Wellposedness and regularity estimates for stochastic Cahn–Hilliard equation with unbounded noise diffusion

Jianbo Cui and Jialin Hong

Abstract. In this article, we consider the one dimensional stochastic Cahn–Hilliard equation driven by multiplicative space-time white noise with diffusion coefficient of sublinear growth. By introducing the spectral Galerkin method, we obtain the well-posedness of the approximated equation in finite dimension. Then with help of the semigroup theory and the factorization method, the approximation processes are shown to possess many desirable properties. Further, we show that the approximation process is strongly convergent in a certain Banach space with an explicit algebraic convergence rate. Finally, the global existence and regularity estimates of the unique solution process are proven by means of the strong convergence of the approximation process, which fills a gap on the global existence of the mild solution for stochastic Cahn–Hilliard equation when the diffusion coefficient satisfies a growth condition of order $\alpha \in (\frac{1}{3}, 1)$.

1. Introduction

In this article, we consider the following stochastic Cahn–Hilliard equation with multiplicative space-time white noise

$$dX(t) + A(AX(t) + F(X(t)))dt = G(X(t))dW(t), \quad t \in (0, T] \tag{1.1}$$
$$X(0) = X_0.$$

Here $0 < T < \infty$, $H := L^2(\mathcal{O})$ with $\mathcal{O} = (0, L), L > 0$, $-A : D(A) \subset H \to H$ is the Laplacian operator under homogenous Dirichlet or Neumann boundary condition, and $\{W(t)\}_{t\geq 0}$ is a generalized Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The nonlinearity F is assumed to be the Nemytskii operator of f', where f is a polynomial of degree 4, i.e., $c_4\xi^4 + c_3\xi^3 + c_2\xi^2 + c_1\xi + c_0$ with $c_i \in \mathbb{R}$, $i = 0, \cdots, 4$, $c_4 > 0$. A typical example is the double well potential $f = \frac{1}{4}(\xi^2 - 1)^2$. For more general drift nonlinearities, we refer to [**14**] and references therein. The diffusion coefficient G is assumed to be the Nemytskii operator

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of g, where g is a globally Lipschitz continuous function with the sublinear growth condition $|g(\xi)| \le C(1+|\xi|^{\alpha}), \alpha < 1$. When $G = I$, Eq. (1.1) corresponds to the Cahn–Hilliard–Cook equation. This equation is used to describe the complicated phase separation and coarsening phenomena in a melted alloy that is quenched to a temperature at which only two different concentration phases can exist stably (see e.g. [**1, 3, 18**]).

The existence and uniqueness of the solution to Eq. (1.1) have already been proven by [**12**] in the case of $G = I$. Moreover, for Eq. (1.1) in dimension $d = 2, 3$, the driving noise should be more regular than the space-time white noise. When G is a bounded diffusion coefficient, the authors in [**4**] obtain the global existence and path regularity of the solution in $d = 1$, and the local existence of the solution in higher dimension $d = 2, 3$. Recently, the authors in [**2**] extend the results on the local existence and uniqueness of the solution in the case that $|g(\xi)| \le C(1+|\xi|^{\alpha})$, $\alpha \in (0, 1]$, $d \le 3$. Meanwhile, the global existence of the solution is achieved under the restriction that $\alpha < \frac{1}{3}$, $d = 1$. However, for the global existence of the solution, it is still unknown whether the sublinear growth condition $\alpha < \frac{1}{3}$ could be extended to the general sublinear growth condition, i.e., $|g(\xi)| \le C(1+|\xi|^{\alpha}), \alpha \in (0, 1)$, which is one main motivation of this article.

To study such a problem, instead of introducing an appropriated cut-off SPDE (see e.g. [**4, 2**]), we use the spectral Galerkin method to discretize Eq. (1.1) and get the spectral Galerkin approximation

(1.2)
$$\begin{aligned} dX^N(t) + A(AX^N(t) + P^N F(X^N(t)))dt &= P^N G(X^N(t))dW(t), \quad t \in (0, T] \\ X^N(0) &= P^N X_0, \end{aligned}$$

where $N \in \mathbb{N}^+$. Then by making use of the factorization formula and the equivalent random form of the semi-discrete equation, we show the well-posedness of the semi-discrete equation (1.2), as well as its uniform a priori estimate and regularity estimate. Furthermore, we show that the limit of the solution of the spectral Galerkin method exists globally and is the unique mild solution of Eq. (1.1). As a consequence, the exponential integrability property, the optimal temporal and spatial regularity estimates of the exact solution are proven. Meanwhile, with help of the Sobolev interpolation equality and the smoothing effect of the semigroup $S(t) := e^{-A^2 t}$, the sharp spatial strong convergence rate of the spectral Galerkin method is established under homogenous Dirichlet boundary condition. To the best of our knowledge, this is not only a new result on the global existence and regularity estimates of the solution, but also the first result on the strong convergence rate of numerical approximation for the stochastic Cahn–Hilliard equation driven by multiplicative space-time white noise.

The rest of this article is organized as follows. In Section 2 the setting and assumptions used are formulated. In Section 3, we prove several uniform a priori estimates and regularity estimates of the spatial spectral Galerkin method. The strong convergence analysis of the spatial spectral Galerkin method is presented in Section 4. Our main result which states existence, uniqueness and regularity of solutions of Eq. (1.1) with nonlinear multiplicative noise is presented in Section 5.

2. Preliminaries

In this section, we present some preliminaries and notations, as well as the assumptions on Eq. (1.1).

Given two separable Hilbert spaces $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ and $(\widetilde{H}, \|\cdot\|_{\widetilde{H}})$, $\mathcal{L}(\mathcal{H}, \widetilde{H})$ and $\mathcal{L}_1(\mathcal{H}, \widetilde{H})$ are the Banach spaces of all linear bounded operators and the nuclear operators from $\mathcal{H}$ to $\widetilde{H}$, respectively. The trace of an operator $\mathcal{T} \in \mathcal{L}_1(\mathcal{H})$ is $tr[\mathcal{T}] = \sum_{k\in\mathbb{N}^+} \langle \mathcal{T} f_k, f_k\rangle_{\mathcal{H}}$, where $\{f_k\}_{k\in\mathbb{N}^+}$ ($\mathbb{N}^+ = \{1, 2, \cdots\}$) is any orthonormal basis of $\mathcal{H}$. In particular, if $\mathcal{T} \geq 0$, $tr[\mathcal{T}] = \|\mathcal{T}\|_{\mathcal{L}_1}$. Denote by $\mathcal{L}_2(\mathcal{H}, \widetilde{H})$ the space of Hilbert–Schmidt operators from $\mathcal{H}$ into $\widetilde{H}$, equipped with the usual norm given by $\|\cdot\|_{\mathcal{L}_2(\mathcal{H},\widetilde{H})} = (\sum_{k\in\mathbb{N}^+} \|\cdot f_k\|^2_{\widetilde{H}})^{\frac{1}{2}}$. The following useful property and inequality hold

$$
\begin{aligned}
&\|\mathcal{S}\mathcal{T}\|_{\mathcal{L}_2(\mathcal{H},\widetilde{H})} \leq \|\mathcal{S}\|_{\mathcal{L}_2(\mathcal{H},\widetilde{H})} \|\mathcal{T}\|_{\mathcal{L}(\mathcal{H})}, \quad \mathcal{T} \in \mathcal{L}(\mathcal{H}),\ \ \mathcal{S} \in \mathcal{L}_2(\mathcal{H}, \widetilde{H}), \\
&tr[\mathcal{Q}] = \|\mathcal{Q}^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})} = \|\mathcal{T}\|^2_{\mathcal{L}_2(\widetilde{H},\mathcal{H})}, \quad \mathcal{Q} = \mathcal{T}\mathcal{T}^*,\ \ \mathcal{T} \in \mathcal{L}_2(\widetilde{H}, \mathcal{H}),
\end{aligned} \tag{2.1}
$$

where $\mathcal{T}^*$ is the adjoint operator of $\mathcal{T}$.

Given a Banach space $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ and $\mathcal{T}$ a linear operator from $\mathcal{H}$ to $\mathcal{E}$, we denote by $\gamma(\mathcal{H}, \mathcal{E})$ the space of γ-radonifying operators endowed with the norm $\|\mathcal{T}\|_{\gamma(\mathcal{H},\mathcal{E})} = (\widetilde{\mathbb{E}}\| \sum_{k\in\mathbb{N}^+} \gamma_k \mathcal{T} f_k\|^2_{\mathcal{E}})^{\frac{1}{2}}$, where $(\gamma_k)_{k\in\mathbb{N}}$ is a Rademacher sequence on a probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$. For convenience, let $L^q = L^q(\mathcal{O})$, $1 \leq q < \infty$ and $E = \mathcal{C}(\mathcal{O})$ equipped with the usual inner product and norm. We also need the following Burkholder inequality in L^q, $q \in [2, \infty)$ (see e.g. [**23**]),

$$
\begin{aligned}
\Big\| \sup_{t\in[0,T]} \Big\| \int_0^t \phi(r) d\widetilde{W}(r) \Big\|_{L^q} \Big\|_{L^p(\Omega)} &\leq C_p \|\phi\|_{L^p(\Omega; L^2([0,T]; \gamma(H; L^q))} \\
&\leq C_p \Big(\mathbb{E}\Big(\int_0^T \Big\| \sum_{k\in\mathbb{N}^+} (\phi(t)e_k)^2 \Big\|_{L^{\frac{q}{2}}} dt \Big)^{\frac{p}{2}} \Big)^{\frac{1}{p}},
\end{aligned} \tag{2.2}
$$

where $\widetilde{W}$ is the H-valued cylindrical Wiener process, $\{e_k\}_{k\in\mathbb{N}^+}$ is any orthonormal basis of H and $\phi \in L^p(\Omega; L^2([0,T]; \gamma(H; L^q)), p \geq 1$, is a predictable process. Next, we introduce some assumptions and spaces associated with A. We denote by $H^k := H^k(\mathcal{O})$ the standard Sobolev space. For convenience, we mainly focus on the well-posedness and numerical approximation for Eq. (1.1) under homogenous Dirichlet boundary condition. We would like to mention that the approach for proving the global existence of the unique solution is also available for Eq. (1.1) under homogenous Neumann boundary condition. Denote $A = -\Delta$ the Dirichlet Laplacian operator with

$$D(A) = \left\{ v \in H^2(\mathcal{O}) : v = 0 \ \text{ on } \ \partial\mathcal{O} \right\}.$$

It is known that A is a positive definite, self-adjoint and unbounded linear operator on H and that there exists an orthonormal eigensystem $\{(\lambda_j, e_j)\}_{j\in\mathbb{N}}$ such that $0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots$ with $\lambda_j \sim j^2$ and $\sup_{j\in\mathbb{N}^+} \|e_j\|_E < \infty$. For any $\alpha \geq 0$, let the operator $A^{\frac{\alpha}{2}} : D(A^{\frac{\alpha}{2}}) \subset \mathbb{H} \to \mathbb{H}$ be given by

$$A^{\frac{\alpha}{2}} x = \sum_{n=1}^{\infty} \lambda_n^{\frac{\alpha}{2}} \langle x, e_n\rangle e_n$$

for all

$$x \in D(A^{\frac{\alpha}{2}}) = \Big\{ x \in \mathbb{H} : \|x\|^2_{\mathbb{H}^\alpha} := \sum_{n=1}^{\infty} \lambda_n^\alpha \langle x, e_n \rangle^2 < \infty \Big\}.$$

