https://doi.org/10.1016/j.sysconle.2023.105545

Mean-Variance Portfolio Selection under No-Shorting Rules: A BSDE Approach

Liangquan Zhang[1]*, Xun Li[2]†

1. School of Mathematics
Renmin University of China, Beijing 100872, China
2. Department of Applied Mathematics
The Hong Kong Polytechnic University, Hong Kong, China

January 27, 2023

Abstract

This paper revisits the mean-variance portfolio selection problem in continuous-time within the framework of short-selling of stocks is prohibited via backward stochastic differential equation approach. To *relax* the strong condition in Li et al. [28], the above issue is formulated as a stochastic recursive optimal linear-quadratic control problem. Due to no-shorting rules (namely, the portfolio taking non-negative values), the well-known "completion of squares" no longer applies directly. To overcome this difficulty, we study the corresponding Hamilton-Jacobi-Bellman (HJB, for short) equation inherently and derive the two groups of Riccati equations. On one hand, the value function constructed via Riccati equations is shown to be a viscosity solution of the HJB equation mentioned before; On the other hand, by means of these Riccati equations and backward semigroup, we are able to get explicitly the efficient frontier and efficient investment strategies for the recursive utility mean-variance portfolio optimization problem.

Keywords- continuous-time, mean-variance portfolio selection, short-selling prohibition, efficient frontier, stochastic LQ control, HJB equation, recursive utility, viscosity solution

AMS subject classifications. 91B28, 93E20

Abbreviated Title. A BSDE to Mean-Variance under No-Shorting Constraints.

1 Introduction

Portfolio selection was traced back to Markowitz's funtamental work [30] on mean-variance efficient portfolios for a single-period investment in 1950s. He applied the quantitative and scientific approaches to risk management and analysis. If the short-selling of stocks is not allowed, efficient

*L. Zhang acknowledges the financial support partly by the National Nature Science Foundation of China (Grant No. 12171053, 11701040, 11871010 & 61871058) and the Fundamental Research Funds for the Central Universities (No. 2019XD-A11).

†X. Li acknowledges the financial support by the Hong Kong General Research Fund under grants 15216720, 15221621 and 15226922. E-mail: li.xun@polyu.edu.hk

portfolios are available by solving a quadratic programming problem. Subsequently, Merton [32] obtained an analytical solution to the single-period mean-variance problem assuming that the covariance matrix is positive definite and short-selling is allowed. While it is natural to generalize the previous work to multi-period and continuous-time portfolio selections, see, e.g., [2, 12, 16, 33, 35] for the multi-period case, and [5, 8, 10, 15, 21, 31] for the continuous-time case.

Zhou and Li [44] considered the continuous-time mean-variance problem where short-selling of stocks is allowed by incorporating the embedding technique adopted in Li and Ng [25]. The concentration of [44] is stochastic linear-quadratic (LQ, for short) optimal control, in order to solve certain finance problems including the mean-variance portfolio selection. From then on, the indefinite stochastic LQ control theory has been studied largely (cf. [3, 4, 26, 41]), and provides a powerful tool for solving some finance problems (see [23, 27, 44]).

The constrained portfolio selection issue has been widely studied (see, e.g., [7, 19, 37, 39, 40, 28]). In particular, Xu and Shreve in their two-part paper [39, 40] investigated a utility maximization problem with a no-shorting constraint using a duality analysis. In [7, 22], the duality results of [39, 40] are extended to a general class of portfolio selection problems in incomplete markets, including those with constraints. The main results in [7, 22] establish the existence of a solution to the dual problem, and show how it can be used to construct a solution of the original portfolio optimization problem. Li et al. [28] investigated the continuous-time mean-variance portfolio selection in the case where short-selling the stocks is not allowed (however, shorting the riskless asset-the bond-is still allowed). The authors employed the stochastic LQ control to study the constrained mean-variance portfolio problem. Compared with [27, 44], the distinctive feature is that shorting is prohibited. As a consequence, a major difficulty comes from the fact the control (portfolio) is constrained, while the LQ theory typically requires the control to be unconstrained (since the optimal control constructed via the Riccati equation may not satisfy the control constraint). Therefore, the elegant Riccati approach does not apply directly.

In this paper, we extend the above mean-variance portfolio optimization problem to a recursive utility portfolio optimization problem. The recursive utility here means that the utility at time t is a function of the future utility. As matter of fact, the recursive utility can be postulated to satisfy certain controlled backward stochastic differential equation (BSDE, for short). As we know, stochastic control systems driven by forward-backward stochastic differential equations (FBSDEs, for short) as a tool are broadly applied in mathematical economics and mathematical finance, which contain a forward SDEs as a special case. They are encountered in stochastic recursive utility optimization problems and principal-agent problems. Pardoux and Peng 1990 [49] proved the well-posedness for nonlinear BSDE as follows:

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))\mathrm{d}s - \int_t^T Z(s)\mathrm{d}W(s). \tag{1}$$

Duffie and Epstein [9] introduced the notion of recursive utilities in continuous-time situation, which is a type of BSDE where the generator f is independent of Z. El Karoui, Peng and Quenez (see [20]) extended the recursive utility to the case where f contains Z. The term Z can be interpreted as an ambiguity aversion term in the market (see Remark ? for the impact of Z in finance). Moreover a kind of non-linear g-expectation can be defined via a linear BSDE (see Chen and Epstein [46] for more details). In particular, the celebrated Black-Scholes formula indeed provides an effective way of representing the option price in finance, which is the solution to a kind of linear BSDE. Out of question, BSDE has been extensively studied in the areas of applied probability and optimal stochastic controls, in financial engineering (see reference therein) etc.

As claimed before, the Riccati approach cannot be applied directly. We overcome this obstacle by focusing on the Hamilton-Jacobi-Bellman (HJB, for short) equation[1]. However, the HJB equation has no classical (i.e. smooth) solutions due to the control constraints. Similar to [28], we cope with this issue by conjecturing a continuous solution to the HJB equation via *two* groups of Riccati equations, and then show that it is indeed the *viscosity solution* to the second order partial differential equation. By virtue of viscosity verification theorem presented in [53], we get the optimal strategy explicitly along with the efficient frontier. In contrast to the classical LQ issue, one of the significant features in recursive framework, embodies that the admissible control set needs to satisfy more than the square integrability due to the terminal condition satisfied by BSDE (see Example 2.1 below for details).

This paper is organized as follows. In Section 2, we recall some results on BSDE and formulate a stochastic recursive LQ control problem under control constraints. In Section 3, we study a stochastic LQ control problem, and we obtain the viscosity solution to the corresponding HJB equation along with the optimal feedback control. Section 4, we discuss a LQ recursive utility portfolio optimization problem in the financial engineering and get the derivation of the efficient investment strategies and the efficient frontier for the portfolio selection problem under a short-selling prohibition. Finally, Section 5 concludes the paper. Some issues on viscosity solution, g-expectation and technique lemmas are scheduled in the Appendix.

2 Problem Formulation and Preliminaries

2.1 Notation

We make use of the following notation:

M'	:	the transpose of any matrix or vector M;
$\|M\|$	:	$\sqrt{\sum_{i,j} m_{ij}^2}$ for any matrix or vector $M = (m_{ij})$;
$\mathbb{R}^n$	:	n dimensional real Euclidean space;
$\mathbb{R}^n_+$	:	the subset of $\mathbb{R}^n$ consisting of elements with nonnegative components.

The uncertainty is generated by a fixed filtered complete probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ on which is defined a standard $\{\mathcal{F}_t\}_{t\geq 0}$-adapted m-dimensional Brownian motion $W(t) \equiv (W^1(t), \cdots, W^m(t))'$. Given a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t | a \leq t \leq b\} (-\infty \leq a < b \leq +\infty)$, a Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|_{\mathcal{H}}$, define the Banach space

$$L^2_{\mathcal{F}}(0,T;\mathcal{H}) \triangleq \left\{ \varphi(\cdot) \middle| \begin{array}{l} \varphi(\cdot) \text{ is an } \mathcal{F}_t\text{-adapted, } \mathcal{H}\text{-valued measurable process on } [a,b] \\ \text{and } \mathbb{E}\int_a^b \|\varphi(t,\omega)\|^2_{\mathcal{H}} \mathrm{d}t < +\infty \end{array} \right\}$$

with the norm

$$\|\varphi(\cdot)\|_{\mathcal{F},2} = \left[\mathbb{E}\int_a^b \|\varphi(t,\omega)\|^2_{\mathcal{H}} \mathrm{d}t\right]^{\frac{1}{2}} < +\infty.$$

Besides, let $\mathcal{H}^p\,[0,T] = \mathcal{L}^p_{\mathcal{F}}(\Omega; C([0,T]\,;\mathbb{R}^n))$ with

$$\|\varphi(\cdot)\|_{\mathcal{H}^p} = \left[\mathbb{E}\left(\sup_{t\in[0,T]} |\varphi(t)|^p\right)\right]^{1\wedge\frac{1}{p}}, \quad \forall \varphi \in \mathcal{H}^p\,[0,T]\,.$$

[1]Recall that the Riccati equation *is* essentially the HJB equation after separating the time and spatial variables.

Let us briefly recall some well-known results on BSDE (1) by Pardox and Peng ([49], 1992). Assume that $\xi \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R})$. Under additional conditions, for instance, the generator $f : [0,T]\times\mathbb{R}\times\mathbb{R}^m \to \mathbb{R}$ satisfies Lipschitz condition in Y and Z, uniformly in t, and $f(t;0;0) \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ for $t\in[0,T]$. Then BSDE (1) admits a unique strong adapted solution $(Y(\cdot),Z(\cdot)) \in L^2_{\mathcal{F}}(0,T;\mathbb{R})\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$. Also, we need the following estimations for BSDE, whose proofs can be seen in Proposition 3.2 of Briand et al. [1].

