

## STOCHASTIC LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEMS WITH RANDOM COEFFICIENTS AND MARKOVIAN REGIME SWITCHING SYSTEM\*

JIAQIANG WEN<sup>†</sup>, XUN LI<sup>‡</sup>, JIE XIONG<sup>†</sup>, AND XIN ZHANG<sup>§</sup>

**Abstract.** This paper thoroughly investigates stochastic linear-quadratic optimal control problems with the Markovian regime switching system, where the coefficients of the state equation and the weighting matrices of the cost functional are random. We prove the solvability of the stochastic Riccati equation under the uniform convexity condition and obtain the closed-loop representation of the open-loop optimal control using the unique solvability of the corresponding stochastic Riccati equation. Moreover, by applying Itô's formula with jumps, we get a representation of the cost functional on a Hilbert space, characterized as the adapted solutions of some forward-backward stochastic differential equations. We show that the necessary condition of the open-loop optimal control is the convexity of the cost functional, and the sufficient condition of the open-loop optimal control is the uniform convexity of the cost functional. In addition, we study the properties of the stochastic value flow of the stochastic linear-quadratic optimal control problem. Finally, as an application, we present a continuous-time mean-variance portfolio selection problem and prove its unique solvability.

**Key words.** stochastic linear-quadratic optimal control, Markovian regime switching, random coefficient, stochastic Riccati equation, mean-variance portfolio selection

**MSC codes.** 49N10, 93E20

**DOI.** 10.1137/22M1481415

**1. Introduction.** The stochastic linear-quadratic (SLQ) optimal control problem plays an important role in control theory, which has developed rapidly in recent decades due to its wide range of applications. In the classical setting, the SLQ optimal control problem can be solved elegantly via the Riccati equation under some mild conditions on the weighting coefficients (see Yong and Zhou [23, Chapter 6]). Chen, Li, and Zhou [3] investigated SLQ optimal control problems with indefinite weighting control matrix and their applications in solving continuous-time mean-variance portfolio selection problems in the 1990s. From then on, there has been increasing interest in the so-called indefinite SLQ optimal control problems as well as in addressing their applications (see Ait Rami, Moore, and Zhou [1], Li, Zhou, and Lim [13], and Zhang and Yin [25]).

---

\* Received by the editors February 28, 2022; accepted for publication (in revised form) September 15, 2022; published electronically April 28, 2023.

<https://doi.org/10.1137/22M1481415>

**Funding:** The first author is supported by National Key R&D Program of China grant 2022YFA1006102, National Natural Science Foundation of China grant 12101291, and Guangdong Basic and Applied Basic Research Foundation grant 2022A1515012017. The second author is supported by RGC of Hong Kong grants 15215319, 15216720, and 15221621, and partially from PolyU 4-ZZHX. The third author is supported by National Key R&D Program of China 2022YFA1006102, and by the National Natural Science Foundation of China grant 11831010. The fourth author is supported by National Natural Science Foundation of China grant 12171086 and by Fundamental Research Funds for the Central Universities grant 2242021R41082.

<sup>†</sup> Department of Mathematics and SUSTech International Center for Mathematics, Southern University of Science and Technology, Shenzhen, Guangdong, 518055, China (wenjq@sustech.edu.cn, xiongj@sustech.edu.cn).

<sup>‡</sup> Department of Applied Mathematics, Hong Kong Polytechnic University, Hong Kong, China (li.xun@polyu.edu.hk).

<sup>§</sup> Corresponding author. School of Mathematics, Southeast University, Nanjing, Jiangsu Province, 211189, China (x.zhang.seu@gmail.com).

To tackle SLQ optimal control problems with random coefficients, Bismut [2] initially derived the existence and uniqueness of the solution of the stochastic Riccati equation (SRE) using techniques of functional analysis. Kohlmann and Tang [11] studied the multidimensional backward SREs and gave an application to the stochastic optimal control problem. Tang [19] studied general SLQ optimal control problems with random coefficients and proved the existence and uniqueness of related backward SREs. Sun, Xiong, and Yong [17] studied the SLQ optimal control problems with random coefficients, proved the solvability of the corresponding SRE, and obtained the closed-loop representation of the open-loop optimal control. Li, Wu, and Yu [12] analyzed a special type of indefinite SLQ problem with random coefficients and studied related SREs. For more details about the efforts devoted to the SRE and its connection with SLQ optimal control problems, we refer interested readers to Kohlmann and Tang [10], Tang [20], etc.

Recently, there has been dramatically increasing interest in the SLQ optimal control problems with random jumps, such as Poisson jumps or the regime switching jumps, which are of practical importance in various fields, such as economics, financial management, science, and engineering. In the past few years, researchers have focused on models of financial markets whose key parameters are described by Markov processes, such as stock returns, interest rates, and volatility. In particular, one could face two market regimes in financial markets, one of which stands for a bull market with price rises, while the other for a bear market with price drops. We call such a formulation the regime switching model, where the market parameters depend on market modes that switch among a finite number of regimes. More recently, applications of SLQ optimal control problems with regime switching models or Poisson jumps have been extensively developed. For instance, Ji and Chizeck [9] formulated a class of continuous-time linear-quadratic optimal control problems with Markovian jumps. Zhou and Yin [28] studied a mean-variance portfolio selection with regime switching. Liu, Yin, and Zhou [14] studied the near-optimal controls of regime switching linear-quadratic control problems. Hu and Oksendal [7] discussed SLQ optimal control problems with Poisson jumps and partial information using the technique of completing squares. Zhang, Elliott, and Siu [26] studied a stochastic maximum principle for a Markov regime switching model and gave applications to finance. Yu [24] studied a kind of backward SLQ optimal control problem with the infinite horizon jump-diffusion. Song, Tang, and Wu [16] established the maximum principle for progressive stochastic optimal control problems with random jumps. Zhang, Li, and Xiong [27] investigated open-loop and closed-loop solvabilities for SLQ optimal control problems with a Markovian regime switching system. Hu, Shi, and Xu [8] applied the constrained SLQ control with regime switching to a portfolio problem. For some other important works, we refer the readers to [4, 6, 15, 21], and the references therein.

In a real market, besides the Markov chain, it is more reasonable to allow the market parameters to depend on the Brownian motion, due to the fact that the interest rates, stock rates, and volatilities are affected by the uncertainties caused by the Brownian motion. However, up to now, few results have been obtained on this topic. In this paper, inspired by the continuous-time mean-variance portfolio selection problems, we are interested in studying this topic, i.e., the SLQ optimal control problems with the Markovian regime switching system and random coefficients. Further, by developing some ideas of Sun, Xiong, and Yong [17] and Li, Zhou, and Lim [13], we establish the existence of an open-loop optimal control, prove the unique solvability of the associated SRE, and apply the theoretical results shown in this paper to treat

a continuous-time mean-variance portfolio selection problem. Next, we present our main results and difficulties in detail.

- (i) We first introduce the associated SRE of Problem (M-SLQ) (see (2)) and prove that the optimal state process of Problem (M-SLQ) is invertible under the uniform convexity condition (7). Then, we show that a bounded process  $\hat{P}(\cdot, \alpha(\cdot))$ , together with two square-integrable processes  $\hat{\Lambda}(\cdot)$  and  $\hat{\zeta}(\cdot)$ , uniquely satisfies the associated SRE (5), thus establishing the unique solvability of the associated SRE (see Theorem 3.1). Moreover, we derive a closed-loop representation for the open-loop optimal control of Problem (M-SLQ) using the unique solvability of the SRE (see Theorem 3.2).
- (ii) In order to prove the above theoretical results, we prove some auxiliary results. We establish the equivalence between Problem (M-SLQ) and Problem (M-SLQ)<sub>0</sub>, i.e., a control  $u^*(\cdot) \in \mathcal{U}[t, T]$  is optimal for Problem (M-SLQ)<sub>0</sub> if and only if it is optimal for Problem (M-SLQ) (see Proposition 3.6), and based on the equivalence, we analyze the time-consistency of the optimal control (see Corollary 3.8). Then, we obtain a quadratic representation of the stochastic value flow in terms of a bounded, left continuous, and  $\mathbb{S}^n$ -valued process (see Theorems 3.11 and 3.12). In addition, by the technique of Itô's formula with jumps, we represent the cost functional of Problem (M-SLQ) as a bilinear form, in terms of the adapted solutions of some forward-backward stochastic differential equations (FBSDEs) in a suitable Hilbert space (see Proposition 5.4).
- (iii) As a financial application, we present an example of the continuous-time mean-variance portfolio selection problem and prove the unique solvability of the related mean-variance problem. Also, we derive the representation of the unique optimal investment strategy (see Theorem 6.1), which further develops the work of Li, Zhou, and Lim [13] to the Markovian regime switching system with random coefficients.

Compared with Sun, Xiong, and Yong [17], the difficulties of this paper mainly come from the solvability of the associated SRE, due to the presence of the Markovian regime switching jumps.

- (iv) First, due to the presence of random coefficients, the Riccati equation associated with Problem (M-SLQ) becomes a nonlinear BSDE, usually referred to as the backward SRE. Furthermore, the backward SRE is driven by both the Brownian motion and the martingales  $(\tilde{N}_{kl}(\cdot))_{k,l \in \mathcal{S}}$  generated by the Markov chain thanks to the occurrence of Markovian regime switching jumps in the model. Thus, the solvability of this BSDE is more complicated than that of the model with deterministic coefficients and without the Markovian regime switching jumps.
- (v) Second, let  $(X_j(s), Y_j(s), Z_j(s), \Gamma^j(s))$  be the solution of the FBSDE composed by the open-loop optimal state process and adjoint equation corresponding to initial state  $(t, e_j, \vartheta)$ . It is worth mentioning that the construction of the fourth component  $\mathbf{\Gamma}(s)$  for the solution of a matrix-valued FBSDE (14) does not directly extend the dimensional by combining  $\Gamma^j(s)$  as  $(\Gamma^1(s), \dots, \Gamma^n(s))$ . Furthermore, the  $\mathbf{\Gamma}$ -term of the solution to (10) associated with the initial state  $(t, \xi, \vartheta)$  is represented as  $\mathbf{\Gamma}(s) \circ \xi$ , which is different from that of  $(Y(s), Z(s)) = (\mathbf{Y}(s)\xi, \mathbf{Z}(s)\xi)$ . See more details in Proposition 3.10.
- (vi) Third, the existence of process  $P(t)$  that appeared in (42) of Sun, Xiong, and Yong [17] for the representation of the value function follows directly

by its definition  $\mathbb{E}[\mathbf{M}(T) + \int_t^T \mathbf{N}(s)ds | \mathcal{F}_t]$ . In our case, we have a similar representation for the value function but with  $\mathbf{M}(T)$  and  $\mathbf{N}(s)$  replaced by functions that depend on the Markov chain  $\alpha(\cdot)$  and  $\mathcal{F}_t$  replaced by the  $\sigma$ -field generated by the Brownian motion and the Markov chain  $\alpha(\cdot)$ . To solve our problem, we need to prove that there exists a process  $P: [0, T] \times \mathcal{S} \times \Omega \rightarrow \mathbb{S}^n$  such that

$$P(t, \alpha(t)) = \mathbb{E} \left[ \mathbf{M}(T, \alpha(T)) + \int_t^T \mathbf{N}(s, \alpha(s)) ds \middle| \mathcal{F}_t \right].$$

This cannot be obtained directly from the definition of conditional expectation and we give proof for this (see the proof of Theorem 3.11). In our model, although the evolution of  $\mathbf{X}(s)$  and  $\mathbf{Y}(s)$  depends on the Markov chain  $\alpha(\cdot)$ ,  $\mathbf{Y}(s)\mathbf{X}(s)^{-1}$  may not equal  $P(s, \alpha(s))$ . We further prove the equality of  $P(s, \alpha(s)) = \mathbf{Y}(s)\mathbf{X}(s)^{-1}$  (see the proof of Theorem 3.1), which plays an important role in solving our problem.

The paper is organized as follows. In section 2, we present some preliminaries and formulate Problem (M-SLQ) with random coefficients and regime switching. In section 3, we state our main results, i.e., the solvability of the corresponding SRE, the closed-loop representation of the open-loop optimal control, and some auxiliary results. In section 4, we prove the invertibility of the optimal state process, the solvability of the corresponding SRE, and the closed-loop representation of the open-loop optimal control. In section 5, we strictly prove some auxiliary results in detail. In section 6, we present an example of the continuous-time mean-variance portfolio selection problem under the Markovian regime switching system with random coefficients. In section 7, we conclude the results.

**2. Preliminaries.** Let  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  be a complete filtered probability space on which a standard one-dimensional Brownian motion  $\{W(t)\}_{t \geq 0}$  and a continuous-time and finite-state Markov chain  $\{\alpha(t)\}_{t \geq 0}$  are defined, where the processes  $W(\cdot)$  and  $\alpha(\cdot)$  are independent and  $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$  is the natural filtration of them with  $\mathcal{F}_0$  containing all  $\mathbb{P}$ -null sets of  $\mathcal{F}$ . Let  $\mathbb{F}^W = \{\mathcal{F}_t^W\}_{t \geq 0}$  be the filtration generated by  $W(\cdot)$  and  $\mathbb{F}^\alpha = \{\mathcal{F}_t^\alpha\}_{t \geq 0}$  be the filtration generated by  $\alpha(\cdot)$ . We identify the state space of the Markov chain  $\alpha(\cdot)$  with a finite set  $\mathcal{S} \triangleq \{1, 2, \dots, D\}$ , where  $D \in \mathbb{N}$ . Furthermore, the generator of the Markov chain  $\alpha(\cdot)$  under  $\mathbb{P}$  is denoted by  $\lambda(t) \triangleq [\lambda_{kl}(t)]_{k, l \in \mathcal{S}}$ , where  $\lambda_{kl}(t)$  is the constant transition intensity of the Markov chain from state  $k$  to state  $l$  at time  $t$ . For each fixed  $k, l \in \mathcal{S}$ , we let  $N_{kl}(t)$  be the number of jumps from state  $k$  into state  $l$  up to time  $t$  and set  $\tilde{\lambda}_{kl}(t) \triangleq \int_0^t \lambda_{kl}(s) I_{\{\alpha(s-) = k\}} ds$ . Let  $N(t) \triangleq (N_{kl}(t))_{k, l \in \mathcal{S}}$  and  $\tilde{N}(t) \triangleq (\tilde{N}_{kl}(t))_{k, l \in \mathcal{S}}$ , where  $\tilde{N}_{kk}(t) \equiv 0$  and  $\tilde{N}_{kl}(t) = N_{kl}(t) - \tilde{\lambda}_{kl}(t)$  when  $k \neq l$ .

Let  $T > 0$  be a fixed terminal time. The trace of a square matrix  $M$  is denoted by  $\text{tr}[M]$ , the set of all  $n \times n$  symmetric matrices is denoted by  $\mathbb{S}^n$ , the set of all  $D \times D$  matrices  $M \triangleq (M_{kl})$  with  $M_{kl} \in \mathbb{S}^n$  is denoted by  $\mathbb{M}_D(\mathbb{S}^n)$ , and the set of all  $D \times D$  matrices  $M \triangleq (M_{kl})$  with  $M_{kl} \in \mathbb{R}^{n \times m}$  is denoted by  $\mathbb{M}_D(\mathbb{R}^{n \times m})$ . For  $\mathbb{H} = \mathbb{R}^n, \mathbb{R}^{n \times m}$ , or  $\mathbb{S}^n$ , denote by  $L^2_{\mathcal{F}}(\Omega; \mathbb{H})$  (resp.,  $L^\infty_{\mathcal{F}}(\Omega; \mathbb{H})$ ) the set of all  $\mathcal{F}$ -measurable,  $\mathbb{H}$ -valued, and square integrable (resp., bounded) random variables. Denote by  $L^2_{\mathbb{F}}(t, T; \mathbb{H})$  (resp.,  $L^\infty_{\mathbb{F}}(t, T; \mathbb{H})$ ) the set of all  $\mathbb{H}$ -valued,  $\mathbb{F}$ -progressive measurable stochastic processes  $\phi(s)$  with  $\mathbb{E} \int_t^T |\phi(s)|^2 ds < \infty$  (resp.,  $\text{esssup}_{s \in [t, T]} |\phi(s)| < \infty$ ), and denote by  $L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{H}))$  the set of all  $\mathbb{H}$ -valued,  $\mathbb{F}$ -adapted, and continuous processes  $\phi(s)$  with  $\mathbb{E}[\sup_{s \in [t, T]} |\phi(s)|^2] < \infty$ . Moreover, for  $\mathbb{G} = \mathbb{M}_D(\mathbb{R}^{n \times m})$  or  $\mathbb{M}_D(\mathbb{S}^n)$ , we denote by  $L^2_{\mathbb{F}}(t, T; \mathbb{G})$  the set of all  $\mathbb{G}$ -valued  $\mathbb{F}$ -progressively measurable

processes  $\phi(s) \triangleq (\phi_{kl}(s)) \in \mathbb{G}$  with  $\mathbb{E} \int_t^T \sum_{k,l=1}^D |\phi_{kl}(s)|^2 \lambda_{kl}(s) I_{\{\alpha(s)=k\}} ds < \infty$ . For  $\eta(s) \triangleq (\eta_{kl}(s)) \in \mathcal{M}_D(\mathbb{R}^{n \times n})$ , we further define

$$\eta(s) \bullet d\tilde{N}(s) \triangleq \sum_{k,l=1}^D \eta_{kl}(s) d\tilde{N}_{kl}(s).$$

We now introduce the following state equation, which is the controlled Markovian regime switching linear stochastic differential equation (SDE) over a finite time horizon  $[t, T]$ :

$$(1) \quad \begin{cases} dX(s) = [A(s, \alpha(s))X(s) + B(s, \alpha(s))u(s)] ds \\ \quad + [C(s, \alpha(s))X(s) + D(s, \alpha(s))u(s)] dW(s), \quad s \in [t, T], \\ X(t) = \xi, \quad \alpha(t) = \vartheta, \end{cases}$$

where  $A(t, \omega, i)$ ,  $B(t, \omega, i)$ ,  $C(t, \omega, i)$ , and  $D(t, \omega, i)$  are given  $\mathcal{F}_t^W$ -measurable processes for each  $i \in \mathcal{S}$ . We call  $(t, \xi, \vartheta)$  an *initial triple*, which comes from the following set:

$$\mathcal{D} = \{(t, \xi, \vartheta) \mid t \in [0, T], \xi \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n), \vartheta \in L^2_{\mathcal{F}_t^\alpha}(\Omega; \mathcal{S})\}.$$

In the state equation (1), the process  $u(\cdot)$  comes from the control space  $\mathcal{U}[t, T] \triangleq L^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$ , called the *control process*, and the solution  $X(\cdot)$  of (1) is called the *state process* with  $(t, \xi, \vartheta)$  and  $u(\cdot)$ . Let us state our SLQ optimal control problem.