By setting $\mathbb{H}^\alpha = D(A^{\frac{\alpha}{2}})$ equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}^\alpha} = \langle A^{\frac{\alpha}{2}} \cdot, A^{\frac{\alpha}{2}} \cdot \rangle$, we get a separable Hilbert space $(\mathbb{H}^\alpha, \langle \cdot, \cdot \rangle_{\mathbb{H}^\alpha})$ for $\alpha \geq 0$. We define

$$\mathbb{H}^{-\alpha} = \Big\{ x = \sum_{n=1}^{\infty} x_n e_n : x_n \in \mathbb{R}, n = 1, 2, \cdots, \text{ s.t. } \|x\|^2_{\mathbb{H}^{-\alpha}} = \sum_{n=1}^{\infty} \lambda_k^{-\alpha} x_n^2 < \infty \Big\}$$

and

$$A^{-\frac{\alpha}{2}} x = \sum_{n=1}^{\infty} \lambda_n^{-\frac{\alpha}{2}} x_n e_n$$

for $x \in \mathbb{H}^{-\alpha}$. It follows that $\mathbb{H}^{-\alpha}$ is the largest set such that $A^{-\frac{\alpha}{2}}$ maps into $\mathbb{H}$ and that the dual space of $\mathbb{H}^\alpha$ is isometric to $\mathbb{H}^{-\alpha}$. In this sense, $\mathbb{H}^{-\alpha} = D(A^{-\frac{\alpha}{2}})$. As a consequence, we could endow $\mathbb{H}^{-\alpha}$ with the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}^{-\alpha}}$. Notice that $H = \mathbb{H}$. For convenience, we denote $\| \cdot \| = \| \cdot \|_{\mathbb{H}}$. The following smoothing effect of the analytical semigroup $S(t) = e^{-A^2 t}, t > 0$ (see e.g. [**16**]),

$$\|A^\beta S(t) v\| \leq C t^{-\frac{\beta}{2}} \|v\|, \ \beta > 0, \ v \in \mathbb{H} \tag{2.3}$$

and the contractivity property of $S(t)$ (see e.g. [**19**, Appendix B]),

$$\begin{aligned} \|S(t)v\|_{L^q} &\leq C t^{-\frac{1}{4}(\frac{1}{p} - \frac{1}{q})} \|v\|_{L^p}, \ 1 \leq p \leq q \leq \infty, \ v \in L^p, \\ \|S(t)v\|_E &\leq C t^{-\frac{1}{4p}} \|v\|_{L^p}, v \in L^p, \end{aligned} \tag{2.4}$$

will be used frequently. The above contractivity property of the semigroup for the parabolic equation could be obtained if the compatibility condition on boundary (see e.g. [**15**]) holds. For the considered case, it has been pointed out in [**19**, Section 2.5] (see (2.2) in [**19**] with $m = 1$), the elliptic differential operator with the Dirichlet boundary condition satisfies the compatibility condition. One could also use the series expansion based on $\{e_n\}_{n=1}^\infty$ and the interpolation arguments to prove (2.4). We present a short proof in the appendix.

Due to the polynomial assumption, the nonlinearity $F : L^6 \to \mathbb{H}$ is a deterministic mapping, i.e.,

$$F(u)(\xi) = 4c_4 u(\xi)^3 + 3c_3 u(\xi)^2 + 2c_2 u(\xi) + c_1, \xi \in \mathcal{O}, u \in L^6.$$

The following properties of F,

$$\begin{aligned} &- \langle F(u) - F(v), u - v \rangle \leq C \|u - v\|^2, \ u, v \in L^6, \\ &\|F(u) - F(v)\| \leq C \|u - v\| (1 + \|u\|_E^2 + \|v\|_E^2), u, v \in E, \\ &(F'(u)v)(\xi) = (12 c_4 (u(\xi))^2 + 6 c_3 u(\xi) + 2 c_2) v(\xi), \xi \in \mathcal{O}, u, v \in L^6, \end{aligned} \tag{2.5}$$

will be frequently used in this paper.

Throughout this article, the Wiener process W is assumed to be a cylindrical Wiener process in $\mathbb{H}$, which implies that for any $\gamma \in (0, \frac{3}{2})$, $\|A^{\frac{\gamma - 2}{2}} Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathbb{H})} < \infty$. We denote by C a generic constant which may depend on several parameters but never on the projection parameter N and may change from occurrence to occurrence. We also remark that the approach for proving the global existence of the unique solution is available for stochastic Cahn–Hilliard equations in higher dimension with more regular Q-Wiener process.

3. A priori estimate and regularity estimate of the spectral Galerkin method

In this section, we give the a priori estimate and regularity estimate of the solution of Eq. (1.2). Notice that Eq. (1.2) is equivalent to the following random PDE and the equation of the discrete stochastic convolution Z^N,

$$dY^N(t) + A(AY^N(t) + P^N F(Y^N(t) + Z^N(t)))dt = 0,\ Y^N(0) = P^N X_0, \tag{3.1}$$

$$dZ^N(t) + A^2 Z^N(t)dt = P^N G(Y^N(t) + Z^N(t))dW(t),\ Z^N(0) = 0. \tag{3.2}$$

We will use the decomposition that $X^N = Y^N + Z^N$ based on (3.1) and (3.2). This is inspired by [**5**] where the author used similar decomposition to show the well-posedness of stochastic reaction-diffusion systems. In the following, we present the a priori and regularity estimates of Z^N and Y^N. Throughout this paper, we assume that X_0 is deterministic.

LEMMA 3.1. Let $X_0 \in \mathbb{H}$, $T > 0$ and $q \geq 1$. There exists a unique solution X^N of Eq. (1.2) satisfying

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\|X^N(t)\|_{\mathbb{H}^{-1}}^q\Big] \leq C(X_0, T, q), \tag{3.3}$$

where $C(X_0, T, q)$ is a positive constant.

PROOF. Thanks to the fact all the norms in finite dimensional normed linear spaces are equivalent, the norm $\|\cdot\|$ and $\|\cdot\|_{\mathbb{H}^{-1}}$ in $P^N(\mathbb{H})$ are equivalent up to constants depending on N. The existence of a unique strong solution for Eq. (1.2) in $\mathbb{H}^{-1}$ can be obtained by the arguments in [**20**, Chapter 3]. However, the moment bound of the exact solution will depend on N by this method. To prove (3.3), we need to find a proper Lyapunov functional and to derive the a priori estimate independent of N. According to Eq. (3.1), by using the chain rule and integration by parts, we have for any $t \leq T$,

$$\begin{aligned}\|Y^N(t)\|_{\mathbb{H}^{-1}}^2 &= \|Y^N(0)\|_{\mathbb{H}^{-1}}^2 - 2\int_0^t \langle \nabla Y^N(s), \nabla Y^N(s)\rangle ds \\ &\quad - 2\int_0^t \langle F(Y^N(s) + Z^N(s)), Y^N(s)\rangle ds \\ &= \|Y^N(0)\|_{\mathbb{H}^{-1}}^2 - 2\int_0^t \|\nabla Y^N(s)\|^2 ds \\ &\quad - 2\int_0^t \langle F(Y^N(s) + Z^N(s)), Y^N(s)\rangle ds.\end{aligned}$$

Young's inequality and (2.5) imply that for arbitrary $\epsilon > 0$,

$$\begin{aligned}&\|Y^N(t)\|_{\mathbb{H}^{-1}}^2 + 2\int_0^t \|\nabla Y^N(s)\|^2 ds + 8(c_4 - \epsilon)\int_0^t \|Y^N(s)\|_{L^4}^4 ds \\ &\leq \|Y^N(0)\|_{\mathbb{H}^{-1}}^2 + C(\epsilon)\int_0^t (1 + \|Z^N(s)\|_{L^4}^4) ds.\end{aligned} \tag{3.4}$$

Thus it suffices to deduce the a priori estimate of $\int_0^t \|Z^N(s)\|_{L^4}^4 ds$. From the mild form of Z^N, the Burkholder inequality and (2.4), it follows that for $q \geq p \geq 2$,

$\mathbb{E}[\|Z^N(s)\|_{L^p}^q]$

$$
\begin{aligned}
&= \mathbb{E}\Big[\Big\| \int_0^s S(s-r)P^N G(Y^N(r)+Z^N(r))dW(r)\Big\|_{L^p}^q\Big] \\
&\le C\mathbb{E}\Big[\Big(\int_0^s \Big\|S(s-r)P^N G(Y^N(r)+Z^N(r))\Big\|_{\gamma(\mathbb{H},L^p)}^2 dr\Big)^{\frac{q}{2}}\Big] \\
&\le C\mathbb{E}\Big[\Big(\int_0^s \sum_{k=1}^\infty \Big\|S(s-r)P^N(G(Y^N(r)+Z^N(r))e_k)\Big\|_{L^p}^2 dr\Big)^{\frac{q}{2}}\Big] \\
&\le C\mathbb{E}\Big[\Big(\int_0^s (s-r)^{-\frac12(\frac12-\frac1p)} \sum_{k=1}^\infty \Big\|S(\frac{s-r}{2})P^N(G(Y^N(r)+Z^N(r))e_k)\Big\|^2 dr\Big)^{\frac{q}{2}}\Big].
\end{aligned}
$$

Applying Parseval's equality, the fact that

$$
\sum_{k=1}^\infty e^{-\lambda_k^2 t} \le Ct^{-\frac14}, t>0,
$$

the sublinear growth of G and Hölder's inequality, we obtain that for $q \ge 4$,

$$
\begin{aligned}
&\mathbb{E}[\|Z^N(s)\|_{L^p}^q] \\
&\le C\mathbb{E}\Big[\Big(\int_0^s (s-r)^{-\frac12(\frac12-\frac1p)} \sum_{j,k=1}^\infty \Big\langle G(Y^N(r)+Z^N(r))e_k, e^{-\frac12\lambda_j^2(s-r)}e_j\Big\rangle^2 dr\Big)^{\frac{q}{2}}\Big] \\
&= C\mathbb{E}\Big[\Big(\int_0^s (s-r)^{-\frac12(\frac12-\frac1p)} \sum_{j=1}^\infty e^{-\lambda_j^2(s-r)}\|G(Y^N(r)+Z^N(r))e_j\|^2 dr\Big)^{\frac{q}{2}}\Big] \\
&\le C\mathbb{E}\Big[\Big(\int_0^s (s-r)^{-\frac12(\frac12-\frac1p)} \sum_{j=1}^\infty e^{-\lambda_j^2(s-r)}\|G(Y^N(r)+Z^N(r))\|^2 dr\Big)^{\frac{q}{2}}\Big] \\
&\le C\mathbb{E}\Big[\Big(\int_0^s (s-r)^{-\frac12+\frac{1}{2p}}\|G(Y^N(r)+Z^N(r))\|^2 dr\Big)^{\frac{q}{2}}\Big] \\
&\le C\mathbb{E}\Big[\Big(\int_0^s (s-r)^{\frac{-p+1}{2p}}\Big(1+\|Y^N(r)\|^{2\alpha}+\|Z^N(r)\|^{2\alpha}\Big)dr\Big)^{\frac{q}{2}}\Big] \\
&\le C(\int_0^s (s-r)^{\frac{-p+1}{p}}dr)^{\frac{q}{4}}\mathbb{E}\Big[\Big(\int_0^s \Big(1+\|Y^N(r)\|^{4\alpha}+\|Z^N(r)\|^{4\alpha}\Big)dr\Big)^{\frac{q}{4}}\Big].
\end{aligned}
$$

Using the Young inequality, we obtain for $0 \le s \le t$,

$$
\begin{aligned}
\mathbb{E}[\|Z^N(s)\|_{L^p}^q] &\le Cs^{\frac{q}{4p}}\Big(1+\mathbb{E}[(\int_0^s \|Y^N(r)\|^{4\alpha}dr)^{\frac{q}{4}}]+\int_0^s \mathbb{E}[\|Z^N(r)\|^q]dr\Big) \\
&\le Cs^{\frac{q}{4p}}\Big(1+\mathbb{E}[(\int_0^s \|Y^N(r)\|_{L^p}^{4\alpha}dr)^{\frac{q}{4}}]+\int_0^s \mathbb{E}[\|Z^N(r)\|_{L^p}^q]dr\Big).
\end{aligned}
$$