Lemma 2.1 *Let (y^i,z^i), $i=1,2$, be the solution to the following*

$$y^i(t)=\xi^i+\int_t^T f^i(s,y^i(s),z^i(s))\mathrm{d}s-\int_t^T z^i(s)\mathrm{d}W(s), \tag{2}$$

where $\mathbb{E}\left[\left|\xi^i\right|^\beta\right]<\infty$, $f^i=f^i(s,y^i,z^i):[0,T]\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ satisfies Lipschitz in (y^i,z^i) and

$$\mathbb{E}\left[\left(\int_t^T \left|f^i(s,y^i(s),z^i(s))\right|\mathrm{d}s\right)^\beta\right]<\infty.$$

Then, for some $\beta>1$, there exists a positive constant C_β (depending on β, T and Lipschitz constant) such that

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}\left|y^1(t)-y^2(t)\right|^\beta+\left(\int_0^T\left|z^1(s)-z^2(s)\right|^2\mathrm{d}s\right)^{\frac{\beta}{2}}\right]\\ \le\ & C_\beta\mathbb{E}\left[\left|\xi^1-\xi^2\right|^\beta+\left(\int_t^T\left|f^1(s,y^1(s),z^1(s))-f^2(s,y^2(s),z^2(s))\right|\mathrm{d}s\right)^\beta\right].\end{aligned}$$

In particular, whenever putting $\xi^2=0$, $f^2=0$, one has

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|y^1(t)\right|^\beta+\left(\int_0^T\left|z^1(s)\right|^2\mathrm{d}s\right)^{\frac{\beta}{2}}\right]\le C_\beta\mathbb{E}\left[\left|\xi^1\right|^\beta+\left(\int_t^T\left|f^1(s,0,0)\right|\mathrm{d}s\right)^\beta\right].$$

Before studying the formulation of control problem, we first look at an interesting example to explain the difference with the classical forward LQ problem. Due to the cost functional defined through the solution of BSDE, which requires the terminal random variable ξ satisfies certain conation, for instance, $\mathbb{E}\left[|\xi|^p\right]<\infty$, for $p>1$.

Remark 2.1 *For $p=1$, let us recall a result obtained in Briand et al. [1] focusing on the well-posedness for BSDE with $\mathbb{E}\left[|\xi|\right]<\infty$. To this end, we introduce the following hypothesis: there exists an $\alpha\in(0,1)$ such that the map g in (2) satisfies $|f(z)|\le C(1+|z|^\alpha), \forall z\in\mathbb{R}$ and Lipschitz condition. In such a case, BSDE is well-defined for all $\xi\in L^1_{\mathcal{F}_T}$. However, it is unfortunate that $g(z)=\mu z, \mu\in\mathbb{R}$ does not satisfy this condition.*

As well-known, the process Y in (1) satisfies the "backward semigroup" property. For any given $t\le T$ and $\eta\in L^2(\Omega,\mathcal{F}_t,P;\mathbb{R})$, we consider the following BSDE which is defined on the interval $[0,t]$:

$$y(t)=\eta+\int_r^t f(s,y(s),z(s))\mathrm{d}s-\int_r^t z(s)\mathrm{d}W(s),\ r\in[0,t]\,. \tag{3}$$

Then, for every $r \leq t$ we define

$$\mathbb{G}_{r,t}[\eta] \triangleq y_r : L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}) \to L^2(\Omega, \mathcal{F}_r, P; \mathbb{R}).$$

From the uniqueness of BSDE, we know immediately that for $t_1 \leq r \leq t$

$$\mathbb{G}_{t_1,t}[\eta] = \mathbb{G}_{t_1,r}[y_r] = \mathbb{G}_{t_1,r}[\mathbb{G}_{r,t}[\eta]].$$

Example 2.1 *Consider the follow systems ($d = 1$):*

$$\begin{cases} \mathrm{d}X(s) &= u(s)\mathrm{d}W(s), \\ -\mathrm{d}Y(s) &= \Big[\langle Q(s)X(s), X(s)\rangle + \langle R(s)u(s), u(s)\rangle + \kappa Z(s)\Big]\mathrm{d}s \\ & \quad -Z(s)\mathrm{d}W(s), \\ X(0) &= x, \quad Y(T) = \langle GX(T), X(T)\rangle, \end{cases} \tag{4}$$

where $\kappa \in \mathbb{R}$. In order to get the explicit solution of $Y(\cdot)$, we introduce the following SDE:

$$\begin{cases} \mathrm{d}\Psi(s) &= \kappa\Psi(s)\mathrm{d}W(s), \\ \Psi(0) &= 1. \end{cases} \tag{5}$$

It is easy to obtain the solution to (5) is

$$\Psi(s) = \exp\left\{-\frac{1}{2}\kappa^2 s + \kappa W(s)\right\}.$$

Applying Itô's formula to $\Psi(s)Y(s)$, we get

$$Y(0) = \mathbb{E}\left[\Psi(T)\langle GX(T), X(T)\rangle + \int_0^T \Psi(s)(\langle Q(s)X(s), X(s)\rangle + \langle R(s)u(s), u(s)\rangle)\mathrm{d}s\right]. \tag{6}$$

Define

$$\begin{aligned} \tilde{G} &= \Psi(T)G, \\ \tilde{Q}(s) &= \Psi(s)Q(s), \\ \tilde{R}(s) &= \Psi(s)R(s), \quad s \in [0,T]. \end{aligned}$$

Observe that (6) is still a quadratic functional. On one hand, the coefficients $\tilde{Q}$ and $\tilde{R}$ are $\mathcal{F}_s$-adapted. Moreover, these coefficients are unbounded. Therefore, we need to restrict $u(\cdot) \in \mathcal{U}_{ad}^p[0,T]$ with $p > 2$ to guarantee the finiteness of the cost functional $Y(0)$. On the other hand, to ensure $\left\langle \tilde{G}X(T), X(T)\right\rangle \in L^p_{\mathcal{F}_T}(\Omega)$, $p > 1$, we also need $u(\cdot) \in \mathcal{U}_{ad}^p[0,T]$ with $p > 2$.

A set $\mathcal{U}_{ad}^{p,+}[0,T]$ of admissible controls is defined by

$$\mathcal{U}_{ad}^{p,+}[0,T] \triangleq \left\{u(\cdot) : u(t) \in \mathbb{R}_+^m \text{ is an } \mathcal{F}_t\text{-adapted and } \left[\mathbb{E}\left(\int_0^T |u(s)|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right]^{1\wedge\frac{1}{p}} < \infty\right\}. \tag{7}$$

Now consider the controlled linear forward-backward stochastic differential equations (FBSDEs in short)

$$\begin{cases} \mathrm{d}X^{t,x;u(\cdot)}(s) &= \big[A(s)X^{t,x;u(\cdot)}(s)+B(s)u(s)+b(s)\big]\,\mathrm{d}s+\sum_{j=1}^m D_j(s)u(s)\mathrm{d}W^j(s),\\ -\mathrm{d}Y^{t,x;u(\cdot)}(s) &= \Big(\frac{1}{2}Q(s)X^{t,x;u(\cdot)}(s)^2+q(s)X^{t,x;u(\cdot)}(s)\\ &\quad +F(s)Y^{t,x;u(\cdot)}(s)+\sum_{j=1}^m O_j(s)Z_j^{t,x;u(\cdot)}(s)+\ell(s)\Big)\mathrm{d}s\\ &\quad -\sum_{j=1}^m Z_j^{t,x;u(\cdot)}(s)\mathrm{d}W^j(s),\quad s\in[t,T],\\ X^{t,x;u(\cdot)}(t) &= x\in\mathbb{R},\quad Y^{t,x;u(\cdot)}(T)=\frac{1}{2}HX^{t,x;u(\cdot)}(T)^2+hX^{t,x;u(\cdot)}(T)+m \end{cases} \tag{8}$$

where $b(t)\in\mathbb{R}$ are scalars, $B(t)'\in\mathbb{R}^m$ and $D_j(t)'\in\mathbb{R}^m$ $(j=1,\cdots,m)$ are column vectors. In addition, we assume that the matrix $\sum_{j=1}^m D_j(t)'D_j(t)$ is non-singular.

The class of admissible controls associated with (8) is the set $\mathcal{U}_{ad}^{p,+}[0,T]$. Given $u(\cdot)\in\mathcal{U}_{ad}^{p,+}[0,T]$, the pair $(X^{t,x;u(\cdot)}(\cdot),Y^{t,x;u(\cdot)}(\cdot),Z^{t,x;u(\cdot)}(\cdot),u(\cdot))$ is referred to as an admissible quadruple if

$$(X^{t,x;u(\cdot)}(\cdot),Y^{t,x;u(\cdot)}(\cdot),Z^{t,x;u(\cdot)}(\cdot))\in L^2_{\mathcal{F}}(t,T;\mathbb{R})\times L^2_{\mathcal{F}}(t,T;\mathbb{R})\times L^2_{\mathcal{F}}(t,T;\mathbb{R}^m)$$

is a solution of FBSDEs (8) associated with $u(\cdot)\in\mathcal{U}_{ad}^{p,+}[0,T]$.

Our objective is to seek an optimal $u(\cdot)\in\mathcal{U}_{ad}^{p,+}[0,T]$ that minimizes the following functional

$$J(t,x;u(\cdot))=Y^{t,x;u(\cdot)}(s)|_{s=t}. \tag{9}$$

The value function associated with the LQ problem (8) is defined by

$$V(t,x)=\inf_{u(\cdot)\in\,\mathcal{U}_{ad}^{p,+}[0,T]}J(t,x;u(\cdot)). \tag{10}$$

Problem (LQ-FBSDEs). Seek an admissible control $u(\cdot)\in\mathcal{U}[t,T]$ such that

$$J(\bar{u}(\cdot))=\inf_{u(\cdot)\in\,\mathcal{U}_{ad}^{p,+}[0,T]}J(t,x;u(\cdot)),$$

subject to (8).

Remark 2.2 *If $F(\cdot)=0$, then cost functional (9) can be rewritten as*

$$\begin{aligned} J(t,x;u(\cdot)) &= \mathcal{E}_g^t\bigg\{\frac{1}{2}HX^{t,x;u(\cdot)}(T)^2+hx(T)+m\\ &\quad+\int_t^T\Big[\frac{1}{2}Q(s)X^{t,x;u(\cdot)}(s)^2+q(s)X^{t,x;u(\cdot)}(s)+\ell(s)\Big]\,\mathrm{d}s\bigg\}, \end{aligned}$$

where $\mathcal{E}_g^t(\cdot)$ is g-expectation (see Appendix B).

3 Main result

In this section, we adopt the HJB equation to get the optimal feedback control for Problem (LQ-FBSDEs) by means of verification theorem (see [53]). Meanwhile, to overcome the control constraints, we employ some results from convex analysis (see Lemma C.1). At last, we should verify the control process belonging to appropriate space.