**Problem (M-SLQ).** For any given initial triple  $(t, \xi, \vartheta) \in \mathcal{D}$ , find a control  $u^*(\cdot) \in \mathcal{U}[t, T]$  such that

$$(2) \quad J(t, \xi, \vartheta; u^*(\cdot)) = \operatorname{ess\,inf}_{u(\cdot) \in \mathcal{U}[t, T]} J(t, \xi, \vartheta; u(\cdot)) \triangleq V(t, \xi, \vartheta),$$

where the cost functional is given as the following quadratic form:

$$(3) \quad \begin{aligned} J(t, \xi, \vartheta; u(\cdot)) \triangleq & \mathbb{E}_t \left[ \langle G(\alpha(T))X(T), X(T) \rangle \right. \\ & \left. + \int_t^T \left\langle \begin{pmatrix} Q(s, \alpha(s)) & S(s, \alpha(s))^\top \\ S(s, \alpha(s)) & R(s, \alpha(s)) \end{pmatrix} \begin{pmatrix} X(s) \\ u(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \right\rangle ds \right]. \end{aligned}$$

Note that  $\mathbb{E}_t[\cdot] \triangleq \mathbb{E}[\cdot | \mathcal{F}_t]$  represents the conditional expectation with respect to (w.r.t.)  $\mathcal{F}_t$ . For the initial triple  $(t, \xi, \vartheta)$ , we call the control process  $u^*(\cdot)$  an *open-loop optimal control* of Problem (M-SLQ) if it satisfies (2), call the corresponding state process  $X^*(\cdot) \equiv X(\cdot; t, \xi, \vartheta, u^*(\cdot))$  an *open-loop optimal state process*, and call the state-control pair  $(X^*(\cdot), u^*(\cdot))$  an *open-loop optimal pair*. We call  $(t, \xi, \vartheta) \mapsto V(t, \xi, \vartheta)$  the *stochastic value flow* of Problem (M-SLQ), because the space  $L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$  becomes larger when  $t$  increases.

*Remark 2.1.* In the cost functional (3), when the conditional expectation  $\mathbb{E}_t[\cdot]$  degenerates to the expectation  $\mathbb{E}$ , we denote the related problem, cost functional, and value function by Problem (M-SLQ)<sub>0</sub>,  $J_0(t, \xi, \vartheta; u(\cdot))$ , and  $V_0(t, \xi, \vartheta)$ , respectively. Clearly,  $V_0(t, \xi, \vartheta) = \mathbb{E}[V(t, \xi, \vartheta)]$ .

**DEFINITION 2.2.** Problem (M-SLQ) is said to be (*uniquely*) *open-loop solvable for the initial triple*  $(t, \xi, \vartheta) \in \mathcal{D}$  if there exists a (*unique*)  $u^*(\cdot) = u^*(\cdot; t, \xi, \vartheta) \in \mathcal{U}[t, T]$  (depending on  $(t, \xi, \vartheta)$ ) such that  $J(t, \xi, \vartheta; u^*(\cdot)) \leq J(t, \xi, \vartheta; u(\cdot))$ , a.s.,  $\forall u(\cdot) \in \mathcal{U}[t, T]$ , and is said to be (*uniquely*) *open-loop solvable* if it is (*uniquely*) *open-loop solvable for any*  $(t, \xi, \vartheta) \in \mathcal{D}$ .

(H) For any choice of  $i \in \mathcal{S}$ ,  $A(t, \omega, i), C(t, \omega, i) \in L_{\mathbb{F}^W}^\infty(0, T; \mathbb{R}^{n \times n})$ ,  $B(t, \omega, i), D(t, \omega, i) \in L_{\mathbb{F}^W}^\infty(0, T; \mathbb{R}^{n \times m})$ ,  $G(\omega, i) \in L_{\mathcal{F}_T^W}^\infty(\Omega; \mathbb{S}^n)$ ,  $Q(t, \omega, i) \in L_{\mathbb{F}^W}^\infty(0, T; \mathbb{S}^n)$ ,  $S(t, \omega, i) \in L_{\mathbb{F}^W}^\infty(0, T; \mathbb{R}^{m \times n})$ , and  $R(t, \omega, i) \in L_{\mathbb{F}^W}^\infty(0, T; \mathbb{S}^m)$ .

Under the condition (H), for any initial triple  $(t, \xi, \vartheta) \in \mathcal{D}$  and any control  $u(\cdot) \in \mathcal{U}[t, T]$ , the classical theory of SDEs (see Sun and Yong [18, Proposition 2.1]) implies that the state equation (1) has a unique solution  $X(\cdot) \equiv X(\cdot; t, \xi, \vartheta, u(\cdot))$ , which is square-integrable and whose path is continuous. Moreover, to simplify our further analysis, we introduce the following BSDE:

$$(4) \quad \begin{cases} dM(s) = - [M(s)A(s, \alpha(s)) + A(s, \alpha(s))^\top M(s) \\ \quad + C(s, \alpha(s))^\top M(s)C(s, \alpha(s)) + \Phi(s)C(s, \alpha(s)) + C(s, \alpha(s))^\top \Phi(s) \\ \quad + Q(s, \alpha(s))] ds + \Phi(s)dW(s) + \eta(s) \bullet d\tilde{N}(s), \quad s \in [t, T], \\ M(T) = G(\alpha(T)), \quad \alpha(t) = \vartheta. \end{cases}$$

The solution of the above BSDE is denoted by the triple  $(M(\cdot), \Phi(\cdot), \eta(\cdot))$ , where  $\eta(\cdot) \triangleq (\eta_{kl}(\cdot))_{k,l \in \mathcal{S}} \in \mathcal{M}_D(\mathbb{R}^{n \times n})$ . Note that the terminal value  $G(\cdot)$  is bounded, so the theory of BSDEs combining Itô's formula with jumps deduces the following result, i.e.,  $M(\cdot)$  is bounded too.

**PROPOSITION 2.3.** *Under (H), the process  $M(\cdot)$  is bounded, where  $(M(\cdot), \Phi(\cdot), \eta(\cdot))$  is the adapted solution of BSDE (4).*

**3. The main results.** In this section, we state our main results, such as the solvability of the corresponding SRE, the closed-loop representation of the open-loop optimal control, and some auxiliary results.

**3.1. Solvability of stochastic Riccati equation.** As shown in Sun, Xiong, and Yong [17], the solvability of the corresponding SRE deduces the closed-loop representation of the open-loop optimal control, which is important to deal with the SLQ problem. Therefore, the SRE plays a crucial role in studying our Problem (M-SLQ), and our core goal is to establish the solvability of the corresponding SRE. For this, we introduce the following SRE: for  $s \in [0, T]$ ,

$$(5) \quad \begin{cases} d\hat{P}(s, \alpha(s)) = - [\hat{Q}(s, \alpha(s)) + \hat{S}(s, \alpha(s))^\top \Theta(s, \alpha(s))] ds + \hat{\Lambda}(s)dW(s) + \hat{\zeta}(s) \bullet d\tilde{N}(s), \\ \hat{P}(T, \alpha(T)) = G(\alpha(T)), \quad \alpha(0) = i_0, \end{cases}$$

where  $i_0 \in \mathcal{S}$  is the initial state of  $\alpha(\cdot)$ , and for any  $(s, \omega, i) \in [0, T] \times \Omega \times \mathcal{S}$ ,

$$(6) \quad \begin{aligned} \hat{Q}(s, i) &\triangleq \hat{P}(s, i)A(s, i) + A(s, i)^\top \hat{P}(s, i) + C(s, i)^\top \hat{P}(s, i)C(s, i) \\ &\quad + \hat{\Lambda}(s)C(s, i) + C(s, i)^\top \hat{\Lambda}(s) + Q(s, i), \\ \hat{S}(s, i) &\triangleq B(s, i)^\top \hat{P}(s, i) + D(s, i)^\top \hat{P}(s, i)C(s, i) + D(s, i)^\top \hat{\Lambda}(s) + S(s, i), \\ \hat{R}(s, i) &\triangleq R(s, i) + D(s, i)^\top \hat{P}(s, i)D(s, i), \quad \Theta(s, i) \triangleq -\hat{R}(s, i)^{-1}\hat{S}(s, i). \end{aligned}$$

Suppose that the cost functional  $J(0, 0, i_0; u(\cdot))$  is uniformly convex in  $u(\cdot)$ , i.e., for any choice of  $i_0 \in \mathcal{S}$ , there is a positive constant  $\varepsilon$  such that

$$(7) \quad J(0, 0, i_0; u(\cdot)) \geq \varepsilon \mathbb{E} \int_0^T |u(s)|^2 ds \quad \forall u(\cdot) \in \mathcal{U}[0, T].$$

THEOREM 3.1. Under conditions (H) and (7), SRE (5) admits a unique adapted solution  $(\hat{P}(\cdot, \alpha(\cdot)), \hat{\Lambda}(\cdot), \hat{\zeta}(\cdot)) \in L_{\mathbb{F}}^{\infty}(0, T; \mathbb{S}^n) \times L_{\mathbb{F}}^2(0, T; \mathbb{S}^n) \times L_{\mathbb{F}}^2(0, T; \mathbb{M}_D(\mathbb{S}^n))$ . In addition,

(8)

$$\hat{R}(s, \alpha(s)) = R(s, \alpha(s)) + D(s, \alpha(s))^{\top} \hat{P}(s, \alpha(s)) D(s, \alpha(s)) \geq \varepsilon I_m \quad \text{a.e. on } [0, T], \quad \text{a.s.}$$

The above theorem shows the solvability of SRE (5), based on which the following closed-loop representation of the open-loop optimal control can be derived.

THEOREM 3.2. Under conditions (H) and (7), Problem (M-SLQ) is uniquely open-loop solvable and the unique open-loop optimal control  $\{u^*(s)\}_{s \in [t, T]}$  w.r.t. the initial triple  $(t, \xi, \vartheta) \in \mathcal{D}$  has the following linear state feedback:

$$(9) \quad u^*(s) = \Theta(s, \alpha(s)) X^*(s), \quad s \in [t, T],$$

where  $\Theta(\cdot)$  is defined in (6) and  $\{X^*(s)\}_{s \in [t, T]}$  is the solution of the following closed-loop system:

$$\begin{cases} dX^*(s) = [A(s, \alpha(s)) + B(s, \alpha(s))\Theta(s, \alpha(s))] X^*(s) ds \\ \quad + [C(s, \alpha(s)) + D(s, \alpha(s))\Theta(s, \alpha(s))] X^*(s) dW(s), \quad s \in [t, T], \\ X^*(t) = \xi, \quad \alpha(t) = \vartheta. \end{cases}$$

Remark 3.3. Sun, Xiong, and Yong [17] studied SLQ optimal control problems with random coefficients, proved the solvability of the corresponding SRE, and obtained the closed-loop representation of the open-loop optimal control. Compared with [17], on one hand, the above theorems further extend the results in [17] to the framework within the Markovian regime switching, which is important in continuous-time mean-variance portfolio selection problems (see section 6). On the other hand, due to the presence of the Markovian regime switching jump in SRE (5), it is difficult to directly prove the solvability of SRE (5). To overcome it, we derive some auxiliary results first.

**3.2. Some auxiliary results.** To simplify the notation, from now on, for any  $(s, i) \in [0, T] \times \mathcal{S}$ ,  $x, y, z \in \mathbb{R}^n$ , and  $u \in \mathbb{R}^m$ , let

$$\begin{aligned} F(s, i, x, y, z, u) &\triangleq B(s, i)^{\top} y + D(s, i)^{\top} z + S(s, i)x + R(s, i)u, \\ \tilde{F}(s, i, x, y, z, u) &\triangleq A(s, i)^{\top} y + C(s, i)^{\top} z + Q(s, i)x + S(s, i)^{\top} u, \\ F_0(s, i, x, y, z) &\triangleq F(s, i, x, y, z, 0), \quad \tilde{F}_0(s, i, x, y, z) \triangleq \tilde{F}(s, i, x, y, z, 0). \end{aligned}$$

The following BSDE is called the associated adjoint equation of the state equation (1):

$$(10) \quad \begin{cases} dY(s) = -\tilde{F}(s, \alpha(s), X(s), Y(s), Z(s), u(s)) ds + Z(s) dW(s) \\ \quad + \Gamma(s) \bullet d\tilde{N}(s), \quad s \in [t, T], \\ Y(T) = G(\alpha(T))X(T), \quad \alpha(t) = \vartheta, \end{cases}$$

where  $(X(\cdot), u(\cdot))$  comes from (1). In order to prove the solvability of SRE (5), we present an alternative characterization of Problem (M-SLQ)<sub>0</sub> in terms of the state equation (1) and the adjoint equation (10), and then show that Problems (M-SLQ) and (M-SLQ)<sub>0</sub> are equivalent.

**THEOREM 3.4.** *Let (H) hold and let the initial triple  $(t, \xi, \vartheta) \in \mathcal{D}$  be given. A process  $u^*(\cdot) \in \mathcal{U}[t, T]$  is an open-loop optimal control of Problem (M-SLQ)<sub>0</sub> w.r.t.  $(t, \xi, \vartheta)$  if and only if the following two conditions hold:*

- (i) *The mapping  $u(\cdot) \mapsto J_0(t, 0, \vartheta; u(\cdot))$  is convex, i.e.,  $J_0(t, 0, \vartheta; u(\cdot)) \geq 0 \forall u(\cdot) \in \mathcal{U}[t, T]$ .*
- (ii) *The following stationarity condition holds:*

$$(11) \quad F(s, \alpha(s), X^*(s), Y^*(s), Z^*(s), u^*(s)) = 0, \quad \text{a.e. } s \in [t, T] \quad \text{a.s.},$$

where the quadruple  $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), \Gamma^*(\cdot))$  is the solution of SDE (1) and BSDE (10) with  $u(\cdot)$  replaced by  $u^*(\cdot)$ .

**Remark 3.5.** In the assertion (ii) of Theorem 3.4, the quadruple  $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), \Gamma^*(\cdot))$  is essentially the solution of the following decoupled FBSDE:

$$(12) \quad \begin{cases} dX^*(s) = [A(s, \alpha(s))X^*(s) + B(s, \alpha(s))u^*(s)]ds \\ \quad + [C(s, \alpha(s))X^*(s) + D(s, \alpha(s))u^*(s)]dW(s), \\ dY^*(s) = -\tilde{F}(s, \alpha(s), X^*(s), Y^*(s), Z^*(s), u^*(s))ds \\ \quad + Z(s)dW(s) + \Gamma^*(s) \bullet d\tilde{N}(s), \quad s \in [t, T], \\ X^*(t) = \xi, \quad \alpha(t) = \vartheta, \quad Y^*(T) = G(\alpha(T))X^*(T). \end{cases}$$

**PROPOSITION 3.6.** *Under (H), for any given initial triple  $(t, \xi, \vartheta) \in \mathcal{D}$ , a control  $u^*(\cdot) \in \mathcal{U}[t, T]$  is optimal for Problem (M-SLQ)<sub>0</sub> if and only if it is optimal for Problem (M-SLQ).*

Based on Theorem 3.4 and Proposition 3.6, we have the following results, which are useful to prove Theorem 3.1 later. For this, denote by  $\tau$  an  $\mathbb{F}$ -stopping time with values in  $[0, T]$  and denote by  $\mathcal{T}[a, b]$  the set of all  $\mathbb{F}$ -stopping times valued in the interval  $[a, b]$  with  $a, b \in [0, T]$ .

**COROLLARY 3.7.** *Let (H) hold and suppose that  $u^*(\cdot) \in \mathcal{U}[t, T]$  is an open-loop optimal control w.r.t. the initial triple  $(t, \xi, \vartheta) \in \mathcal{D}$ ; then  $V(t, \xi, \vartheta) = J(t, \xi, \vartheta; u^*(\cdot)) = \langle Y^*(t), \xi \rangle$ , where the quadruple  $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), \Gamma^*(\cdot))$  is the adapted solution of FBSDE (12) w.r.t.  $u^*(\cdot)$ .*

**COROLLARY 3.8.** *Let (H) hold and suppose  $u^*(\cdot) \in \mathcal{U}[t, T]$  is an open-loop optimal control w.r.t. the initial triple  $(t, \xi, \vartheta) \in \mathcal{D}$ ; then for any stopping time  $\tau \in \mathcal{T}[t, T]$ , the restriction  $u^*(\cdot)|_{[\tau, T]} \triangleq \{u^*(s); s \in [\tau, T]\}$  remains optimal w.r.t.  $(\tau, X^*(\tau), \alpha(\tau))$ , where  $X^*(\cdot)$  is the solution of the forward equation of (12).*

**PROPOSITION 3.9.** *Let (H) and (7) hold; then for any  $\tau \in \mathcal{T}[0, T]$ ,*

$$J_0(\tau, 0, \alpha(\tau); u(\cdot)) \geq \varepsilon \mathbb{E} \int_{\tau}^T |u(s)|^2 ds \quad \forall u(\cdot) \in \mathcal{U}[\tau, T].$$

As a direct consequence, Problem (M-SLQ) is uniquely solvable.

Next, we state some properties for the stochastic value flow  $V(t, \xi, \vartheta)$ , which are important to prove Theorem 3.1 later. Recall that  $\{e_j\}_{j=1}^n$  is the standard basis of  $\mathbb{R}^n$ .

**PROPOSITION 3.10.** *Let (H) hold. Suppose that Problem (M-SLQ) is solvable at the initial triple  $(t, e_j, \vartheta)$  with  $1 \leq j \leq n$ . Let the state-control pair  $\{(X_j(s), u_j(s))\}_{s \in [t, T]}$  of SDE (1) be an open-loop optimal pair w.r.t.  $(t, e_j, \vartheta)$ , and let  $\{(Y_j(s), Z_j(s), \Gamma^j(s))\}_{s \in [t, T]}$  be the solution of the associated adjoint equation (10). Denote  $\Gamma_{kl}(s) \triangleq (\Gamma_{kl}^1(s), \dots, \Gamma_{kl}^n(s))$  and*



$$(13) \quad \begin{aligned} \mathbf{X}(s) &\triangleq (X_1(s), \dots, X_n(s)), & \mathbf{U}(s) &\triangleq (u_1(s), \dots, u_n(s)), & s \in [t, T], \\ \mathbf{Y}(s) &\triangleq (Y_1(s), \dots, Y_n(s)), & \mathbf{Z}(s) &\triangleq (Z_1(s), \dots, Z_n(s)), & \mathbf{\Gamma}(s) \triangleq (\mathbf{\Gamma}_{kl}(s))_{k,l \in \mathcal{S}}. \end{aligned}$$

Then the quintuple  $(\mathbf{X}(\cdot), \mathbf{U}(\cdot), \mathbf{Y}(\cdot), \mathbf{Z}(\cdot), \mathbf{\Gamma}(\cdot))$  satisfies the following FBSDE:

$$(14) \quad \begin{cases} d\mathbf{X}(s) = [A(s, \alpha(s))\mathbf{X}(s) + B(s, \alpha(s))\mathbf{U}(s)]ds \\ \quad + [C(s, \alpha(s))\mathbf{X}(s) + D(s, \alpha(s))\mathbf{U}(s)]dW(s), \\ d\mathbf{Y}(s) = -\tilde{F}(s, \alpha(s), \mathbf{X}(s), \mathbf{Y}(s), \mathbf{Z}(s), \mathbf{U}(s))ds + \mathbf{Z}(s)dW(s) \\ \quad + \mathbf{\Gamma}(s) \bullet d\tilde{N}(s), & s \in [t, T], \\ \mathbf{X}(t) = I_n, \quad \alpha(t) = \vartheta, \quad \mathbf{Y}(T) = G(\alpha(T))\mathbf{X}(T), \end{cases}$$

and

$$(15) \quad F(s, \alpha(s), \mathbf{X}(s), \mathbf{Y}(s), \mathbf{Z}(s), \mathbf{U}(s)) = 0, \quad \text{a.e. } s \in [t, T], \text{ a.s.}$$

In addition, for every  $\xi \in L^\infty_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ , the state-control pair  $(\mathbf{X}\xi, \mathbf{U}\xi) = \{(\mathbf{X}(s)\xi, \mathbf{U}(s)\xi)\}_{s \in [t, T]}$  is optimal w.r.t.  $(t, \xi, \vartheta)$  and the triple  $(\mathbf{Y}\xi, \mathbf{Z}\xi, \mathbf{\Gamma} \circ \xi) = \{(\mathbf{Y}(s)\xi, \mathbf{Z}(s)\xi, \mathbf{\Gamma}(s) \circ \xi)\}_{s \in [t, T]}$  with  $\mathbf{\Gamma}(s) \circ \xi \triangleq (\mathbf{\Gamma}_{kl}(s)\xi)_{k,l \in \mathcal{S}}$  solves the adjoint BSDE (10) associated with the pair  $(\mathbf{X}\xi, \mathbf{U}\xi)$ .

