right.
$$
$$
\left. + \| Z^{\theta,(n)} - Z^{(n)} \|_{M^p} + \| Z^{0,\theta,(n)} - Z^{0,(n)} \|_{M^p} \right) = 0,
$$

$$(4.8)$$

and

$$
\varphi^{\theta,(n)} = \frac{\varphi^\theta - \varphi^{\bar{o}} - \sum_{i=1}^{n-1} \theta^i \varphi^{(i)}}{\theta^n}, \qquad \varphi = \widehat{X}, Y, Z, Z^0. \qquad (4.9)
$$

It remains to be shown that the above results also hold for $n + 1$. Define

$$
\varphi^{\theta,(n+1)} = \frac{\varphi^{\theta,(n)} - \varphi^{(n)}}{\theta}, \quad \varphi = \widehat{X}, Y, Z, Z^0,
$$

which implies by (4.9)

$$\varphi^{\theta,(n+1)} = \frac{\varphi^\theta - \varphi^{\bar o} - \sum_{i=1}^n \theta^i \varphi^{(i)}}{\theta^{n+1}}, \qquad \varphi = \widehat{X}, Y, Z, Z^0.$$

It can also be verified directly that $(\widehat{X}^{\theta,(n+1)}, Y^{\theta,(n+1)}, Z^{\theta,(n+1)}, Z^{0,\theta,(n+1)})$ satisfies (4.5) with n replaced by $n+1$. By the argument in the proof of Lemma 4.1(1), (4.6) is wellposed with n replaced by $n+1$. Denote the unique solution to (4.6) by $(\widehat{X}^{(n+1)}, Y^{(n+1)}, Z^{(n+1)}, Z^{0,(n+1)})$. By (4.8) and the same argument in the proof of Lemma 4.1(2), we have for each $p > 1$

$$\lim_{\theta \to 0} \Big(\|\widehat{X}^{\theta,(n+1)} - \widehat{X}^{(n+1)}\|_{S^p} + \|Y^{\theta,(n+1)} - Y^{(n+1)}\|_{S^p}$$
$$+ \|Z^{\theta,(n+1)} - Z^{(n+1)}\|_{M^p} + \|Z^{0,\theta,(n+1)} - Z^{0,(n+1)}\|_{M^p} \Big) = 0.$$

Thus, there exists Δ_θ with $\lim_{\theta \to 0} \Delta_\theta = 0$ such that

$$\varphi^{\theta,(n+1)} = \varphi^{(n+1)} + \Delta_\theta, \qquad \varphi = \widehat{X}, Y, Z, Z^0,$$

which implies that by the definition of $\varphi^{\theta,(n+1)}$

$$\varphi^{\theta,(n)} = \varphi^{(n)} + \theta \varphi^{(n+1)} + \theta \Delta_\theta.$$

By the induction assumption (4.9), it holds that

$$\varphi^\theta = \varphi^{\bar o} + \sum_{i=1}^{n-1} \theta^i \varphi^{(i)} + \theta^n \varphi^{(n)} + \theta^{n+1} \varphi^{(n+1)} + \theta^{n+1} \Delta_\theta,$$

which implies (4.7) with $n+1$.   □

5 Comments on N-Player Games

In this section, we comment on the N-player game (1.2)-(1.3) introduced in the Introduction. By the same argument as in Proposition 2.2, the NE of the N-player game is equivalent to a multidimensional FBSDE, whose wellposedness can be obtained by the same argument in Section 3, and poses no essential difference other than notational complexity. Therefore, we will omit the detailed proof for the solvability of the FBSDE with general market parameters, but instead, we discuss the case in which all market parameters are deterministic functions and connect our result with the N-player games studied in [8, 9, 22].

By the same argument as in Proposition 2.2, the NE $(\pi^{1,*}, \ldots, \pi^{N,*})$ of the N-player game with power utility functions (1.2)-(1.3) is equivalent to the

following multidimensional FBSDE

$$
\begin{cases}
d\widehat{X}_t^i = \dfrac{h_t^i + \sigma_t^i Z_t^{ii} + \sigma_t^{i0} Z_t^{i0}}{(1-\gamma^i)((\sigma_t^i)^2 + (\sigma_t^{i0})^2)} \\
\qquad \times \left\{ \left(h_t^i - \dfrac{h_t^i + \sigma_t^i Z_t^{ii} + \sigma_t^{i0} Z_t^{i0}}{2(1-\gamma^i)} \right) dt + \sigma_t^i \, dW_t^i + \sigma_t^{i0} \, dW_t^0 \right\}, \\
-dY_t^i = \left\{ \dfrac{(Z_t^{ii})^2 + (Z_t^{i0})^2}{2} + \dfrac{\sum_{j\neq i}(Z_t^{ij})^2}{2} + \dfrac{\gamma^i \left(h_t^i + \sigma_t^i Z_t^{ii} + \sigma_t^{i0} Z_t^{i0} \right)^2}{2(1-\gamma^i)((\sigma_t^i)^2 + (\sigma_t^{i0})^2)} \right\} dt \\
\qquad - Z_t^{ii} \, dW_t^i - Z_t^{i0} \, dW_t^0 - \sum_{j\neq i} Z_t^{ij} \, dW_t^j, \\
\widehat{X}_0^i = \log x^i, \quad Y_T^i = -\dfrac{\theta^i \gamma^i}{N-1} \sum_{j\neq i} \widehat{X}_T^j,
\end{cases}
\tag{5.1}
$$

with $\widehat{X}^i = \log X^i$ denoting the logarithm of the wealth process, and

$$
\pi^{i,*} = \frac{h^i + \sigma^i Z^{ii} + \sigma^{i0} Z^{i0}}{(1-\gamma^i)((\sigma^i)^2 + (\sigma^{i0})^2)}.
\tag{5.2}
$$