3.1 HJB Equation

We observe that the Riccati equation approach is not applicable directly in this case. Therefore, we study the corresponding HJB equation instead, which is the following partial differential equation (PDE, for short):

$$\begin{cases} v_t(t,x) + v_x(t,x)\left[b(t)+A(t)x\right] + F(t)v(t,x) + \frac{1}{2}Q(t)x^2 + q(t)x + \ell(t) \\ \quad + \inf\limits_{u\geq 0}\left[v_x(t,x)\left(B(t)+\sum_{j=1}^m O_j(t)D_j(t)\right)u + \dfrac{1}{2}v_{xx}(t,x)u'D(t)'D(t)u\right] = 0, \\ v(T,x) = \dfrac{1}{2}Hx^2 + hx + m, \end{cases} \tag{11}$$

where $D(t)' = (D_1(t)', \cdots, D_m(t)')$ and $Q(t) \geq 0$. Unfortunately, owing essentially to the non-negativity constraint of the control, the HJB equation does not have a smooth solution, as opposed to the unconstrained case where the solution to the HJB equation is a quadratic function which can be constructed via the Riccati equation. The idea here is to construct a function, show that it is a *viscosity* solution (see Appendix for the definition) to the HJB equation, and then employ the verification theorem to construct the optimal control.

As a matter of fact, HJB equation (11) can be equivalently turned into

$$\begin{cases} \bar{v}_t(t,x) + \inf\limits_{u\geq 0}\Big\{\bar{v}_x(t,x)\left[\left(B(t)+\sum_{j=1}^m O_j(t)D_j(t)\right)u + b(t) + A(t)x\right] \\ +\dfrac{1}{2}\bar{v}_{xx}(t,x)u'D(t)'D(t)u\Big\} + e^{\int_0^t F(s)\mathrm{d}s}\left(\dfrac{1}{2}Q(t)x^2 + q(t)x + \ell(t)\right) = 0, \\ \bar{v}(T,x) = e^{\int_0^T F(s)\mathrm{d}s}\left(\dfrac{1}{2}Hx^2 + hx + m\right). \end{cases} \tag{12}$$

Indeed, multiplying both sides of the first equality in (11) by $e^{\int_0^t F(s)\mathrm{d}s}$, we have

$$\begin{aligned} & e^{\int_0^t F(s)\mathrm{d}s}v_t(t,x) + e^{\int_0^t F(s)\mathrm{d}s}F(t)v(t,x) + e^{\int_0^t F(s)\mathrm{d}s}\left(\tfrac{1}{2}Q(t)x^2 + q(t)x + \ell(t)\right) \\ & +\inf_{u\geq 0}\Big\{e^{\int_0^t F(s)\mathrm{d}s}v_x(t,x)\left[\left(B(t)+\sum_{j=1}^m O_j(t)D_j(t)\right)u + b(t) + A(t)x\right] \\ & \frac{1}{2}e^{\int_0^t F(s)\mathrm{d}s}v_{xx}(t,x)u'D(t)'D(t)u\Big\} = 0. \end{aligned}$$

Taking

$$\bar{v}(t,x) = e^{\int_0^t F(s)\mathrm{d}s}v(t,x),$$

we get the desired result.

In order to employ a result from convex analysis, we need the following assumption. For that, we set

$$\mathcal{B}(t) = B(t) + \sum_{j=1}^m O_j(t)D_j(t).$$

We put the following assumptions:

(A1) The coefficients $A(\cdot), B(\cdot), b(\cdot), D_j, j = 1, \ldots, m, F(\cdot), O(\cdot), \ell(\cdot), Q(\cdot), q(\cdot)$ and the terminal coefficients H, h are deterministic and bounded processes.

(A2) Assume that $\mathcal{B}(t)' \in \mathbb{R}^m_+$.

Remark 3.1 *Note that whenever $F(\cdot) = 0$, $O(\cdot) = 0$, Problem (LQ-FBSDEs) will be turned into the classical LQ problem in Yong and Zhou [42].*

Remark 3.2 *In fact, in [28], the authors particularly emphasize that $B(t)' \in \mathbb{R}^m_+$. Hence, Assumption (A2) actually relaxes this condition.*

Now let us consider the above LQ problem (8)–(9).

Set

$$\bar{z}(t) := \underset{z(t)\in[0,\infty)^m}{\operatorname{argmin}} \frac{1}{2}\|(D(t)')^{-1}z(t) + (D(t)')^{-1}\mathcal{B}(t)'\|^2, \tag{13}$$

and

$$\bar{\xi}(t) := (D(t)')^{-1}\bar{z}(t) + (D(t)')^{-1}\mathcal{B}(t)'. \tag{14}$$

Note that $\bar{\xi}(t)$ is a column vector *independent* of x.

Remark 3.3 *The authors construct the value function of HJB equation (11) in (14) using $\bar{\xi}(t)$, if $\mathcal{B}(t) \equiv B(t)$. The similar role can be found in Theorem 2.3 of Xu and Shreve [40] (existence of optimal consumption).*

Let $P_1(t)$, $g_1(t)$ and $c_1(t)$, respectively, denote the solutions of the following differential equations (the first being a special Riccati equation)

$$\begin{cases} \dot{P}_1(t) = \left[-2A(t) + \|\bar{\xi}(t)\|^2\right] P_1(t) - e^{\int_0^t F(s)\mathrm{d}s}Q(t), \\ P_1(T) = He^{\int_0^T F(s)\mathrm{d}s}, \\ P_1(t) > 0, \qquad \forall t \in [0,T], \end{cases} \tag{15}$$

$$\begin{cases} \dot{g}_1(t) = \left[-A(t) + \|\bar{\xi}(t)\|^2\right] g_1(t) - e^{\int_0^t F(s)\mathrm{d}s}q(t) - b(t)P_1(t), \\ g_1(T) = he^{\int_0^T F(s)\mathrm{d}s}, \end{cases} \tag{16}$$

$$\begin{cases} \dot{c}_1(t) = -e^{\int_0^t F(s)\mathrm{d}s}\ell(t) + \frac{1}{2}\|\bar{\xi}(t)\|^2 P_1(t)^{-1}g_1(t)^2 - b(t)g_1(t), \\ c_1(T) = me^{\int_0^T F(s)\mathrm{d}s}, \end{cases} \tag{17}$$

and $P_2(t)$, $g_2(t)$ and $c_2(t)$, respectively denote the solutions of the following differential equations (the first being *another* special Riccati equation)

$$\begin{cases} \dot{P}_2(t) = -2A(t)P_2(t) - e^{\int_0^t F(s)\mathrm{d}s}Q(t), \\ P_2(T) = He^{\int_0^T F(s)\mathrm{d}s}, \\ P_2(t) > 0, \qquad \forall t \in [0,T], \end{cases} \tag{18}$$

$$\begin{cases} \dot{g}_2(t) = -A(t)g_2(t) - e^{\int_0^t F(s)\mathrm{d}s}q(t) - b(t)P_2(t), \\ g_2(T) = he^{\int_0^T F(s)\mathrm{d}s}, \end{cases} \tag{19}$$

$$\begin{cases} \dot{c}_2(t) = -e^{\int_0^t F(s)\mathrm{d}s}\ell(t) - b(t)g_2(t), \\ c_2(T) = me^{\int_0^T F(s)\mathrm{d}s}. \end{cases} \tag{20}$$

In the next subsection, we shall show that

$$V(t,x)=\begin{cases}\frac{1}{2}P_1(t)x^2+g_1(t)x+c_1(t), & \text{if } x+\dfrac{g_1(t)}{P_1(t)}\le 0,\\ \\ \frac{1}{2}P_2(t)x^2+g_2(t)x+c_2(t), & \text{if } x+\dfrac{g_2(t)}{P_2(t)}>0\end{cases} \tag{21}$$

is a viscosity solution of the HJB equation (12), and

$$u^*(t,x) \;=\; \begin{cases}-D(t)^{-1}\bar{\xi}(t)\Big(x+\dfrac{g_1(t)}{P_1(t)}\Big), & \text{if } x+\dfrac{g_1(t)}{P_1(t)}\le 0,\\ \\ 0, & \text{if } x+\dfrac{g_2(t)}{P_2(t)}>0\end{cases} \tag{22}$$

is the associated optimal feedback control.

Note that all the coefficients are bounded, from the expression of $u^*(t,x)$ in (22), $X\in\mathcal{H}^p[0,T]$ for any $p\ge 1$ and thus $u\in\mathcal{U}_{ad}^{p,+}[0,T]$, for any $p\ge 1$. From this, we claim that BSDE in (8) admits a unique adapted solution $(Y^{t,x;u(\cdot)}(\cdot),Z^{t,x;u(\cdot)}(\cdot))$. Consequently, $Y^{t,x;u(\cdot)}(t)$ provides the corresponding possible optimal value of the cost.

Remark 3.4 *The ODEs (15)–(17) can be derived by assuming the value function to be a quadratic form (as in (21)), substituting into the HJB equation (12), and eventually comparing the terms of x^2, x and the constant respectively. See [42, pp. 317–318] for a complete discussion. Normally, these equations can be encountered in stochastic LQ problems with nonhomogeneous terms.*

3.2 Target control

In this subsection, we verify the aforementioned results. At the beginning, we prove that V constructed in (21) is a viscosity solution to the HJB equation (12).

We start with equation (15).