Based on the above results, we have the following theorems, which present a quadratic form of the stochastic value flow into a bounded and left-continuous process.

**THEOREM 3.11.** *Let (H) hold. For any given  $t \in [0, T]$  and  $\vartheta \in L^2_{\mathcal{F}_t^\alpha}(\Omega; \mathcal{S})$ , if Problem (M-SLQ) is solvable at the initial time  $t$ , then there is a process  $P : [0, T] \times \mathcal{S} \times \Omega \rightarrow \mathbb{S}^n$  such that*

$$(16) \quad V(t, \xi, \vartheta) = \langle P(t, \vartheta)\xi, \xi \rangle \quad \forall \xi \in L^\infty_{\mathcal{F}_t}(\Omega; \mathbb{R}^n).$$

**THEOREM 3.12.** *Let conditions (H) and (7) hold. Then the process  $P = \{P(t, i); (t, i) \in [0, T] \times \mathcal{S}\}$  that appears in (16) is bounded and left-continuous.*

Finally, based on Theorems 3.11 and 3.12, we introduce a stopped SLQ problem and present some results for it, which is useful to the proof of Theorem 3.1 too. Recall that  $\tau \in \mathcal{T}(0, T]$  is an  $\mathbb{F}$ -stopping time, and set

$$\mathcal{D}^\tau \triangleq \{(\sigma, \xi, \vartheta) \mid \sigma \in \mathcal{T}[0, \tau], \xi \in L^2_{\mathcal{F}_\sigma}(\Omega; \mathbb{R}^n), \vartheta \in L^2_{\mathcal{F}_\sigma}(\Omega; \mathcal{S})\}.$$

**Problem (M-SLQ) $^\tau$ .** For any given  $(\sigma, \xi, \vartheta) \in \mathcal{D}^\tau$ , find a control  $u^*(\cdot) \in \mathcal{U}[\sigma, \tau]$  such that

$$J^\tau(\sigma, \xi, \vartheta; u^*(\cdot)) = \operatorname{ess\,inf}_{u(\cdot) \in \mathcal{U}[\sigma, \tau]} J^\tau(\sigma, \xi, \vartheta; u(\cdot)) \triangleq V^\tau(\sigma, \xi, \vartheta),$$

where the cost functional

$$J^\tau(\sigma, \xi, \vartheta; u(\cdot)) \triangleq \mathbb{E}_\sigma \left[ \langle P(\tau, \alpha(\tau))X(\tau), X(\tau) \rangle + \int_\sigma^\tau \left\langle \begin{pmatrix} Q(s, \alpha(s)) & S(s, \alpha(s))^\top \\ S(s, \alpha(s)) & R(s, \alpha(s)) \end{pmatrix} \begin{pmatrix} X(s) \\ u(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \right\rangle ds \right]$$

with  $X(\cdot)$  being the solution of (1) w.r.t.  $(\sigma, \xi, \vartheta)$  over  $[\sigma, \tau]$ . Similar to Remark 2.1, when the above conditional expectation  $\mathbb{E}_\sigma[\cdot]$  degenerates to  $\mathbb{E}$ , we denote the related problem, cost functional, and value function by Problem (M-SLQ) $^\tau_0$ ,  $J^\tau_0(t, \xi, \vartheta; u(\cdot))$ , and  $V^\tau_0(t, \xi, \vartheta)$ , respectively.

PROPOSITION 3.13. *Let conditions (H) and (7) hold. Then*

- (i) *Problem (M-SLQ) $^\tau$  is uniquely solvable at every  $\sigma \in \mathcal{T}[0, \tau]$ ;*
- (ii) *if  $u^*(\cdot) \in \mathcal{U}[\sigma, T]$  is an optimal control of Problem (M-SLQ) w.r.t.  $(\sigma, \xi, \vartheta) \in \mathcal{T}[0, \tau] \times L^\infty_{\mathcal{F}_\sigma}(\Omega; \mathbb{R}^n) \times L^2_{\mathcal{F}_\sigma}(\Omega; \mathcal{S})$ , then the restriction  $.u^*(\cdot)|_{[\sigma, \tau]}$  of  $u^*(\cdot)$  on the interval  $[\sigma, \tau]$  is also an optimal control of Problem (M-SLQ) $^\tau$  w.r.t. the same initial triple  $(\sigma, \xi, \vartheta)$ ;*
- (iii) *the stochastic value flow  $V^\tau(\cdot)$  of Problem (M-SLQ) $^\tau$  has the form  $V^\tau(\sigma, \xi, \vartheta) = \langle P(\sigma, \vartheta)\xi, \xi \rangle \forall (\sigma, \xi, \vartheta) \in \mathcal{T}[0, \tau] \times L^\infty_{\mathcal{F}_\sigma}(\Omega; \mathbb{R}^n) \times L^2_{\mathcal{F}_\sigma}(\Omega; \mathcal{S})$ , where  $P(\cdot, \alpha(\cdot))$  is the process that appears in (16).*

**4. Proof of the solvability of SRE.** In this section, based on the auxiliary results of subsection 3.2, we prove the solvability of SRE (5). First, we prove that SRE (5) is uniquely solvable, and then we prove that the first component  $\hat{P}(\cdot, \alpha(\cdot))$  of SRE (5) is exactly the process  $P(\cdot, \alpha(\cdot))$  that appears in (16). Finally, as a by-product, the open-loop optimal control is represented as linear feedback of the state. Next, we use several lemmas to prove Theorem 3.1. Recall that SDE (1) is the state equation, BSDE (10) is the associated adjoint equation, and  $\{e_j\}_{j=1}^n$  is the standard basis of  $\mathbb{R}^n$ .

LEMMA 4.1. *Suppose the conditions (H) and (7) hold. Denote by  $\{X_j(s)\}_{s \in [0, T]}$  the (unique) open-loop optimal state process corresponding to the initial triple  $(t, \xi, \vartheta) = (0, e_j, i_0)$ . Then the  $\mathbb{R}^{n \times n}$ -valued process  $\{\mathbf{X}(s)\}_{s \in [0, T]}$  with  $\mathbf{X}(s) \triangleq (X_1(s), \dots, X_n(s))$  is invertible.*

*Proof.* Let  $u_j(\cdot) \in \mathcal{U}[0, T]$  be the unique optimal control w.r.t.  $(0, e_j, i_0)$  so that for  $s \in [0, T]$ ,

$$\begin{cases} dX_j(s) = [A(s, \alpha(s))X_j(s) + B(s, \alpha(s))u_j(s)]ds \\ \quad + [C(s, \alpha(s))X_j(s) + D(s, \alpha(s))u_j(s)]dW(s), \\ X(0) = e_j, \quad \alpha(0) = i_0. \end{cases}$$

Then, with  $\mathbf{U}(s) = (u_1(s), \dots, u_n(s))$ , one has for  $s \in [0, T]$ ,

$$(17) \quad \begin{cases} d\mathbf{X}(s) = [A(s, \alpha(s))\mathbf{X}(s) + B(s, \alpha(s))\mathbf{U}(s)]ds \\ \quad + [C(s, \alpha(s))\mathbf{X}(s) + D(s, \alpha(s))\mathbf{U}(s)]dW(s), \\ \mathbf{X}(0) = I_n, \quad \alpha(0) = i_0. \end{cases}$$

We define the stopping time (at which, for the first time,  $\mathbf{X}(\cdot)$  is not invertible):

$$\theta(\omega) = \inf \{s \in [0, T]; \det(\mathbf{X}(s, \omega)) = 0\},$$

where we use the convention that the infimum of the empty set is infinity. It should be pointed out that if we prove that  $\mathbb{P}(\theta = \infty) = 1$ , i.e., the set  $\mathbb{O} \triangleq \{\omega \in \Omega : \theta(\omega) \leq T\}$  has probability zero, then we can get that  $X(\cdot)$  is invertible.

Suppose the contrary and set  $\tau = \theta \wedge T$ . Then  $\tau$  is in  $\mathcal{T}(0, T]$  too. When  $\tau = \theta$  on  $\mathbb{O}$ , we know that  $\mathbf{X}(\tau)$  is not invertible on  $\mathbb{O}$  by the definition of  $\theta$ . Thus, we can choose an  $\mathcal{F}_\tau$ -measurable,  $\mathbb{S}^n$ -valued, positive semidefinite random matrix  $H$  with  $|H| = 1$  on  $\mathbb{O}$  such that  $H(\omega)\mathbf{X}(\tau(\omega), \omega) = 0$  for any  $\omega \in \Omega$ . Note that  $P = \{P(t, i); (t, i) \in [0, T] \times \mathcal{S}\}$  is bounded, is left-continuous, and satisfies (16) according to Theorems 3.11 and 3.12. For  $\sigma \in \mathcal{T}[0, \tau]$  with  $s \in [\sigma, \tau]$ , we consider the equation

$$(18) \quad \begin{cases} dX(s) = [A(s, \alpha(s))X(s) + B(s, \alpha(s))u(s)]ds \\ \quad + [C(s, \alpha(s))X(s) + D(s, \alpha(s))u(s)]dW(s), \\ X(\sigma) = \xi, \quad \alpha(\sigma) = \vartheta, \end{cases}$$

and the following auxiliary cost functional:

$$(19) \quad J^H(\sigma, \xi, \vartheta; u(\cdot)) \triangleq J^\tau(\sigma, \xi, \vartheta; u(\cdot)) + \mathbb{E}_\sigma[\langle HX(\tau), X(\tau) \rangle].$$

Consider the problem of minimizing the cost functional (19) subject to the state equation (18), called Problem (M-SLQ)<sub>H</sub>. Moreover, denote by  $V_H(\cdot)$  the associated stochastic value flow. Then the following two assertions hold:

(i) For any choice of  $\sigma \in \mathcal{T}[0, \tau)$  and  $\vartheta \in L^2_{\mathcal{F}_\sigma^\alpha}(\Omega; \mathcal{S})$  with  $\alpha(\sigma) = \vartheta$ ,

$$\mathbb{E}[J^H(\sigma, 0, \vartheta; u(\cdot))] \geq \mathbb{E}[J^\tau(\sigma, 0, \vartheta; u(\cdot))] \geq \varepsilon \mathbb{E}\left[\int_\sigma^\tau |u(s)|^2 ds\right] \quad \forall u(\cdot) \in \mathcal{U}[\sigma, \tau].$$

(ii) The restriction  $u_j^\tau(\cdot) = .u_j(\cdot)|_{[0, \tau]}$  over the time interval  $[0, \tau]$  is optimal for both Problem (M-SLQ)<sup>τ</sup> and Problem (M-SLQ)<sub>H</sub> w.r.t. the same initial triple  $(0, e_j, i_0)$ .

For the above assertion (i), in fact, the first inequality is true since  $H$  is positive semi-definite, and the second inequality comes from the proof of assertion (i) of Proposition 3.13. Then, by Proposition 3.9, both Problems (M-SLQ)<sup>τ</sup> and (M-SLQ)<sub>H</sub> are uniquely solvable at any  $\sigma \in \mathcal{T}[0, \tau)$ .

For the above assertion (ii), on one hand, by assertion (ii) of Proposition 3.13, we see that  $u_j^\tau(\cdot)$  is optimal for Problem (M-SLQ)<sup>τ</sup> w.r.t. the initial triple  $(0, e_j, i_0)$ . On the other hand, according to Theorem 3.4, in order to show that  $u_j^\tau(\cdot)$  is an optimal control for Problem (M-SLQ)<sub>H</sub> w.r.t.  $(0, e_j, i_0)$ , it suffices to prove that the solution  $(X_j^\tau(\cdot), Y_j^\tau(\cdot), Z_j^\tau(\cdot), \Gamma^{j, \tau}(\cdot))$  of the FBSDE

$$(20) \quad \begin{cases} dX_j^\tau(s) = [A(s, \alpha(s))X_j^\tau(s) + B(s, \alpha(s))u_j^\tau(s)] ds \\ \quad + [C(s, \alpha(s))X_j^\tau(s) + D(s, \alpha(s))u_j^\tau(s)] dW(s), \\ dY_j^\tau(s) = -\tilde{F}(s, \alpha(s), X_j^\tau(s), Y_j^\tau(s), Z_j^\tau(s), u_j^\tau(s)) ds \\ \quad + Z_j^\tau(s) dW(s) + \Gamma^{j, \tau}(s) \bullet d\tilde{N}(s), \quad s \in [0, \tau], \\ X_j^\tau(0) = e_j, \quad \alpha(0) = i_0, \quad Y_j^\tau(\tau) = [P(\tau, \alpha(\tau)) + H]X_j^\tau(\tau), \end{cases}$$

satisfies the following stationarity condition:

$$(21) \quad F(s, \alpha(s), X_j^\tau(s), Y_j^\tau(s), Z_j^\tau(s), u_j^\tau(s)) = 0, \quad \text{a.e. } s \in [0, \tau], \text{ a.s.}$$

Note that  $X_j^\tau(s) = \mathbf{X}(s)e_j$  for  $0 \leq s \leq \tau$ . Thus, by the choice of  $H$ , we have

$$(22) \quad HX_j^\tau(\tau) = H\mathbf{X}(\tau)e_j = 0.$$

It follows that the terminal value  $Y_j^\tau(\tau) = P(\tau, \alpha(\tau))X_j^\tau(\tau)$ , which implies that FBSDE (20) is exactly the FBSDE associated with Problem (M-SLQ)<sup>τ</sup>. Then from Theorem 3.4 again, the stationarity condition (21) follows from the fact that  $u_j^\tau(\cdot)$  is an open-loop optimal control of Problem (M-SLQ)<sup>τ</sup> w.r.t.  $(0, e_j, i_0)$ .

Now, for Problem (M-SLQ)<sub>H</sub>, by Theorem 3.11 and the above assertion (i), there is a bounded, left-continuous, and  $\mathbb{S}^n$ -valued process  $P_H(\cdot, i)$  such that

$$(23) \quad V_H(\sigma, \xi, \vartheta) = \langle P_H(\sigma, \vartheta)\xi, \xi \rangle \quad \forall (\sigma, \xi, \vartheta) \in \mathcal{T}[0, \tau) \times L^\infty_{\mathcal{F}_\sigma}(\Omega; \mathbb{R}^n) \times L^2_{\mathcal{F}_\sigma}(\Omega; \mathcal{S}).$$

Next, we prove that  $P(\cdot, \alpha(\cdot))$  appearing in (16) equals to  $P_H(\cdot, \alpha(\cdot))$  appearing in (23), i.e.,

$$(24) \quad P(t, \alpha(t)) = P_H(t, \alpha(t)), \quad t \in [0, \tau).$$

By the above assertion (ii), we see that  $\{(X_j^\tau(s), u_j^\tau(s))\}_{s \in [0, \tau]}$  is the optimal state-control pair for both Problem (M-SLQ) $^\tau$  and Problem (M-SLQ) $_H$  w.r.t. the initial triple  $(0, e_j, i_0)$ . Let

$$\mathbf{X}^\tau(s) \triangleq (X_1^\tau(s), \dots, X_n^\tau(s)), \quad \mathbf{U}^\tau(s) \triangleq (u_1^\tau(s), \dots, u_n^\tau(s)), \quad s \in [0, \tau],$$

and take an arbitrary  $x \in \mathbb{R}^n$ . From Proposition 3.10, we have that the pair  $(\mathbf{X}^\tau x, \mathbf{U}^\tau x)$  is the optimal state-control for both Problem (M-SLQ) $^\tau$  and Problem (M-SLQ) $_H$  w.r.t. the initial triple  $(0, x, i_0)$ . Moreover, by Corollary 3.8, the pair  $(\mathbf{X}^\tau(s)x, \mathbf{U}^\tau(s)x)$  with  $s \in [t, \tau]$  remains optimal w.r.t.  $(t, \mathbf{X}^\tau(t)x, \alpha(t))$  for every  $t \in [0, \tau)$ . Thus, note that  $H\mathbf{X}^\tau(\tau) = 0$  by (22),

$$\begin{aligned} V_H(t, \mathbf{X}^\tau(t)x, \alpha(t)) &= J_H(t, \mathbf{X}^\tau(t)x, \alpha(t); \mathbf{U}^\tau(t)x) \\ &= J(t, \mathbf{X}^\tau(t)x, \alpha(t); \mathbf{U}^\tau(t)x) + \mathbb{E} \langle H\mathbf{X}^\tau(\tau)x, \mathbf{X}^\tau(\tau)x \rangle \\ &= J(t, \mathbf{X}^\tau(t)x, \alpha(t); \mathbf{U}^\tau(t)x) = V(t, \mathbf{X}^\tau(t)x, \alpha(t)). \end{aligned}$$

Noting that  $\mathbf{X}^\tau(t) = \mathbf{X}(t)$  when  $t \in [0, \tau)$ , we deduce from the above that

$$\begin{aligned} \langle P_H(t, \alpha(t))\mathbf{X}(t)x, \mathbf{X}(t)x \rangle &= V_H(t, \mathbf{X}^\tau(t)x, \alpha(t)) \\ &= V(t, \mathbf{X}^\tau(t)x, \alpha(t)) = \langle P(t, \alpha(t))\mathbf{X}(t)x, \mathbf{X}(t)x \rangle. \end{aligned}$$

Since  $x \in \mathbb{R}^n$  is arbitrary, it follows that

$$\mathbf{X}(t)^\top P(t, \alpha(t))\mathbf{X}(t) = \mathbf{X}(t)^\top P_H(t, \alpha(t))\mathbf{X}(t), \quad t \in [0, \tau).$$

From the definition of  $\tau$ , we see that  $\mathbf{X}$  is invertible on  $[0, \tau)$ , which implies that the relation (24) holds. Thus, for some choice of  $i \in \mathcal{S}$ , one has

$$(25) \quad P(t, i) = P_H(t, i), \quad t \in [0, \tau).$$

However, on the other hand,  $P_H(\tau, i) = P(\tau, i) + H$ , and both  $P(\cdot, i)$  and  $P_H(\cdot, i)$  are left-continuous. Finally, in (25), letting  $t \uparrow \tau$  yields a contradiction,  $P(\tau, i) = P(\tau, i) + H$ , since  $|H| = 1$  on  $\mathbb{O}$ .  $\square$

Note that under the conditions of Theorem 3.1, Problem (M-SLQ) is uniquely solvable by Proposition 3.9. Let  $\{(X_j(s), u_j(s))\}_{s \in [0, T]}$  be the optimal pair and  $\{(Y_j(s), Z_j(s), \Gamma^j(s))\}_{s \in [t, T]}$  be the adapted solution of the adjoint BSDE corresponding to the initial triple  $(0, e_j, i_0)$  for each  $j = 1, 2, \dots, n$ , respectively. Now, with  $\Gamma_{kl}(s) \triangleq (\Gamma_{kl}^1(s), \dots, \Gamma_{kl}^n(s))$ , Proposition 3.10 implies that the matrix-valued processes  $(\mathbf{X}(\cdot), \mathbf{U}(\cdot), \mathbf{Y}(\cdot), \mathbf{Z}(\cdot), \mathbf{\Gamma}(\cdot))$  defined by (13) satisfy FBSDE (14) with the initial triple  $(0, I_n, i_0)$  over  $[0, T]$ . Moreover, the following stationarity holds:

$$(26) \quad F(s, \alpha(s), \mathbf{X}(s), \mathbf{Y}(s), \mathbf{Z}(s), \mathbf{U}(s)) = 0, \quad \text{a.e. } s \in [0, T], \text{ a.s.}$$

In addition, on one hand, from Lemma 4.1 we see that  $\mathbf{X}(\cdot)$  is invertible. On the other hand, from Theorems 3.11 and 3.12, there is a process  $P : [0, T] \times \mathcal{S} \times \Omega \rightarrow \mathbb{S}^n$ , which is left-continuous and bounded, that satisfies (16).