Since the moment bounds of Z^N and Y^N are finite depending on N, we can apply the Gronwall's inequality and get that for $0 \le s \le T$,

$$
\mathbb{E}[\|Z^N(s)\|_{L^p}^q] \le C(T)\Big(1+\mathbb{E}\Big[(\int_0^s \|Y^N(r)\|_{L^p}^{4\alpha}dr)^{\frac{q}{4}}\Big]\Big). \tag{3.5}
$$

Now taking the kth moment, $k \in \mathbb{N}^+$ on (3.4) and letting $p=4$, $q=4k$, we have

$$
\begin{aligned}
&\mathbb{E}\Big[(\int_0^t \|Y^N(s)\|_{L^4}^4 ds)^k\Big] \\
&\le C(\epsilon,k)\|Y^N(0)\|_{\mathbb{H}^{-1}}^{2k} + C(\epsilon,k)\int_0^t (1+\mathbb{E}[\|Z^N(s)\|_{L^4}^{4k}])ds
\end{aligned}
$$

$$\leq C(\epsilon,k)\|Y^N(0)\|_{\mathbb{H}^{-1}}^{2k}+C(\epsilon,k,T)\Big(C(\epsilon_1)+\epsilon_1\int_0^t\mathbb{E}[(\int_0^s\|Y^N(r)\|_{L^4}^4dr)^k]ds\Big),$$

where $\epsilon_1>0$ is a small number such that $C(\epsilon,k,T)\epsilon_1 T<\frac{1}{2}$. The above estimation leads to

$$\mathbb{E}\Big[(\int_0^t\|Y^N(s)\|_{L^4}^4ds)^k\Big]\leq C\|Y^N(0)\|_{\mathbb{H}^{-1}}^{2k}+C(k,\epsilon,\epsilon_1,T),$$

which in turns yields that for $k\in\mathbb{N}^+$,

$$\mathbb{E}\Big[\|Y^N(t)\|_{\mathbb{H}^{-1}}^{2k}\Big]+\mathbb{E}\Big[(\int_0^t\|\nabla Y^N(s)\|^2ds)^k\Big]+\mathbb{E}\Big[(\int_0^t\|Y^N(s)\|_{L^4}^4ds)^k\Big] \leq C(X_0,T,k). \tag{3.6}$$

Based on the a priori estimates of Z^N and Y^N in L^p and $\mathbb{H}^{-1}$, we complete the proof. □

LEMMA 3.2. Let $X_0\in\mathbb{H}$, $T>0$ and $q\geq1$. There exists a positive constant $C(X_0,T,q)$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big\|Z^N(t)\big\|_E^q\Big]\leq C(X_0,T,q). \tag{3.7}$$

PROOF. By using the factorization formula in [**13**, Proposition 5.9 and Theorem 5.10], we have that for $\frac{3}{8}-\frac{\alpha}{4}>\alpha_1>\frac{1}{p}+\gamma$, large enough $p>1$, $\gamma=\frac{1}{8}$,

$$\mathbb{E}\Big[\sup_{s\in[0,T]}\|Z^N(s)\|_E^q\Big]\leq C(q,T)\mathbb{E}\Big[\|Y_{\alpha_1,N}\|_{L^p(0,T;\mathbb{H})}^q\Big],$$

where $Y_{\alpha_1,N}(s)=\int_0^s(s-r)^{-\alpha_1}S(s-r)P^NG(Y^N(r)+Z^N(r))dW(r)$. The factorization formula is applicable here since the condition (5.14) in [**13**, Theorem 5.10] holds for any $t\in[0,T]$. Indeed, by using the fact that $\sum_{i=1}^\infty e^{-\lambda_i^2t}\leq Ct^{-\frac{1}{4}}$ and $\sup_{i\in\mathbb{N}^+}\|e_i\|_E<\infty$, we have

$$\int_0^t(t-s)^{\alpha_1-1}\Big(\int_0^s(s-\sigma)^{-2\alpha_1}\sum_{i=1}^\infty\mathbb{E}\Big[\Big\|S(s-\sigma)P^NG(Y^N(\sigma)+Z^N(\sigma))e_i\Big\|^2\Big]d\sigma\Big)^{\frac{1}{2}}ds$$
$$\leq C\int_0^t(t-s)^{\alpha_1-1}\Big(\int_0^s(s-\sigma)^{-(2\alpha_1+\frac{1}{4})}\mathbb{E}\Big[1+\|Y^N(\sigma)\|^{2\alpha}+\|Z^N(\sigma)\|^{2\alpha}\Big]d\sigma\Big)^{\frac{1}{2}}ds.$$

Since $\alpha<1$, we could take $l>2$ which is close to 2 such that $2\alpha l<4$. For convenience, we assume that $2\alpha l=4-\epsilon$ for some $\epsilon>0$. As a consequence $\frac{l}{l-1}>\frac{2}{2-\alpha}$. By applying Hölder's inequality and (3.6), we obtain that

$$\int_0^t(t-s)^{\alpha_1-1}\Big(\int_0^s(s-\sigma)^{-(2\alpha_1+\frac{1}{4})}\mathbb{E}\Big[1+\|Y^N(\sigma)\|^{2\alpha}+\|Z^N(\sigma)\|^{2\alpha}\Big]d\sigma\Big)^{\frac{1}{2}}ds$$
$$\leq C\int_0^t(t-s)^{\alpha_1-1}\Big(\int_0^s(s-\sigma)^{-(2\alpha_1+\frac{1}{4})\frac{l}{l-1}}d\sigma\Big)^{\frac{l-1}{2l}}$$
$$\times\Big(\int_0^s\mathbb{E}\Big[1+\|Y^N(\sigma)\|^{2\alpha l}+\|Z^N(\sigma)\|^{2\alpha l}\Big]d\sigma\Big)^{\frac{1}{2l}}ds$$
$$\leq C\int_0^t(t-s)^{\alpha_1-1}\Big(\int_0^s(s-\sigma)^{-(2\alpha_1+\frac{1}{4})\frac{l}{l-1}}d\sigma\Big)^{\frac{l-1}{2l}}ds.$$

The right term in the last estimate is finite if and only if $(2\alpha_1+\frac{1}{4})\frac{l}{l-1}<1$. This is allowed since the assumption that $\frac{3}{8}-\frac{\alpha}{4}>\alpha_1>\frac{1}{p}+\gamma$ for large enough $p>1$,

$\gamma = \frac{1}{8}$. Indeed, we only need to take a small enough ϵ_1 depending on α and ϵ such that $\frac{3}{8} - \frac{\alpha}{4} - \epsilon_1 > \alpha_1 > \frac{1}{p} + \gamma$ and $(2\alpha_1 + \frac{1}{4})\frac{l}{l-1} \le (1 - \frac{\alpha}{2} - 2\epsilon_1)\frac{1}{1-\frac{\alpha}{2-\frac{\epsilon}{2}}} < 1$. Thus it suffices to estimate $\mathbb{E}\Big[\|Y_{\alpha_1,N}\|^q_{L^p(0,T;\mathbb{H})}\Big]$. From the Hölder and Burkholder inequalities, it follows that for $q \ge \max(p,2)$,

$$
\begin{aligned}
&\mathbb{E}\Big[\|Y_{\alpha_1,N}\|^q_{L^p(0,T;\mathbb{H})}\Big]\\
&= \mathbb{E}\Big[\Big(\int_0^T \Big\|\int_0^s (s-r)^{-\alpha_1} S(s-r)P^N G(Y^N(r)+Z^N(r))dW(r)\Big\|^p ds\Big)^{\frac{q}{p}}\Big]\\
&\le C(T,q)\int_0^T \mathbb{E}\Big[\Big\|\int_0^s (s-r)^{-\alpha_1} S(s-r)P^N G(Y^N(r)+Z^N(r))dW(r)\Big\|^q\Big]ds\\
&\le C(T,q)\int_0^T \mathbb{E}\Big[\Big(\int_0^s (s-r)^{-2\alpha_1}\sum_{i\in\mathbb{N}^+}\Big\|S(s-r)P^N G(Y^N(r)+Z^N(r))e_i\Big\|^2 dr\Big)^{\frac{q}{2}}\Big]ds\\
&\le C(T,q)\int_0^T \mathbb{E}\Big[\Big(\int_0^s (s-r)^{-(2\alpha_1+\frac{1}{4})}(1+\|Y^N(r)\|^{2\alpha}+\|Z^N(r)\|^{2\alpha})dr\Big)^{\frac{q}{2}}\Big]ds.
\end{aligned}
$$

Since $\alpha < 1$, one can choose a positive number $l > 2$ and a large enough number p such that $2\alpha l < 4$ and $(2\alpha_1 + \frac{1}{4})\frac{l}{l-1} < 1$. Then by using a priori estimates (3.5) and (3.6), we obtain that for $\frac{3}{8} - \frac{\alpha}{4} > \alpha_1 > \frac{1}{p} + \gamma$, large enough $q \ge p > 1$, $\gamma = \frac{1}{8}$,

$$
\begin{aligned}
\mathbb{E}\Big[\|Y_{\alpha,N}\|^q_{L^p(0,T;H)}\Big] &\le C(T,q,\alpha)\int_0^T \Big(\int_0^s (s-r)^{-(2\alpha_1+\frac{1}{4})\frac{l}{l-1}}dr\Big)^{\frac{q(l-1)}{2l}}\\
&\quad\times \mathbb{E}\Big[\Big(\int_0^s (1+\|Y^N(r)\|^{2\alpha l}+\|Z^N(r)\|^{2\alpha l})dr\Big)^{\frac{q}{2l}}\Big]ds\\
&\le C(T,q,\alpha,X_0),
\end{aligned}
$$

which implies that $\mathbb{E}\Big[\sup_{s\in[0,T]}\|Z^N(s)\|^q_E\Big] \le C(T,q,\alpha,X_0)$. □

Corollary 3.1. Let $X_0 \in \mathbb{H}$, $T > 0$ and $q \ge 1$. Then the solution X^N of Eq. (1.2) satisfies

$$
\mathbb{E}\Big[\sup_{t\in[0,T]}\big\|X^N(t)\big\|^q_{\mathbb{H}^{-1}}\Big] \le C(X_0,T,q), \tag{3.8}
$$

where $C(X_0,T,q)$ is a positive constant.

Proof. Similar arguments as in the proof of (3.6) yield that for any $k \ge 1$,

$$
\mathbb{E}\Big[\sup_{t\in[0,T]}\|Y^N(t)\|^{2k}_{\mathbb{H}^{-1}}\Big] \le C(X_0,T,k).
$$

Combining this estimate with Lemma 3.2, we complete the proof. □

Thanks to the above a priori estimates of Y^N and Z^N, we are now in a position to deduce the a priori estimate of X^N in $\mathbb{H}$.