Clearly,

$$P_1(t)=e^{\int_t^T(2A(z)-\|\bar{\xi}(z)\|^2)\mathrm{d}z}\Big[He^{\int_0^T F(s)\mathrm{d}s}+\int_t^T e^{\int_0^v F(s)\mathrm{d}s}Q(v)e^{\int_v^T(-2A(z)+\|\bar{\xi}(z)\|^2)\mathrm{d}z}\mathrm{d}v\Big]. \tag{23}$$

is the solution of (15). Note in particular that the constraint $P_1(t)>0$ is automatically satisfied. Define $\eta_1(t):=\frac{g_1(t)}{P_1(t)}$, it follows from (15) and (16) that

$$\begin{aligned}\dot{\eta}_1(t) &= \frac{P_1(t)\dot{g}_1(t)-\dot{P}_1(t)g_1(t)}{P_1(t)^2}\\ &= \Big[A(t)+e^{\int_0^t F(s)\mathrm{d}s}Q(t)P_1(t)^{-1}\Big]\eta_1(t)-\Big[P_1(t)^{-1}e^{\int_0^t F(s)\mathrm{d}s}q(t)+b(t)\Big].\end{aligned}$$

It is easy to get

$$
\begin{aligned}
\eta_1(t) &= \exp\left[-\int_t^T (A(z)+e^{\int_0^z F(s)\mathrm{d}s}Q(z)P_1(z)^{-1})\mathrm{d}z\right] \\
&\cdot\left\{\frac{h}{H}-\int_t^T (b(v)+P_1(v)^{-1}e^{\int_0^v F(s)\mathrm{d}s}Q(v))\right. \\
&\left.\cdot\exp\left[-\int_v^T (A(z)+e^{\int_0^z F(s)\mathrm{d}s}Q(z)P_1(z)^{-1})\mathrm{d}z\right]\mathrm{d}v\right\}. \qquad (24)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
g_1(t) &= P_1(t)\eta_1(t) \\
&= e^{\int_t^T (2A(z)-\|\bar{\xi}(z)\|^2)\mathrm{d}z}\Big[e^{\int_0^T F(s)\mathrm{d}s}H \\
&+\int_t^T e^{\int_0^v F(s)\mathrm{d}s}Q(v)e^{\int_v^T(-2A(z)+\|\bar{\xi}(z)\|^2)\mathrm{d}z}dv\Big] \\
&\cdot\exp\left[-\int_t^T \Big(A(z)+e^{\int_0^z F(s)\mathrm{d}s}Q(z)P_1(z)^{-1}\Big)\mathrm{d}z\right] \\
&\cdot\left\{\frac{h}{H}-\int_t^T \Big(b(v)+P_1(v)^{-1}e^{\int_0^v F(s)\mathrm{d}s}Q(v)\Big)\right. \\
&\left.\cdot\exp\left[-\int_v^T \Big(A(z)+e^{\int_0^z F(s)\mathrm{d}s}Q(z)P_1(z)^{-1}\Big)\mathrm{d}z\right]\mathrm{d}v\right\}. \qquad (25)
\end{aligned}
$$

Now we present

$$
c_1(t) = me^{\int_0^T F(s)\mathrm{d}s}-\int_t^T\left(-e^{\int_0^v F(s)\mathrm{d}s}\ell(v)+\frac{1}{2}\|\bar{\xi}(v)\|^2\eta_1(v)g_1(v)-b(v)g_1(v)\right)\mathrm{d}v,
$$

where $\eta_1(v)$ and $g_1(v)$ are obtained in (24) and (25).

Now we define the region Γ_1 in the (t,x)-plane as follows:

$$
\Gamma_1 := \left\{(t,x)\in[0,T]\times\mathbb{R}\;\middle|\;x+\eta_1(t)<0\right\}.
$$

In Γ_1, V as given by (21) is sufficiently smooth for the terms in (21) to be well defined, with

$$
\begin{aligned}
&V_t(t,x)=\frac{1}{2}\dot{P}_1(t)x^2+\dot{g}_1(t)x+\dot{c}_1(t),\\
&V_x(t,x)=P_1(t)x+g_1(t),\\
&V_{xx}(t,x)=P_1(t).
\end{aligned}
$$

Inserting them into the left-hand side (LHS) of (12), we get

$$
\begin{aligned}
\text{LHS} &= V_t(t,x) + e^{\int_0^t F(s)\mathrm{d}s}\left[\tfrac{1}{2}Q(t)x^2 + q(t)x + \ell(t)\right] + V_x(t,x)\left[A(t)x + b(t)\right] \\
&\quad + \inf_{u\geq 0}\left[\tfrac{1}{2}V_{xx}(t,x)u'D(t)'D(t)u + V_x(t,x)\mathcal{B}(t)u\right] \\
&= \left[\tfrac{1}{2}\dot{P}_1(t)x^2 + \dot{g}_1(t)x + \dot{c}_1(t)\right] \\
&\quad + \left[P_1(t)x + g_1(t)\right]\left[A(t)x + b(t)\right] + e^{\int_0^t F(s)\mathrm{d}s}(\tfrac{1}{2}Q(t)x^2 + q(t)x + \ell(t)) \\
&\quad + \inf_{u\geq 0}\left\{\tfrac{1}{2}P_1(t)u'D(t)'D(t)u + [P_1(t)x + g_1(t)]\mathcal{B}(t)u\right\} \\
&= \tfrac{1}{2}\left[\dot{P}_1(t) + 2A(t)P_1(t) + e^{\int_0^t F(s)\mathrm{d}s}Q(t)\right]x^2 \\
&\quad + \left[\dot{g}_1(t) + A(t)g_1(t) + e^{\int_0^t F(s)\mathrm{d}s}q(t) + P_1(t)b(t)\right]x \\
&\quad + \left[\dot{c}_1(t) + e^{\int_0^t F(s)\mathrm{d}s}\ell(t) + g_1(t)b(t)\right] \\
&\quad + P_1(t)\inf_{u\geq 0}\left\{\tfrac{1}{2}u'D(t)'D(t)u + [x + \eta_1(t)]\mathcal{B}(t)u\right\}.
\end{aligned}
\tag{26}
$$

Applying Lemma C.2 with $\alpha = -[x+\eta_1(t)] > 0$, it follows that the minimizer of (26) is achieved by

$$
u^*(t,x) = -D(t)^{-1}\bar{\xi}(t)[x + \eta_1(t)].
$$

Inserting $u^*(t,x)$ back into (26) and noting (15), (16) and (17), it immediately yields that LHS $= 0$. This implies that V satisfies the HJB equation (12) in Γ_1.

Remark 3.5 *Observe that $\bar{\xi}(\cdot)$ is not explicitly analytical since it contains the term $\bar{z}(\cdot)$. Nevertheless, it can be easily attained numerically by solving the quadratic program in (13) offline.*

Next we are going to deal with the region Γ_2 defined by

$$
\Gamma_2 := \left\{(t,x) \in [0,T] \times \mathbb{R} \;\middle|\; x + \eta_2(t) > 0\right\}.
$$

Repeating to the derivations used above, we obtain

$$
\begin{cases}
P_2(t) = e^{2\int_t^T A(s)\mathrm{d}s}\left[He^{\int_0^T F(s)\mathrm{d}s} + \int_t^T e^{\int_0^z F(s)\mathrm{d}s}Q(z)e^{-\int_z^T 2A(s)\mathrm{d}s}\mathrm{d}z\right], \\[2ex]
g_2(t) = e^{\int_t^T A(s)\mathrm{d}s}\left\{he^{\int_0^T F(s)\mathrm{d}s} + \int_t^T \left[(e^{\int_0^z F(s)\mathrm{d}s}q(z) + b(z)P_2(z))e^{-\int_z^T A(s)\mathrm{d}s}\right]\mathrm{d}z\right\}, \\[2ex]
c_2(t) = me^{\int_0^T F(s)\mathrm{d}s} + \displaystyle\int_t^T \left[e^{\int_0^z F(s)\mathrm{d}s}\ell(z) + b(z)g_2(z)\right]\mathrm{d}z, \\[2ex]
\eta_2(t) = \frac{g_2(t)}{P_2(t)}.
\end{cases}
$$

In Γ_2, V is once again sufficiently smooth for the derivatives in (21) to be well defined, and

$$
\begin{aligned}
&V_t(t,x) = \frac{1}{2}\dot{P}_2(t)x^2 + \dot{g}_2(t)x + \dot{c}_2(t), \\
&V_x(t,x) = P_2(t)x + g_2(t), \\
&V_{xx}(t,x) = P_2(t).
\end{aligned}
$$

Substituting into the left-hand side (LHS) of (12), we have

$$
\begin{aligned}
\text{LHS} &= V_t(t,x) + e^{\int_0^t F(s)\mathrm{d}s}\left[\tfrac{1}{2}Q(t)x^2 + q(t)x + \ell(t)\right] + V_x(t,x)\left[A(t)x + b(t)\right] \\
&\quad + \inf_{u\geq 0}\Big[\tfrac{1}{2}V_{xx}(t,x)u'D(t)'D(t)u + V_x(t,x)\mathcal{B}(t)u\Big] \\
&= \Big[\tfrac{1}{2}\dot{P}_2(t)x^2 + \dot{g}_2(t)x + \dot{c}_2(t)\Big] + e^{\int_0^t F(s)\mathrm{d}s}\left[\tfrac{1}{2}Q(t)x^2 + q(t)x + \ell(t)\right] \\
&\quad + \left[P_2(t)x + g_2(t)\right]\left[A(t)x + b(t)\right] + \inf_{u\geq 0}\Big\{\tfrac{1}{2}P_2(t)u'D(t)'D(t)u \\
&\quad + [P_2(t)x + g_2(t)]\mathcal{B}(t)u\Big\} \qquad (27) \\
&= \tfrac{1}{2}\Big[\dot{P}_2(t) + 2A(t)P_2(t) + e^{\int_0^t F(s)\mathrm{d}s}Q(t)\Big]x^2 \\
&\quad + \Big[\dot{g}_2(t) + A(t)g_2(t) + e^{\int_0^t F(s)\mathrm{d}s}q(t) + b(t)P_2(t)\Big]x \\
&\quad + \Big[\dot{c}_2(t) + e^{\int_0^t F(s)\mathrm{d}s}\ell(t) + g_2(t)b(t)\Big] \\
&\quad + P_2(t)\inf_{u\geq 0}\Big\{\tfrac{1}{2}u'D(t)'D(t)u + [x + \eta_2(t)]\mathcal{B}(t)u\Big\}.
\end{aligned}
$$

Since $x + \eta_2(t) > 0$, the minimizer of (27) is

$$u^*(t,x) = 0.$$

Substituting $u^*(t,x)$ into (27), it is easy to show that V satisfies the HJB equation (12) in Γ_2.