LEMMA 4.2. *Suppose (H) and (7) hold. Then,*

$$(27) \quad P(s, \alpha(s)) = \mathbf{Y}(s)\mathbf{X}(s)^{-1}, \quad s \in [0, T],$$

where  $P(\cdot, \alpha(\cdot))$  is the process appearing in (16) and the pair  $(\mathbf{X}(\cdot), \mathbf{Y}(\cdot))$  is defined in (13).

*Proof.* For arbitrary  $x \in \mathbb{R}^n$  and  $s \in [0, T]$ , set that

$$\begin{aligned} X^*(s) &= \mathbf{X}(s)x, & u^*(s) &= \mathbf{U}(s)x, \\ Y^*(s) &= \mathbf{Y}(s)x, & Z^*(s) &= \mathbf{Z}(s)x, & \Gamma^*(s) &= \mathbf{\Gamma}(s) \circ x. \end{aligned}$$

Then, on one hand, Proposition 3.10 implies that the pair  $(X^*(\cdot), u^*(\cdot))$  is an open-loop optimal pair w.r.t. the initial triple  $(0, x, i_0)$  and that the triple  $(Y^*(\cdot), Z^*(\cdot), \Gamma^*(\cdot))$  is the adapted solution to the adjoint BSDE associated with  $(X^*(\cdot), u^*(\cdot))$ . On the other hand, for every  $s \in [0, T]$ , Corollary 3.8 implies that the restriction  $(X^*(\cdot)|_{[s, T]}, u^*(\cdot)|_{[s, T]})$  of  $(X^*(\cdot), u^*(\cdot))$  over the interval  $[s, T]$  remains optimal w.r.t.  $(s, X^*(s), \alpha(s))$ . Thus, from Corollary 3.7, we have  $V(s, X^*(s), \alpha(s)) = \langle Y^*(s), X^*(s) \rangle$ . Now, owing to (16), the above equation implies that

$$\begin{aligned} x^\top \mathbf{X}(s)^\top P(s, \alpha(s)) \mathbf{X}(s)x &= \langle P(s, \alpha(s)) \mathbf{X}(s)x, \mathbf{X}(s)x \rangle = \langle P(s, \alpha(s)) X^*(s), X^*(s) \rangle \\ &= V(s, X^*(s), \alpha(s)) = \langle Y^*(s), X^*(s) \rangle = \langle \mathbf{Y}(s)x, \mathbf{X}(s)x \rangle = x^\top \mathbf{X}(s)^\top \mathbf{Y}(s)x. \end{aligned}$$

Since  $x \in \mathbb{R}^n$  is arbitrary, we deduce that  $\mathbf{X}(s)^\top P(s, \alpha(s)) \mathbf{X}(s) = \mathbf{X}(s)^\top \mathbf{Y}(s)$ . Then the desired result follows from the fact that  $\mathbf{X}(\cdot)$  is invertible.  $\square$

In the following, please keep in mind that  $P(\cdot, \alpha(\cdot))$  represents the process that appears in (16) and  $(\mathbf{X}(\cdot), \mathbf{U}(\cdot), \mathbf{Y}(\cdot), \mathbf{Z}(\cdot), \mathbf{\Gamma}(\cdot))$  defined by (13) satisfies FBSDE (14) with  $(0, I_n, i_0)$  over  $[0, T]$ .

LEMMA 4.3. *Suppose (H) and (7) hold. Then, with the relation (27) and the notation*

$$\begin{aligned} (28) \quad \Theta(s, \alpha(s)) &= \mathbf{U}(s) \mathbf{X}(s)^{-1}, & \Lambda(s) &= \Pi(s, \alpha(s)) - P(s, \alpha(s)) [C(s) + D(s) \Theta(s, \alpha(s))], \\ \Pi(s, \alpha(s)) &= \mathbf{Z}(s) \mathbf{X}(s)^{-1}, & \zeta(s) &= (\zeta_{k,l}(s))_{k,l \in \mathcal{S}} \text{ with} \\ \zeta_{kl}(s) &\triangleq \mathbf{\Gamma}_{kl}(s) \mathbf{X}^{-1}(s), & s &\in [0, T], \end{aligned}$$

we have that the triple  $(P(\cdot, \alpha(\cdot)), \Lambda(\cdot), \zeta(\cdot))$  satisfies the following BSDE: for  $s \in [0, T]$ ,

$$(29) \quad \begin{cases} dP(s, \alpha(s)) = - [\hat{Q}(s, \alpha(s)) + \hat{S}(s, \alpha(s))^\top \Theta(s, \alpha(s))] ds + \Lambda(s) dW(s) + \zeta(s) \bullet d\tilde{N}(s), \\ P(T, \alpha(T)) = G(\alpha(T)), \quad \alpha(0) = i_0, \end{cases}$$

where  $\hat{Q}(\cdot)$  and  $\hat{S}(\cdot)$  are defined in (6). Moreover,  $\Lambda(\cdot)$  is symmetric, and

$$(30) \quad \hat{S}(s, \alpha(s)) + \hat{R}(s, \alpha(s)) \Theta(s, \alpha(s)) = 0, \quad \text{a.e. } s \in [0, T], \text{ a.s.}$$

*Proof.* First, by the relation (16), we have

$$\langle G(\alpha(T)) \xi, \xi \rangle = V(T, \xi, \alpha(T)) = \langle P(T, \alpha(T)) \xi, \xi \rangle \quad \forall \xi \in L^\infty_{\mathcal{F}_T}(\Omega; \mathbb{R}^n),$$

which leads to that  $P(T, \alpha(T)) = G(\alpha(T))$ . Note that  $\mathbf{X}(\cdot)$  satisfies SDE (17) and is invertible, so its invertibility (denoted by  $\mathbf{X}^{-1}(\cdot)$ ) exists and satisfies the following SDE:

$$d\mathbf{X}(s)^{-1} = \Xi(s, \alpha(s)) ds + \Delta(s, \alpha(s)) dW(s), \quad s \in [0, T],$$

where

$$\begin{aligned} \Xi(s, \alpha(s)) &= \mathbf{X}^{-1}(s) \{ [C(s, \alpha(s)) + D(s, \alpha(s)) \Theta(s, \alpha(s))]^2 - A(s, \alpha(s)) \\ &\quad - B(s, \alpha(s)) \Theta(s, \alpha(s)) \}, \\ \Delta(s, \alpha(s)) &= -\mathbf{X}^{-1}(s) [C(s, \alpha(s)) + D(s, \alpha(s)) \Theta(s, \alpha(s))]. \end{aligned}$$

Applying Itô's formula to the right-hand side of (27), we have

$$\begin{aligned}
 dP(s, \alpha(s)) &= -F(s, \alpha(s), \mathbf{X}(s), \mathbf{Y}(s), \mathbf{Z}(s), \mathbf{U}(s))\mathbf{X}^{-1}(s)ds + Z(s)\Delta(s)ds \\
 &\quad + \mathbf{Z}(s)\mathbf{X}^{-1}(s)dW(s) + \sum_{k,l=1}^D \Gamma_{kl}(s)\mathbf{X}^{-1}(s)d\tilde{N}_{kl}(s) \\
 &\quad + \mathbf{Y}(s)\Xi(s, \alpha(s))ds + \mathbf{Y}(s)\Delta(s, \alpha(s))dW(s) \\
 &= [-A(s, \alpha(s))^\top P(s, \alpha(s)) - C(s, \alpha(s))^\top \Pi(s, \alpha(s)) \\
 &\quad - Q(s, \alpha(s)) - S(s, \alpha(s))^\top \Theta(s, \alpha(s)) \\
 &\quad + P(s, \alpha(s))[(C(s, \alpha(s)) + D(s, \alpha(s))\Theta(s, \alpha(s)))^2 \\
 &\quad - A(s, \alpha(s)) - B(s, \alpha(s))\Theta(s, \alpha(s))] \\
 &\quad - \Pi(s, \alpha(s))(C(s, \alpha(s)) + D(s, \alpha(s))\Theta(s, \alpha(s))]ds + \sum_{k,l=1}^D \zeta_{kl}(s)d\tilde{N}_{kl}(s) \\
 &\quad + [\Pi(s, \alpha(s)) - P(s, \alpha(s))[C(s, \alpha(s)) + D(s, \alpha(s))\Theta(s, \alpha(s))]]dW(s) \\
 &= -[\hat{Q}(s, \alpha(s)) + \hat{S}(s, \alpha(s))^\top \Theta(s, \alpha(s))]ds + \Lambda(s)dW(s) + \zeta(s) \bullet d\tilde{N}(s).
 \end{aligned}$$

Note that  $P(s, \alpha(s))$  is symmetric, i.e.,  $P(s, \alpha(s)) = P(s, \alpha(s))^\top$  for  $s \in [0, T]$ . Comparing the diffusion coefficients of the above BSDEs satisfied by  $P(\cdot, \alpha(\cdot))$  and  $P^\top(\cdot, \alpha(\cdot))$ , we obtain  $\Lambda(s) = \Lambda(s)^\top$  for  $s \in [0, T]$ . Further combining (26) and (28), we get

$$\hat{S}(s, \alpha(s)) + \hat{R}(s, \alpha(s))\Theta(s, \alpha(s)) = F(s, \alpha(s), \mathbf{X}(s), \mathbf{Y}(s), \mathbf{Z}(s), \mathbf{U}(s))\mathbf{X}^{-1}(s) = 0.$$

This completes the proof. □

LEMMA 4.4. *Suppose (H) holds, and there exists a constant  $\varepsilon > 0$  such that (7) holds. Then*

$$(31) \quad \hat{R}(s, \alpha(s)) = R(s, \alpha(s)) + D(s, \alpha(s))^\top P(s, \alpha(s))D(s, \alpha(s)) \geq \varepsilon I_m, \quad \text{a.e. on } [0, T], \text{ a.s.}$$

*Proof.* The procedure of the proof can be divided into three steps.

*Step 1.* Let us temporarily assume that processes  $\Theta(\cdot) = \{\Theta(s, \alpha(s))\}_{s \in [0, T]}$  and  $\Lambda(\cdot) = \{\Lambda(s)\}_{s \in [0, T]}$  defined by (28) satisfy the following condition:

$$(32) \quad \text{esssup}_{\omega \in \Omega} \int_0^T [|\Theta(s, \omega, \alpha(s))|^2 + |\Lambda(s, \omega)|^2] ds < \infty.$$

Choosing an arbitrary control  $v(\cdot) \in \mathcal{U}[0, T]$ , we consider the following SDE: for  $s \in [t, T]$ ,

$$(33) \quad \begin{cases} dX^v(s) = \{ [A(s, \alpha(s)) + B(s, \alpha(s))\Theta(s, \alpha(s))]X^v(s) + B(s, \alpha(s))v(s) \} ds \\ \quad + \{ [C(s, \alpha(s)) + D(s, \alpha(s))\Theta(s, \alpha(s))]X^v(s) + D(s, \alpha(s))v(s) \} dW(s), \\ X^v(0) = 0, \quad \alpha(0) = i_0. \end{cases}$$

By the standard SDE theory, we see that  $X^v(\cdot)$ , the solution of SDE (33), belongs to space  $L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^n))$ . Hence,

$$(34) \quad u(s) \triangleq \Theta(s, \alpha(s))X^v(s) + v(s) \in \mathcal{U}[0, T].$$

According to  $u(\cdot)$  defined in (34) and the uniqueness of SDEs,  $X(\cdot)$ , the solution of (1) w.r.t. the initial triple  $(0, 0, i_0)$  over  $[0, T]$  coincides with  $X^v(\cdot)$ , the solution of (33), i.e.,  $X^v(s) = X(s)$  for any  $s \in [0, T]$ . Note that  $P(s, \alpha(s))$  satisfies BSDE (29). By applying Itô's formula to  $\langle P(s, \alpha(s))X^v(s), X^v(s) \rangle$  first and then taking expectations on both sides, we can get

$$\begin{aligned}
 & \mathbb{E} \langle G(\alpha(T))X^v(T), X^v(T) \rangle \\
 (35) \quad &= \mathbb{E} \int_0^T \left\{ - \langle [Q(s, \alpha(s)) + \hat{S}(s, \alpha(s))^\top \Theta(s, \alpha(s))] X^v(s), X^v(s) \rangle \right. \\
 & \quad + 2 \langle [\hat{S}(s, \alpha(s)) - S(s, \alpha(s))]^\top u(s), X^v(s, \alpha(s)) \rangle \\
 & \quad \left. + \langle D(s, \alpha(s))^\top P(s, \alpha(s)) D(s, \alpha(s)) u(s), u(s) \rangle \right\} ds.
 \end{aligned}$$

Substituting (35) into (3) with the initial triple  $(0, 0, i_0)$ , we have

$$\begin{aligned}
 J(0, 0, i_0; u(\cdot)) &= \mathbb{E} \int_0^T \left\{ - \langle \hat{S}(s, \alpha(s))^\top \Theta(s, \alpha(s)) X^v(s), X^v(s) \rangle \right. \\
 & \quad \left. + 2 \langle \hat{S}(s, \alpha(s))^\top u(s), X^v(s) \rangle + \langle \hat{R}(s, \alpha(s)) u(s), u(s) \rangle \right\} ds.
 \end{aligned}$$

Combining (30) and (34), we further deduce that

$$\begin{aligned}
 & J(0, 0, i_0; u(\cdot)) \\
 &= \mathbb{E} \int_0^T \langle \hat{R}(s, \alpha(s)) [u(s) - \Theta(s, \alpha(s)) X^v(s)], u(s) - \Theta(s, \alpha(s)) X^v(s) \rangle ds \\
 &= \mathbb{E} \int_0^T \langle \hat{R}(s, \alpha(s)) v(s), v(s) \rangle ds.
 \end{aligned}$$

Finally, the condition (7) implies that  $J(0, 0, i_0; u(\cdot)) \geq 0 \forall u(\cdot) \in \mathcal{U}[0, T]$ . Therefore,

$$(36) \quad \hat{R}(s, \alpha(s)) = R(s, \alpha(s)) + D(s, \alpha(s))^\top P(s, \alpha(s)) D(s, \alpha(s)) \geq 0, \text{ a.e. on } [0, T], \text{ a.s.}$$

*Step 2.* Now we prove that without the additional condition (32), the above result (36) still holds. The key method is to apply a localization technique so that the preceding argument can be applied to a certain stopped SLQ problem. In detail, for each  $k \geq 1$ , we define the following stopping time (with the convention  $\inf \emptyset = \infty$ ):

$$\tau_k = \inf \left\{ t \in [0, T]; \int_0^t [|\Theta(s, \alpha(s))|^2 + |\Lambda(s)|^2] ds \geq k \right\} \wedge T.$$

Take an arbitrary control  $v(\cdot) \in \mathcal{U}[0, T]$  and consider the state equation (33) over the interval  $[0, \tau_k]$ . By the definition of  $\tau_k$ , we have

$$\int_0^{\tau_k} [|\Theta(s, \alpha(s))|^2 + |\Lambda(s)|^2] ds \leq k,$$

which implies that  $X^v(\cdot)$ , the solution of (33) over  $[0, \tau_k]$ , belongs to  $L^2_{\mathbb{F}}(\Omega; C([0, \tau_k]; \mathbb{R}^n))$ . Hence  $u(s) \triangleq \Theta(s, \alpha(s))X^v(s) + v(s) \in \mathcal{U}[0, \tau_k]$ . Then we can proceed as in Step 1 to get that

$$J(0, 0, i_0; u(\cdot)) = \mathbb{E} \int_0^{\tau_k} \langle [R(s, \alpha(s)) + D(s, \alpha(s))^\top P(s, \alpha(s)) D(s, \alpha(s))] v(s), v(s) \rangle ds.$$

By the proof of assertion (i) of Proposition 3.13, we have that for any choice  $i_0 \in \mathcal{S}$ ,  $J(0, 0, i_0; u(\cdot)) \geq 0$  for every  $u(\cdot) \in \mathcal{U}[0, \tau_k]$ , and note that  $v(\cdot) \in \mathcal{U}[0, T]$  is arbitrary, so

$$(37) \quad \hat{R}(s, \alpha(s)) = R(s, \alpha(s)) + D(s, \alpha(s))^\top P(s, \alpha(s))D(s, \alpha(s)) \geq 0, \quad \text{a.e. on } [0, \tau_k], \text{ a.s.}$$

Because the process  $\mathbf{X}^{-1}(\cdot)$  is continuous,  $\mathbf{U}(\cdot)$  and  $\mathbf{Z}(\cdot)$  are square-integrable, and the processes  $P(\cdot, \alpha(\cdot))$ ,  $C(\cdot)$ , and  $D(\cdot)$  are bounded, from (28) we have

$$\int_0^T [|\Theta(s, \alpha(s))|^2 + |\Lambda(s)|^2] ds < \infty, \quad \text{a.s.}$$

This implies that  $\lim_{k \rightarrow \infty} \tau_k = T$  almost surely. Hence (36) still holds by letting  $k \rightarrow \infty$  in (37).