The proof of the wellposedness for (5.1) is the same as that for (2.2); we compare (5.1) with the system when $\theta^i = 0$ and transform the resulting FBSDE to a BSDE by the approach in Proposition 3.3, followed by an *a priori* estimate and fixed point argument as in Lemma 3.5 and Theorem 3.6. Instead of the detailed proof, in the next theorem, we obtain the explicit expression of the NE when market parameters are deterministic.

Theorem 5.1 *When all of the coefficients h^i, σ^i and σ^{i0}, $i = 1, \ldots, N$, are deterministic functions of time, the unique NE is $(\pi^{1,*}, \ldots, \pi^{N,*})$, where the unique bounded optimal strategy for player i is*

$$
\begin{aligned}
\pi_t^{i,*} = {}& \frac{h_t^i}{(1-\gamma^i)(\sigma_t^i)^2 + \left(1 - \gamma^i - \frac{\theta^i \gamma^i}{N-1}\right)(\sigma_t^{i0})^2} \\
& - \frac{\theta^i \gamma^i \sigma_t^{i0}}{(1-\gamma^i)(\sigma_t^i)^2 + \left(1 - \gamma^i - \frac{\theta^i \gamma^i}{N-1}\right)(\sigma_t^{i0})^2} \frac{\varphi_t^{(1),N}}{1 + \varphi_t^{(2),N}}, \\
& t \in [0,T], \quad i = 1, \ldots, N,
\end{aligned}
\tag{5.3}
$$

where

$$
\varphi^{(1),N} = \frac{1}{N-1} \sum_{j=1}^{N} \frac{h^j \sigma^{j0}}{(1-\gamma^j)(\sigma^j)^2 + \left(1 - \gamma^j - \frac{\theta^j \gamma^j}{N-1}\right)(\sigma^{j0})^2}
$$

and

$$
\varphi^{(2),N} = \frac{1}{N-1} \sum_{j=1}^{N} \frac{\theta^j \gamma^j (\sigma^{j0})^2}{(1-\gamma^j)(\sigma^j)^2 + \left(1 - \gamma^j - \frac{\theta^j \gamma^j}{N-1}\right)(\sigma^{j0})^2}.
$$

In addition, if $\{(x^i, \theta^i, \gamma^i, h^i, \sigma^i, \sigma^{i0})\}_{i=1}^N$ are realizations from independent copies of $(x, \theta, \gamma, h, \sigma, \sigma^0)$ in Theorem 3.13, then (5.3) converges to (3.29).

Proof One can show that (5.1) admits at most one solution in $(S^2 \times S^2 \times L^\infty \times L^\infty \times L^\infty)^N$. The unique bounded optimal strategy is constructed as follows. Let $\overline{\widehat{X}}^{-i} = \frac{1}{N-1} \sum_{j \neq i} \widehat{X}^j$ and $\overline{\widehat{x}}^{-i} = \frac{1}{N-1} \sum_{j \neq i} \log x^j$. By taking the average of the forward dynamics of (5.1), we get $\overline{\widehat{X}}^{-i}$, and by taking $\overline{\widehat{X}}^{-i}$ into the backward dynamics of (5.1), it holds that

$$
\begin{aligned}
Y_t^i \\
= &-\frac{\theta^i \gamma^i}{N-1} \sum_{j \neq i} \int_0^T \frac{h_s^j + \sigma_s^j Z_s^{jj} + \sigma_s^{j0} Z_s^{j0}}{(1-\gamma^j)((\sigma_s^j)^2 + (\sigma_s^{j0})^2)} \left(h_s^j - \frac{h_s^j + \sigma_s^j Z_s^{jj} + \sigma_s^{j0} Z_s^{j0}}{2(1-\gamma^j)} \right) ds \\
&+ \int_t^T \left\{ \frac{(Z_s^{ii})^2 + (Z_s^{i0})^2}{2} + \frac{\sum_{j \neq i}(Z_s^{ij})^2}{2} + \frac{\gamma^i \left(h_s^i + \sigma_s^i Z_s^{ii} + \sigma_s^{i0} Z_s^{i0} \right)^2}{2(1-\gamma^i)((\sigma_s^i)^2 + (\sigma_s^{i0})^2)} \right\} ds \\
&- \frac{\theta^i \gamma^i}{N-1} \sum_{j \neq i} \int_0^t \frac{\sigma_s^j Z_s^{jj} + \sigma_s^{j0} Z_s^{j0} + h_s^j}{(1-\gamma^j)((\sigma_s^j)^2 + (\sigma_s^{j0})^2)} \sigma_s^j \, dW_s^j \\
&- \frac{\theta^i \gamma^i}{N-1} \sum_{j \neq i} \int_0^t \frac{\sigma_s^j Z_s^{jj} + \sigma_s^{j0} Z_s^{j0} + h_s^j}{(1-\gamma^j)((\sigma_s^j)^2 + (\sigma_s^{j0})^2)} \sigma_s^{j0} \, dW_s^0 \\
&- \int_t^T Z_s^{ii} \, dW_s^i + \int_t^T \sum_{j \neq i} \left\{ -\frac{\theta^i \gamma^i}{N-1} \frac{\sigma_s^j Z_s^{jj} + \sigma_s^{j0} Z_s^{j0} + h_s^j}{(1-\gamma^j)((\sigma_s^j)^2 + (\sigma_s^{j0})^2)} \sigma_s^j - Z_s^{ij} \right\} dW_s^j \\
&+ \int_t^T \left\{ -\frac{\theta^i \gamma^i}{N-1} \sum_{j \neq i} \frac{\sigma_s^j Z_s^{jj} + \sigma_s^{j0} Z_s^{j0} + h_s^j}{(1-\gamma^j)((\sigma_s^j)^2 + (\sigma_s^{j0})^2)} \sigma_s^{j0} - Z_s^{i0} \right\} dW_s^0 - \theta^i \gamma^i \overline{\widehat{x}}^{-i}.
\end{aligned}
$$