Finally, the switching curve Γ_3 is defined by

$$\Gamma_3 := \Big\{(t,x) \in [0,T]\times\mathbb{R} \,\Big|\, x + \eta_1(t) = x + \eta_2(t) = 0\Big\},$$

where the non-smoothness of V occurs. First, at $(t,x) \in \Gamma_3$, it derives

$$
\begin{cases}
V_t(t,x) &= \dfrac{1}{2}\dot{P}_1(t)x^2 + \dot{g}_1(t)x + \dot{c}_1(t) = \dfrac{1}{2}\dot{P}_2(t)x^2 + \dot{g}_2(t)x + \dot{c}_2(t) \\
&= e^{\int_0^t F(s)\mathrm{d}s}\left[-\tfrac{1}{2}Q(t)x^2 - q(t)x - \ell(t)\right], \\
V_x(t,x) &= P_1(t)x + g_1(t) = P_2(t)x + g_2(t) = 0.
\end{cases}
$$

Clearly, $V(t,x)$ is also continuously differentiable at points on Γ_3. However, V_{xx} does not exist on Γ_3, since $P_1(t) \not\equiv P_2(t)$. This means that V does not provide the necessary smoothness properties to qualify as a classical solution of the HJB equation (12). Therefore, we are required to work within the framework of viscosity solutions. From Definition A.1 in Appendix, it can be shown that for any $(t,x) \in \Gamma_3$,

$$
\begin{cases}
D_{t,x}^{1,2,+}V(t,x) = \Big\{e^{\int_0^t F(s)\mathrm{d}s}\left[-\tfrac{1}{2}Q(t)x^2 - q(t)x - \ell(t)\right]\Big\} \times \{0\} \times [P_2(t), +\infty), \\
D_{t,x}^{1,2,-}V(t,x) = \Big\{e^{\int_0^t F(s)\mathrm{d}s}\left[-\tfrac{1}{2}Q(t)x^2 - q(t)x - \ell(t)\right]\Big\} \times \{0\} \times (-\infty, P_1(t)].
\end{cases}
$$

For the HJB equation (12), we define

$$G(t,x,u,p,P) = pB(t)u + \frac{1}{2}Pu'D(t)'D(t)u + e^{\int_0^t F(s)\mathrm{d}s}\left[\frac{1}{2}Q(t)x^2 + q(t)x + \ell(t)\right].$$

For any $(q,p,P) \in D^{1,2,+}_{t,x}V(t,x)$, where $(t,x) \in \Gamma_3$, we have

$$q + \inf_{u\geq 0} G(t,x,u,p,P) = \inf_{u\geq 0}\left\{\tfrac{1}{2}Pu'D(t)'D(t)u\right\} \geq \inf_{u\geq 0}\left\{\tfrac{1}{2}P_2(t)u'D(t)'D(t)u\right\} = 0.$$

Therefore, V is a viscosity sub-solution of the HJB equation (11).

On the other hand, for $(q,p,P) \in D^{1,2,-}_{t,x}V(t,x)$ where $(t,x) \in \Gamma_3$, we have

$$q + \inf_{u\geq 0} G(t,x,u,p,P) = \inf_{u\geq 0}\left\{\tfrac{1}{2}Pu'D(t)'D(t)u\right\} \leq \inf_{u\geq 0}\left\{\tfrac{1}{2}P_1(t)u'D(t)'D(t)u\right\} = 0.$$

Consequently, V is also a viscosity super-solution of the HJB equation (12). Finally, it is easy to check that the terminal condition

$$V(T,x) = e^{\int_0^T F(s)\mathrm{d}s}\Big(\frac{1}{2}Hx^2 + hx + m\Big)$$

is satisfied. Hence, it follows from Definition A.1 that $V(t,x)$ is a viscosity solution of the HJB equation (12). Moreover, for any $(t,x) \in \Gamma_3$, if we take

$$\begin{aligned}&(q^*(t,x), p^*(t,x), P^*(t,x), u^*(t,x))\\ &:= (-\frac{1}{2}Q(t)x^2 - q(t)x - \ell(t), 0, P_2(t), 0) \in D^{1,2,+}_{t,x}V(t,x) \times \mathcal{U}[s,T],\end{aligned}$$

then

$$q^*(t,x) + G(t,x,u^*(t,x),p^*(t,x),P^*(t,x)) = 0.$$

It then follows from the *verification theorem* developed by Zhang in [53] that $u^*(t,x)$ defined by (22) is the optimal feedback control.

Remark 3.6 *Apparently, the approach developed in this paper, is completely difference from the duality methods in [7, 22] (through convex duality). More previously, the existence and optimality of the candidate portfolio are constructed by the theory of viscosity solutions and the corresponding verification theorem established in [53].*

4 Applications to Financial Portfolio Optimization

We consider a financial market where $m+1$ assets are traded continuously on a finite horizon $[0,T]$. One asset is a bond, whose price $P_0(t)$, $t \geq 0$, evolves according to the differential equation:

$$\begin{cases} \mathrm{d}S_0(t) &= r(t)S_0(t)\mathrm{d}t,\\ S_0(t) &= s_0,\end{cases}$$

where $r(t)$ (> 0) is the interest rate of the bond. The remaining m assets are stocks, and their prices are modeled by the stochastic differential equations:

$$\begin{cases} \mathrm{d}S_i(t) &= S_i(t)\left\{b_i(t)\mathrm{d}t + \sum_{j=1}^m \sigma_{ij}(t)\mathrm{d}W^j(t)\right\}, \quad t \in [0,T],\\ S_i(0) &= s_i > 0,\end{cases}$$

where $b_i(t)$ $(> r(t))$ is the appreciation rate and $\sigma_{ij}(t)$ is the volatility coefficient. Denote $b(t) := (b_1(t), \cdots, b_m(t))'$ and $\sigma(t) := (\sigma_{ij}(t))$. We assume throughout that $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$

are deterministic, Borel-measurable, and bounded on $[0,T]$. In addition, we assume that the non-degeneracy condition

$$\sigma(t)\sigma(t)' \geq \delta I, \quad \forall t \in [0,T],$$

where $\delta > 0$ is a given constant, is satisfied. Also, we define the relative risk coefficient

$$\theta(t) \triangleq \sigma^{-1}(t)(b(t) - r(t)\mathbf{1}),$$

where $\mathbf{1}$ is the m-dimensional column vector with each component equal to 1.

Suppose an agent has an initial wealth $X_0 > 0$ and the total wealth of his position at time $t \geq 0$ is $X(t)$, Then it is well-known that $X(\cdot)$ follows (see, e.g., [44])

$$\begin{cases} \mathrm{d}X(t) &= \Big\{r(t)X(t) + \sum_{j=1}^m (b_i(t) - r(t)u_i(t))\Big\}\,\mathrm{d}t + \sum_{i,j=1}^m \sigma_{ij}(t)u_i(t)\mathrm{d}W^j(t), \\ X(0) &= X_0, \end{cases} \tag{28}$$

where $u_i(t)$, $i = 0, 1, \cdots, m$, denotes the total market value of the agent's wealth in the i-th bond/stock. We call $u(t) := (u_1(t), \cdots, u_m(t))$ the portfolio (which changes over time t). An important restriction considered in this paper is the prohibition of short-selling the stocks, i.e., it must be satisfied that $u_i(t) \geq 0$, $i = 1, \cdots, m$. On the other hand, borrowing from the money market (at the interest rate $r(t)$) is still allowed; that is, $u_0(t)$ is not explicitly constrained.

Mean-variance portfolio selection refers to the problem of finding an allowable investment policy (i.e., a dynamic portfolio satisfying all the constraints) such that the expected terminal wealth satisfies $\mathbb{E}[X(T)] = \kappa$ while the risk measured by the variance of the terminal wealth

$$\mathrm{Var}(X(T)) = \mathbb{E}[X(T) - \mathbb{E}X(T)]^2 = \mathbb{E}[X(T) - \kappa]^2$$

is minimized.

We impose throughout this paper the following assumption.

(A3) The value of the expected terminal wealth κ satisfies $\kappa \geq X_0 e^{\int_0^T r(s)ds}$.

Remark 4.1 *Assumption (A3) states that the investor's expected terminal wealth κ cannot be less than $X_0 e^{\int_0^T r(s)ds}$ which coincides with the amount that he/she would earn if all of the initial wealth is invested in the bond for the entire investment period. Clearly, this is a reasonable assumption, for the solution of the problem under $\kappa < X_0 e^{\int_0^T r(s)ds}$ is foolish for rational investors.*

Definition 4.1 *A portfolio $u(\cdot)$ is said to be admissible if $u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m_+)$.*

Definition 4.2 *The mean-variance portfolio selection problem is formulated as the following optimization problem parameterized by $\kappa \geq X_0 e^{\int_0^T r(s)ds}$:*

$$\begin{array}{ll} \min & \mathrm{Var}(X(T)) \equiv \mathbb{E}[X(T) - \kappa]^2, \\ \text{subject to} & \begin{cases} \mathbb{E}[X(T)] = \kappa, \\ u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m_+), \\ (X(\cdot), u(\cdot)) \text{ satisfies equation (28).} \end{cases} \end{array} \tag{29}$$

Moreover, the optimal control of (29) is called an efficient strategy, and $(\mathrm{Var}(X(T)), \kappa)$, where $\mathrm{Var}(X(T))$ is the optimal value of (29) corresponding to κ, is called an efficient point. The set of all efficient points, when the parameter κ runs over $[X_0 e^{\int_0^T r(s)ds}, +\infty)$, is called the efficient frontier.

Since (29) is a convex optimization problem, the equality constraint $\mathbb{E}[X(T)] = \kappa$ can be dealt with by introducing a Lagrange multiplier $\mu \in \mathbb{R}$. Therefore, the portfolio problem (29) can be solved via the following optimal stochastic control problem (for every fixed μ). Note that

$$\begin{aligned} & \mathbb{E}\Big\{[X(T)^2 - \mu^2 - 2\mu[X(T) - \kappa]\Big\} \\ = \ & \mathbb{E}\left[|X(T) - \mu|^2\right] - (\mu - \kappa)^2, \ \lambda \in \mathbb{R}. \end{aligned}$$

We have

Problem (L(μ))

$$\begin{aligned} & \min_{u(\cdot)\geq 0} \ \mathbb{E}\Big\{[X(T) - \mu]^2\Big\} - (\mu - \kappa)^2, \\ & \text{subject to} \ \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m_+), \\ (X(\cdot), u(\cdot)) \text{ satisfy equation (28)}, \end{cases} \end{aligned} \tag{30}$$

where the factor 2 in front of the multiplier μ is introduced in the objective function just for convenience.

Set

$$x(t) := X(t) - \mu e^{-\int_t^T r(s)\mathrm{d}s}.$$

Problem $L(\mu)$ is equivalent to the following problem

$$\min_{u(\cdot)\geq 0} \ \mathbb{E}\left[x(T)^2 - (\mu - \kappa)^2\right],$$

subject to

$$\begin{cases} \mathrm{d}x(t) = [A(t)x(t) + B(t)u(t)]\,\mathrm{d}t + \sum_{j=1}^m D_j(t)u(t)\mathrm{d}W^j(t), \\ x(0) = X_0 - \mu e^{-\int_0^T r(s)\mathrm{d}s}, \end{cases}$$

where $u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m_+)$ and

$$\begin{aligned} A(t) &= r(t), \quad B(t) = (b_1(t) - r(t), \cdots, b_m(t) - r(t)), \\ D_j(t) &= (\sigma_{1j}(t), \cdots, \sigma_{mj}(t)). \end{aligned}$$

In this paper, we extend the above classical mean-variance portfolio optimization problem to a recursive utility portfolio optimization problem. The recursive utility means that the utility at time t is a function of the future utility. In fact, within our framework, the recursive utility can be described by the controlled BSDE. Suppose a small investor, endowed with initial wealth $x_0 > 0$, chooses at each time t his/her portfolio $u(\cdot)$. The investor wants to choose an optimal portfolio to minimize the following recursive utility functional with generator:

$$f(t, x, y, z, u) = \beta(t)y + \langle \gamma(t), z \rangle, \tag{31}$$

where $\beta(t) \in \mathbb{R}_-$, $\gamma(t) \in \mathbb{R}^d$, $\forall t \in [0, T]$. The term z can be interpreted as an ambiguity aversion term in the market (see Chen and Epstein 2002, [46]). The recursive utility functional (31) defined above depicts a kind of additive utility of recursive type. It is a meaningful and nontrivial generalization of the classical standard additive utility and has found many applications in mathematical economics and mathematical finance. For more details on utility functions, see Duffie and Epstein [9], Section 1.4 of El Karoui et al. [20] or Schroder and Skiadas [34].