Lemma 3.3. Let $X_0 \in \mathbb{H}$, $T > 0$ and $q \ge 1$. There exists a positive constant $C(X_0,T,q)$ such that

$$
\mathbb{E}\Big[\sup_{t\in[0,T]}\big\|X^N(t)\big\|^q_{\mathbb{H}}\Big] \le C(X_0,T,q). \tag{3.9}
$$

Proof. By applying the chain rule, (2.5), and using integration by parts, Young's and Hölder's inequalities, we have that for any small $\epsilon > 0$,

$$\begin{aligned}
&\|Y^N(t)\|^2\\
&= \|Y^N(0)\|^2 - \int_0^t 2\|(-A)Y^N(s)\|^2 ds + \int_0^t \langle (-A)\big(8c_4(Y^N(s)+Z^N(s))^3\\
&+ 6c_3(Y^N(s)+Z^N(s))^2 + 4c_2(Y^N(s)+Z^N(s))\big), Y^N(s)\rangle ds\\
&\le \|Y^N(0)\|^2 - \int_0^t 2\|(-A)Y^N(s)\|^2 ds - \int_0^t \langle \nabla 8c_4(Y^N(s)+Z^N(s))^3, \nabla Y^N(s)\rangle ds\\
&+ \frac{\epsilon}{2}\int_0^t \|(-A)Y^N(s)\|^2 ds + C(\epsilon, c_3, c_2)\int_0^t \Big(1 + \|Z^N(s)\|_{L^4}^4 + \|Y^N(s)\|_{L^4}^4\Big) ds\\
&\le \|Y^N(0)\|^2 - \int_0^t (2 - \frac{\epsilon}{2})\|(-A)Y^N(s)\|^2 ds - \int_0^t \langle \nabla 8c_4(Y^N(s))^3, \nabla Y^N(s)\rangle ds\\
&+ \int_0^t \langle 8c_4\Big(3(Y^N(s))^2 Z^N(s) + 3Y^N(s)(Z^N(s))^2 + (Z^N(s))^3\Big), (-A)Y^N(s)\rangle ds\\
&+ C(\epsilon, c_3, c_2)\int_0^t \Big(1 + \|Z^N(s)\|_{L^4}^4 + \|Y^N(s)\|_{L^4}^4\Big) ds\\
&\le \|Y^N(0)\|^2 - \int_0^t (2-\epsilon)\|(-A)Y^N(s)\|^2 ds\\
&+ C(\epsilon, c_3, c_2)\int_0^t \Big(1 + \|Z^N(s)\|_{L^4}^4 + \|Y^N(s)\|_{L^4}^4\Big) ds\\
&+ C(\epsilon, c_4)\int_0^t \Big(1 + \|Y^N(s)\|_{L^4}^4 \|Z^N(s)\|_E^2 + \|Z^N(s)\|_E^6\Big) ds.
\end{aligned}$$

Taking the pth moment and using the a priori estimates (3.6) and (3.7), we have that for $p \ge 1$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]} \|Y^N(t)\|^{2p}\Big] + \mathbb{E}\Big[\int_0^T \|(-A)Y^N(s)\|^{2p} ds\Big] \qquad (3.10)\\
&\le C(p,T)\Big(\|X_0^N\|^{2p} + \mathbb{E}\Big[(1 + \sup_{s\in[0,T]} \|Z^N(s)\|_E^{2p})(\int_0^T \|Y^N(s)\|_{L^4}^4 ds)^p\Big]\\
&\quad + \mathbb{E}\Big[\Big(\int_0^T (1 + \|Y^N(s)\|_{L^4}^4 + \|Z^N(s)\|_E^6) ds\Big)^p\Big]\Big)\\
&\le C(T, X_0^N, p),
\end{aligned}$$

which, together with (3.7) and the Hölder inequality, completes the proof. □

Based on the a priori estimate of $\|X^N\|$, we are in a position to deduce the regularity estimate of X^N. Before that, we first give the regularity estimate of Z^N.

Lemma 3.4. Let $X_0 \in \mathbb{H}$, $q \ge 1$ and $\gamma \in (0, \frac{3}{2})$. Then the discrete stochastic convolution Z^N satisfies

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big\|Z^N(t)\big\|_{\mathbb{H}^\gamma}^q\Big] \le C(X_0, T, q) \qquad (3.11)$$

for a positive constant $C(X_0, T, q)$.

Proof. By the factorization method in [**13**, Proposition 5.9 and Theorem 5.10] and Lemma 3.3, we have for $\frac{3}{8} > \alpha_1 > \frac{1}{p} + \beta$, $p > 1$, $\beta = \frac{\gamma}{4}$,

$$\mathbb{E}\Big[\sup_{s\in[0,T]} \|Z^N(s)\|^q_{\mathbb{H}^\gamma}\Big] \le C(T,q)\mathbb{E}\Big[\|Y_{\alpha_1,N}\|^q_{L^p(0,T;\mathbb{H})}\Big],$$

where $Y_{\alpha_1,N}(s) = \int_0^s (s-r)^{-\alpha_1} S(s-r)P^N G(Y^N(r) + Z^N(r))dW(r)$. The factorization formula is applicable here since the condition (5.14) in [**13**, Theorem 5.10] holds for any $t \in [0,T]$. Indeed, by using the fact that $\sum_{i=1}^\infty e^{-\lambda_i^2 t} \le Ct^{-\frac{1}{4}}$ and $\sup_{i=1}^\infty \|e_i\|_E < \infty$, and Lemma 3.3, we have that for $\alpha_1 < \frac{3}{8}$,

$$\begin{aligned}
&\int_0^t (t-s)^{\alpha_1 - 1}\Big(\int_0^s (s-\sigma)^{-2\alpha_1} \sum_{i=1}^\infty \mathbb{E}\Big[\Big\|S(s-\sigma)P^N G(Y^N(\sigma) + Z^N(\sigma))e_i\Big\|^2\Big]d\sigma\Big)^{\frac{1}{2}} ds \\
&\le C\int_0^t (t-s)^{\alpha_1-1}\Big(\int_0^s (s-\sigma)^{-(2\alpha_1+\frac{1}{4})}\mathbb{E}\Big[1 + \|Y^N(\sigma)\|^{2\alpha} + \|Z^N(\sigma)\|^{2\alpha}\Big]d\sigma\Big)^{\frac{1}{2}} ds \\
&\le C\int_0^t (t-s)^{\alpha_1-1}\int_0^s (s-\sigma)^{-(2\alpha_1+\frac{1}{4})} ds < \infty.
\end{aligned}$$

From the Hölder and Burkholder inequalities, the estimates (3.6) and (3.9), it follows that for $q \ge \max(p,2)$,

$$\begin{aligned}
&\mathbb{E}\Big[\|Y_{\alpha_1,N}\|^q_{L^p(0,T;\mathbb{H})}\Big] \\
&\le C(T,q,p)\mathbb{E}\Big[\int_0^T \Big\|\int_0^s (s-r)^{-\alpha_1} S(s-r)P^N G(Y^N(r) + Z^N(r))dW(r)\Big\|^q ds\Big] \\
&\le C(T,q,p)\int_0^T \mathbb{E}\Big[\Big(\int_0^s (s-r)^{-2\alpha_1} \sum_{i\in\mathbb{N}^+} \Big\|S(s-r)P^N G(Y^N(r) + Z^N(r))e_i\Big\|^2 dr\Big)^{\frac{q}{2}}\Big] ds \\
&\le C(T,q,p)\int_0^T \mathbb{E}\Big[\Big(\int_0^s (s-r)^{-(2\alpha_1+\frac{1}{4})}(1 + \|Y^N(r)\|^{2\alpha} + \|Z^N(r)\|^{2\alpha})dr\Big)^{\frac{q}{2}}\Big] ds \\
&\le C(T,q,p)\Big(1 + \mathbb{E}\Big[\sup_{r\in[0,T]} \|Y^N(r)\|^{2\alpha q}\Big] + \mathbb{E}\Big[\sup_{r\in[0,T]} \|Z^N(r)\|^{2\alpha q}\Big]\Big) \\
&\quad \times \int_0^T \Big(\int_0^s (s-r)^{-(2\alpha_1+\frac{1}{4})} dr\Big)^{\frac{q}{2}} ds \\
&\le C(X_0,T,q,p,\alpha)\int_0^T \Big(\int_0^s (s-r)^{-(2\alpha_1+\frac{1}{4})} dr\Big)^{\frac{q}{2}} ds.
\end{aligned}$$

Since $\gamma < \frac{3}{2}$ and $\frac{3}{8} > \alpha_1 > \frac{1}{p} + \frac{\gamma}{4}$, one can choose a large enough number p such that $\frac{2}{p} + \frac{\gamma}{2} + \frac{1}{4} < 2\alpha_1 + \frac{1}{4} < 1$. Thus we obtain

$$\mathbb{E}\Big[\|Y_{\alpha_1,N}\|^q_{L^p(0,T;H)}\Big] \le C(X_0,T,q,p,\gamma),$$

which completes the proof. □

Next, we deduce the following uniform regularity estimate of X^N.

Proposition 3.1. Let $X_0 \in \mathbb{H}^\gamma$, $\gamma \in [1,\frac{3}{2})$, $T > 0$, $q \ge 1$ and $N \in \mathbb{N}^+$. Then the unique mild solution X^N of Eq. (1.2) satisfies

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big\|X^N(t)\big\|^q_{\mathbb{H}^\gamma}\Big] \le C(X_0,T,q) \tag{3.12}$$

for a positive constant $C(X_0,T,q)$.

Proof. Due to (3.11), it suffices to give the regularity estimate for Y^N. Before that, we give the following estimate of $\|Y^N(t)\|_{L^6}$. The Sobolev embedding theorem, the contractivity (2.4) from L^6 to L^2, the smoothing effect (2.3), and the Gagliardo–Nirenberg inequality yield that

$$\begin{aligned}
&\|Y^N(t)\|_{L^6}\\
&\le \|S(t)X_0^N\|_{L^6}+\int_0^t \|S(\frac{t-s}{2})(S(\frac{t-s}{2})A)P^NF(Y^N(s)+Z^N(s))\|_{L^6}ds\\
&\le C\|X_0^N\|_{L^6}+C\int_0^t (t-s)^{-\frac{1}{12}}\|S(\frac{t-s}{2})A\|\|F(Y^N(s)+Z^N(s))\|ds\\
&\le C\|X_0^N\|_{L^6}+C\int_0^t (t-s)^{-\frac{7}{12}}\Big(1+\|Z^N(s)\|_{L^6}^3+\|Y^N(s)\|_{L^6}^3\Big)ds\\
&\le C\|X_0^N\|_{\mathbb{H}^1}+C\int_0^t (t-s)^{-\frac{7}{12}}\Big(1+\|Z^N(s)\|_{L^6}^3+\|AY^N(s)\|^{\frac{1}{2}}\|Y^N(s)\|^{\frac{5}{2}}\Big)ds.
\end{aligned}$$

From the Hölder and Young inequalities, the estimates (3.6), (3.9) and (3.11), it follows that for any $q\ge 1$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\|Y^N(t)\|_{L^6}^q\Big]\le C(q)\|X_0^N\|_{\mathbb{H}^1}^q\\
&+C(q)\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\int_0^t (t-s)^{-\frac{7}{12}}\Big(1+\|Z^N(s)\|_{L^6}^3+\|AY^N(s)\|^{\frac{1}{2}}\sup_{r\in[0,T]}\|Y^N(r)\|^{\frac{5}{2}}\Big)ds\Big)^q\Big]\\
&\le C(q)\|X_0^N\|_{\mathbb{H}^1}^q+C(q)\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\int_0^t (t-s)^{-\frac{7}{12}}\Big(1+\|Z^N(s)\|_{L^6}^3\Big)ds\Big)^q\Big]\\
&+C(q)\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\int_0^t (t-s)^{-\frac{7}{12}}\|AY^N(s)\|^{\frac{1}{2}}\sup_{r\in[0,T]}\|Y^N(r)\|^{\frac{5}{2}}ds\Big)^q\Big]\\
&\le C(q)\|X_0^N\|_{\mathbb{H}^1}^q+C(q)(\int_0^T (t-s)^{-\frac{7}{12}}ds)^q\Big(1+\mathbb{E}\Big[\sup_{s\in[0,T]}\|Z^N(s)\|_E^{3q}\Big]\Big)\\
&\quad+C(q)\mathbb{E}\Big[\Big(\int_0^T\|AY^N(s)\|^2ds\Big)^{\frac{q}{2}}\Big]\\
&\quad+C(q)(\int_0^T (t-s)^{-\frac{7}{9}}ds)^{\frac{3q}{2}}\mathbb{E}\Big[\sup_{s\in[0,T]}\|Y^N(s)\|^{5q}\Big]\\
&\le C(X_0,T,q).
\end{aligned}$$