*Step 3.* In order to obtain the stronger property (31), we take an arbitrary but fixed  $\epsilon \in (0, \varepsilon)$  and consider the following stochastic LQ problem of minimizing:

$$\begin{aligned} J_\epsilon(t, \xi, \vartheta; u(\cdot)) &= \mathbb{E}_t \left[ \langle G(\alpha(T))X(T), X(T) \rangle \right. \\ &\quad \left. + \int_t^T \left\langle \begin{pmatrix} Q(s, \alpha(s)) & S(s, \alpha(s))^\top \\ S(s, \alpha(s)) & R(s, \alpha(s)) - \epsilon I_m \end{pmatrix} \begin{pmatrix} X(s) \\ u(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \right\rangle ds \right], \end{aligned}$$

where  $X(\cdot)$  is the solution of state equation (1). Clearly, with  $\varepsilon$  replaced by  $\varepsilon - \epsilon$ , the conditions of Theorem 3.1 still hold for  $J_\epsilon(t, \xi, \vartheta; u(\cdot))$ . Thus, there is a process  $P_\epsilon(\cdot)$  such that

$$\begin{aligned} V_\epsilon(t, \xi, \vartheta) &\triangleq \inf_{u(\cdot) \in \mathcal{U}[t, T]} J_\epsilon(t, \xi, \vartheta; u(\cdot)) = \langle P_\epsilon(t, \vartheta)\xi, \xi \rangle \\ \forall(t, \xi, \vartheta) &\in [0, T] \times L^\infty_{\mathcal{F}_t}(\Omega; \mathbb{R}^n) \times L^2_{\mathcal{F}_t^\alpha}(\Omega; \mathcal{S}). \end{aligned}$$

Then, by the previous discussion, we have

$$R(s, \alpha(s)) - \epsilon I_m + D(s, \alpha(s))^\top P_\epsilon(s, \alpha(s))D(s, \alpha(s)) \geq 0, \quad \text{a.e. on } [0, T], \text{ a.s.}$$

Now, by the definition of  $J_\epsilon(t, \xi, \vartheta; u(\cdot))$ , we deduce that

$$\begin{aligned} V(t, \xi, \vartheta) &= \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, \xi, \vartheta; u(\cdot)) \\ &\geq \inf_{u(\cdot) \in \mathcal{U}[t, T]} J_\epsilon(t, \xi, \vartheta; u(\cdot)) \\ &= V_\epsilon(t, \xi, \vartheta) \quad \forall(t, \xi, \vartheta) \in [0, T] \times L^\infty_{\mathcal{F}_t}(\Omega; \mathbb{R}^n) \times L^2_{\mathcal{F}_t^\alpha}(\Omega; \mathcal{S}), \end{aligned}$$

from which we see that  $P(t, \vartheta) \geq P_\epsilon(t, \vartheta)$  for any  $t \in [0, T]$  and  $\vartheta \in L^2_{\mathcal{F}_t^\alpha}(\Omega; \mathcal{S})$ , and therefore

$$\hat{R}(s, \alpha(s)) \geq R(s, \alpha(s)) + D(s, \alpha(s))^\top P_\epsilon(s, \alpha(s))D(s, \alpha(s)) \geq \varepsilon I_m, \quad \text{a.e. on } [0, T], \text{ a.s.}$$

Finally, note that  $\epsilon \in (0, \varepsilon)$  is arbitrary, so property (31) holds. This completes the proof.  $\square$

Based on the above preparations, we now can prove Theorems 3.1 and 3.2.

*Proof of Theorem 3.1.* On one hand, from Lemma 4.3, we see that the bounded process  $P(\cdot, \alpha(\cdot))$  in (16) and the processes  $\Lambda(\cdot)$  and  $\zeta(\cdot)$  defined by (28) satisfy BSDE (29) and relation (30). On the other hand, Lemma 4.4 implies that



$\hat{R}(s, \alpha(s)) = R(s, \alpha(s)) + D(s, \alpha(s))^\top P(s, \alpha(s))D(s, \alpha(s)) \geq \varepsilon I_m$ , a.e. on  $[0, T]$ , a.s.,

which, together with relation (30), deduces that

$$(38) \quad \Theta(s, \alpha(s)) = -\hat{R}(s, \alpha(s))^{-1} \hat{S}(s, \alpha(s)), \quad \text{a.e. on } [0, T], \text{ a.s.}$$

Substituting (38) into BSDE (29) yields that

$$(39) \quad \begin{cases} dP(s, \alpha(s)) = [\hat{\Psi}(s, \alpha(s)) - \hat{Q}(s, \alpha(s))]ds + \Lambda(s)dW(s) + \zeta(s) \bullet d\tilde{N}(s), & s \in [0, T], \\ P(T, \alpha(T)) = G(\alpha(T)), & \alpha(0) = i_0, \end{cases}$$

where  $\hat{Q}(\cdot)$  is defined by (6), and

$$\hat{\Psi}(s, \alpha(s)) \triangleq \hat{S}(s, \alpha(s))^\top \hat{R}(s, \alpha(s))^{-1} \hat{S}(s, \alpha(s)), \quad s \in [0, T].$$

Then SRE (5) follows easily from (39), and  $(\hat{P}(\cdot, \alpha(\cdot)), \hat{\Lambda}(\cdot), \hat{\zeta}(\cdot))$ , the solution of SRE (5), coincides with  $(P(\cdot, \alpha(\cdot)), \Lambda(\cdot), \zeta(\cdot))$ , the solution of BSDE (39), i.e.,

$$\hat{P}(s, \alpha(s)) = P(s, \alpha(s)), \quad \hat{\Lambda}(s) = \Lambda(s), \quad \hat{\zeta}(s) = \zeta(s), \quad s \in [0, T].$$

In the following, we still adopt  $(P(\cdot, \alpha(\cdot)), \Lambda(\cdot), \zeta(\cdot))$  in order to keep the consistency of symbols.

It remains to prove that the processes  $\Lambda(\cdot)$  and  $\zeta(\cdot)$  are square-integrable. Note that in BSDE (39), the matrix-valued processes  $A(\cdot)$ ,  $C(\cdot)$ ,  $Q(\cdot)$ , and  $P(\cdot, \alpha(\cdot))$  are all bounded and the process  $\hat{\Psi}(\cdot)$  is positive semidefinite, so we can choose a positive constant  $K$  such that

$$(40) \quad \begin{cases} \text{tr}[P(s, \alpha(s))] + |P(s, \alpha(s))|^2 \leq K, & \text{tr}[\hat{Q}(s, \alpha(s))] \leq K[1 + |\Lambda(s)|], \\ \text{tr}[P(s, \alpha(s))\hat{Q}(s, \alpha(s))] \leq |P(s, \alpha(s))| \cdot |\hat{Q}(s, \alpha(s))| \leq K[1 + |\Lambda(s)|], \\ \text{tr}[-P(s, \alpha(s))\hat{\Psi}(s, \alpha(s))] \leq \lambda_{\max}[-P(s, \alpha(s))] \text{tr}[\hat{\Psi}(s, \alpha(s))] \leq K \text{tr}[\hat{\Psi}(s, \alpha(s))], \end{cases}$$

for Lebesgue almost every  $s$ ,  $\mathbb{P}$ -a.s. In the last inequality, we adopt Theorem 7.4.1.1 of Horn and Johnson [5]. In the following, we denote by the same letter  $K$  a generic positive constant whose value may be different from line to line. Define for each  $m \geq 1$  the stopping time (with the convention  $\inf \emptyset = \infty$ )

$$(41) \quad \lambda_m = \inf \left\{ t \in [0, T]; \int_0^t [|\Lambda(s)|^2 + \sum_{k,l=1}^D |\zeta_{kl}(s)|^2 \lambda_{kl}(s) I_{\{\alpha(s^-)=k\}}] ds \geq m \right\},$$

which implies that  $\lim_{m \rightarrow \infty} \lambda_m = \infty$  almost surely. Then we have

$$(42) \quad \begin{aligned} P(t \wedge \lambda_m, \alpha(t \wedge \lambda_m)) &= P(0, i_0) + \int_0^{t \wedge \lambda_m} [\hat{\Psi}(s, \alpha(s)) - \hat{Q}(s, \alpha(s))] ds \\ &\quad + \int_0^{t \wedge \lambda_m} \Lambda(s) dW(s) + \int_0^{t \wedge \lambda_m} \zeta(s) \bullet d\tilde{N}(s). \end{aligned}$$

From the definition of  $\lambda_m$ , it is easy to see that the processes

$$\begin{aligned} \left\{ \int_0^{t \wedge \lambda_m} \Lambda(s) dW(s) \right\}_{t \in [0, T]} &= \left\{ \int_0^t \Lambda(s) \mathbf{1}_{\{s \leq \lambda_m\}} dW(s) \right\}_{t \in [0, T]}, \\ \left\{ \int_0^{t \wedge \lambda_m} \zeta(s) \bullet d\tilde{N}(s) \right\}_{t \in [0, T]} &= \left\{ \int_0^t \zeta(s) \mathbf{1}_{\{s \leq \lambda_m\}} \bullet d\tilde{N}(s) \right\}_{t \in [0, T]} \end{aligned}$$

are matrices of square-integrable martingales w.r.t. the filtration  $\mathbb{F} = \{\mathcal{F}_t; 0 \leq t < \infty\}$  and  $\mathbb{F}^\alpha = \{\mathcal{F}_t^\alpha; 0 \leq t < \infty\}$ , respectively. Therefore, taking expectations on both sides of (42), we have

$$\mathbb{E}[P(t \wedge \lambda_m, \alpha(t \wedge \lambda_m))] = P(0, i_0) + \mathbb{E} \int_0^{t \wedge \lambda_m} [\hat{\Psi}(s, \alpha(s)) - \hat{Q}(s, \alpha(s))] ds.$$

Thus, combining with (40), we see that

$$\begin{aligned} (43) \quad & \mathbb{E} \int_0^{t \wedge \lambda_m} \text{tr}[\hat{\Psi}(s, \alpha(s))] ds \\ &= \mathbb{E} \text{tr} [P(t \wedge \lambda_m, \alpha(t \wedge \lambda_m)) - P(0, i_0)] + \mathbb{E} \int_0^{t \wedge \lambda_m} \text{tr}[\hat{Q}(s, \alpha(s))] ds \\ &\leq K \left[ 1 + \mathbb{E} \int_0^{t \wedge \lambda_m} |\Lambda(s)| ds \right]. \end{aligned}$$

On the other hand, for BSDE (39), applying Itô's formula to  $P(s, \alpha(s))^2$  and denoting  $\zeta^2(s) \triangleq (\zeta_{kl}(s)^2)$ , we have

$$\begin{aligned} & d[P(s, \alpha(s))]^2 \\ &= \left\{ P(s, \alpha(s)) [\hat{\Psi}(s, \alpha(s)) - \hat{Q}(s, \alpha(s))] + [\hat{\Psi}(s, \alpha(s)) - \hat{Q}(s, \alpha(s))] P(s, \alpha(s)) \right. \\ &\quad \left. + \Lambda(s)^2 + \sum_{k,l=1}^D \zeta_{kl}^2(s) \lambda_{kl}(s) I_{\{\alpha(s-)=k\}} \right\} ds \\ &\quad + [P(s, \alpha(s)) \Lambda(s) + \Lambda(s) P(s, \alpha(s))] dW(t) \\ &\quad + \sum_{k,l=1}^D 2P(s-) \zeta_{kl}(s) d\tilde{N}_{kl}(s) + \zeta^2(s) \bullet d\tilde{N}(s). \end{aligned}$$

Now, note (41), a similar argument shows that

$$\begin{aligned} \mathbb{E}[P(s, \alpha(s))]^2 &= P(0, i_0) + \mathbb{E} \int_0^{t \wedge \lambda_m} \left\{ P(s, \alpha(s)) [\hat{\Psi}(s, \alpha(s)) - \hat{Q}(s, \alpha(s))] \right. \\ &\quad \left. + [\hat{\Psi}(s, \alpha(s)) - \hat{Q}(s, \alpha(s))] P(s, \alpha(s)) + \Lambda(s)^2 \right. \\ &\quad \left. + \sum_{k,l=1}^D \zeta_{kl}(s)^2 \lambda_{kl}(s) I_{\{\alpha(s-)=k\}} \right\} ds. \end{aligned}$$

Combining with (40)–(43) and recalling the Frobenius norm, we have

$$\begin{aligned} & \mathbb{E} \int_0^{t \wedge \lambda_m} |\Lambda(s)|^2 ds + \mathbb{E} \int_0^{t \wedge \lambda_m} \sum_{k,l=1}^D |\zeta_{kl}(s)|^2 \lambda_{kl}(s) I_{\{\alpha(s-)=k\}} ds \\ &= \text{tr} \left[ \mathbb{E} \int_0^{t \wedge \lambda_m} |\Lambda(s)|^2 ds + \mathbb{E} \int_0^{t \wedge \lambda_m} \sum_{k,l=1}^D [\zeta_{kl}(s)]^2 \lambda_{kl}(s) I_{\{\alpha(s-)=k\}} ds \right] \\ &= \mathbb{E} |P(t \wedge \lambda_m, \alpha(t \wedge \lambda_m))|^2 - |P(0, i_0)|^2 + 2\mathbb{E} \int_0^{t \wedge \lambda_m} \text{tr}[P(s, \alpha(s)) \hat{Q}(s, \alpha(s))] ds \\ &\quad + 2\mathbb{E} \int_0^{t \wedge \lambda_m} \text{tr}[-P(s, \alpha(s)) \hat{\Psi}(s, \alpha(s))] ds \end{aligned}$$

$$\begin{aligned} &\leq K + K\mathbb{E} \int_0^{t \wedge \lambda_m} [1 + |\Lambda(s)|] ds + K\mathbb{E} \int_0^{t \wedge \lambda_m} \text{tr}[\hat{\Psi}(s, \alpha(s))] ds \\ &\leq K + K\mathbb{E} \int_0^{t \wedge \lambda_m} |\Lambda(s)| ds \leq K + 2K^2 + \frac{1}{2}\mathbb{E} \int_0^{t \wedge \lambda_m} |\Lambda(s)|^2 ds, \end{aligned}$$

where in the last inequality we employ the Cauchy–Schwarz inequality. Hence, we obtain that

$$\frac{1}{2}\mathbb{E} \int_0^{t \wedge \lambda_m} |\Lambda(s)|^2 ds + \mathbb{E} \int_0^{t \wedge \lambda_m} \sum_{k,l=1}^D |\zeta_{kl}(s)|^2 \lambda_{kl}(s) I_{\{\alpha(s-)=k\}} ds \leq K + 2K^2.$$

Because  $\lim_{m \rightarrow \infty} \lambda_m = \infty$  almost surely and  $K$  does not depend on  $m$  and  $t$ , we conclude that the processes  $\Lambda(\cdot)$  and  $\zeta(\cdot)$  are square-integrable by letting  $m \rightarrow \infty$  first and then  $t \uparrow T$ .  $\square$

*Proof of Theorem 3.2.* Note that Propositions 3.9 and 3.10 imply that Problem (M-SLQ) is uniquely open-loop solvable at any initial time  $t < T$ , and the open-loop optimal control  $u^*(\cdot)$  w.r.t.  $(t, \xi, \vartheta)$  is given by

$$u^*(s) = (u_1(s), \dots, u_n(s)) \xi, \quad s \in [t, T].$$

Therefore, for the sake of the optimal control w.r.t. any initial triple  $(t, \xi, \vartheta) \in \mathcal{D}$ , it is sufficient to determine the open-loop optimal control  $u_j(\cdot) = \{u_j(s); s \in [0, T]\}$  w.r.t.  $(0, e_j, i_0)$  for each  $j = 1, \dots, n$ . From Lemma 4.1, we see that the process  $\mathbf{X}(\cdot) = \{\mathbf{X}(s); 0 \leq s \leq T\}$  is invertible. On the other hand, Lemma 4.3 tells us that finding the open-loop optimal controls  $u_1, \dots, u_n$  is equivalent to finding

$$\Theta(s, \alpha(s)) = \mathbf{U}(s)\mathbf{X}(s)^{-1}, \quad s \in [0, T].$$

The latter can be accomplished by solving SRE (5), whose solvability is obtained by Theorem 3.1. In fact, from the proof of Theorem 3.1, we see that  $\Theta(\cdot)$  is actually determined by (38). Summarizing these observations, we conclude the closed-loop representation (9).  $\square$

**5. Proof of some auxiliary results.** In this section, we give the proofs to some auxiliary results listed in subsection 3.2. For this, we let  $(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot), \tilde{\Gamma}(\cdot))$  and  $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), \bar{\Gamma}(\cdot))$  be the adapted solutions of the following decoupled FBSDEs, respectively:

$$(44) \quad \begin{cases} d\tilde{X}(s) = [A(s, \alpha(s))\tilde{X}(s) + B(s, \alpha(s))u(s)] ds \\ \quad + [C(s, \alpha(s))\tilde{X}(s) + D(s, \alpha(s))u(s)] dW(s), \\ d\tilde{Y}(s) = -\tilde{F}(s, \alpha(s), \tilde{X}(s), \tilde{Y}(s), \tilde{Z}(s), u(s)) ds \\ \quad + \tilde{Z}(s) dW(s) + \tilde{\Gamma}(s) \bullet d\tilde{N}(s), \quad s \in [t, T], \\ \tilde{X}(t) = 0, \quad \alpha(t) = \vartheta, \quad \tilde{Y}(T) = G(\alpha(T))\tilde{X}(T), \end{cases}$$

$$(45) \quad \begin{cases} d\bar{X}(s) = A(s, \alpha(s))\bar{X}(s) ds + C(s, \alpha(s))\bar{X}(s) dW(s), \quad s \in [t, T], \\ d\bar{Y}(s) = -\bar{F}_0(s, \alpha(s), \bar{X}(s), \bar{Y}(s), \bar{Z}(s)) ds + \bar{Z}(s) dW(s) + \bar{\Gamma}(s) \bullet d\tilde{N}(s), \\ \bar{X}(t) = \xi, \quad \alpha(t) = \vartheta, \quad \bar{Y}(T) = G(\alpha(T))\bar{X}(T). \end{cases}$$

Then the adapted solution  $(X(\cdot), Y(\cdot), Z(\cdot), \Gamma(\cdot))$  of SDE (1) and BSDE (10) could be written as the sum of  $(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot), \tilde{\Gamma}(\cdot))$  and  $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), \bar{\Gamma}(\cdot))$ , i.e., for  $s \in [t, T]$ ,

$$(46) \quad X(s) = \tilde{X}(s) + \bar{X}(s), \quad Y(s) = \tilde{Y}(s) + \bar{Y}(s), \quad Z(s) = \tilde{Z}(s) + \bar{Z}(s), \quad \Gamma(s) = \tilde{\Gamma}(s) + \bar{\Gamma}(s).$$

In what follows, for any  $u(\cdot), v(\cdot) \in \mathcal{U}[t, T]$ , we set

$$[[u, v]]_t = \mathbb{E}_t \int_t^T \langle u(s), v(s) \rangle ds, \quad [[u, v]] = \mathbb{E} \int_t^T \langle u(s), v(s) \rangle ds.$$

Now, for any  $u(\cdot) \in \mathcal{U}[t, T]$  and  $\xi \in \mathcal{X}_t \triangleq L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ , we define two linear operators  $\mathcal{N}_t$  and  $\mathcal{L}_t$  as follows: for  $s \in [t, T]$ ,

$$(47) \quad \begin{aligned} [\mathcal{N}_t u](s, \alpha(s)) &= F(s, \alpha(s), \tilde{X}(s), \tilde{Y}(s), \tilde{Z}(s), u(s)), \\ [\mathcal{L}_t \xi](s, \alpha(s)) &= F_0(s, \alpha(s), \tilde{X}(s), \tilde{Y}(s), \tilde{Z}(s)). \end{aligned}$$

First, we present a representation of the cost functional, which is characterized as a bilinear form on a Hilbert space in terms of the adapted solutions of some FBSDEs. Let us present the following two lemmas first, whose proofs can be found in [22].