Now, we consider the recursive framework as follows

$$\left\{\begin{array}{rcl} \mathrm{d}x(t) &=& [A(t)x(t)+B(t)u(t)]\,\mathrm{d}t+\sum_{j=1}^m D_j(t)u(t)\mathrm{d}W^j(t), \\ -\mathrm{d}y(t) &=& [\beta(t)y(t)+\langle\gamma(t),z(t)\rangle]\,\mathrm{d}t-\sum_{j=1}^m z_j(t)\mathrm{d}W^j(t), \\ x(0) &=& X_0-\mu e^{-\int_0^T r(s)\mathrm{d}s}, \quad y(T)=x(T)^2-(\mu-\kappa)^2. \end{array}\right. \tag{32}$$

Problem (FBSDE). For any $u(\cdot)\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m_+)$, the investor's utility functional is defined by

$$\mathcal{J}(0,X_0-\mu e^{-\int_0^T r(s)\mathrm{d}s};u(\cdot)) = y(s)\big|_{s=0},$$

subject to (32). The portfolio optimization problem can be rewritten as

$$\mathcal{J}(0,X_0-\mu e^{-\int_0^T r(s)\mathrm{d}s};\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}^p_{ad}[0,T]} \mathcal{J}(0,X_0-\mu e^{-\int_0^T r(s)\mathrm{d}s};u(\cdot)). \tag{33}$$

Now, corresponding to (13) and (14), set

$$\bar{\pi}(t) := \underset{\pi(t)\in[0,\infty)^m}{\operatorname{argmin}} \frac{1}{2}\left\|\sigma(t)^{-1}\pi(t)+\sigma(t)^{-1}\left[(b(t)-r(t)\mathbf{1})+\sum_{j=1}^m\gamma_j(t)D_j(t)'\right]\right\|^2 \tag{34}$$

and

$$\bar{\theta}(t) := \sigma(t)^{-1}\bar{\pi}(t)+\sigma(t)^{-1}\left[b(t)-r(t)\mathbf{1}+\sum_{j=1}^m\gamma_j(t)D_j(t)'\right].$$

Before analyzing the efficient frontier of the original portfolio selection problem (29), we first present the optimal investment strategy for the problem Problem (FBSDEs). The optimal control obtained in (22) translates into the following strategy:

$$\begin{array}{rcl} u^*(t,X) &\equiv& (u_1^*(t,X),\ldots,u_m^*(t,X))' \\ &=& \left\{\begin{array}{ll} (\sigma(t)')^{-1}\bar{\theta}(t)\left[X_0-\mu e^{-\int_0^T r(s)\mathrm{d}s}\right], & \text{if } X_0-\mu e^{-\int_0^T r(s)\mathrm{d}s}\le 0, \\ 0, & \text{if } X_0-\mu e^{-\int_0^T r(s)\mathrm{d}s}>0. \end{array}\right. \end{array} \tag{35}$$

Consider the following BSDEs:

$$\left\{\begin{array}{rcl} -\mathrm{d}y^1(t) &=& \left[\beta(t)y^1(t)+\left\langle\gamma(t),z^1(t)\right\rangle\right]\mathrm{d}t-\sum_{j=1}^m z_j^1(t)\mathrm{d}W^j(t), \\ -\mathrm{d}y^2(t) &=& \left[\beta(t)y^2(t)+\left\langle\gamma(t),z^2(t)\right\rangle\right]\mathrm{d}t-\sum_{j=1}^m z_j^2(t)\mathrm{d}W^j(t), \\ y^1(T) &=& x(T)^2,\ y^2(T)=-(\mu-\kappa)^2. \end{array}\right. \tag{36}$$

Clearly, due to the linear property of BSDE, we derive that

$$y^2(t) = -(\mu-\kappa)^2 e^{\int_t^T\beta(s)\mathrm{d}s}.$$

For simplicity, let $\Pi(\mu)\triangleq X_0-\mu e^{-\int_0^T r(s)\mathrm{d}s}$. We now seek the efficient frontier for the portfolio selection problem (29). At the beginning, for every fixed μ, we have

$$\begin{aligned}
&\min_{u(\cdot)\in\mathcal{U}_{ad}^p[0,T]} \mathbb{G}_{0,T}\left[x(T)^2-(\mu-\kappa)^2\right] \\
=\ &\min_{u(\cdot)\in\mathcal{U}_{ad}^p[0,T]} \mathbb{G}_{0,T}\left[x(T)^2\right]-(\mu-\kappa)^2 e^{\int_0^T \beta(s)\mathrm{d}s} \\
=\ &V(0,x(0))-(\mu-\kappa)^2 e^{\int_0^T \beta(s)\mathrm{d}s} \\
=\ &\frac{1}{2}P(0)x(0)^2+g(0)x(0)+c(0)-(\mu-\kappa)^2 e^{\int_0^T \beta(s)\mathrm{d}s} \\
=\ &\frac{1}{2}P(0)\Pi(\mu)^2+g(0)\Pi(\mu)+c(0)-(\mu-\kappa)^2 e^{\int_0^T \beta(s)\mathrm{d}s},
\end{aligned}$$

where $\mathbb{G}_{0,T}$ is the backward semigroup associated with BSDE in (32), moreover, $P(\cdot), g(\cdot)$ and $c(\cdot)$ can be selected via either $P_1(\cdot), g_1(\cdot)$ and $c_1(\cdot)$ or $P_2(\cdot), g_2(\cdot)$ and $c_2(\cdot)$, respectively, depending on whether $\Pi(\mu)\le 0$ or otherwise (see (21)). Indeed, under the optimal strategy (35), we get

$$\begin{aligned}
&\min_{u(\cdot)\in\mathcal{U}_{ad}^p[0,T]} \mathbb{G}_{0,T}\left[x(T)^2-(\mu-\kappa)^2\right] \\
=\ &\begin{cases}
\left[e^{\int_0^T (2r(z)-\|\bar{\xi}(z)\|^2)\mathrm{d}z}\Pi(\mu)^2-(\mu-\kappa)^2\right] e^{\int_0^T \beta(s)\mathrm{d}s}, & \text{if } \Pi(\mu)\le 0,\\
\left[e^{\int_0^T 2r(s)\mathrm{d}s}\Pi(\mu)^2-(\mu-\kappa)^2\right] e^{\int_0^T \beta(s)\mathrm{d}s}, & \text{if } \Pi(\mu)> 0.
\end{cases}
\end{aligned} \tag{37}$$

We deal with the first term in (37), which is a quadratic function in μ,

$$\begin{aligned}
&e^{\int_0^T (2r(z)-\|\bar{\xi}(z)\|^2)\mathrm{d}z}\left[X_0^2-2X_0\cdot\mu e^{-\int_0^T r(s)\mathrm{d}s}+\mu^2 e^{-2\int_0^T r(s)\mathrm{d}s}\right]-(\mu-\kappa)^2 \\
=\ &(e^{\int_0^T -\|\bar{\xi}(z)\|^2\mathrm{d}z}-1)\mu^2+2\Big(\kappa-X_0 e^{\int_0^T (r(z)-\|\bar{\xi}(z)\|^2)\mathrm{d}z}\Big)\mu \\
&+X_0^2 e^{\int_0^T (2r(z)-\|\bar{\xi}(z)\|^2)\mathrm{d}z}-\kappa^2,
\end{aligned} \tag{38}$$

and the second term becomes a linear function in μ

$$\begin{aligned}
&e^{\int_0^T 2r(s)\mathrm{d}s}(X_0-\mu e^{-\int_0^T r(s)\mathrm{d}s})^2-(\mu-\kappa)^2 \\
=\ &2\Big(\kappa-X_0 e^{\int_0^T r(s)\mathrm{d}s}\Big)\mu+X_0^2 e^{\int_0^T 2r(s)\mathrm{d}s}-\kappa^2.
\end{aligned} \tag{39}$$

Observe that the above equalities (38)-(39) still depend on the Lagrange multiplier μ. In order to seek the optimal value and optimal strategy for the original portfolio selection problem (29) one needs to maximize the value in (37) over $\mu\in\mathbb{R}$ by virtue of the Lagrange duality theorem [29]. A simple computation indicates that (37) attains its maximum value

$$\frac{e^{-\int_0^T \|\bar{\xi}(z)\|^2\mathrm{d}z}(\kappa-X_0 e^{\int_0^T r(z)\mathrm{d}z})^2}{1-e^{-\int_0^T \|\bar{\xi}(z)\|^2\mathrm{d}z}}\cdot e^{\int_0^T \beta(s)\mathrm{d}s}$$

at

$$\mu^*=\frac{\kappa-X_0 e^{\int_0^T (r(z)-\|\bar{\xi}(z)\|^2)\mathrm{d}z}}{1-e^{-\int_0^T \|\bar{\xi}(z)\|^2\mathrm{d}z}}.$$

Finally, one can get that

$$
\begin{aligned}
& X_0 - \mu^* e^{-\int_0^T r(s)\mathrm{d}s} \\
= \; & \frac{X_0 - X_0 e^{-\int_0^T \|\bar{\xi}(z)\|^2 \mathrm{d}z} - \kappa e^{-\int_0^T r(s)\mathrm{d}s} + X_0 e^{\int_0^T (r(z)-\|\bar{\xi}(z)\|^2)\mathrm{d}z} e^{-\int_0^T r(s)\mathrm{d}s}}{1 - e^{-\int_0^T \|\bar{\xi}(z)\|^2 \mathrm{d}z}} \\
= \; & \frac{X_0 - \kappa e^{-\int_0^T r(s)\mathrm{d}s}}{1 - e^{-\int_0^T \|\bar{\xi}(z)\|^2 \mathrm{d}z}} \\
\leq \; & 0,
\end{aligned}
$$

where the last inequality is verified by (A3).