The mild form of $Y^N(t)$ and (2.3) lead to

$$\begin{aligned}
\|Y^N(t)\|_{\mathbb{H}^\gamma}&\le \|S(t)X_0^N\|_{\mathbb{H}^\gamma}+\int_0^t \big\|S(t-s)AF(Y^N(s)+Z^N(s))\big\|_{\mathbb{H}^\gamma}ds\\
&\le C\|X_0^N\|_{\mathbb{H}^\gamma}+C\int_0^t (t-s)^{-\frac{1}{2}}\big\|S(\frac{t-s}{2})F(Y^N(s)+Z^N(s))\big\|_{\mathbb{H}^\gamma}ds\\
&\le C\|X_0\|_{\mathbb{H}^\gamma}+C\int_0^t (t-s)^{-\frac{1}{2}-\frac{\gamma}{4}}\big(1+\|Y^N(s)\|_{L^6}^3+\|Z^N(s)\|_{L^6}^3\big)ds.
\end{aligned}$$

By taking qth moment and making use of the a priori estimates of $\|Y^N\|_{L^6}$ and $\|Z^N\|_{\mathbb{H}^\gamma}$, we finish the proof. $\square$

REMARK 3.1. If $X_0 \in \mathbb{H}^\gamma$, $\gamma \in (0,1)$, the estimate (3.12) also holds for $\gamma \in (0,1)$. The key ingredient of the proof is the application of the contractivity of $S(t)$ to deal with the term $\|S(t)X_0^N\|_{L^6}$. Indeed, (2.4) yields that

$$\|S(t)X_0^N\|_{L^6} \le Ct^{-\frac{1}{12}}\|X_0\|.$$

From the Hölder inequality, it follows that there exist p_1, q_1 satisfying $\frac{1}{p_1}+\frac{1}{q_1}=1$, $(\frac{1}{2}+\frac{\gamma}{4})p_1 < 1$ and $q_1 < 4$, such that

$$\int_0^t (t-s)^{-\frac{1}{2}-\frac{\gamma}{4}}\|Y^N(s)\|_{L^6}^3 ds \le (\int_0^t (t-s)^{-(\frac{1}{2}+\frac{\gamma}{4})p_1} ds)^{\frac{1}{p_1}} (\int_0^t \|Y^N(s)\|_{L^6}^{3q_1} ds)^{\frac{1}{q_1}}.$$

Based on the above estimate and similar arguments as in the proof of Proposition 3.1, we obtain the desired result.

Now, we are in a position to answer the well-posedness problem of Eq. (1.1). Before that, we give a useful lemma whose proof is similar to that of [**11**, Lemma 4].

LEMMA 3.5. Let $g : L^4 \to H$ be the Nemytskii operator of a polynomial of second degree. Then for any $\beta \in (0,1)$, it holds that

$$\|g(x)y\|_{\mathbb{H}^{-1}} \le C\big(1+\|x\|_E^2+\|x\|_{\mathbb{H}^\beta}^2\big)\|y\|_{\mathbb{H}^{-\beta}},$$

where $x \in E, x \in \mathbb{H}^\beta$ and $y \in \mathbb{H}$.

PROPOSITION 3.2. Let $\sup\limits_{N\in\mathbb{N}^+}\|X_0^N\|_E \le C(X_0)$, $T>0$ and $q \ge 1$. Then the unique solution X^N of Eq. (1.2) satisfies

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big\|X^N(t)\big\|_E^q\Big] \le C(X_0,T,q) \tag{3.13}$$

for a positive constant $C(X_0,T,q)$.

PROOF. Due to Lemma 3.2, it remains to bound $\mathbb{E}\Big[\sup\limits_{t\in[0,T]}\big\|Y^N(t)\big\|_E^q\Big]$. The mild form of Y^N, combined with (2.4), (2.3), the boundedness of $S(t)$ in E (see (2.4) whose proof is shown in appendix or [**22**]), and the estimation of $\|Y^N\|_{L^6}$, yields that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\|Y^N(t)\|_E^q\Big]\\
&\le \mathbb{E}\Big[\sup_{t\in[0,T]}\|S(t)X_0^N\|_E^q\Big] + C\mathbb{E}\Big[\Big(\int_0^T (t-s)^{-\frac{5}{8}}\|F(Y^N+Z^N)\|ds\Big)^q\Big]\\
&\le C(X_0,T,q),
\end{aligned}$$

which completes the proof. □

4. Strong convergence analysis of the spectral Galerkin method

The main idea of our approach to proving the global existence of the solution is to show the uniform convergence of the sequence $\{(Y^N,Z^N)\}_{N\in\mathbb{N}^+}$ and then to prove the limit process is the unique mild solution of Eq. (1.1). In the following, we first present the strong convergence analysis of the spectral Galerkin approximation in $\mathbb{H}^{-1}$. We would like to mention that there already exists some convergence result of finite dimensional approximation for Eq. (1.1) driving by additive space-time white noise (see e.g. [**11**]). Different from the additive case, the convergence rate analysis of finite dimensional approximation for Eq. (1.1) driving by multiplicative space-time noise is more involved and has not been studied yet.

PROPOSITION 4.1. Let $X_0 \in \mathbb{H}^\gamma$, $\gamma \in (0, \frac{3}{2})$, $T > 0$, $p \geq 1$ and $\sup_{N \in \mathbb{N}^+} \|X_0^N\|_E \leq C(X_0)$. Assume that X^N and X^M are the spectral Galerkin approximations with different parameters $N, M \in \mathbb{N}^+, N < M$. Then it holds that

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\|X^N(t) - X^M(t)\|_{\mathbb{H}^{-1}}^{2p}\Big] \leq C(T, X_0, p)\lambda_N^{-\gamma p}, \tag{4.1}$$

where $C(T, X_0, p)$ is a positive constant.

PROOF. Due to Proposition 3.1, we obtain that for $t \in [0, T]$, $p \geq 1$ and $\gamma \in (0, \frac{3}{2})$,

$$\begin{aligned} \mathbb{E}\Big[\|(I - P^N)X^M(t)\|_{\mathbb{H}^{-1}}^p\Big] &\leq \mathbb{E}\Big[\|(I - P^N)A^{-\frac{1}{2}-\frac{\gamma}{2}}A^{\frac{\gamma}{2}}X^M(t)\|^p\Big] \\ &\leq C(X_0, T, p, \gamma)\lambda_N^{-\frac{p}{2}-\frac{\gamma p}{2}}. \end{aligned}$$

Thus it remains to estimate $\|X^N - P^N X^M\|_{\mathbb{H}^{-1}}$. From the Taylor expansion and Itô formula, it follows that for $p \geq 1$,

$$\begin{aligned} &\|X^N(t) - P^N X^M(t)\|_{\mathbb{H}^{-1}}^{2p} \\ &= -2p\int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2}\|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^1}^2 ds \\ &\quad - 2p\int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2}\Big\langle A\big(P^N F(X^N(s)) - P^N F(X^M(s))\big), \\ &\quad X^N(s) - P^N X^M(s)\Big\rangle_{\mathbb{H}^{-1}} ds \\ &\quad + 2p\int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2}\langle X^N(s) - P^N X^M(s), \\ &\quad (G(X^N(s)) - G(X^M(s)))dW(s)\rangle_{\mathbb{H}^{-1}} \\ &\quad + p\int_0^t \sum_{i\in\mathbb{N}^+} \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2}\|P^N(G(X^N(s)) - G(X^M(s)))e_i\|_{\mathbb{H}^{-1}}^2 ds \\ &\quad + p(2p-2)\int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-4} \sum_{i\in\mathbb{N}^+} |\langle X^N(s) - P^N X^M(s), \\ &\quad P^N(G(X^N(s)) - G(X^M(s)))e_i\rangle_{\mathbb{H}^{-1}}|^2 ds \\ &=: -2p\int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2}\|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^1}^2 ds \\ &\quad + I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned}$$

The monotonicity of $-F$, i.e., for $u^N, v^N \in P^N(\mathbb{H})$,

$$\langle -F(u^N) + F(v^N), u^N - v^N\rangle \leq C\|u^N - v^N\|^2,$$

which is obtained by $c_4 > 0$, and (2.5) yield that

$$I_1 \leq C\int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2}\|X^N(s) - P^N X^M(s)\|^2 ds$$

$$+ 2\int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2} \Big\langle A^{-\frac{1}{2}} \int_0^1 F'(\theta X^N(s) + (1-\theta)X^M(s))d\theta$$
$$(I - P^N)X^M(s), A^{\frac{1}{2}}(X^N(s) - P^N X^M(s))\Big\rangle ds.$$

From Lemma 3.5 and Young's inequality, it follows that for $\beta \in (0,1)$ and small $\epsilon > 0$,

$$\begin{aligned}
I_1 &\le \epsilon \int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2} \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^1}^2 ds \\
&\quad + C(\epsilon) \int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p} ds \\
&\quad + C(\epsilon) \int_0^t \Big\|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2} \\
&\qquad \Big\|A^{-\frac{1}{2}} \int_0^1 F'(\theta X^N(s) + (1-\theta)X^M(s))d\theta (I - P^N)X^M(s)\Big\|^2 ds \\
&\le \epsilon \int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2} \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^1}^2 ds \\
&\quad + C(\epsilon) \int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p} ds \\
&\quad + C(\epsilon) \lambda_N^{-\gamma-\beta} \int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2} \Big(1 + \|X^N(s)\|_{\mathbb{H}^\beta}^4 \\
&\qquad + \|X^M(s)\|_{\mathbb{H}^\beta}^4 + \|X^N(s)\|_E^4 + \|X^M(s)\|_E^4\Big) \|X^M(s)\|_{\mathbb{H}^\gamma}^2 ds,
\end{aligned}$$

where we have used the estimate

$$\|(I - P^N)v\| \le C\lambda_N^{-\frac{\kappa}{2}} \|v\|_{\mathbb{H}^\kappa} \tag{4.2}$$

for $v \in \mathbb{H}^\kappa, \kappa > 0$ in the last inequality. The uniform boundedness of $\{e_j\}_{j\in\mathbb{N}^+}$ and the Young inequality yield that for small $\epsilon > 0$,

$$\begin{aligned}
&\mathbb{E}\Big[I_3 + I_4\Big] \\
&\le C\mathbb{E}\Big[\int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2} \sum_{i\in\mathbb{N}^+} \|P^N((G(X^N(s)) - G(X^M(s)))e_i)\|_{\mathbb{H}^{-1}}^2 ds\Big] \\
&\le C\mathbb{E}\Big[\int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2} \sum_{j\in\mathbb{N}^+} \|(G(X^N(s)) - G(X^M(s)))e_j\|^2 \lambda_j^{-1} ds\Big] \\
&\le C(\epsilon)\mathbb{E}\Big[\int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p} ds\Big] \\
&\quad + \epsilon\mathbb{E}\Big[\int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2} \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^1}^2 ds\Big] \\
&\quad + C\mathbb{E}\Big[\int_0^t \|X^N(s) - P^N X^M(s)\|_{\mathbb{H}^{-1}}^{2p-2} \|(I - P^N)X^M(s)\|^2 ds\Big].
\end{aligned}$$