LEMMA 5.1. *Under (H), for any initial triple  $(t, \xi, \vartheta) \in \mathcal{D}$  and a control  $u(\cdot) \in \mathcal{U}[t, T]$ , we have*

$$J(t, \xi, \vartheta; u(\cdot)) = \langle Y(t), \xi \rangle + \mathbb{E} \left[ \int_t^T \langle F(s, \alpha(s), X(s), Y(s), Z(s), u(s)), u(s) \rangle ds \Big| \mathcal{F}_t \right],$$

where the quadruple  $(X(\cdot), Y(\cdot), Z(\cdot), \Gamma(\cdot))$  is the solution of SDE (1) and BSDE (10).

LEMMA 5.2. *Let (H) hold; then the linear operator  $\mathcal{N}_t$  defined in (47) is a bounded self-adjoint operator and the linear operator  $\mathcal{L}_t$  defined in (47) is a bounded operator. Moreover, there exists a positive constant  $K$ , independent of  $(t, \xi, \vartheta)$ , such that*

$$(48) \quad [[\mathcal{L}_t \xi, \mathcal{L}_t \xi]] \leq K \mathbb{E} |\xi|^2 \quad \forall \xi \in \mathcal{X}_t.$$

Remark 5.3. Denote by  $(\mathcal{X}(\cdot), \mathcal{Y}(\cdot), \mathcal{Z}(\cdot), \check{\Gamma}(\cdot))$  the adapted solution of the following FBSDE:

$$\begin{cases} d\mathcal{X}(s) = A(s, \alpha(s))\mathcal{X}(s)ds + C(s, \alpha(s))\mathcal{X}(s)dW(s), \\ d\mathcal{Y}(s) = -\tilde{F}_0(s, \alpha(s), \mathcal{X}(s), \mathcal{Y}(s), \mathcal{Z}(s))ds \\ \quad + \mathcal{Z}(s)dW(s) + \check{\Gamma}(s) \bullet d\tilde{N}(s), \quad s \in [0, T], \\ \mathcal{X}(0) = I_n, \quad \alpha(0) = i_0, \quad \mathcal{Y}(T) = G(\alpha(T))\mathcal{X}(T). \end{cases}$$

It is easy to check that the process  $\mathcal{X}(\cdot)$  is invertible (denote by  $\mathcal{X}^{-1}(\cdot)$ ) with  $\mathcal{X}^{-1}(\cdot)$  satisfying the following equation:

$$\begin{cases} d\mathcal{X}^{-1}(s) = \mathcal{X}^{-1}(s)[C(s, \alpha(s))^2 - A(s, \alpha(s))]ds \\ \quad - \mathcal{X}^{-1}(s)C(s, \alpha(s))dW(s), \quad s \in [0, T], \\ \mathcal{X}^{-1}(0) = I_n, \quad \alpha(0) = i_0. \end{cases}$$

For every  $\xi \in L^\infty_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$  and  $\eta(s) \triangleq (\eta_{kl}(s)) \in \mathcal{M}_D(\mathbb{R}^{n \times n})$ , we set  $\eta(s) \circ \xi \triangleq (\eta_{kl}(s)\xi)$ . Now, on the other hand, the processes

$$\mathcal{X}(s)\mathcal{X}^{-1}(t)\xi, \quad \mathcal{Y}(s)\mathcal{X}^{-1}(t)\xi, \quad \mathcal{Z}(s)\mathcal{X}^{-1}(t)\xi, \quad \check{\Gamma}(s) \circ \mathcal{X}^{-1}(t)\xi$$

are all square-integrable and satisfy FBSDE (45). Therefore, by the uniqueness of the adapted solutions, one has that for  $s \in [t, T]$ ,

$$\bar{X}(s) = \mathcal{X}(s)\mathcal{X}^{-1}(t)\xi, \quad \bar{Y}(s) = \mathcal{Y}(s)\mathcal{X}^{-1}(t)\xi,$$

$$\bar{Z}(s) = Z(s)\mathcal{X}^{-1}(t)\xi, \quad \bar{\Gamma}(s) = \check{\Gamma}(s) \circ \mathcal{X}^{-1}(t)\xi.$$

Hence, for every  $\xi \in L^\infty_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ , the operator  $\mathcal{L}_t\xi$  could be represented in terms of the quadruple  $(\mathcal{X}(\cdot), \mathcal{Y}(\cdot), Z(\cdot), \check{\Gamma}(\cdot))$  as follows:

$$(49) \quad [\mathcal{L}_t\xi](s, \alpha(s)) = F_0(s, \alpha(s), \mathcal{X}(s), \mathcal{Y}(s), Z(s))\mathcal{X}^{-1}(t)\xi, \quad s \in [t, T].$$

This relation is useful in proving Theorem 3.12.

Now, we are ready to show the representation to the cost functional  $J_0(t, \xi, \vartheta; u(\cdot))$ . Note that  $J_0(t, \xi, \vartheta; u(\cdot))$  and  $J(t, \xi, \vartheta; u(\cdot))$  have the relation  $J_0(t, \xi, \vartheta; u(\cdot)) = \mathbb{E}J(t, \xi, \vartheta; u(\cdot))$ , and from Proposition 2.3, the first component  $M(\cdot)$  of BSDE (4) is bounded.

PROPOSITION 5.4. *Let  $\mathcal{N}_t$  and  $\mathcal{L}_t$  be defined in (47) and  $M(\cdot)$  be the first component of the solution of BSDE (4). Then, under (H), the cost functional  $J(t, \xi, \vartheta; u(\cdot))$  has the following representation:*

$$(50) \quad J(t, \xi, \vartheta; u(\cdot)) = [[\mathcal{N}_t u, u]]_t + 2[[\mathcal{L}_t\xi, u]]_t + \langle M(t)\xi, \xi \rangle \quad \forall (t, \xi, \vartheta) \in \mathcal{D}.$$

Furthermore, the cost functional  $J_0(t, \xi, \vartheta; u(\cdot))$  has the following representation:

$$(51) \quad J_0(t, \xi, \vartheta; u(\cdot)) = [[\mathcal{N}_t u, u]] + 2[[\mathcal{L}_t\xi, u]] + \mathbb{E}\langle M(t)\xi, \xi \rangle \quad \forall (t, \xi, \vartheta) \in \mathcal{D}.$$

*Proof.* Fix any  $(t, \xi, \vartheta) \in \mathcal{D}$  and  $u(\cdot) \in \mathcal{U}[t, T]$ . Let  $(\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot), \tilde{\Gamma}(\cdot))$ ,  $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), \bar{\Gamma}(\cdot))$ , and  $(X(\cdot), Y(\cdot), Z(\cdot), \Gamma(\cdot))$  be the adapted solutions of FBSDE (44), FBSDE (45), and SDE (1) and BSDE (10), respectively. Then the relation (46) holds between them. By Lemma 5.1 and the definitions of  $\mathcal{N}_t$  and  $\mathcal{L}_t$ , we have

$$(52) \quad J(t, \xi, \vartheta; u(\cdot)) = \langle \tilde{Y}(t), \xi \rangle + \langle \bar{Y}(t), \xi \rangle + \mathbb{E}_t \int_t^T \langle [\mathcal{N}_t u](s) + [\mathcal{L}_t\xi](s), u(s) \rangle ds.$$

On one hand, applying Itô's formula to  $\langle \tilde{Y}(s), \bar{X}(s) \rangle$  on  $[t, T]$  implies that

$$\begin{aligned} & \mathbb{E}_t \langle G(\alpha(T))\tilde{X}(T), \bar{X}(T) \rangle - \langle \tilde{Y}(t), \xi \rangle \\ &= -\mathbb{E}_t \int_t^T [\langle Q(s, \alpha(s))\tilde{X}(s), \bar{X}(s) \rangle + \langle S(s, \alpha(s))\bar{X}(s), u(s) \rangle] ds. \end{aligned}$$

On the other hand, applying Itô's formula to  $\langle \bar{Y}(s), \tilde{X}(s) \rangle$  on  $[t, T]$  gives that

$$\begin{aligned} & \mathbb{E}_t \langle G(\alpha(T))\bar{X}(T), \tilde{X}(T) \rangle \\ &= \mathbb{E}_t \int_t^T [\langle B(s, \alpha(s))^\top \bar{Y}(s) + D(s, \alpha(s))^\top \bar{Z}(s), u(s) \rangle - \langle Q(s, \alpha(s))\bar{X}(s), \tilde{X}(s) \rangle] ds. \end{aligned}$$

So we have

$$\langle \tilde{Y}(t), \xi \rangle = \mathbb{E}_t \int_t^T [\langle F_0(s, \alpha(s), \bar{X}(s), \bar{Y}(s), \bar{Z}(s)), u(s) \rangle] ds = \mathbb{E}_t \int_t^T \langle [\mathcal{L}_t\xi](s), u(s) \rangle ds.$$

Moreover, applying Itô's formula to  $M(s)\bar{X}(s)$ , we have

$$(53) \quad \begin{aligned} & d[M(s)\bar{X}(s)] \\ &= -[A(s, \alpha(s))^\top M(s)\bar{X}(s) + C(s, \alpha(s))^\top (M(s)C(s, \alpha(s)) \end{aligned}$$

$$\begin{aligned}
 & + \Phi(s))\bar{X}(s) + Q(s, \alpha(s))\bar{X}(s)] ds \\
 & + (M(s)C(s, \alpha(s)) + \Phi(s))\bar{X}(s)dW(s) + (\eta(s) \circ \bar{X}(s)) \bullet d\tilde{N}(s).
 \end{aligned}$$

Since  $M(T)\bar{X}(T) = \bar{Y}(T)$ , we see that the triple  $(M\bar{X}, (MC + \Phi)\bar{X}, \eta \circ \bar{X})$  satisfies the same BSDE as  $(\bar{Y}(\cdot), \bar{Z}(\cdot), \bar{\Gamma}(\cdot))$ . Thus, by comparing (45) and (53), we have

$$\begin{aligned}
 \bar{Y}(s) &= M(s)\bar{X}(s), \quad \bar{Z}(s) = (M(s)C(s, \alpha(s)) + \Phi(s))\bar{X}(s), \\
 \bar{\Gamma}(s) &= \eta(s) \circ \bar{X}(s), \quad s \in [t, T].
 \end{aligned}$$

It follows that  $\langle \bar{Y}(t), \xi \rangle = \langle M(t)\xi, \xi \rangle$ . Substituting this relation into (52) implies result (50). By taking expectation to (50), we can easily obtain (51).  $\square$

**COROLLARY 5.5.** *Under the condition (H), a control  $u^*(\cdot) \in \mathcal{U}[t, T]$  is optimal for Problem (M-SLQ)<sub>0</sub> w.r.t.  $(t, \xi, \vartheta) \in \mathcal{D}$  if and only if*

$$(54) \quad \mathcal{N}_t \geq 0, \quad \text{and} \quad \mathcal{N}_t u^* + \mathcal{L}_t \xi = 0.$$

Moreover, if  $\mathcal{N}_t$  is invertible and satisfies the positivity condition  $\mathcal{N}_t \geq 0$ , then Problem (M-SLQ)<sub>0</sub> is uniquely solvable at  $t$ , and the unique optimal control  $u^*(\cdot)$  w.r.t.  $(t, \xi, \vartheta) \in \mathcal{D}$  is given by

$$(55) \quad u^*(s) = -[\mathcal{N}_t^{-1} \mathcal{L}_t \xi](s), \quad s \in [t, T].$$

*Proof.* By Definition 2.2, we see that  $u^*(\cdot) \in \mathcal{U}[t, T]$  is optimal for Problem (M-SLQ)<sub>0</sub> w.r.t. the initial triple  $(t, \xi, \vartheta)$  if and only if

$$(56) \quad J_0(t, \xi, \vartheta; u^*(\cdot) + \lambda v(\cdot)) - J_0(t, \xi, \vartheta; u^*(\cdot)) \geq 0 \quad \forall v(\cdot) \in \mathcal{U}[t, T], \lambda \in \mathbb{R}.$$

According to the representation (51), one has

$$J_0(t, \xi, \vartheta; u^*(\cdot) + \lambda v(\cdot)) - J_0(t, \xi, \vartheta; u^*(\cdot)) = \lambda^2 [[\mathcal{N}_t v, v]] + 2\lambda [[\mathcal{N}_t u^*, v]] + 2\lambda [[\mathcal{L}_t \xi, v]].$$

Hence, (56) is equivalent to

$$\lambda^2 [[\mathcal{N}_t v, v]] + 2\lambda [[\mathcal{N}_t u^* + \mathcal{L}_t \xi, v]] \geq 0 \quad \forall v(\cdot) \in \mathcal{U}[t, T], \forall \lambda \in \mathbb{R}.$$

Therefore, we must have

$$[[\mathcal{N}_t v, v]] \geq 0 \quad \text{and} \quad [[\mathcal{N}_t u^* + \mathcal{L}_t \xi, v]] = 0 \quad \forall v \in \mathcal{U}[t, T],$$

which implies that (54) holds. The converse assertion is obvious. Finally, the representation (55) is a direct consequence of (54). This completes the proof.  $\square$

*Remark 5.6.* It is noteworthy that in Corollary 5.5, the assumptions that  $\mathcal{N}_t \geq 0$  and  $\mathcal{N}_t$  is invertible are equivalent to that  $\mathcal{N}_t$  is uniformly positive, i.e., there is a constant  $\varepsilon > 0$  such that

$$(57) \quad [[\mathcal{N}_t u, u]] \geq \varepsilon [[u, u]] \quad \forall u(\cdot) \in \mathcal{U}[t, T].$$

From Corollary 5.5, we see that the condition  $\mathcal{N}_t \geq 0$  (or equivalently,  $[[\mathcal{N}_t u, u]] \geq 0$  for every  $u(\cdot) \in \mathcal{U}[t, T]$ ) is necessary for the existence of an open-loop optimal control of Problem (M-SLQ)<sub>0</sub>, and the condition (57), slightly stronger than  $\mathcal{N}_t \geq 0$ , is sufficient for the existence of an open-loop optimal control of Problem (M-SLQ)<sub>0</sub>. Moreover, by (51), we have  $[[\mathcal{N}_t u, u]] = J_0(t, 0, \vartheta; u(\cdot))$ .

Based on the above discussions, we are in a position to prove some auxiliary results presented in subsection 3.2.

*Proof of Theorem 3.4.* By Corollary 5.5, we see that  $u^*(\cdot) \in \mathcal{U}[t, T]$  is an open-loop optimal control of Problem (M-SLQ)<sub>0</sub> w.r.t.  $(t, \xi, \vartheta)$  if and only if  $\mathcal{N}_t \geq 0$  and  $\mathcal{N}_t u^* + \mathcal{L}_t \xi = 0$  hold. According to (51), we see that  $\mathcal{N}_t \geq 0$  is equivalent to  $J_0(t, 0, \vartheta; u(\cdot)) = [[\mathcal{N}_t u, u]] \geq 0$  for any  $u(\cdot) \in \mathcal{U}[t, T]$ , which is exactly the condition of the assertion (i) of Theorem 3.4. From the definitions of  $\mathcal{N}_t$  and  $\mathcal{L}_t$ , one has

$$[\mathcal{N}_t u^* + \mathcal{L}_t \xi](s, \alpha(s)) = F(s, \alpha(s), X^*(s), Y^*(s), Z^*(s), u^*(s)), \quad s \in [t, T],$$

where  $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), \Gamma^*(\cdot))$  is the solution of FBSDE (12). Therefore,  $\mathcal{N}_t u^* + \mathcal{L}_t \xi = 0$  is equivalent to stationarity condition (11).  $\square$

*Proof of Proposition 3.6.* The sufficiency is obvious. Next we prove the necessity. Suppose that  $u^*(\cdot) \in \mathcal{U}[t, T]$  is optimal for Problem (M-SLQ)<sub>0</sub>, and let  $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), \Gamma^*(\cdot))$  be the adapted solution of FBSDE (12). In order to prove that  $u^*(\cdot)$  is also optimal for Problem (M-SLQ), it suffices to show that for any set  $\Lambda \in \mathcal{F}_t$ ,

$$(58) \quad \mathbb{E}[L(t, \xi, \vartheta; u^*(\cdot))\mathbf{1}_\Lambda] \leq \mathbb{E}[L(t, \xi, \vartheta; u(\cdot))\mathbf{1}_\Lambda] \quad \forall u(\cdot) \in \mathcal{U}[t, T],$$

where for any  $(t, \xi, \vartheta) \in \mathcal{D}$  and  $u(\cdot) \in \mathcal{U}[t, T]$ ,

$$(59) \quad L(t, \xi, \vartheta; u(\cdot)) \triangleq \langle G(\alpha(T))X(T), X(T) \rangle + \int_t^T \left\langle \begin{pmatrix} Q(s, \alpha(s)) & S(s, \alpha(s))^\top \\ S(s, \alpha(s)) & R(s, \alpha(s)) \end{pmatrix} \begin{pmatrix} X(s) \\ u(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \right\rangle ds.$$

Note that in (59),  $X(\cdot)$  is the solution of the state equation (1) w.r.t.  $u(\cdot)$ . For this, we would like to fix an arbitrary control  $u(\cdot) \in \mathcal{U}[t, T]$  and an arbitrary set  $\Lambda \in \mathcal{F}_t$ . Define

$$\widehat{\xi}(\omega) = \xi(\omega)\mathbf{1}_\Lambda(\omega), \quad \widehat{\vartheta}(\omega) = \vartheta(\omega)\mathbf{1}_\vartheta(\omega), \quad \widehat{u}(s) = u(s)\mathbf{1}_\Lambda(\omega), \quad \widehat{u}^*(s) = u^*(s)\mathbf{1}_\Lambda(\omega),$$

and consider the following FBSDE: for  $s \in [t, T]$ ,

$$\begin{cases} d\widehat{X}^*(s) = [A(s, \alpha(s))\widehat{X}^*(s) + B(s, \alpha(s))\widehat{u}^*(s)]ds \\ \quad + [C(s, \alpha(s))\widehat{X}^*(s) + D(s, \alpha(s))\widehat{u}^*(s)]dW(s), \\ d\widehat{Y}^*(s) = -\widehat{F}(s, \alpha(s), \widehat{X}^*(s), \widehat{Y}^*(s), \widehat{Z}^*(s), \widehat{u}^*(s))ds \\ \quad + \widehat{Z}^*(s)dW(s) + \widehat{\Gamma}^*(s) \bullet d\widetilde{N}(s), \\ \widehat{X}^*(t) = \widehat{\xi}, \quad \alpha(t) = \widehat{\vartheta}, \quad \widehat{Y}^*(T) = G(\alpha(T))\widehat{X}^*(T). \end{cases}$$

Then the solution  $(\widehat{X}^*(\cdot), \widehat{Y}^*(\cdot), \widehat{Z}^*(\cdot), \widehat{\Gamma}^*(\cdot))$  of the above FBSDE is given by

$$\widehat{X}^*(\cdot) = X^*(\cdot)\mathbf{1}_\Lambda(\omega), \quad \widehat{Y}^*(\cdot) = Y^*(\cdot)\mathbf{1}_\Lambda(\omega), \quad \widehat{Z}^*(\cdot) = Z^*(\cdot)\mathbf{1}_\Lambda(\omega), \quad \widehat{\Gamma}^*(\cdot) = \Gamma^*(\cdot)\mathbf{1}_\Lambda(\omega).$$

Now, by Theorem 3.4,  $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), \Gamma^*(\cdot))$  satisfies (11). Hence, multiplying on both sides of (11) by  $\mathbf{1}_\Lambda$ , one has that  $F(s, \alpha(s), \widehat{X}^*(s), \widehat{Y}^*(s), \widehat{Z}^*(s), \widehat{u}^*(s)) = 0$  for any  $s \in [t, T]$ . Again, applying Theorem 3.4 to the initial triple  $(t, \widehat{\xi}, \widehat{\vartheta})$ , we conclude that  $\widehat{u}^*(\cdot)$  is an open-loop optimal control of Problem (M-SLQ)<sub>0</sub> w.r.t. the initial triple  $(t, \widehat{\xi}, \widehat{\vartheta})$ . Therefore,  $\mathbb{E}[L(t, \widehat{\xi}, \widehat{\vartheta}; \widehat{u}^*(\cdot))] \leq \mathbb{E}[L(t, \widehat{\xi}, \widehat{\vartheta}; \widehat{u}(\cdot))]$ .