Remark 4.2 *If $\beta = 0$ and $\gamma = 0$, we get the result obtained in Li et al. [28]. If we examine the derivation of (34), we instantly observe that the term Z in BSDE (1) puts some impact on $\bar{\pi}$, which weakens the condition on $B(\cdot)$ (see [28], page 1544).*

5 Conclusion

This paper studies a continuous-time recursive mean-variance portfolio selection problem where the short-selling is not allowable. By technique of BSDE and viscosity solution theory, we derive the efficient strategies and efficient frontier explicitly. An interesting problem is to extend the results to the case where all the market coefficients are random processes. Of course, a backward stochastic partial differential equation must be employed within the framework of stochastic viscosity solution theory (cf. [24]).

A Appendix: Viscosity Solutions

We list here some basic terminologies from the theory of viscosity solutions which are referred to in the paper.

Let

$$G(t, x, v, p, P, u) = \frac{1}{2}\sigma(t, x, u)^\top P \sigma(t, x, u) + p^\top b(t, x, u) - f(t, x, v, \sigma^\top p, u),$$

where

$$
\begin{aligned}
\sigma &: [0, T) \times \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}^n, \\
b &: [0, T) \times \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}^n \\
f &: [0, T) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{n_u} \to \mathbb{R}.
\end{aligned}
$$

Consider the second-order partial differential equation (PDE in short)

$$
\begin{cases}
v_t + \inf_{u \geq 0} G(t, x, u, v, v_x, v_{xx}) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n, \\
v(T, x) = g(x),
\end{cases}
\tag{40}
$$

where $g : \mathbb{R}^n \to \mathbb{R}$.

Clearly the HJB equation (11) is a special case of (40). It is well-known that (40) does not in general have classical (smooth) solutions. A generalized concept of solution, called a viscosity solution, is introduced in [6]. The main result in [42] is that under certain mild conditions, there exists a unique viscosity solution in the first order case. In the second-order case, uniqueness is proven in [42]. See also [11, 42] for more details about viscosity solution and application in stochastic control.

Definition A.1 *Let $v \in C([0,T] \times \mathbb{R}^n)$ and $(t_0, x_0) \in (0,T) \times \mathbb{R}^n$. Then the second-order super-differential of v at (t_0, x_0) is defined by*

$$\begin{aligned} D_{t,x}^{1,2,+} v(t_0, x_0) = \Big\{ (\varphi_t(t_0,x_0),\, \varphi_x(t_0,x_0), \varphi_{xx}(t_0,x_0)) \,\Big|\, \\ \varphi \in C^\infty((0,T)\times\mathbb{R}^n) \text{ and } v - \varphi \text{ has a local maximum at } (t_0,x_0) \Big\}, \end{aligned} \tag{41}$$

and the second order sub-differential of v is defined by

$$\begin{aligned} D_{t,x}^{1,2,-} v(t_0, x_0) = \Big\{ (\varphi_t(t_0,x_0),\, \varphi_x(t_0,x_0), \varphi_{xx}(t_0,x_0)) \,\Big|\, \\ \varphi \in C^\infty((0,T)\times\mathbb{R}^n) \text{ and } v - \varphi \text{ has a local minimum at } (t_0,x_0) \Big\}. \end{aligned} \tag{42}$$

Moreover, v is a viscosity solution of (40) if

$$v(T,x) = g(x), \quad \forall\, x \in \mathbb{R}^n, \tag{43}$$

and

$$q + \inf_{u\in U} G(t,x,v,p,P,u) \;\geq\; 0, \quad \forall (q,p,P) \in D_{t,x}^{1,2,+} v(t,x), \tag{44}$$

$$q + \inf_{u\in U} G(t,x,v,p,P,u) \;\leq\; 0, \quad \forall (q,p,P) \in D_{t,x}^{1,2,-} v(t,x), \tag{45}$$

for all $(t,x) \in [0,T) \times \mathbb{R}^n$.

In particular, v is called a *viscosity sub-solution* if it satisfies (43)-(44), and a *viscosity super-solution* if it satisfies (43) and (45).

B Appendix: g-expectation

For reader's convenience, let us state briefly a kind of generalized expectation, *g-expectation*. As we have known that sometimes the objective expectation does not quite represent people's preferences [1, 8]. An effective way is to employ the so-called generalized expectation [50] which is subjective, in some sense. More precisely, consider the following the so-called backward stochastic differential equation (BSDE in short):

$$\begin{cases} \mathrm{d}\zeta(s) = -g(\kappa(s))\mathrm{d}s + \kappa(s)\mathrm{d}W(s), \\ \zeta(T) = \xi, \qquad 0 \leq s \leq T, \end{cases} \tag{46}$$

where $g : \mathbb{R}^d \to \mathbb{R}$ is a given map for which we introduce the following assumption.

(H1) The map $g : \mathbb{R}^d \to \mathbb{R}$ satisfies the following conditions:

$$\left|g(\kappa) - g(\kappa')\right| \le C \left|\kappa - \kappa'\right|, \quad t \in, \ \kappa, \kappa' \in \mathbb{R}^d, \text{ a.s.}, \tag{47}$$

and

$$g(\kappa) = 0 \quad \Longleftrightarrow \quad \kappa = 0. \tag{48}$$

From Pardoux and Peng [49], when (H1) holds, for any $\xi \in L^2_{\mathcal{F}_T}(0, T; \mathbb{R})$, BSDE (47) admits a unique adapted strong solution $(\zeta(s), \kappa(\cdot)) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$. Further, if (48) holds, we may define

$$\mathcal{E}_g(\xi) = \zeta(0). \tag{49}$$

It is easy to show that the map $\xi \to \mathcal{E}_g(\xi)$ keeps all the properties that $\mathbb{E}$ has, except possibly for the linearity. We call $\mathcal{E}_g(\xi)$ the generalized expectation of ξ associated with g (also called g-expectation [50]). Further, it is clear that whenever $g(\cdot, \cdot) = 0$, $\mathcal{E}_g^t$ is reduced to the original (linear) conditional expectation. A typical examples of g satisfying (A1) as follows:

$$g(\kappa) = \langle b, \kappa \rangle, \ \forall b, \ \kappa \in \mathbb{R}^d. \tag{50}$$

Clearly, (50)[2] is positively homogeneous.

The g-expectation, a nonlinear expectation introduced by Peng [50] via a nonlinear BSDE above, can be considered as a nonlinear extension of the well-known Girsanov transformations. The original motivation for studying g-expectation comes from the theory of expected utility, which is the foundation of modern mathematical economics. Chen and Epstein [46] gave an application of g-expectation to recursive utility. The g-expectation, apart from their own theoretical values, have found important applications in various areas especially in finance. For example, the super- and sub-pricing of contingent claims in an incomplete market can both be captured by the g-probability. Ambiguity in financial modeling can be described by the g-expectation. Indeed, Chen and Epstein [46] introduced a k-ignorance model involving the g-probability to study ambiguity aversion. The g-expectations have also been found to have intimate connection with the rapidly developed risk measure theory.

With the above-defined g-expectation on hand, it is quite natural to introduce the following cost functional: for any $u(\cdot) \in \mathcal{U}_{ad}$,

$$J_g(u(\cdot)) = \mathcal{E}_g \left\{ \frac{1}{2} \int_0^T \left[\langle Q(t)x(t), x(t) \rangle \, \mathrm{d}t + \frac{1}{2} \langle Hx(T), x(T) \rangle \right] \right\},$$

for some suitable g satisfying (H1). We currently can formulate our stochastic LQ problem with generalized expectation as follows.

We now formulate the Problems $(\mathrm{LQ})_g$ in a different view of point. For any $u(\cdot) \in \mathcal{U}_{ad}$, let $(\zeta(\cdot), \kappa(\cdot))$ be the adapted solution of (46).

Put

$$\begin{cases} y^1(t) &= \zeta(t) - \int_0^t \langle Q(s)x(s), x(s) \rangle \, \mathrm{d}s, \\ z^1(t) &= \kappa(t), \ t \in [0, T]. \end{cases}$$

[2] In fact, this kind of function can be employed to study the relationship between the Choquet expectation and g-expectation (see [47] for more details).

Then $(y^1(\cdot), z^1(\cdot))$ is the unique adapted solution to the following BSDE:

$$\left\{\begin{array}{rcl} -\mathrm{d}y^1(s) & = & \left[\frac{1}{2}(\langle Q(s)x(s), x(s)\rangle) + g(s, z^1(s))\right] \mathrm{d}s - z^1(s)\mathrm{d}W(s), \\ y^1(T) & = & \frac{1}{2}\langle Hx(T), x(T)\rangle, \ 0 \le s \le T. \end{array}\right. \tag{51}$$

Thus, Problem $(\mathrm{LQ})_g$ can also be formulated as follows. The state equation takes the following form which are (decoupled) FBSDEs:

$$\left\{\begin{array}{rcl} \mathrm{d}x(s) & = & [B(s)u(s) + b(s)]\,\mathrm{d}s + \sum_{i=1}^m D_i(s)u(s)\mathrm{d}W^j(s), \\ -\mathrm{d}y^1(s) & = & \left[\frac{1}{2}\langle Q(s)x(s), x(s)\rangle + g(z^1(s))\right]\mathrm{d}s - z^1(s)\mathrm{d}W(s), \\ x(0) & = & x_0, \ 0 \le s \le T, \ y^1(T) = \frac{1}{2}\langle Hx(T), x(T)\rangle. \end{array}\right. \tag{52}$$

This kind of FBSDEs have been first studied by Yong [52].

C Appendix: Technique Lemmas

We recall some results from convex analysis from [51].

Lemma C.1 *Let s be a continuous, strictly convex quadratic function*

$$s(z) \triangleq \frac{1}{2}\left\|(\mathcal{D}')^{-1}z + (\mathcal{D}')^{-1}\mathcal{B}'\right\|^2 \tag{53}$$

over $z \in [0,\infty)^m$, where $\mathcal{B}' \in \mathbb{R}^m_+$, $\mathcal{D} \in \mathbb{R}^{m\times m}$ and $\mathcal{D}'\mathcal{D} > 0$. Then s has a unique minimizer $\bar z \in [0,\infty)^m$, i.e.,

$$\left\|(\mathcal{D}')^{-1}\bar z + (\mathcal{D}')^{-1}\mathcal{B}'\right\|^2 \le \left\|(\mathcal{D}')^{-1}z + (\mathcal{D}')^{-1}\mathcal{B}'\right\|^2, \quad \forall z \in [0,\infty)^m.$$

The Kuhn-Tucker conditions for the minimization of s in (53) over $[0,\infty)^m$ lead to the Lagrange multiplier vector $\bar\nu \in [0,\infty)^m$ such that $\bar\nu = \nabla s(\bar z) = (\mathcal{D}'\mathcal{D})^{-1}\bar z + (\mathcal{D}'\mathcal{D})^{-1}\mathcal{B}'$ and $\bar\nu'\bar z = 0$.