The above estimations, combined with the Young inequality, the martingale property of the stochastic integral I_2 and 4.2, yield that for $\beta \in (0,1)$ and small $\epsilon > 0$,

$$\mathbb{E}\Big[\|X^N(t) - P^N X^M(t)\|_{\mathbb{H}^{-1}}^{2p}\Big]$$

$$
\begin{aligned}
&\leq -2p\int_0^t \mathbb{E}\Big[\|X^N(s)-P^NX^M(s)\|_{\mathbb{H}^{-1}}^{2p-2}\|X^N(s)-P^NX^M(s)\|_{\mathbb{H}^1}^2\Big]ds\\
&\quad+\mathbb{E}\Big[I_1(t)+I_2(t)+I_3(t)+I_4(t)\Big]\\
&\leq C(\epsilon)\int_0^t \mathbb{E}\Big[\|X^N(s)-P^NX^M(s)\|_{\mathbb{H}^{-1}}^{2p}\Big]ds\\
&\quad+C(\epsilon)\lambda_N^{-\gamma-\beta}\int_0^t \mathbb{E}\Big[\|X^N(s)-P^NX^M(s)\|_{\mathbb{H}^{-1}}^{2p-2}\Big(1+\|X^N\|_{\mathbb{H}^\beta}^4\\
&\quad+\|X^M\|_{\mathbb{H}^\beta}^4+\|X^N\|_E^4+\|X^M\|_E^4\Big)\|X^M(s)\|_{\mathbb{H}^\gamma}^2\Big]ds\\
&\quad+C\int_0^t \mathbb{E}\Big[\|X^N(s)-P^NX^M(s)\|_{\mathbb{H}^{-1}}^{2p-2}\|(I-P^N)X^M(s)\|^2\Big]ds\\
&\leq C(\epsilon)\int_0^t \mathbb{E}\Big[\|X^N(s)-P^NX^M(s)\|_{\mathbb{H}^{-1}}^{2p}\Big]ds+C(\epsilon)\lambda_N^{-\gamma p-\beta p}\int_0^t \mathbb{E}\Big[\Big(1+\|X^N\|_{\mathbb{H}^\beta}^4\\
&\quad+\|X^M\|_{\mathbb{H}^\beta}^4+\|X^N\|_E^4+\|X^M\|_E^4\Big)^p\|X^M(s)\|_{\mathbb{H}^\gamma}^{2p}\Big]ds\\
&\quad+C\lambda_N^{-\gamma p}\int_0^t \mathbb{E}\Big[\|X^M(s)\|_{\mathbb{H}^\gamma}^{2p}\Big]ds.
\end{aligned}
$$

Combining the regularity estimates of X^N and X^M in Propositions 3.1 and 3.2, we complete the proof by using the Gronwall inequality. □

Now, we are in the position to deduce the error estimate in $\mathbb{H}$, which implies that $\{X^N\}_{N\in\mathbb{N}^+}$ is a Cauchy sequence in $L^p(\Omega;C([0,T];\mathbb{H}))$.

THEOREM 4.1. Let $X_0\in\mathbb{H}^\gamma$, $\gamma\in(0,\frac{3}{2})$, $T>0$, $p\geq 1$ and $\sup_{N\in\mathbb{N}^+}\|X_0^N\|_E\leq C(X_0)$. Assume that X^N and X^M are the spectral Galerkin approximations with different parameters $N,M\in\mathbb{N}^+, N<M$. Then for $\tau\in(0,\gamma)$, it holds that

$$
\mathbb{E}\Big[\sup_{t\in[0,T]}\|X^N(t)-X^M(t)\|^{2p}\Big]\leq C(T,X_0,p)\lambda_N^{-\tau p}. \tag{4.3}
$$

for a positive constant $C(T,X_0,p)$.

PROOF. From the mild form of X^N and P^NX^M, (2.3) and Hölder's inequality, together with Lemma 3.5 and the interpolation inequality, i.e, for any $\beta\in(0,1)$,

$$
\|v\|_{\mathbb{H}^{-\beta}}\leq C\|v\|_{\mathbb{H}^{-1}}^\beta\|v\|^{1-\beta},\ \text{for } v\in\mathbb{H},
$$

it follows that for $p\geq\frac{l}{2}$, $l>4$ and $\beta\in(0,1)$,

$$
\begin{aligned}
&\mathbb{E}\Big[\|X^N(t)-P^NX^M(t)\|^{2p}\Big]\\
&\leq C(p,T)\mathbb{E}\Big[\int_0^t(t-s)^{-\frac{3}{4}}\Big\|A^{-\frac{1}{2}}(F(X^N(s))-F(X^M(s)))\Big\|^{2p}ds\Big]\\
&+C(p,T)\mathbb{E}\Big[\Big\|\int_0^t S(t-s)P^N(G(X^N(s))-G(X^M(s)))dW(s)\Big\|^{2p}\Big]\\
&\leq C(p,T)\Big(\int_0^T(t-s)^{-\frac{3l}{4(l-1)}}ds\Big)^{\frac{2p(l-1)}{l}}\mathbb{E}\Big[\Big(\int_0^T\big(1+\|X^N(s)\|_E^{2l}+\|X^M(s)\|_E^{2l}\\
&\quad+\|X^N(s)\|_{\mathbb{H}^\beta}^{2l}+\|X^M(s)\|_{\mathbb{H}^\beta}^{2l}\big)\big\|X^N(s)-X^M(s)\big\|_{\mathbb{H}^{-1}}^{\beta l}\big\|X^N(s)-X^M(s)\big\|^{(1-\beta)l}ds\Big)^{\frac{2p}{l}}\Big]
\end{aligned}
$$

$$+ C(p,T)\mathbb{E}\Big[\Big\|\int_0^t S(t-s)P^N(G(X^N(s))-G(X^M(s)))dW(s)\Big\|^{2p}\Big].$$

The Burkholder inequality, Parseval's inequality, the fact that $\sum_{i=1}^{\infty} e^{-\lambda_i^2 t} \le Ct^{-\frac{1}{4}}$, Proposition 4.1, (4.2), (3.12) and (3.13) yield that

$$\begin{aligned}
&\mathbb{E}\Big[\|X^N(t)-P^NX^M(t)\|^{2p}\Big]\\
&\le C(p,T,X_0,\beta)\lambda_N^{-\beta p\gamma}\Big(\mathbb{E}\Big[\int_0^T \big(1+\|X^N(s)\|_E^{8p}+\|X^M(s)\|_E^{8p}+\|X^N(s)\|_{\mathbb{H}^\beta}^{8p}\\
&\quad+\|X^M(s)\|_{\mathbb{H}^\beta}^{8p}\big)\big\|X^N(s)-X^M(s)\big\|^{(1-\beta)4p}ds\Big]\Big)^{\frac{1}{2}}\\
&+C(p,T)\mathbb{E}\Big[\Big(\int_0^t\sum_{i\in\mathbb{N}^+}\|S(t-s)P^N(G(X^N(s))-G(X^M(s)))e_i\|^2ds\Big)^p\Big]\\
&\le C(p,T,X_0,\beta)\lambda_N^{-\beta p\gamma}+C(p,T)\mathbb{E}\Big[\Big(\int_0^t\sum_{i\in\mathbb{N}^+}e^{-\lambda_i^2(t-s)}\|G(X^N(s))-G(X^M(s))\|^2ds\Big)^p\Big]\\
&\le C(p,T,X_0,\beta)\lambda_N^{-\beta p\gamma}+C(p,T)\mathbb{E}\Big[\Big(\int_0^t\|X^N(s)-P^NX^M(s)\|^4ds\Big)^{\frac{p}{2}}\Big]\\
&+C(p,T)\mathbb{E}\Big[\Big(\int_0^t\|(I-P^N)X^M(s)\|^4ds\Big)^{\frac{p}{2}}\Big]\\
&\le C(p,T,X_0,\beta)(\lambda_N^{-\beta p\gamma}+\lambda_N^{-p\gamma})+C(p,T)\int_0^t\mathbb{E}\Big[\|X^N(s)-P^NX^M(s)\|^{2p}\Big]ds.
\end{aligned}$$

From the Gronwall inequality and $\beta\in(0,1)$, it follows that

$$\sup_{t\in[0,T]}\mathbb{E}\Big[\|X^N(t)-P^NX^M(t)\|^{2p}\Big]\le C(X_0,T,p,\gamma)\lambda_N^{-\beta p\gamma}. \tag{4.4}$$

Furthermore, taking supreme over $t\in[0,T]$, similar arguments yield that for $p\ge\frac{l}{2}$, $l>4$ and $\beta\in(0,1)$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\|X^N(t)-P^NX^M(t)\|^{2p}\Big]\\
&\le C(p,T,X_0,\beta)\lambda_N^{-\beta p\gamma}\\
&+C(p)\mathbb{E}\Big[\sup_{t\in[0,T]}\Big\|\int_0^t S(t-s)P^N(G(X^N(s))-G(X^M(s)))dW(s)\Big\|^{2p}\Big].
\end{aligned}$$

The factorization method in [**13**, Proposition 5.9 and Theorem 5.10] yields that for $\frac{3}{8}>\alpha_1>\frac{1}{q}$, $q>1$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\Big\|\int_0^t S(t-s)P^N(G(X^N(s))-G(X^M(s)))dW(s)\Big\|^{2p}\Big]\\
&\le C(p,q,T)\mathbb{E}\Big[\|Z_{\alpha_1,N,M}\|_{L^q([0,T];\mathbb{H})}^{2p}\Big],
\end{aligned}$$

where $Z_{\alpha_1,N,M}(s)=\int_0^s(s-r)^{-\alpha_1}S(s-r)P^N(G(X^N(r))-G(X^M(r)))dW(r)$. Thus it suffices to estimate $\mathbb{E}\Big[\|Z_{\alpha_1,N,M}\|_{L^q([0,T];H)}^{2p}\Big]$. From the Burkholder and Hölder inequalities, and the estimate (4.4), it follows that for $2p\ge q$, $(2\alpha_1+\frac{1}{4})\frac{l}{l-1}<1$

and $p \geq l$,

$$
\begin{aligned}
&\mathbb{E}\Big[\|Z_{\alpha_1,N,M}\|^{2p}_{L^q([0,T];\mathbb{H})}\Big]\\
&= \mathbb{E}\Big[\Big(\int_0^T \Big\|\int_0^s (s-r)^{-\alpha_1} S(s-r)P^N(G(X^N(s)) - G(X^M(s)))dW(r)\Big\|^q ds\Big)^{\frac{2p}{q}}\Big]\\
&\leq C(T,p)\int_0^T \mathbb{E}\Big[\Big\|\int_0^s (s-r)^{-\alpha_1} S(s-r)P^N(G(X^N(s)) - G(X^M(s)))dW(r)\Big\|^{2p}\Big]ds\\
&\leq C\int_0^T \mathbb{E}\Big[\Big(\int_0^s (s-r)^{-2\alpha_1}\sum_{i\in\mathbb{N}^+}\Big\|S(s-r)P^N(G(X^N(s)) - G(X^M(s)))e_i\Big\|^2 dr\Big)^p\Big]ds\\
&\leq C\int_0^T \Big(\int_0^s (s-r)^{-(2\alpha_1+\frac14)\frac{l}{l-1}}dr\Big)^{\frac{(l-1)p}{l}}\mathbb{E}\Big[\Big(\int_0^s \|X^N(s) - P^N X^M(s)\|^{2l}dr\Big)^{\frac{p}{l}}ds\Big]\\
&+ C\mathbb{E}\Big[\int_0^T \Big(\int_0^s (s-r)^{-(2\alpha_1+\frac14)}\|(I-P^N)X^M(t)\|^2 dr\Big)^p ds\Big].
\end{aligned}
$$

Combining the above estimates, using Hölder's inequality, (3.12), (4.2) and (4.4), we complete the proof. □

REMARK 4.1. If $X_0 \in \mathbb{H}^\gamma$, $\gamma > \frac{1}{2}$, then $\|X_0^N\|_E \leq C(X_0)$ holds for every $N \in \mathbb{N}^+$. If the bound of $\|X_0^N\|_E$ is not uniform, then by using (2.4), we have that

$$\mathbb{E}\big[\|X^N(t)\|_E^q\big] \leq C(X_0,T,q)(1+t^{-\frac{q}{8}}).$$

As a result, it could be checked that Proposition 4.1, Theorem 4.1 and Proposition 5.1 still hold with $p=1$, which is helpful for establishing the wellposedness result under mild assumptions.