To construct an adapted Y^i, let

$$
\begin{cases}
0 = Z^{ii} \\
0 = -\dfrac{\theta^i \gamma^i}{N-1} \dfrac{\sigma^j Z^{jj} + \sigma^{j0} Z^{j0} + h^j}{(1-\gamma^j)((\sigma^j)^2 + (\sigma^{j0})^2)} \sigma^j - Z^{ij} \\
0 = -\dfrac{\theta^i \gamma^i}{N-1} \sum_{j \neq i} \dfrac{\sigma^j Z^{jj} + \sigma^{j0} Z^{j0} + h^j}{(1-\gamma^j)((\sigma^j)^2 + (\sigma^{j0})^2)} \sigma^{j0} - Z^{i0}.
\end{cases} \tag{5.4}
$$

The first and the third equalities in (5.4) yield that

$$
\begin{aligned}
&\left(1 - \frac{\theta^i \gamma^i}{N-1} \frac{(\sigma^{i0})^2}{(1-\gamma^i)((\sigma^i)^2 + (\sigma^{i0})^2)} \right) Z^{i0} \\
= &-\frac{\theta^i \gamma^i}{N-1} \sum_{j=1}^N \frac{(\sigma^{j0})^2 Z^{j0}}{(1-\gamma^j)((\sigma^j)^2 + (\sigma^{j0})^2)} - \frac{\theta^i \gamma^i}{N-1} \sum_{j \neq i} \frac{h^j \sigma^{j0}}{(1-\gamma^j)((\sigma^j)^2 + (\sigma^{j0})^2)},
\end{aligned}
\tag{5.5}
$$

which further implies by multiplying $\frac{\frac{(\sigma^{i0})^2}{(1-\gamma^i)((\sigma^i)^2+(\sigma^{i0})^2)}}{1-\frac{\theta^i\gamma^i}{N-1}\frac{(\sigma^{i0})^2}{(1-\gamma^i)((\sigma^i)^2+(\sigma^{i0})^2)}}$ and taking the sum from 1 to N on both sides

$$\sum_{i=1}^{N}\frac{(\sigma^{i0})^2 Z^{i0}}{(1-\gamma^i)((\sigma^i)^2+(\sigma^{i0})^2)}$$

$$=\sum_{i=1}^{N}\frac{-\frac{\theta^i\gamma^i}{N-1}\frac{(\sigma^{i0})^2}{(1-\gamma^i)((\sigma^i)^2+(\sigma^{i0})^2)}}{1-\frac{\theta^i\gamma^i}{N-1}\frac{(\sigma^{i0})^2}{(1-\gamma^i)((\sigma^i)^2+(\sigma^{i0})^2)}}\sum_{j=1}^{N}\frac{(\sigma^{j0})^2 Z^{j0}}{(1-\gamma^j)((\sigma^j)^2+(\sigma^{j0})^2)}$$

$$+\sum_{i=1}^{N}\frac{-\frac{\theta^i\gamma^i}{N-1}\frac{(\sigma^{i0})^2}{(1-\gamma^i)((\sigma^i)^2+(\sigma^{i0})^2)}}{1-\frac{\theta^i\gamma^i}{N-1}\frac{(\sigma^{i0})^2}{(1-\gamma^i)((\sigma^i)^2+(\sigma^{i0})^2)}}\sum_{j\neq i}\frac{h^j\sigma^{j0}}{(1-\gamma^j)((\sigma^j)^2+(\sigma^{j0})^2)}.$$