Lemma C.2 *Let h be a continuous, strictly convex quadratic function*

$$h(z) \triangleq \frac{1}{2}z'\mathcal{D}'\mathcal{D}z - \alpha\mathcal{B}z$$

over $z \in [0,\infty)^m$, where $\mathcal{B}' \in \mathbb{R}^m_+$, $\mathcal{D} \in \mathbb{R}^{m\times m}$ and $\mathcal{D}'\mathcal{D} > 0$.

(i) For every $\alpha > 0$, h has the unique minimizer $\alpha\mathcal{D}^{-1}\bar\xi \in [0,\infty)^m$, where $\bar\xi = (\mathcal{D}'^{-1}\bar z + (\mathcal{D}'^{-1}\mathcal{B}'$. Here $\bar z$ is the minimizer of $s(z)$ specified in Lemma C.1. Furthermore, $\bar z'\mathcal{D}^{-1}\bar\xi = 0$ and

$$h(\alpha\bar\nu) = h(\alpha\mathcal{D}^{-1}\bar\xi) = -\frac{1}{2}\alpha^2\|\bar\xi\|^2.$$

(ii) For every $\alpha < 0$, h has the unique minimizer 0.

Lemma C.1 and Lemma C.2-(i) are proved in Section 5.2 and Lemma 3.2 of [40], while Lemma C.2-(ii) is obvious.

Remark C.1 *Note that the vector $\bar\xi$ is independent of the parameter α.*

References

[1] P. Briand, B. Delyon, Y. Hu, E. Pardoux and L. Stoica. L^p solutions of backward stochastic differential equations. *Stochastic Processes and Applications* 108:109–129, 2003.

[2] J.Y. Campbell, A.W. Lo and A.C. MacKinlay. *The Econometrics of Financial Markets.* Princeton University Press, New Jersey, 1997.

[3] S. Chen, X. Li and X.Y. Zhou. Stochastic linear quadratic regulators with indefinite control weight costs. *SIAM Journal on Control and Optimization*, 36:1685–1702, 1998.

[4] S. Chen and X. Zhou. Stochastic linear quadratic regulators with indefinite control weight costs. II. *SIAM Journal on Control and Optimization*, 39:1065–1081, 2000.

[5] J. Cox and C.F. Huang. Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory*, 49:33–83, 1989.

[6] M.G. Crandall and P.L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Transactions of The American Mathematical Society*, 277:1–42, 1983.

[7] J. Cvitanic and I. Karatzas. Convex duality in constrained portfolio optimization. *The Annals of Applied Probability*, 2:767–818, 1992.

[8] D. Duffie and H. Richardson. Mean-variance hedging in continuous time. *The Annals of Applied Probability*, 14:1–15, 1991.

[9] D. Duffie and L. G. Epstein, Stochastic differential utility. *Econometrica*. 60(1992) 353-394.

[10] B. Dumas and E. Luciano. An exact solution to a dynamic portfolio choice problem under transactions costs. *Journal of Finance*, 46:577–595, 1991.

[11] W.H. Fleming and H.M. Soner. *Controlled Markov Processes and Viscosity Solutions.* Springer-Verlag, New York, 1993.

[12] J.C. Francis. *Investments: Analysis and Management.* McGraw-Hill, New York, 1976.

[13] R.R. Grauer. A comparison of growth optimal and mean-variance investment policies. *Journal of Financial and Quantitative Analysis*, 16:1–21, 1981.

[14] R.R. Grauer. and N.H. Hakansson. On the use of mean-variance and quadratic approximations in implementing dynamic investment strategies: A comparison of the returns and investment policies. *Management Science*, 39:856–871, 1993.

[15] S.J. Grossman and Z. Zhou. Equilibrium analysis of portfolio insurance. *Journal of Finance*, 51:1379–1403, 1996.

[16] N.H. Hakansson. Capital growth and the mean-variance approach to portfolio selection. *Journal of Financial and Quantitative Analysis*, 6:517–557, 1971.

[17] Y. Hu, X. Zhou, Constrained stochastic LQ control with random coefficients, and application to portfolio selection. *SIAM J. Control Optim.* 44 (2005) 444-466.

[18] R. Jensen. The maximum principle for viscosity solutions of second order fully nonlinear partial differential equations. *Archive for Rational Mechanics and Analysis*, 101:1–27, 1988.

[19] I. Karatzas and S.G. Kou. Hedging American contingent claims with constrained portfolios. *Finance and Stochastics*, 2:215–258, 1998.

[20] N.E. Karoui, S. Peng and M.C. Quenez, Backward stochastic differential equations in finance. *Math. Financ.*, 7 (1997) 1-71.

[21] I. Karatzas, J.P. Lehoczky and S.E. Shreve. Optimal portfolio and consumption decisions for a small investor on a finite horizon. *SIAM Journal on Control and Optimization*, 25:1157–1186, 1987.

[22] I. Karatzas and S.E. Shreve. *Methods of Mathematical Finance.* Springer-Verlag, New York, 1999.

[23] M. Kohlmann and X. Y. Zhou. Relationship between backward stochastic differential equations and stochastic controls: A linear-quadratic approach. *SIAM Journal on Control and Optimization*, 38:1392–1407, 2000.

[24] J. Qiu, Viscosity solutions of stochastic Hamilton-Jacobi-Bellman equations. *SIAM J. Control Optim.* 56, no. 5, 3708–3730, 2018,.

[25] D. Li and W.L. Ng. Optimal dynamic portfolio selection: Multi-period mean-variance formulation. *Mathematical Finance*, 10:387–406, 2000.

[26] A.E.B. Lim and X.Y. Zhou. Optimal stochastic LQR control with integral quadratic constraints and indefinite control weights. *IEEE Transactions on Automatic Control*, 44:1359–1369, 1999.

[27] A.E.B. Lim and X.Y. Zhou. Mean-variance portfolio selection with random parameters. To appear in *Mathematics of Operations Research*, 2001.

[28] X. Li, X.Y. Zhou and A.E.B. Lim . Dynamic Mean-Variance Portfolio Selection with No-Shorting Constraints, *SIAM Journal on Control and Optimization,* Vol. 40, No. 5, 1540–1555

[29] D.G. Luenberger. *Optimization by Vector Space Methods.* Wiley, New York, 1968.

[30] H. Markowitz. Portfolio selection. *Journal of Finance*, 7:77–91, 1959.

[31] R.C. Merton. Lifetime portfolio selection under uncertainty: the continuous-time case. *The Review of Economics and Statistics*, 51:247–257, 1969.

[32] R.C. Merton. An analytical derivation of the efficient portfolio frontier. *Journal of Financial and Economics Analysis*, 7:1851–1872, 1972.

[33] J. Mossin. Optimal multi-period portfolio policies. *Journal of Business*, 41:215–229, 1968.

[34] M. Schroder and C. Skiadas, Optimal consumption and portfolio selection with stochastic differential utility, *Journal of Economic Theory*, vol. 89, no. 1, pp. 68–126, 1999.

[35] P.A. Samuelson. Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics*, 51: 236–246, 1969.

[36] W.F. Sharpe. Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance*, 19:425-442, 1964.

[37] H. Shirakawa. Optimal consumption and portfolio selection with incomplete markets and upper and lower bound constraints. *Mathematical Finance*, 4:1-24, 1994.

[38] J. Tobin. Liquidity preference as behavior towards risk. *Review of Economic Statistics*, 25:65–86, 1958.

[39] G.L. Xu and S.E. Shreve. A duality method for optimal consumption and investment under short-selling prohibition: I. general market coefficients. *The Annals of Applied Probability*, 2:87–112, 1992.

[40] G.L. Xu and S.E. Shreve. A duality method for optimal consumption and investment under short-selling prohibition: II. constant market coefficients. *The Annals of Applied Probability*, 2:314– 328, 1992.

[41] D. Yao, S. Zhang and X.Y. Zhou. Stochastic LQ control via semidefinite programming. To appear in *SIAM Journal on Control and Optimization*, 2001.

[42] J. Yong and X.Y. Zhou. *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Springer-Verlag, New York, 1999.

[43] Y.G. Zhao and W.T. Ziemba. Mean-variance versus expected utility in dynamic investment analysis. Working paper, Faculty of Commerce and Business Administration, University of British Columbia, 2000.

[44] X.Y. Zhou and D. Li. Continuous time mean-variance portfolio selection: A stochastic LQ framework, *Applied Mathematics and Optimization*, 42:19–33, 2000.

[45] X.Y. Zhou, J. Yong and X. Li. Stochastic verification theorems within the framework of viscosity solutions. *SIAM Journal on Control and Optimization*, 35:243–253, 1997.

[46] Z. Chen and L Epstein, Ambiguity, risk, and asset returns in continuous time. *Econometrica*, 70 (2002), 1403-1443.

[47] Z. Chen, T. Chen and M. Davison, Choquet expectation and Peng's g-expectation, *The Annals of Probability*, Vol. 33, No. 3, (2005), 1179-1199.

[48] F. Gozzi, A. Swiech, X. Zhou, Erratum: a corrected proof of the stochastic verification theorem within the framework of viscosity solutions", *SIAMJ.Control Optim.* 48 (6) (2010) 4177–4179.

[49] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation. *Syst. Control Lett.*, 14 (1990), 55-61.

[50] S. Peng, Nonlinear expectations, nonlinear evaluations and risk measures. Stochastic Methods in Finance, Lecture Notes in Math., Vol. 1856. Springer, Berlin, 165-253, (2004) n

[51] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, New Jersey, 1970.

[52] J. Yong, A Stochastic Linear Quadratic Optimal Control Problem with Generalized Expectation, *Stochastic Analysis and Applications*, 26: (2008), 1136-1160.

[53] L. Zhang, Stochastic verification theorem of forward-backward controlled systems for viscosity solutions. *Syst. & Control Lett.*, 61(5), 649-654 (2012).

[54] L. Zhang, Optimal Control of Foward-backward Stochastic Differential Equations, *Science Press*, (2019).