5. Global existence and regularity estimate

Based on the convergence of the approximate process X^N, we are in the position to show the global existence of the unique solution for Eq. (1.1) driven by multiplicative space-time white noise.

PROPOSITION 5.1. Let $T>0$, $X_0 \in \mathbb{H}^\gamma$, $\gamma \in (0,\frac{3}{2})$, $p \geq 1$ and $\sup\limits_{N\in\mathbb{N}^+}\|X_0^N\|_E \leq C(X_0)$. Then Eq. (1.1) possesses a unique mild solution X in $L^{2p}(\Omega; C(0,T;\mathbb{H}))$.

PROOF. We first show the local uniqueness of the mild solution for Eq. (1.1) which is based on the Lipschitz continuity of G and the local Lipschitz continuity of F. More precisely, assume that we have two different mild solutions X_1 and X_2 for Eq. (1.1) in $L^{2p}(\Omega; C(0,T;\mathbb{H}))$ with the same initial datum X_0. We aim to prove the local uniqueness, i.e., $X_1(t) = X_2(t)$ for $t \in [0, \tau_1^R \wedge \tau_2^R), a.s.$, where $\tau_i^R := \inf\big\{t \geq 0 : \sup\limits_{r\in[0,t]}\|X_i(r)\| \geq R\big\}, i = 1,2$, for a large enough $R>0$. The stopping time τ_i^R is well-defined and non-decreasing since $X_i \in C(0,T;\mathbb{H}), a.s.$, is $\mathcal{F}_t$-measurable for $i=1,2$. For each process X_i, we could consider the decomposition $X_i = Y_i + Z_i$, $i=1,2$, satisfying

$$
\begin{aligned}
&dZ_i(t) = -A^2 Z_i(t) + G(X_i(t))dW(t), Z_i(0) = 0,\\
&dY_i(t) = -A^2 Y_i(t) - AF(X_i(t))dt, Y_i(0) = X(0),
\end{aligned}
$$

where Z_i is the stochastic convolution and Y_i is the mild solution of the second equation. The existence of Y_i is guaranteed by $Y_i = X_i - Z_i$. Then it follows that

$$\begin{aligned}
&d(Y_1(t) - Y_2(t)) = -A^2(Y_1(t) - Y_2(t))dt - A(F(X_1(t)) - F(X_2(t)))dt,\\
&Y_1(0) - Y_2(0) = 0,
\end{aligned}$$

and

$$\begin{aligned}
&d(Z_1(t) - Z_2(t)) = -A^2(Z_1(t) - Z_2(t))dt + (G(X_1(t)) - G(X_2(t)))dW(t),\\
&Z_1(0) - Z_2(0) = 0.
\end{aligned}$$

The factorization method in [**13**, Proposition 5.9 and Theorem 5.10] yields that for $\frac{3}{8} > \alpha_1 > \frac{1}{q}$, $q > 1$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,\tau_1^R\wedge\tau_2^R\wedge T]} \|Z_1(t) - Z_2(t)\|^{2p}\Big]\\
&= \mathbb{E}\Big[\sup_{t\in[0,\tau_1^R\wedge\tau_2^R\wedge T]} \Big\|\int_0^t S(t-s)(G(X_1(s)) - G(X_2(s)))dW(s)\Big\|^{2p}\Big]\\
&\le C(p,q,T)\mathbb{E}\Big[\|Z_{\alpha_1}\|^{2p}_{L^q([0,\tau_1^R\wedge\tau_2^R\wedge T];\mathbb{H})}\Big],
\end{aligned}$$

where $Z_{\alpha_1}(s) = \int_0^s (s-r)^{-\alpha_1} S(s-r)(G(X_1(r)) - G(X_2(r)))dW(r)$. From the Burkholder and Hölder inequalities, and the estimate (4.4), it follows that for $2p \ge q$, $(2\alpha_1 + \frac{1}{4})\frac{l}{l-1} < 1$ and $p \ge l$,

$$\begin{aligned}
&\mathbb{E}\Big[\|Z_{\alpha_1}\|^{2p}_{L^q([0,\tau_1^R\wedge\tau_2^R\wedge T];\mathbb{H})}\Big]\\
&= \mathbb{E}\Big[\Big(\int_0^{\tau_1^R\wedge\tau_2^R\wedge T} \Big\|\int_0^s (s-r)^{-\alpha_1} S(s-r)(G(X_1(s)) - G(X_2(s)))dW(r)\Big\|^q ds\Big)^{\frac{2p}{q}}\Big]\\
&\le C(T,p)\int_0^T \mathbb{E}\Big[\mathbb{I}_{\{s\le\tau_1^R\wedge\tau_2^R\wedge T\}}\Big\|\int_0^s (s-r)^{-\alpha_1} S(s-r)(G(X_1(s)) - G(X_2(s)))dW(r)\Big\|^{2p}\Big]ds\\
&\le C\int_0^T \mathbb{E}\Big[\mathbb{I}_{\{s\le\tau_1^R\wedge\tau_2^R\wedge T\}}\Big(\int_0^s (s-r)^{-2\alpha_1}\sum_{i\in\mathbb{N}^+}\Big\|S(s-r)(G(X_1(s)) - G(X_2(s)))e_i\Big\|^2 dr\Big)^p\Big]ds\\
&\le C\int_0^T \Big(\int_0^s (s-r)^{-(2\alpha_1+\frac{1}{4})\frac{l}{l-1}}dr\Big)^{\frac{(l-1)p}{l}}\mathbb{E}\Big[\Big(\mathbb{I}_{\{s\le\tau_1^R\wedge\tau_2^R\wedge T\}}\int_0^s \|X_1(s) - X_2(s)\|^{2l}dr\Big)^{\frac{p}{l}}ds\Big]\\
&\le C(p,T)\mathbb{E}\Big[\sup_{t\in[0,\tau_1^R\wedge\tau_2^R\wedge T]}\Big\|X_1(t) - X_2(t)\Big\|^{2p}\Big].
\end{aligned}$$

Here $\mathbb{I}$ is the indicator function.

In the following, we will take a smooth approximation $\{X^n(t)\}_{n=1}^\infty$ of $X(t)$ in $\mathbb{H}$ such that we can apply the chain rule to the equation of $Y_1^n(t) - Y_2^n(t)$. For instance, we could take $Y_i^n(t) = P_i^n Y(t)$ or the spectral Galerkin approximation $Y_i^N(t)$. In the end, we will take $n \to \infty$ or $N \to \infty$ and get the desired estimate for $Y(t) \in \mathbb{H}$. For convenience, we omit the subindex n in the proof. The chain rule, together with the Young inequality and the Gagliardo–Nirenberg inequality, i.e,

$$\|u\|_E \le C\|\Delta u\|^{\frac{1}{4}}_{L^2}\|u\|^{\frac{3}{4}}_{L^2} + C\|u\|_{L^2},$$

yields that for $t \in [0, \tau_1^R \wedge \tau_2^R \wedge T]$ and for a small $\epsilon \in (0,1)$,

$$\|Y_1(t) - Y_2(t)\|^2$$

$$
\begin{aligned}
&\le -\int_0^t 2\|A(Y_1(s)-Y_2(s))\|^2 ds \\
&\quad + 2\int_0^t \langle -A(F(X_1(s))-F(X_2(s))), Y_1(t)-Y_2(t)\rangle ds \\
&\le -\int_0^t (2-\epsilon)\|A(Y_1(s)-Y_2(s))\|^2 ds + C(\epsilon)\int_0^t \|F(X_1(s))-F(X_2(s))\|^2 ds \\
&\le C(\epsilon)\int_0^t \|X_1(s)-X_2(s)\|^2(1+\|Y_1(s)\|_E^4+\|Y_2(s)\|_E^4+\|Z_1(s)\|_E^4+\|Z_2(s)\|_E^4)ds \\
&\le C(\epsilon)\int_0^t \|X_1(s)-X_2(s)\|^2(1+\|AY_1(s)\|^2+\|AY_2(s)\|^2+\|Y_1(s)\|^6 \\
&\qquad +\|Y_2(s)\|^6+\|Z_1(s)\|_E^4+\|Z_2(s)\|_E^4)ds.
\end{aligned}
$$

The finiteness of the right hand side of the above estimate could be obtained as follows. Following the similar arguments proving (3.4) and (3.10), one can get

$$
\begin{aligned}
&\|Y_i(t)\|^2 + \int_0^t \|(-A)Y_i(s)\|^2 ds \\
&\le C(t)\Big(\|Y_i(0)\|^2 + (1+\sup_{s\in[0,t]}\|Z(s)\|_E^2)\int_0^T \|Y_i(s)\|_{L^4}^4 ds + \int_0^t \|Z_i(s)\|_E^6 ds\Big) \\
&\le C(t)\Big(\|Y_i(0)\|^2 + (1+\sup_{s\in[0,t]}\|Z(s)\|_E^2)\int_0^T \|Z_i(s)\|_{L^4}^4 ds + \int_0^t \|Z_i(s)\|_E^6 ds\Big).
\end{aligned}
$$

The analogous arguments as in the proof of Lemma 3.2 yield that for $\alpha \in (0,1)$ and large enough $q>1$,

$$
\mathbb{E}\Big[\|Z_i\|_{C(0,t;E)}^q\Big] \le C(T,q,\alpha)\mathbb{E}\Big[\int_0^t (1+\|X_i(r)\|^{\alpha q})dr\Big].
$$

By Gronwall's inequality, we conclude that there exists a constant $C(R,T,X_0) > 0$ such that for $t \le \tau_1^R \wedge \tau_2^R \wedge T$,

$$
\|Y_1(t)-Y_2(t)\|^2 \le \exp(C(R,T,X_0))\int_0^t \|X_1(s)-X_2(s)\|^2 ds.
$$

From the above estimates, we conclude that for $0\le s\le t\le \tau_1^R\wedge\tau_2^R\wedge T$ and $p\ge 1$,

$$
\begin{aligned}
&\|X_1(s)-X_2(s)\|^{2p} \\
&\le C_p\|Y_1(s)-Y_2(s)\|^{2p} + C_p\|Z_1(s)-Z_2(s)\|^{2p} \\
&\le C_p\exp(C(R,T,X_0))\int_0^s \|X_1(s)-X_2(s)\|^{2p}ds + C_p\|Z_1(s)-Z_2(s)\|^{2p},
\end{aligned}
$$

which, together with Gronwall's inequality, yields that

$$
\|X_1(s)-X_2(s)\|^{2p} \le \exp(C(p,T)\exp(C(R,T,X_0)))\|Z_1(s)-Z_2(s)\|^{2p}.
$$

Taking expectation, applying the factorization formula in [**13**, Proposition 5.9 and Theorem 5.10]], and using the Burkholder and Hölder inequalities, as well as the fact that $\sup_{i\in\mathbb{N}^+}\|e_i\|_E \le C$ and that $\sum_{i\in\mathbb{N}^+} e^{-\lambda_i^2 t} \le Ct^{-\frac14}$, we have for a large enough $q>1$, $\frac38 > \alpha_1 > \frac1q$ and $2p \ge q$,

$$
\mathbb{E}\Big[\sup_{s\in[0,t]}\|X_1(s)-X_2(s)\|^{2p}\Big]
$$

$$\le \exp(C(p,T)\exp(C(R,T,X_0)))\mathbb{E}\Big[\sup_{s\in[0,t]}\|Z_1(s)-Z_2(s)\|^{2p}\Big]$$

$$\le C(R,T,p,X_0)\mathbb{E}\Big[\Big(\int_0^t\Big(\int_0^s(s-r)^{-\alpha_1}S(s-r)(G(X_1(r))-G(X_2(r)))dW(r)\Big)^q ds\Big)^{\frac{2p}{q}}\Big]$$

$$\le C(R,T,p,X_0)\mathbb{E}\Big[\int_0^t\Big(\int_0^s(s-r)^{-\frac{1}{4}-2\alpha_1}\|G(X_1(r))-G(X_2(r))\|^2dr\Big)^p dt\Big]$$

$$\le C(R,T,p,X_0)\int_0^t\mathbb{E}\Big[\sup_{r\in[0,s]}\|X_1(r))-X_2(r)\|^{2p}\Big]ds,$$

where $\Gamma(s)=\int_0^s(s-r)^{-\alpha_1}S(s-r)(G(X_1(s))-G(X_2(s)))dW(r)$.