Note that the state process  $X^*(\cdot) = X^*(\cdot; t, \xi, \vartheta, u^*(\cdot))$  and the state process  $\widehat{X}^*(\cdot) = X^*(\cdot; t, \widehat{\xi}, \widehat{\vartheta}, \widehat{u}^*(\cdot))$  are related by  $X^*(\cdot; t, \xi, \vartheta, u^*(\cdot))\mathbf{1}_\Lambda = \widehat{X}^*(\cdot; t, \widehat{\xi}, \widehat{\vartheta}, \widehat{u}^*(\cdot))$ . It

follows that  $L(t, \xi, \vartheta; u^*(\cdot))\mathbf{1}_\Lambda = L(t, \widehat{\xi}, \widehat{\vartheta}; \widehat{u}^*(\cdot))$ . Similarly, we have  $L(t, \xi, \vartheta; u(\cdot))\mathbf{1}_\Lambda = L(t, \widehat{\xi}, \widehat{\vartheta}; \widehat{u}(\cdot))$ . Thus,

$$\mathbb{E}[L(t, \xi, \vartheta; u^*(\cdot))\mathbf{1}_\Lambda] = \mathbb{E}[L(t, \widehat{\xi}, \widehat{\vartheta}; \widehat{u}^*(\cdot))] \leq \mathbb{E}[L(t, \widehat{\xi}, \widehat{\vartheta}; \widehat{u}(\cdot))] = \mathbb{E}[L(t, \xi, \vartheta; u(\cdot))\mathbf{1}_\Lambda],$$

from which we see that (58) holds. This completes the proof.  $\square$

Based on Theorem 3.4 and Proposition 3.6, we have Corollaries 3.7 and 3.8, whose proofs can be found in [22].

*Proof of Proposition 3.9.* For any  $u(\cdot) \in \mathcal{U}[\tau, T]$ , we define  $u_e(s) = 0$  when  $s \in [t, \tau]$  and  $u_e(s) = u(s)$  when  $s \in [\tau, T]$ . Using the same argument as in the proof of Corollary 3.8 with  $t = 0$ , one has

$$J_0(\tau, 0, \alpha(\tau); u(\cdot)) = J_0(0, 0, i_0; u_e(\cdot)) \geq \varepsilon \mathbb{E} \int_0^T |u_e(s)|^2 ds = \varepsilon \mathbb{E} \int_\tau^T |u(s)|^2 ds.$$

Hence, by setting  $\tau \equiv t$  and from Remark 5.6, Corollary 5.5, and Proposition 3.6, we obtain that Problem (M-SLQ) is uniquely solvable.  $\square$

*Proof of Proposition 3.10.* By Theorem 3.4, it is easy to see that the first assertion holds. Now, we consider the second assertion. Note that  $\xi \in L^\infty_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$  is bounded, so the pair  $(\mathbb{X}(s), \mathbb{U}(s)) \triangleq (\mathbf{X}(s)\xi, \mathbf{U}(s)\xi)$  is square-integrable and satisfies the following state equation: for  $s \in [t, T]$ ,

$$(60) \quad \begin{cases} d\mathbb{X}(s) = [A(s, \alpha(s))\mathbb{X}(s) + B(s, \alpha(s))\mathbb{U}(s)] ds \\ \quad + [C(s, \alpha(s))\mathbb{X}(s) + D(s, \alpha(s))\mathbb{U}(s)] dW(s), \\ \mathbb{X}(t) = \xi, \quad \alpha(t) = \vartheta. \end{cases}$$

Similarly, we see that the triple  $(\mathbb{Y}(s), \mathbb{Z}(s), \check{\Gamma}(s)) \triangleq (\mathbf{Y}(s)\xi, \mathbf{Z}(s)\xi, \mathbf{\Gamma}(s) \circ \xi)$  is the adapted solution to the following adjoint BSDE associated with  $(\mathbb{X}(\cdot), \mathbb{U}(\cdot))$ :

$$\begin{cases} d\mathbb{Y}(s) = -\tilde{F}(s, \alpha(s), \mathbb{X}(s), \mathbb{Y}(s), \mathbb{Z}(s), \mathbb{U}(s)) ds \\ \quad + \mathbb{Z}(s) dW(s) + \check{\Gamma}(s) \bullet d\tilde{N}(s), \quad s \in [t, T], \\ \mathbb{Y}(T) = G(\alpha(T))\mathbb{X}(T), \quad \alpha(t) = \vartheta. \end{cases}$$

Furthermore, (15) deduces that

$$\begin{aligned} & F(s, \alpha(s), \mathbb{X}(s), \mathbb{Y}(s), \mathbb{Z}(s), \mathbb{U}(s)) \\ & = F(s, \alpha(s), \mathbf{X}(s), \mathbf{Y}(s), \mathbf{Z}(s), \mathbf{U}(s))\xi = 0, \quad \text{a.e. } s \in [t, T], \text{ a.s.} \end{aligned}$$

Thus, combining Theorem 3.4, the pair  $(\mathbb{X}(\cdot), \mathbb{U}(\cdot))$  is optimal w.r.t. the initial triple  $(t, \xi, \vartheta)$ .  $\square$

*Proof of Theorem 3.11.* Let  $\{(X_j(s), u_j(s))\}_{s \in [t, T]}$  and  $\{(\mathbf{X}(s), \mathbf{U}(s))\}_{s \in [t, T]}$  be as in Theorem 3.10; then the pair  $\{(\mathbf{X}(s)\xi, \mathbf{U}(s)\xi)\}_{s \in [t, T]}$  is optimal w.r.t. the initial triple  $(t, \xi, \vartheta)$ . For simplicity of presentation, we denote

$$\begin{aligned} \mathbf{M}(T, \alpha(T)) & \triangleq \mathbf{X}(T)^\top G(\alpha(T))\mathbf{X}(T), \\ \mathbf{N}(s, \alpha(s)) & \triangleq \begin{pmatrix} \mathbf{X}(s) \\ \mathbf{U}(s) \end{pmatrix}^\top \begin{pmatrix} Q(s, \alpha(s)) & S(s, \alpha(s))^\top \\ S(s, \alpha(s)) & R(s, \alpha(s)) \end{pmatrix} \begin{pmatrix} \mathbf{X}(s) \\ \mathbf{U}(s) \end{pmatrix}, \end{aligned}$$

and then we could rewrite

$$(61) \quad L(t, \xi, \vartheta; \mathbf{U}\xi) = \langle \mathbf{M}(T, \alpha(T))\xi, \xi \rangle + \int_t^T \langle \mathbf{N}(s, \alpha(s))\xi, \xi \rangle ds,$$



where  $L(\cdot)$  is defined in (59). Now, on one hand, since the pair  $\{(\mathbf{X}(s)\xi, \mathbf{U}(s)\xi)\}_{s \in [t, T]}$  is optimal w.r.t.  $(t, \xi, \vartheta)$ , from the definition (2) we have  $V(t, \xi, \vartheta) = \mathbb{E}[L(t, \xi, \vartheta; \mathbf{U}\xi) | \mathcal{F}_t]$ . On the other hand, note that SDE (60) works over  $[t, T]$  where the initial value  $\xi$  is  $\mathcal{F}_t$ -measurable and  $\alpha(t) = \vartheta$  is  $\mathcal{F}_t^\alpha$ -measurable, hence there is a process  $P : [0, T] \times \mathcal{S} \times \Omega \rightarrow \mathbb{S}^n$  such that

$$(62) \quad P(t, \alpha(t)) = \mathbb{E} \left[ \mathbf{M}(T, \alpha(T)) + \int_t^T \mathbf{N}(s, \alpha(s)) ds \middle| \mathcal{F}_t \right].$$

In fact, for the special case of  $C = G = 1$  and  $B = D = Q = S = R = 0$ , one can calculate the solution of SDE (60) to obtain that

$$\mathbb{X}(s) = \xi \exp \left\{ \int_t^s \left[ A(r, \alpha(r)) - \frac{1}{2} \right] dr + W(s) - W(t) \right\}, \quad s \in [t, T].$$

Note that  $\mathbb{X}(s) = \mathbf{X}(s)\xi$ , and in this case  $\mathbf{M}(T, \alpha(T)) = \mathbf{X}(T)^\top \mathbf{X}(T)$  and  $\mathbf{N}(s, \alpha(s)) \equiv 0$ , so

$$\mathbb{E}[\mathbf{M}(T, \alpha(T)) | \mathcal{F}_t] = \mathbb{E} \left[ \exp \left\{ \int_t^T [2A(r, \alpha(r)) - 1] dr \right\} \middle| \mathcal{F}_t \right].$$

Thus (62) holds because  $\alpha(\cdot)$  is a Markov chain. Moreover, for the general situation, one can still prove that (62) holds using a similar argument. Finally, by combining (61)–(62) and noting that  $\alpha(t) = \vartheta$ , we have

$$\begin{aligned} V(t, \xi, \vartheta) &= \mathbb{E}[L(t, \xi, \vartheta; \mathbf{U}\xi) | \mathcal{F}_t] \\ &= \left\langle \mathbb{E} \left[ \mathbf{M}(T, \alpha(T)) + \int_t^T \mathbf{N}(s, \alpha(s)) ds \middle| \mathcal{F}_t \right] \xi, \xi \right\rangle = \langle P(t, \vartheta)\xi, \xi \rangle. \end{aligned}$$

This completes the proof. □

*Proof of Theorem 3.12.* First, we prove the boundedness of the process  $P = \{P(t, i); (t, i) \in [0, T] \times \mathcal{S}\}$ . From Proposition 3.9, we have that for every  $t \in [0, T]$ , the operator  $\mathcal{N}_t$  defined in (47) satisfies

$$(63) \quad [[\mathcal{N}_t u, u]] = J_0(t, 0, \vartheta; u(\cdot)) \geq \varepsilon \mathbb{E} \int_t^T |u(s)|^2 ds = \varepsilon [[u, u]] \quad \forall u(\cdot) \in \mathcal{U}[t, T],$$

which implies that the operator  $\mathcal{N}_t$  is positive and invertible. Moreover, on one hand, for any initial state  $\xi \in L^\infty_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ , by Corollary 5.5, the related open-loop optimal control is given by

$$u_{t, \xi}^*(s) = -[\mathcal{N}_t^{-1} \mathcal{L}_t \xi](s), \quad s \in [t, T].$$

Then, by substituting  $u_{t, \xi}^*(\cdot)$  into (51), one has

$$(64) \quad \mathbb{E} \langle P(t, \vartheta)\xi, \xi \rangle = V_0(t, \xi, \vartheta) = \mathbb{E} \langle M(t)\xi, \xi \rangle - [[\mathcal{N}_t^{-1} \mathcal{L}_t \xi, \mathcal{L}_t \xi]],$$

where  $\mathcal{N}_t$  and  $\mathcal{L}_t$  are defined in (47) and  $M(\cdot)$  is the solution of BSDE (4). Therefore,

$$(65) \quad \mathbb{E} \langle P(t, \vartheta)\xi, \xi \rangle \leq \mathbb{E} \langle M(t)\xi, \xi \rangle.$$

On the other hand, combining (63) and (64) leads to

$$(66) \quad \mathbb{E} \langle P(t, \vartheta)\xi, \xi \rangle \geq \mathbb{E} \langle M(t)\xi, \xi \rangle - \varepsilon^{-1} [[\mathcal{L}_t \xi, \mathcal{L}_t \xi]].$$

Thus, from Lemma 5.2, we have

$$(67) \quad \mathbb{E}\langle P(t, \vartheta)\xi, \xi \rangle \geq \mathbb{E}\langle M(t)\xi, \xi \rangle - \varepsilon^{-1}K\mathbb{E}|\xi|^2 = \mathbb{E}\langle [M(t) - \varepsilon^{-1}KI_n] \xi, \xi \rangle,$$

where  $K$  is a positive constant as in Lemma 5.2. Note that  $\xi \in L^\infty_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$  is bounded and arbitrary, which, together with (65) and (67), implies that

$$M(t) - \varepsilon^{-1}KI_n \leq P(t, \vartheta) \leq M(t), \quad (t, \vartheta) \in [0, T] \times L^2_{\mathcal{F}_t}(\Omega; \mathcal{S}).$$

Therefore, the process  $P = \{P(t, i); (t, i) \in [0, T] \times \mathcal{S}\}$  is bounded as follows from the above inequality and the fact that  $M(\cdot)$  is bounded (see Proposition 2.3).

In the following, we prove that it is left-continuous. For simplicity but without loss of generality, we only study the left-continuity at  $t = T$ , and the situation of  $t \in (0, T)$  could be proved similarly by considering the related Problem (M-SLQ). Note that, owing to (65) and (66), for every initial triple  $(t, \xi, \vartheta) \in [0, T] \times L^\infty(\Omega; \mathbb{R}^n) \times L^2_{\mathcal{F}_t}(\Omega; \mathcal{S})$ , we have

$$(68) \quad \mathbb{E}\langle M(t)\xi, \xi \rangle - \varepsilon^{-1}[[\mathcal{L}_t\xi, \mathcal{L}_t\xi]] \leq \mathbb{E}\langle P(t, \vartheta)\xi, \xi \rangle \leq \mathbb{E}\langle M(t)\xi, \xi \rangle.$$

Noting representation (49), we can rewrite  $[[\mathcal{L}_t\xi, \mathcal{L}_t\xi]]$  as follows:

$$(69) \quad [[\mathcal{L}_t\xi, \mathcal{L}_t\xi]] = \mathbb{E} \int_t^T \left\langle [F_0(s, \alpha(s), \mathcal{X}(s), \mathcal{Y}(s), \mathcal{Z}(s))\mathcal{X}^{-1}(t)]^\top \cdot [F_0(s, \alpha(s), \mathcal{X}(s), \mathcal{Y}(s), \mathcal{Z}(s))\mathcal{X}^{-1}(t)]\xi, \xi \right\rangle ds.$$

Again, note that  $\xi \in L^\infty_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$  is bounded and arbitrary, which, together with (68) and (69), implies that for any  $(t, \vartheta) \in [0, T] \times L^2_{\mathcal{F}_t}(\Omega; \mathcal{S})$ ,

$$\begin{aligned} & M(t) - \varepsilon^{-1}[\mathcal{X}^{-1}(t)]^\top \\ & \quad \times \int_t^T F_0(s, \alpha(s), \mathcal{X}(s), \mathcal{Y}(s), \mathcal{Z}(s))^\top F_0(s, \alpha(s), \mathcal{X}(s), \mathcal{Y}(s), \mathcal{Z}(s)) ds \mathcal{X}^{-1}(t) \\ & \leq P(t, \vartheta) \leq M(t). \end{aligned}$$

Note that  $\mathcal{X}^{-1}(t)$  and  $M(\cdot)$  are  $\mathcal{F}_t$ -measurable, and  $P(\cdot, \alpha(\cdot)) \in L^\infty_{\mathbb{F}}(0, T; \mathbb{S}^n)$ . Hence, by taking the conditional expectations w.r.t.  $\mathcal{F}_t$  on both sides of the above inequality, one has

$$\begin{aligned} & M(t) - \frac{1}{\varepsilon}[\mathcal{X}^{-1}(t)]^\top \\ & \quad \times \mathbb{E} \left[ \int_t^T F_0(s, \alpha(s), \mathcal{X}(s), \mathcal{Y}(s), \mathcal{Z}(s))^\top F_0(s, \alpha(s), \mathcal{X}(s), \mathcal{Y}(s), \mathcal{Z}(s)) ds \middle| \mathcal{F}_t \right] \mathcal{X}^{-1}(t) \\ & \leq P(t, i) \leq M(t). \end{aligned}$$

Finally, letting  $t \uparrow T$  and using the dominated convergence theorem, we get that

$$\lim_{t \uparrow T} P(t, i) = M(T) = G(i) = P(T, i).$$

Hence, the second purpose is obtained. □

*Proof of Proposition 3.13.* First, we prove the assertion (i). For any fixed initial triple  $(\sigma, \xi, \vartheta) \in \mathcal{T}[0, \tau] \times L^\infty_{\mathcal{F}_\sigma}(\Omega; \mathbb{R}^n) \times L^2_{\mathcal{F}_\sigma}(\Omega; \mathcal{S})$  and  $u(\cdot) \in \mathcal{U}[\sigma, \tau]$ , denote by

$\{X_1(s)\}_{s \in [\sigma, \tau]}$  the corresponding solution of state equation (1) w.r.t.  $(\sigma, \xi, \vartheta)$  over  $[\sigma, \tau]$ . Consider Problem (M-SLQ) for the initial triple  $(\tau, X_1(\tau), \alpha(\tau))$ . By the assumptions, there exists  $\varepsilon > 0$  such that (7) holds. Hence, from Proposition 3.9 and representation (16), Problem (M-SLQ) is solvable at  $\tau$  and  $V(\tau, X_1(\tau), \alpha(\tau)) = \langle P(\tau, \alpha(\tau))X_1(\tau), X_1(\tau) \rangle$ .