From the above linear equation for $\sum_{i=1}^{N}\frac{(\sigma^{i0})^2 Z^{i0}}{(1-\gamma^i)((\sigma^i)^2+(\sigma^{i0})^2)}$, we get

$$\sum_{i=1}^{N}\frac{(\sigma^{i0})^2 Z^{i0}}{(1-\gamma^i)((\sigma^i)^2+(\sigma^{i0})^2)}$$

$$=\frac{-1}{1+\sum_{i=1}^{N}\frac{\frac{\theta^i\gamma^i}{N-1}\frac{(\sigma^{i0})^2}{(1-\gamma^i)((\sigma^i)^2+(\sigma^{i0})^2)}}{1-\frac{\theta^i\gamma^i}{N-1}\frac{(\sigma^{i0})^2}{(1-\gamma^i)((\sigma^i)^2+(\sigma^{i0})^2)}}}$$

$$\times\sum_{i=1}^{N}\frac{\frac{\theta^i\gamma^i}{N-1}\frac{(\sigma^{i0})^2}{(1-\gamma^i)((\sigma^i)^2+(\sigma^{i0})^2)}}{1-\frac{\theta^i\gamma^i}{N-1}\frac{(\sigma^{i0})^2}{(1-\gamma^i)((\sigma^i)^2+(\sigma^{i0})^2)}}\sum_{j\neq i}\frac{h^j\sigma^{j0}}{(1-\gamma^j)((\sigma^j)^2+(\sigma^{j0})^2)}.$$

Taking the equality back into (5.5), we obtain

$$Z^{i0}=-\frac{\frac{\theta^i\gamma^i}{N-1}}{1-\frac{\theta^i\gamma^i(\sigma^{i0})^2}{(N-1)(1-\gamma^i)\{(\sigma^{i0})^2+(\sigma^0)^2\}}}\sum_{j\neq i}\frac{\sigma^{j0}h^j}{(1-\gamma^j)\{(\sigma^j)^2+(\sigma^{j0})^2\}}$$

$$+\frac{\frac{\theta^i\gamma^i}{N-1}}{1-\frac{\theta^i\gamma^i(\sigma^{i0})^2}{(N-1)(1-\gamma^i)\{(\sigma^i)^2+(\sigma^{i0})^2\}}}\frac{1}{1+\varphi^{(2),N}}$$

$$\times\sum_{i=1}^{N}\frac{\frac{\theta^i\gamma^i(\sigma^{i0})^2}{(N-1)(1-\gamma^i)\{(\sigma^i)^2+(\sigma^{i0})^2\}}}{1-\frac{\theta^i\gamma^i(\sigma^{i0})^2}{(N-1)(1-\gamma^i)\{(\sigma^i)^2+(\sigma^{i0})^2\}}}\sum_{j\neq i}\frac{\sigma^{j0}h^j}{(1-\gamma^j)\{(\sigma^j)^2+(\sigma^{j0})^2\}},$$

where $\varphi^{(2),N}$ is defined in the statement of the theorem. Consequently, the optimal strategy (5.3) can be obtained from (5.2). The convergence from (5.3) to (3.29) is obtained by the law of large numbers. $\qquad\square$

Remark 5.2 This remark discusses the link between our N-player game and the N-player games studied by Espinosa and Touzi in [8], where all players trade common stocks; in our model, the stock prices are driven by both idiosyncratic noise and common noise. First, when there is no trading constraint, Espinosa and Touzi obtained a unique NE by convex duality approaches for

general utility functions; refer to [8, Theorem 3.3]. Note that the analysis in [8] can cover the case of power utility functions as in our paper, although power utility functions do not satisfy the Inada conditions in [8, (2.4)]. Second, when there is a general trading constraint, Espinosa and Touzi obtained a unique NE for exponential utility functions by assuming the market parameters to be deterministic and continuous; refer to [8, Theorem 4.8]. This NE was called *deterministic Nash equilibrium,* which was also studied in [9]. When there is a linear trading constraint, [8, Theorem 5.2] obtained an NE in closed form for exponential utility functions and deterministic market parameters. Thus, our paper partially recovers [8, Theorem 5.2] given Remark 2.4. Third, [8, Example 5.12–5.16] obtained a closed form NE under various trading constraints by assuming the price processes of risky assets to be independent. The N-player game with independence assumption is similar to our game when the individual volatility $\sigma^i = 0$ for all i; however, we have no trading constraint. Fourth, in our paper, both idiosyncratic noise and common noise are one-dimensional, so we cannot study *correlated investments* as in [8, Example 5.17], although generalization to multi-dimension can be expected. Finally, [8] also investigated convergence from N-player games to MFGs under deterministic coefficients; refer to [8, Proposition 4.12, Proposition 5.7, Example 5.8 and Example 5.9].

Remark 5.3 (Comments on closed-loop NE) The closed-loop NE may not exist. However, once it exists, the closed-loop NE must be identical to the open-loop NE. Indeed, If we assume all competing players except player i use a closed-loop strategy $\pi^j(t, \theta, \gamma, \boldsymbol{X})$ with $\boldsymbol{X} = (X^1, \ldots, X^N)$, then the same argument leading to (5.1) implies that $\pi^{i,*}$ has the same expression as (5.2). If the market parameters $(h^i, \sigma^i, \sigma^{i0})$ are progressively measurable with respect to the filtration generated by the Browian motions, $\{Z^{ij}\}_{j=0,1,\ldots,N}$ are not necessarily deterministic functions of $\boldsymbol{X}$ since the nonsingularity of the term $(\pi^{i,*})^2((\sigma^i)^2 + (\sigma^{i0})^2)$ cannot be guaranteed. Consequently, the closed-loop equilibrium may not exist. Furthermore, under the assumptions in Theorem 5.1, the open-loop NE (5.3) is also a closed-loop one.

    This comment also applies to the MFG (1.4).

As a corollary of Theorem 5.1, we recover the N-player games in [22] and conclude that the constant equilibrium is unique in L^∞.

Corollary 5.4 (Lacker & Zariphopoulou's N-player games revisited) *Assume that all of the coefficients h^i, σ^i and σ^{i0} are constants, then the NE obtained in (5.3) is unique in L^∞. Furthermore, (5.3) is consistent with [22, Theorem 14] by taking [22, Remark 16] into account.*

6 Conclusion

In this paper mean field portfolio games with random market parameters are studied. We establish a one-to-one correspondence between the NE of the portfolio game and the solution to some mean field FBSDE. The unique NE is

obtained by solving the FBSDE under a weak interaction assumption. When the market parameters do not depend on the Brownian paths, we get the NE in closed form. Our result partially generalizes the results in [8] and completely generalizes the results in [22]. Moreover, motivated by the weak interaction assumption, we establish an asymptotic expansion result in powers of the competition parameter θ. In particular, we expand the log-value function and the optimal investment in powers of θ into any order. This result allows us to obtain the value function and the optimal investment based only on the benchmark model without competition. Although our paper focuses on the case of power utility functions, our analysis can also be extended to cases of exponential and log utility functions.

A θ-Dependent Terms in the BSDE (3.13)

In this section, we summarize the cumbersome θ-dependent terms in the BSDE (3.13). To facilitate the presentation, we introduce the following notation

$$f^\sigma = \frac{\sigma h}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)}, \quad f^{\sigma^0} = \frac{\sigma^0 h}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)}, \quad f^h = \frac{h^2}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)},$$

$$\psi = \frac{\sigma\sigma^0}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)}, \quad \psi^\sigma = \frac{\sigma^2}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)}, \quad \psi^{\sigma^0} = \frac{(\sigma^0)^2}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)},$$

$$\phi^{(1)} = \frac{h\sigma^0}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} + \frac{\sigma\sigma^0 Z^{\bar\sigma}}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} + \frac{(\sigma^0)^2 Z^{0,\bar\sigma}}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)},$$

$$\phi^{(2)} = \frac{h\sigma}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} + \frac{\sigma^2 Z^{\bar\sigma}}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)} + \frac{\sigma\sigma^0 Z^{0,\bar\sigma}}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)},$$

$$\phi^{(3)} = Z^{0,\bar\sigma} + \frac{\gamma\sigma^0(h + \sigma Z^{\bar\sigma} + \sigma^0 Z^{0,\bar\sigma})}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)}, \quad \phi^{(4)} = Z^{\bar\sigma} + \frac{\gamma\sigma(h + \sigma Z^{\bar\sigma} + \sigma^0 Z^{0,\bar\sigma})}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)},$$

$$\phi^\sigma = 1 + \frac{\gamma\sigma^2}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)}, \quad \phi^{\sigma^0} = 1 + \frac{\gamma(\sigma^0)^2}{(1-\gamma)(\sigma^2 + (\sigma^0)^2)},$$

$$g. = -\mathbb{E}\left[\left. \frac{\theta\gamma(\sigma_\cdot^0)^2}{(1-\gamma)(\sigma_\cdot^2 + (\sigma_\cdot^0)^2)} \right| \mathcal{F}_\cdot^0 \right].$$

The terms that are dependent on θ follow

$$\mathcal{J}_2(t; \widetilde{Z}, \widetilde{Z}^0, \theta) = \mathcal{I}_1(t; \widetilde{Z}) + \mathcal{I}_2(t; \widetilde{Z}^0) + \mathcal{I}_3(t; \widetilde{Z}, \widetilde{Z}^0) + \mathcal{I}_4(t),$$