Now for Problem (M-SLQ), let  $v^*(\cdot) \in \mathcal{U}[\tau, T]$  be an open-loop optimal control w.r.t. the initial triple  $(\tau, X_1(\tau), \alpha(\tau))$  and let  $\{X_2^*(s)\}_{s \in [\tau, T]}$  be the corresponding open-loop optimal state process. Define

$$[u \oplus v^*](s) = \begin{cases} u(s), & s \in [\sigma, \tau], \\ v^*(s), & s \in [\tau, T], \end{cases} \quad \text{and} \quad \tilde{X}(s) = \begin{cases} X_1(s), & s \in [\sigma, \tau], \\ X_2^*(s), & s \in [\tau, T]. \end{cases}$$

Then, it is easy to see that  $[u \oplus v^*](\cdot) \in \mathcal{U}[\sigma, T]$  and  $\tilde{X}(\cdot)$  satisfies the following equation:

$$\begin{cases} d\tilde{X}(s) = [A(s, \alpha(s))\tilde{X}(s) + B(s, \alpha(s))[u \oplus v^*](s)] ds \\ \quad + [C(s, \alpha(s))\tilde{X}(s) + D(s, \alpha(s))[u \oplus v^*](s)] dW(s), \\ \tilde{X}(\sigma) = \xi, \quad \alpha(\sigma) = \vartheta. \end{cases}$$

Furthermore, we have

$$\begin{aligned} (70) \quad & J_0(\sigma, \xi, \vartheta; [u \oplus v^*](\cdot)) \\ &= \mathbb{E} \left[ \langle G(\alpha(T))X_2^*(T), X_2^*(T) \rangle \right. \\ &\quad + \int_{\tau}^T \left\langle \begin{pmatrix} Q(s, \alpha(s)) & S(s, \alpha(s))^\top \\ S(s, \alpha(s)) & R(s, \alpha(s)) \end{pmatrix} \begin{pmatrix} X_2^*(s) \\ v^*(s) \end{pmatrix}, \begin{pmatrix} X_2^*(s) \\ v^*(s) \end{pmatrix} \right\rangle ds \Big] \\ &\quad + \mathbb{E} \left[ \int_{\sigma}^{\tau} \left\langle \begin{pmatrix} Q(s, \alpha(s)) & S(s, \alpha(s))^\top \\ S(s, \alpha(s)) & R(s, \alpha(s)) \end{pmatrix} \begin{pmatrix} X_1(s) \\ u(s) \end{pmatrix}, \begin{pmatrix} X_1(s) \\ u(s) \end{pmatrix} \right\rangle ds \right] \\ &= \mathbb{E} \left[ V(\tau, X_1(\tau), \alpha(\tau)) \right. \\ &\quad + \int_{\sigma}^{\tau} \left\langle \begin{pmatrix} Q(s, \alpha(s)) & S(s, \alpha(s))^\top \\ S(s, \alpha(s)) & R(s, \alpha(s)) \end{pmatrix} \begin{pmatrix} X_1(s) \\ u(s) \end{pmatrix}, \begin{pmatrix} X_1(s) \\ u(s) \end{pmatrix} \right\rangle ds \Big] \\ &= J_0^\tau(\sigma, \xi, \vartheta; u(\cdot)), \end{aligned}$$

where in the last equation we employ relation (16) and the definition of  $J_0^\tau$ . In particular, when  $\xi = 0$ , from Proposition 3.9 we have

$$J_0^\tau(\sigma, 0, \vartheta; u(\cdot)) = J_0(\sigma, 0, \vartheta; [u \oplus v^*](\cdot)) \geq \varepsilon \mathbb{E} \int_{\sigma}^T |[u \oplus v^*](s)|^2 ds \geq \varepsilon \mathbb{E} \int_{\sigma}^{\tau} |u(s)|^2 ds,$$

which combining Remark 5.6, Corollary 5.5, and Proposition 3.6 implies assertion (i) holds.

Now, we consider assertions (ii) and (iii). Note that (70) still holds when the expectation  $\mathbb{E}$  replaced by the conditional expectation  $\mathbb{E}_\sigma[\cdot]$ . Therefore,  $J^\tau(\sigma, \xi, \vartheta; u(\cdot)) \geq \langle P(\sigma, \vartheta)\xi, \xi \rangle$ . For Problem (M-SLQ), assume that  $u^*(\cdot) \in \mathcal{U}[\sigma, T]$  is an open-loop optimal control w.r.t.  $(\sigma, \xi, \vartheta)$ , and denote by  $X^* = \{X^*(s); \sigma \leq s \leq T\}$  the related open-loop optimal state process, i.e., for  $s \in [\sigma, T]$ ,

$$\begin{cases} dX^*(s) = [A(s, \alpha(s))X^*(s) + B(s, \alpha(s))u^*(s)]ds \\ \quad + [C(s, \alpha(s))X^*(s) + D(s, \alpha(s))u^*(s)]dW(s), \\ X^*(\sigma) = \xi, \quad \alpha(\sigma) = \vartheta. \end{cases}$$

Then from Corollary 3.8, we see that for Problem (M-SLQ), the restriction  $.u^*(\cdot)|_{[\tau, T]}$  of  $u^*(\cdot)$  on the interval  $[\tau, T]$  is optimal for the initial triple  $(\tau, X^*(\tau), \alpha(\tau))$ . Moreover, in (70), using  $.u^*(\cdot)|_{[\sigma, \tau]}$  and  $.u^*(\cdot)|_{[\tau, T]}$  to replace  $u(\cdot)$  and  $v^*(\cdot)$ , respectively, and noting that  $u^*(\cdot)|_{[\sigma, \tau]} \oplus u^*(\cdot)|_{[\tau, T]} = u^*(\cdot)$ , we deduce

$$J^T(\sigma, \xi, \vartheta; u^*(\cdot)|_{[\sigma, \tau]}) = J(\sigma, \xi, \vartheta; u^*(\cdot)) = \langle P(\sigma, \vartheta)\xi, \xi \rangle.$$

Therefore, the assertions (ii) and (iii) hold.  $\square$

**6. Example.** As presented in the introduction, in a realistic market, it is better to allow the market parameters to depend on both the Markov chain and Brownian motion, due to the interest rates, stock rates, and volatilities being affected by the uncertainties caused by the Brownian motion. In this section, as an application of our main results, we give an example of the continuous-time mean-variance portfolio selection problem, which partially develops the work of Li, Zhou, and Lim [13] to the Markovian regime switching system with random coefficients. For simplicity, we would like to let  $m = n = 1$ .

Suppose there is a market in which two assets are traded over a finite horizon  $[0, T]$ . One of the assets is the bond whose asset price  $S_0(\cdot)$  is subject to the following ordinary differential equation:

$$\begin{cases} dS_0(s) = r(s)S_0(s)ds, & s \in [0, T], \\ S_0(0) = S_0 > 0, \end{cases}$$

where  $r(\cdot)$  is a positive and bounded function, which represents the bond's interest rate. The other one of the assets is stock whose asset price  $S_1(\cdot)$  satisfies the following SDE:

$$\begin{cases} dS_1(s) = S_1(s)\{b(s, \alpha(s))dt + \sigma(s, \alpha(s))dW(s)\}, & s \in [0, T], \\ S_1(0) = S_1 > 0, \quad \alpha(0) = i_0, \end{cases}$$

where  $b(s, \alpha(s))$  is the appreciation rate and  $\sigma(s, \alpha(s))$  is the volatility or the dispersion of the stock. Assume that for any choice of  $i \in \mathcal{S}$ , both  $b(\cdot, i)$  and  $\sigma(\cdot, i)$  are in the space  $L_{\mathbb{F}^W}^\infty(0, T; \mathbb{R})$ , and there is a positive constant  $\delta$  such that  $\sigma(s, \alpha(s))^2 \geq \delta$  for any  $s \in [0, T]$ . Assume that the trading of shares takes place continuously and transaction costs and consumption are not considered. Then, a small investor's self-financing wealth process  $X(\cdot)$  satisfies the following SDE:

$$(71) \quad \begin{cases} dX(s) = \{r(s)X(s) + [b(s, \alpha(s)) - r(s)]u(s)\}ds + \sigma(s, \alpha(s))u(s)dW(s), & s \in [0, T], \\ X(0) = x_0 > 0, \quad \alpha(0) = i_0, \end{cases}$$

where  $u(\cdot) \in L_{\mathbb{F}^W}^2(0, T; \mathbb{R})$  is a portfolio of the investor, which may change over time  $s \in [0, T]$ . Note that  $u(s) = 0$  implies that the investor invests his/her total wealth in the bond at time  $s \in [0, T]$ .

As shown by Li, Zhou, and Lim [13], the mean-variance portfolio selection refers to the problem of finding an allowable investment policy (i.e., a dynamic portfolio satisfying all the constraints) such that the expected terminal wealth satisfies  $\mathbb{E}[X(T)] = d$  while the risk measured by the variance of the terminal wealth is

$$\text{Var}(X(T)) = \mathbb{E}[X(T) - \mathbb{E}[X(T)]]^2 = \mathbb{E}[(X(T) - d)^2].$$

Then we consider the following dynamic stochastic optimization:

$$(72) \quad \begin{cases} \min \mathbb{E}[(X(T) - d)^2], \\ \text{s.t. } \mathbb{E}[X(T)] = d, \\ (X(\cdot), u(\cdot)) \text{ satisfies (7.2)}. \end{cases}$$

Since (72) is a convex optimization problem, the equality constraint  $\mathbb{E}[X(T)] = d$  can be dealt with by introducing a Lagrange multiplier  $\mu \in \mathbb{R}$ . Therefore, we have

$$\mathbb{E}[(X(T) - d)^2] - 2\mu(\mathbb{E}[X(T)] - d) = \mathbb{E}[(X(T) - (d + \mu))^2] - \mu^2 = \mathbb{E}[X(T) - \gamma]^2 - \mu^2,$$

where  $\gamma = d + \mu$ . Now, if we set

$$\tilde{X}(s) = X(s) - \gamma \exp \left\{ - \int_s^T r(s) ds \right\},$$

then (72) can be transferred into the following problem:

$$(73) \quad \begin{cases} \min \mathbb{E}[\tilde{X}(T)^2] \\ \text{s.t. } d\tilde{X}(s) = \left\{ r(s)\tilde{X}(s) + [b(s, \alpha(s)) - r(s)]u(s) \right\} ds \\ \quad + \sigma(s, \alpha(s))u(s)dW(s), \quad s \in [0, T], \\ \tilde{X}(0) = x_0 - \gamma \exp \left\{ - \int_0^T r(s) ds \right\}, \quad \alpha(0) = i_0. \end{cases}$$

It is easy to see that problem (73) is a special case of Problem (M-SLQ) with

$$\begin{aligned} G(\alpha(T)) &= 1, \quad A(s, \alpha(s)) = r(s), \quad B(s, \alpha(s)) = b(s, \alpha(s)) - r(s), \\ D(s, \alpha(s)) &= \sigma(s, \alpha(s)), \quad s \in [0, T], \end{aligned}$$

and other coefficients are zero. Then the SRE associated to problem (73) is, for  $s \in [0, T]$ ,

$$(74) \quad \begin{cases} d\hat{P}(s, \alpha(s)) = - [\hat{Q}(s, \alpha(s)) + \hat{S}(s, \alpha(s))^\top \Theta(s, \alpha(s))] ds + \hat{\Lambda}(s) dW(s) + \hat{\zeta}(s) \bullet d\tilde{N}(s), \\ \hat{P}(T, \alpha(T)) = 1, \quad \alpha(0) = i_0, \end{cases}$$

where  $i_0 \in \mathcal{S}$  and for  $s \in [0, T]$ ,

$$(75) \quad \begin{aligned} \hat{Q}(s, \alpha(s)) &\triangleq 2r(s)\hat{P}(s, \alpha(s)), \\ \hat{S}(s, \alpha(s)) &\triangleq [b(s, \alpha(s)) - r(s)]\hat{P}(s, \alpha(s)) + \sigma(s, \alpha(s))\hat{\Lambda}(s), \\ \hat{R}(s, \alpha(s)) &\triangleq \sigma(s, \alpha(s))^2\hat{P}(s, \alpha(s)), \quad \Theta(s, \alpha(s)) \triangleq -\hat{R}(s, \alpha(s))^{-1}\hat{S}(s, \alpha(s)). \end{aligned}$$

Now, we point out that conditions (H) and (7) hold for this case. In fact, it is trivial to verify condition (H). As for condition (7), by solving the linear SDE of (73), it is easy to verify that there is a positive constant  $\varepsilon$  such that

$$\mathbb{E}[\tilde{X}(T)^2] \geq \varepsilon \mathbb{E} \int_0^T |u(s)|^2 ds \quad \forall u(\cdot) \in \mathcal{U}[0, T].$$

Therefore, Theorems 3.1 and 3.2 deduce the following conclusion concerning the mean-variance portfolio selection problem.

**THEOREM 6.1.** *SRE (74) admits a unique adapted solution  $(\hat{P}(\cdot, \alpha(\cdot)), \hat{\Lambda}(\cdot), \hat{\zeta}(\cdot)) \in L_{\mathbb{F}}^{\infty}(0, T; \mathbb{R}) \times L_{\mathbb{F}}^2(0, T; \mathbb{R}) \times L_{\mathbb{F}}^2(0, T; \mathbb{R})$  and the mean-variance problem (73) is uniquely solvable. Moreover, the unique optimal investment strategy  $u^*(\cdot)$  has the representation*

$$u^*(s) = \Theta(s, \alpha(s))X^*(s), \quad s \in [0, T],$$

where  $\Theta(\cdot)$  is defined in (75) and  $X^*(\cdot)$  is the solution of the closed-loop system: for  $s \in [0, T]$ ,

$$\begin{cases} dX^*(s) = \{r(s) + [b(s, \alpha(s)) - r(s)]\Theta(s, \alpha(s))\}X^*(s)ds \\ \quad + \sigma(s, \alpha(s))\Theta(s, \alpha(s))X^*(s)dW(s), \\ X^*(0) = x(0) - \gamma \exp\left\{-\int_0^T r(s)ds\right\}, \quad \alpha(0) = i_0. \end{cases}$$

*Remark 6.2.* The above result partially develops the mean-variance problems of Li, Zhou, and Lim [13] to the Markovian regime switching system with random coefficients, which is significantly different from that of Sun, Xiong, and Yong [17]. Note that we consider the one-dimensional state case of two assets (one bond and one stock) to be just for simplicity, and the multidimensional case of  $m + 1$  assets (one bond and  $m$  stocks) can be proved using a similar argument.

**7. Conclusion.** This paper develops the work of Sun, Xiong, and Yong [17] to the framework within the Markovian regime switching system, obtains the solvability of SRE (5) with jumps and random coefficients, gets the closed-loop representation of the open-loop optimal control, and gives a financial application of the continuous-time mean-variance portfolio selection problem, which develops the work of Li, Zhou, and Lim [13]. In addition, a new point of view for the uniform convexity of the cost functional is presented, and the equivalence between Problem (M-SLQ)<sub>0</sub> and Problem (M-SLQ) is obtained. Note that it remains open if someone could get the closed-loop solvability in this model.

**Acknowledgments.** The authors would like to thank the associate editor and the anonymous referees for their insightful comments that improved the quality of this paper.

#### REFERENCES

- [1] M. AIT RAMI, J. MOORE, AND X. ZHOU, *Indefinite stochastic linear quadratic control and generalized differential Riccati equation*, SIAM J. Control Optim., 40 (2001), pp. 1296–1311.
- [2] J. M. BISMUT, *Linear quadratic optimal stochastic control with random coefficients*, SIAM J. Control Optim., 14 (1976), pp. 419–444.
- [3] S. CHEN, X. LI, AND X. ZHOU, *Stochastic linear quadratic regulators with indefinite control weight costs*, SIAM J. Control Optim., 36 (1998), pp. 1685–1702.
- [4] C. DONNELLY AND A. J. HEUNIS, *Quadratic risk minimization in a regime-switching model with portfolio constraints*, SIAM J. Control Optim., 50 (2012), pp. 2431–2461.
- [5] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, 2nd ed., Cambridge University Press, New York, 2012.
- [6] Y. HU, G. LIANG, AND S. TANG, *Systems of ergodic BSDEs arising in regime switching forward performance processes*, SIAM J. Control Optim., 58 (2020), pp. 2503–2534.
- [7] Y. HU AND B. OKSENDAL, *Partial information linear quadratic control for jump diffusions*, SIAM J. Control Optim., 47 (2008), pp. 1744–1761.
- [8] Y. HU, X. SHI, AND Z. XU, *Constrained stochastic LQ control with regime switching and application to portfolio selection*, Ann. Appl. Probab., 32 (2022), pp. 426–460.

- [9] Y. JI AND H. CHIZECK, *Controllability, stabilizability, and continuous-time Markovian jump linear quadratic control*, IEEE Trans. Automat. Control, 35 (1990), pp. 777–788.
- [10] M. KOHLMANN AND S. TANG, *New developments in backward stochastic Riccati equations and their applications*, in Mathematical Finance, M. Kohlmann and S. Tang, eds., Birkhäuser, Basel, 2001, pp. 194–214.
- [11] M. KOHLMANN AND S. TANG, *Multidimensional backward stochastic Riccati equations and applications*, SIAM J. Control Optim., 41 (2003), pp. 1696–1721.
- [12] N. LI, Z. WU, AND Z. YU, *Indefinite stochastic linear-quadratic optimal control problems with random jumps and related stochastic Riccati equations*, Sci. China Math., 61 (2018), pp. 563–576.
- [13] X. LI, X. ZHOU, AND A. E. B. LIM, *Dynamic mean-variance portfolio selection with no-shorting constraints*, SIAM J. Control Optim., 40 (2002), pp. 1540–1555.
- [14] Y. LIU, G. YIN, AND X. ZHOU, *Near-optimal controls of random-switching LQ problems with indefinite control weight costs*, Automatica, 41 (2005), pp. 1063–1070.
- [15] H. MEI AND J. YONG, *Equilibrium strategies for time-inconsistent stochastic switching systems*, ESAIM Control Optim. Calc. Var., 25 (2019).
- [16] Y. SONG, S. TANG, AND Z. WU, *The maximum principle for progressive optimal stochastic control problems with random jumps*, SIAM J. Control Optim., 58 (2020), pp. 2171–2187.
- [17] J. SUN, J. XIONG, AND J. YONG, *Indefinite stochastic linear-quadratic optimal control problems with random coefficients: Closed-loop representation of open-loop optimal controls*, Ann. Appl. Probab., 31 (2021), pp. 460–499.
- [18] J. SUN AND J. YONG, *Linear quadratic stochastic differential games: Open-loop and closed-loop saddle points*, SIAM J. Control Optim., 52 (2014), pp. 4082–4121.
- [19] S. TANG, *General linear quadratic optimal stochastic control problems with random coefficients: Linear stochastic Hamilton systems and backward stochastic Riccati equations*, SIAM J. Control Optim., 42 (2003), pp. 53–75.
- [20] S. TANG, *Dynamic programming for general linear quadratic optimal stochastic control with random coefficients*, SIAM J. Control Optim., 53 (2015), pp. 1082–1106.
- [21] J. WEN, X. LI, AND J. XIONG, *Weak closed-loop solvability of stochastic linear quadratic optimal control problems of Markovian regime switching system*, Appl. Math. Optim., 84 (2021), pp. 535–565.
- [22] J. WEN, X. LI, J. XIONG, AND X. ZHANG, *Stochastic Linear-Quadratic Optimal Control Problems with Random Coefficients and Markovian Regime Switching System*, <https://arxiv.org/abs/2202.04880v2>, 2022.
- [23] J. YONG AND X. ZHOU, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, 1999.
- [24] Z. YU, *Infinite horizon jump-diffusion forward-backward stochastic differential equations and their application to backward linear-quadratic problems*, ESAIM Control Optim. Calc. Var., 23 (2017), pp. 1331–1359.
- [25] Q. ZHANG AND G. YIN, *On nearly optimal controls of hybrid LQG problems*, IEEE Trans. Automat. Control, 44 (1999), pp. 2271–2282.
- [26] X. ZHANG, R. J. ELLIOTT, AND T. K. SIU, *A stochastic maximum principle for a Markov regime-switching jump-diffusion model and its application to finance*, SIAM J. Control Optim., 50 (2012), pp. 964–990.
- [27] X. ZHANG, X. LI, AND J. XIONG, *Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems of Markovian regime switching system*, ESAIM Control Optim. Calc. Var., 27 (2021).
- [28] X. ZHOU AND G. YIN, *Markowitz’s mean-variance portfolio selection with regime switching: A continuous-time model*, SIAM J. Control Optim., 42 (2003), pp. 1466–1482.