where the terms involving $\widetilde{Z}$ are given by

$$
\begin{aligned}
&\mathcal{I}_1(t;\widetilde{Z})\\
&= \left\{ \frac{\phi_t^{\sigma^0}\theta^2\gamma^2}{2(1-g_t)^2} + \theta\gamma\mathbb{E}\left[\left.\frac{\theta^2\gamma^2\psi_t^{\sigma^0}}{2(1-\gamma)(1-g_t)^2}\right|\mathcal{F}_t^0\right]\right\}\\
&\quad \times \left(\mathbb{E}[\psi_t\widetilde{Z}_t|\mathcal{F}_t^0]\right)^2 + \theta\gamma\mathbb{E}\left[\left.\frac{\psi_t^\sigma}{2(1-\gamma)}(\widetilde{Z}_t)^2\right|\mathcal{F}^0\right]\\
&\quad - \frac{\theta\gamma^2\psi_t}{1-g_t}\widetilde{Z}_t\mathbb{E}[\psi_t\widetilde{Z}_t|\mathcal{F}_t^0] - \theta\gamma\mathbb{E}\left[\left.\frac{\theta\gamma\psi_t}{(1-\gamma)(1-g_t)}\widetilde{Z}_t\right|\mathcal{F}_t^0\right]\mathbb{E}[\psi_t\widetilde{Z}_t|\mathcal{F}_t^0]\\
&\quad + \left\{\frac{\theta\gamma}{1-g_t}\mathbb{E}[\theta\gamma f_t^{\sigma^0}|\mathcal{F}_t^0] - \frac{\theta\gamma}{1-g_t}\phi_t^{(3)} + \frac{\theta^2\gamma^2}{(1-g_t)^2}\phi_t^{\sigma^0}\mathbb{E}[\phi_t^{(1)}|\mathcal{F}_t^0]\right\}\mathbb{E}[\psi_t\widetilde{Z}_t|\mathcal{F}_t^0]\\
&\quad + \theta\gamma\left\{-\mathbb{E}\left[\left.\frac{\theta\gamma\phi_t^{(1)}}{(1-\gamma)(1-g_t)}\right|\mathcal{F}_t^0\right] + \mathbb{E}\left[\left.\frac{\theta^2\gamma^2\psi_t^{\sigma^0}}{(1-\gamma)(1-g_t)^2}\right|\mathcal{F}_t^0\right]\mathbb{E}[\phi_t^{(1)}|\mathcal{F}_t^0]\right\}\mathbb{E}[\psi_t\widetilde{Z}_t|\mathcal{F}_t^0]\\
&\quad - \theta\gamma\mathbb{E}[f_t^\sigma\widetilde{Z}_t|\mathcal{F}_t^0] + \theta\gamma\mathbb{E}\left[\left.\frac{\phi_t^{(2)}}{1-\gamma}\widetilde{Z}_t\right|\mathcal{F}_t^0\right]\\
&\quad - \theta\gamma\mathbb{E}[\phi_t^{(1)}|\mathcal{F}_t^0]\mathbb{E}\left[\left.\frac{\theta\gamma\psi_t}{(1-\gamma)(1-g_t)}\widetilde{Z}_t\right|\mathcal{F}_t^0\right] + \left\{\phi_t^{(4)} - \frac{\theta\gamma^2}{1-g_t}\psi_t\mathbb{E}[\phi_t^{(1)}|\mathcal{F}_t^0]\right\}\widetilde{Z}_t,
\end{aligned}
$$

the terms involving $\widetilde{Z}^0$ are given by

$$
\begin{aligned}
&\mathcal{I}_2(t;\widetilde{Z}^0)\\
&= \left\{\frac{\phi_t^{\sigma^0}\theta^2\gamma^2}{2(1-g_t)^2} + \theta\gamma\mathbb{E}\left[\left.\frac{\theta^2\gamma^2\psi_t^{\sigma^0}}{2(1-\gamma)(1-g_t)^2}\right|\mathcal{F}_t^0\right]\right\}\left(\mathbb{E}[\psi_t^{\sigma^0}\widetilde{Z}_t^0|\mathcal{F}_t^0]\right)^2 - \frac{\theta\gamma\phi_t^{\sigma^0}}{1-g_t}\widetilde{Z}_t^0\mathbb{E}[\psi_t^{\sigma^0}\widetilde{Z}_t^0|\mathcal{F}_t^0]\\
&\quad + \theta\gamma\mathbb{E}\left[\left.\frac{\psi_t^{\sigma^0}}{2(1-\gamma)}(\widetilde{Z}_t^0)^2\right|\mathcal{F}^0\right] - \theta\gamma\mathbb{E}\left[\left.\frac{\theta\gamma\psi_t^{\sigma^0}}{(1-\gamma)(1-g_t)}\widetilde{Z}_t^0\right|\mathcal{F}_t^0\right]\mathbb{E}[\psi_t^{\sigma^0}\widetilde{Z}_t^0|\mathcal{F}_t^0]\\
&\quad + \frac{\theta\gamma}{1-g_t}\left\{\mathbb{E}[\theta\gamma f_t^{\sigma^0}|\mathcal{F}_t^0] - \phi_t^{(3)} + \frac{\theta\gamma}{1-g_t}\phi_t^{\sigma^0}\mathbb{E}[\phi_t^{(1)}|\mathcal{F}_t^0]\right\}\mathbb{E}[\psi_t^{\sigma^0}\widetilde{Z}_t^0|\mathcal{F}_t^0]\\
&\quad + \theta\gamma\left\{-\mathbb{E}\left[\left.\frac{\theta\gamma\phi_t^{(1)}}{(1-\gamma)(1-g_t)}\right|\mathcal{F}_t^0\right] + \mathbb{E}\left[\left.\frac{\theta^2\gamma^2\psi_t^{\sigma^0}}{(1-\gamma)(1-g_t)^2}\right|\mathcal{F}^0\right]\mathbb{E}[\phi_t^{(1)}|\mathcal{F}_t^0]\right\}\mathbb{E}[\psi_t^{\sigma^0}\widetilde{Z}_t^0|\mathcal{F}_t^0]\\
&\quad + \theta\gamma\mathbb{E}\left[\left.\frac{\phi_t^{(1)}}{1-\gamma}\widetilde{Z}_t^0\right|\mathcal{F}_t^0\right] - \theta\gamma\mathbb{E}[\phi_t^{(1)}|\mathcal{F}_t^0]\mathbb{E}\left[\left.\frac{\theta\gamma\psi_t^{\sigma^0}}{(1-\gamma)(1-g_t)}\widetilde{Z}_t^0\right|\mathcal{F}_t^0\right] - \theta\gamma\mathbb{E}[f_t^{\sigma^0}\widetilde{Z}_t^0|\mathcal{F}_t^0]\\
&\quad + \left\{\phi_t^{(3)} - \frac{\theta\gamma}{1-g_t}\phi_t^{\sigma^0}\mathbb{E}[\phi_t^{(1)}|\mathcal{F}_t^0]\right\}\widetilde{Z}_t^0,
\end{aligned}
$$

the crossing terms involving both $\widetilde{Z}$ and $\widetilde{Z}^0$ are given by

$$
\begin{aligned}
&\mathcal{I}_3(t; \widetilde{Z}, \widetilde{Z}_t^0) \\
&= -\frac{\theta\gamma^2\psi}{1-g_t}\widetilde{Z}_t\mathbb{E}[\psi_t^{\sigma^0}\widetilde{Z}_t^0|\mathcal{F}_t^0] - \frac{\theta\gamma\phi_t^{\sigma^0}}{1-g_t}\mathbb{E}[\psi_t\widetilde{Z}_t|\mathcal{F}_t^0]\widetilde{Z}_t^0 \\
&\quad + \frac{\theta^2\gamma^2\phi_t^{\sigma^0}}{(1-g_t)^2}\mathbb{E}[\psi_t\widetilde{Z}_t|\mathcal{F}_t^0]\mathbb{E}[\psi_t^{\sigma^0}\widetilde{Z}_t^0|\mathcal{F}_t^0] \\
&\quad - \theta\gamma\mathbb{E}\left[\frac{\theta\gamma\psi_t^{\sigma^0}}{(1-\gamma)(1-g_t)}\widetilde{Z}_t^0\bigg|\mathcal{F}_t^0\right]\mathbb{E}[\psi_t\widetilde{Z}_t|\mathcal{F}_t^0] \\
&\quad + \theta\gamma\mathbb{E}\left[\frac{\theta^2\gamma^2\psi_t^{\sigma^0}}{(1-\gamma)(1-g_t)^2}\bigg|\mathcal{F}_t^0\right]\mathbb{E}[\psi_t\widetilde{Z}_t|\mathcal{F}_t^0]\mathbb{E}[\psi_t^{\sigma^0}\widetilde{Z}_t^0|\mathcal{F}_t^0] \\
&\quad + \theta\gamma\mathbb{E}\left[\frac{\psi_t}{1-\gamma}\widetilde{Z}_t\widetilde{Z}_t^0\bigg|\mathcal{F}_t^0\right] - \theta\gamma\mathbb{E}\left[\frac{\theta\gamma\psi_t}{(1-\gamma)(1-g_t)}\widetilde{Z}_t\bigg|\mathcal{F}_t^0\right]\mathbb{E}\left[\psi_t^{\sigma^0}\widetilde{Z}_t^0|\mathcal{F}_t^0\right],
\end{aligned}
$$

and the remaining terms are given by

$$
\begin{aligned}
&\mathcal{I}_4(t) \\
&= \frac{(\theta\gamma)^2\phi_t^{\sigma^0}}{2(1-g_t)^2}(\mathbb{E}[\phi_t^{(1)}|\mathcal{F}_t^0])^2 - \frac{\theta\gamma}{1-g_t}\left(-\mathbb{E}[\theta\gamma f_t^{\sigma^0}|\mathcal{F}_t^0] + \phi_t^{(3)}\right)\mathbb{E}[\phi_t^{(1)}|\mathcal{F}_t^0] \\
&\quad + \theta\gamma\mathbb{E}\left[\frac{(\sigma_t^2 + (\sigma_t^0)^2)|\phi_t^{(1)}|^2}{2}\bigg|\mathcal{F}_t^0\right] - \theta\gamma\mathbb{E}[\phi_t^{(1)}|\mathcal{F}_t^0]\mathbb{E}\left[\frac{\theta\gamma\phi_t^{(1)}}{(1-\gamma)(1-g_t)}\bigg|\mathcal{F}_t^0\right] \\
&\quad + \frac{\theta^3\gamma^3}{(1-g_t)^2}\mathbb{E}\left[\frac{\psi_t^{\sigma^0}}{2(1-\gamma)}\bigg|\mathcal{F}_t^0\right]\left(\mathbb{E}[\phi_t^{(1)}|\mathcal{F}_t^0]\right)^2 \\
&\quad - \theta\gamma\mathbb{E}[f_t^h + f_t^\sigma Z_t^{\bar{o}} + f_t^{\sigma^0}Z_t^{0,\bar{o}}|\mathcal{F}_t^0].
\end{aligned}
$$

B Reverse Hölder's Inequality

We summarize the reverse Hölder's inequality for a general stochastic process and for the stochastic exponential of a stochastic process in the BMO space ([20, Theorem 3.1]), which is used in the main text.

For some $p > 1$, we say that a stochastic process D satisfies R_p if there exists a constant C, such that for any $[0, T]$-valued stopping time τ, it holds that

$$
\mathbb{E}\left[\left|\frac{D_T}{D_\tau}\right|^p\bigg|\mathcal{F}_\tau\right] \leq C.
$$

Let $\Theta \in H_{BMO}^2$ and B be a Brownian motion. Define

$$
\mathcal{E}_t(\Theta) = \mathcal{E}\left(\int_0^t \Theta_s\, dB_s\right).
$$

Let Φ be a function defined on $(1,\infty)$:

$$\Phi(x) = \left\{1 + \frac{1}{x^2}\log\frac{2x-1}{2(x-1)}\right\}^{\frac{1}{2}} - 1,$$

which is a continuous decreasing function satisfying $\lim_{x\searrow 1}\Phi(x) = \infty$ and $\lim_{x\nearrow\infty}\Phi(x) = 0$. Let p_Θ be the unique constant, such that $\Phi(p_\Theta) = \|\Theta\|_{BMO}$. Then we have the following reverse Hölder's inequality.

Lemma B.1 *For any $1 < p < p_\Theta$ and any stopping time τ, it holds that $\mathcal{E}(\Theta)$ satisfies R_p. In particular,*

$$\mathbb{E}\left[\left.\left|\frac{\mathcal{E}_T(\Theta)}{\mathcal{E}_\tau(\Theta)}\right|^p\right|\mathcal{F}_\tau\right] \leq K(p, \|\Theta\|_{BMO}),$$

where

$$K(p, N) = \frac{2}{1 - \frac{2(p-1)}{2p-1}e^{p^2(N^2+2N)}}.$$

References

1. P. Briand and F. Confortola, *BSDEs with stochastic lipschitz conditon and quadratic PDEs in Hilbert spaces*, Stochastic Processes and their Applications, 118 (2008), pp. 818–838.
2. P. Briand and Y. Hu, *Quadratic BSDEs with convex generators and unbounded terminal conditions*, Probability Theory and Related Fields, 141 (2008), pp. 543–567.
3. R. Carmona and F. Delarue, *Probabilistic analysis of mean-field games*, SIAM Journal on Control and Optimization, 51 (2013), pp. 2705–2734.
4. P. Chan and R. Sircar, *Bertrand and cournot mean field games*, Applied Mathematics & Optimization, 71 (2015), pp. 533–569.
5. ———, *Fracking, renewables, and mean field games*, SIAM Review, 59 (2017), pp. 588–615.
6. G. dos Reis and V. Platonov, *Forward utilities and mean-field games under relative performance concerns*, in From Particle Systems to Partial Differential Equations, Springer, 2019, pp. 227–251.
7. ———, *Forward utility and market adjustments in relative investment-consumption games of many players*, SIAM Journal on Financial Mathematics, 13 (2022), pp. 844–876.
8. G. Espinosa and N. Touzi, *Optimal investment under relative performance concerns*, Mathematical Finance, 25 (2015), pp. 221–257.
9. C. Frei and G. dos Reis, *A financial market with interacting investors: does an equilibrium exist?*, Mathematics and Financial Economics, 4 (2011), pp. 161–182.
10. G. Fu, P. Graewe, U. Horst, and A. Popier, *A mean field game of optimal portfolio liquidation*, Mathematics of Operations Research, 46 (2021), pp. 1250–1281.
11. G. Fu and U. Horst, *Mean field leader follower games with terminal state constraint*, SIAM Journal on Control and Optimization, 58 (2020), pp. 2078–2113.
12. G. Fu, X. Su, and C. Zhou, *Mean field exponential utility games: a probabilistic approach*, arXiv:2006.07684, (2020).
13. M. Herdegen, J. Muhle-Karbe, and D. Possamaï, *Equilibrium asset pricing with transaction costs*, in Finance and Stochastics, 25 (2021), pp. 231–275.

14. H. Hibon, Y. Hu, and S. Tang, *Mean-field type quadratic BSDEs*, arXiv:1708.08784, (2017).
15. U. Horst, *Stationary equilibria in discounted stochastic games with weakly interacting players*, Games and Economic Behavior, 52 (2005), pp. 83–108.
16. U. Horst, X. Xia, and C. Zhou, *Portfolio liquidation under factor uncertainty*, The Annals of Applied Probability, 32 (2022), pp. 80–123.
17. R. Hu and T. Zariphopoulou, *N-player and mean-field games in Itô-diffusion markets with competitive or homophilous interaction*, in Stochastic Analysis, Filtering, and Stochastic Optimization, Springer, 2022, pp. 209–237.
18. Y. Hu, P. Imkeller, and M. Müller, *Utility maximization in incomplete markets*, The Annals of Applied Probability, 15 (2005), pp. 1691–1712.
19. M. Huang, R. Malhamé, and P. Caines, *Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle*, Communications in Information and Sytems, 6 (2006), pp. 221–252.
20. N. Kazamaki, *Continuous Exponential Martingales and BMO*, Springer, 2006.
21. D. Lacker and A. Soret, *Many-player games of optimal consumption and investment under relative performance criteria*, Mathematics and Financial Economics, 14 (2020), pp. 263–281.
22. D. Lacker and T. Zariphopoulou, *Mean field and n-agent games for optimal investment under relative performance criteria*, Mathematical Finance, 29 (2019), pp. 1003–1038.
23. J.-M. Lasry and P.-L. Lions, *Mean field games*, Japanese Journal of Mathematics, 2 (2007), pp. 229–260.
24. A. Matoussi, D. Possamaï, and C. Zhou, *Robust utility maximization in non-dominated models with 2BSDEs*, Mathematical Finance, 25 (2015), pp. 258–287.
25. R. Rouge and N. El Karoui, *Pricing via utility maximization and entropy*, Mathematical Finance, 10 (2000), pp. 259–276.
26. R. Tevzadze, *Solvability of backward stochastic differential equations with quadratic growth*, Stochastic processes and their Applications, 118 (2008), pp. 503–515.