From Gronwall's inequality, it follows that for any $t\le\tau_1^R\wedge\tau_2^R\wedge T$,

$$\mathbb{E}\Big[\sup_{s\in[0,t]}\|X_1(s)-X_2(s)\|^{2p}\Big]=0.$$

Thus the local uniqueness of the mild solution holds. As a consequence, if the global existence of the mild solution holds, then we have

$$\mathbb{E}\Big[\sup_{s\in[0,T]}\|X_1(s)-X_2(s)\|^{2p}\Big]\le\lim_{R\to\infty}\mathbb{E}\Big[\sup_{s\in[0,\tau_1^R\wedge\tau_2^R\wedge T]}\|X_1(s)-X_2(s)\|^{2p}\Big]=0.$$

Here we use the fact that $\lim_{R\to\infty}\tau_1^R\wedge\tau_2^R\wedge T=T, a.s.$ This is ensured by Chebyshev's inequality and

$$\begin{aligned}&\lim_{R\to\infty}\mathbb{P}(\tau_1^R\wedge\tau_2^R\wedge T<T)\\ &\le\lim_{R\to\infty}\mathbb{P}(\sup_{t\in[0,T]}\|X_1(t)\|\ge R)+\lim_{R\to\infty}\mathbb{P}(\sup_{t\in[0,T]}\|X_2(t)\|\ge R)\\ &\le\lim_{R\to\infty}\frac{1}{R}\mathbb{E}\Big[\sup_{t\in[0,T]}\|X_1(t)\|\Big]+\lim_{R\to\infty}\frac{1}{R}\mathbb{E}\Big[\sup_{t\in[0,T]}\|X_2(t)\|\Big]=0.\end{aligned}$$

In the following, we show the existence of the global mild solution. According to Theorem 4.1, we have that $\{X^N\}_{N\in\mathbb{N}^+}$ is a Cauchy sequence in $L^{2p}(\Omega;C([0,T];\mathbb{H}))$. From the Sobolev interpolation inequality

$$\|u\|_{\mathbb{H}^{\gamma_1}}\le C\|u\|^{\frac{\gamma-\gamma_1}{\gamma}}\|u\|_{\mathbb{H}^\gamma}^{\frac{\gamma_1}{\gamma}},\ \text{for }\gamma_1\in(0,\gamma),u\in\mathbb{H}^\gamma,$$

and the uniform estimates $\sup_{N\in\mathbb{N}^+}\|X^N\|_{L^{2p}(\Omega;C([0,T];\mathbb{H}^\gamma))}<\infty$ in Proposition 3.1 and Remark 3.1, it follows that for $\gamma_1\in(0,\gamma)$, X^N is also a Cauchy sequence in $L^{2p}(\Omega;C([0,T];\mathbb{H}^{\gamma_1}))$. In the following, we deal with the case that $\gamma>\frac{1}{2}$ for simplicity. When $\gamma\le\frac{1}{2}$, we need more steps to prove that X is a mild solution (see Appendix for more details). Let us denote its limit by $X\in L^{2p}(\Omega;\,C([0,T];\mathbb{H}^{\gamma_1})$ for $\gamma_1\in(0,\gamma)$.

It suffices to prove that X is the mild solution of Eq. (1.1), i.e.,

$$X(t)=S(t)X_0+\int_0^tS(t-s)(-A)F(X(s))ds+\int_0^tS(t-s)G(X(s))dW(s),\ \text{a.s.}$$

We introduce the decomposition $X=Y+Z$, where $Z(t)=\int_0^tS(t-s)G(X(s))dW(s)$. The strong convergence of X^N in $L^{2p}(\Omega;C([0,T];\mathbb{H}^{\gamma_1}))$ implies that the strong convergence Z^N to Z in $L^{2p}(\Omega;C([0,T];\mathbb{H}^{\gamma_1}))$ by applying the Burkholder's inequality, the factorization formula and similar arguments as in the proof of Theorem 4.1.

Thus, we also have the strong convergence of Y^N to Y in $L^{2p}(\Omega; C([0,T]; \mathbb{H}^{\gamma_1}))$ by $Y = X - Z$ and $Y^N = X^N - Z^N$. It remains to show that Y satisfies

$$Y(t) = S(t)X_0 + \int_0^t S(t-s)(-A)F(X(s))ds, \text{ a.s.} \tag{5.1}$$

We claim that all the terms on the right-hand side are finite. The first term $S(t)X_0$ is finite since $X_0 \in \mathbb{H}^\gamma$. The finiteness of the last term is achieved due to (2.3), Hölder's inequality, the Sobolev embedding theorem $\mathbb{H}^{\frac{1}{2}^+} \hookrightarrow L^6$ and Lemma 3.1. Here $\frac{1}{2}^+$ denotes $\frac{1}{2} + \epsilon'$ for any $\epsilon' > 0$. Indeed, for $\frac{1}{2} < \gamma_1 < \gamma$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big\| \int_0^t S(t-s)(-A)F(X(s))ds\Big\|^{2p}\Big] \\
&\le C_p \mathbb{E}\Big[\Big(\int_0^t (t-s)^{-\frac{1}{2}}(1 + \|X(s)\|_{L^6}^3)ds\Big)^{2p}\Big] \\
&\le C_p \mathbb{E}\Big[\sup_{s\in[0,T]} \|X(s)\|_{\mathbb{H}^{\gamma_1}}^{6p}\Big] \le \lim_{N\to\infty} C_p \mathbb{E}\Big[\sup_{s\in[0,T]} \|X^N(s)\|_{\mathbb{H}^{\gamma_1}}^{6p}\Big] < \infty,
\end{aligned}$$

where in the last step we use the convergence of X^N to X.

The mild form of Y^N and the right-hand side in (5.1), and (2.3) yield that

$$\begin{aligned}
&Err := \|S(t)(I - P^N)X_0\|_{C([0,T;\mathbb{H}])} \\
&+ \Big\| \int_0^t S(t-s)A(F(X(s)) - P^N F(X^N(s)))ds\Big\|_{L^{2p}(\Omega; C([0,T];\mathbb{H}))} \\
&\le C(T, X_0)\lambda_N^{-\frac{\gamma}{2}} + \Big\| \int_0^t S(t-s)A(I - P^N)F(X(s))ds\Big\|_{L^{2p}(\Omega; C([0,T];\mathbb{H}))} \\
&+ \Big\| \int_0^t S(t-s)AP^N(F(X(s) - F(X^N(s)))ds\Big\|_{L^{2p}(\Omega; C([0,T];\mathbb{H}))} \\
&\le C(T, X_0, p)\lambda_N^{-\frac{\gamma}{2}} + C(T,p)\lambda_N^{-\frac{\gamma}{2}} \Big\| \int_0^t (t-s)^{-\frac{1}{2}-\frac{\gamma}{4}} \|F(X^N(s))\|_{\mathbb{H}} ds\Big\|_{L^{2p}(\Omega; C([0,T];\mathbb{R}))} \\
&+ C(T,p)\Big\| \int_0^t (t-s)^{-\frac{1}{2}}(1 + \|X(s)\|_E^2 + \|X^N(s)\|_E^2)\|X(s) - X^N(s)\|ds\Big\|_{L^{2p}(\Omega; C([0,T];\mathbb{R}))}.
\end{aligned}$$

According to the Sobolev embedding theorem $\mathbb{H}^{\frac{1}{2}^+} \hookrightarrow E$, Lemma 3.1 and Theorem 4.1, we have that for $\beta \in (0,1)$,

$$Err \le C(T, X_0, p)\lambda_N^{-\frac{\beta\gamma}{2}}.$$

The above estimation implies that

$$S(t)X_0 - \int_0^t S(t-s)AF(X(s))ds + \int_0^t S(t-s)G(X(s))dW(s)$$

is the limit of X^N in $L^{2p}(\Omega; C([0,T]; \mathbb{H}))$. By the uniqueness of the limit in $L^{2p}(\Omega; C([0,T]; \mathbb{H}))$, we conclude that $X(t)$ is the mild solution. □

From the arguments as in the above proof, we immediately get the following well-posedness result under mild assumptions. As a cost, we can not obtain the optimal convergence rate of this Cauchy sequence $\{X^N\}_{N\in\mathbb{N}^+}$.

Theorem 5.1. Let $T > 0$, $X_0 \in \mathbb{H}^\gamma, \gamma > 0$, $p \ge 1$. Then Eq. (1.1) possesses a unique mild solution X in $L^{2p}(\Omega; C([0,T]; \mathbb{H}))$.

PROOF. Since the strong convergence in Theorem 4.1 holds with $p=1$ (see Remark 4.1), we have that $\{X^N\}_{N\in\mathbb{N}^+}$ is a Cauchy sequence in $L^2(\Omega;C([0,T];\mathbb{H}))$, which implies that there exists a subsequence $\{X^{N_k}\}_{k\in\mathbb{N}^+}$ converging to X in $C([0,T];\mathbb{H})$ a.s. Notice that Lemma 3.3 implies that $X^N\in L^{2q}(\Omega;C([0,T];\mathbb{H}))$ for any $q\geq 1$. By using the Hölder inequality, we obtain

$$
\begin{aligned}
&\|X-X^N\|_{L^{2p}(\Omega;C([0,T];\mathbb{H}))}\\
&\leq \|X-X^N\|^{\frac{4p-2}{4p}}_{L^{4p-2}(\Omega;C([0,T];\mathbb{H}))}\|X-X^N\|^{\frac{1}{2p}}_{L^2(\Omega;C([0,T];\mathbb{H}))}\\
&\leq \|X-X^N\|^{\frac{1}{2p}}_{L^2(\Omega;C([0,T];\mathbb{H}))}\big(\|X^N\|_{L^{4p-2}(\Omega;C(0,T;\mathbb{H}))}+\|X\|_{L^{4p-2}(\Omega;C(0,T;\mathbb{H}))}\big)^{\frac{2p-1}{2p}}\\
&\leq C(X_0,T,p)\|X-X^N\|^{\frac{1}{2p}}_{L^2(\Omega;C([0,T];\mathbb{H}))},
\end{aligned}
$$

where in the last step, we have applied Fatou's lemma and Lemma 3.3,

$$
\begin{aligned}
\mathbb{E}\Big[\|X\|^{4p-2}_{C(0,T;\mathbb{H}))}\Big]&=\mathbb{E}\Big[\lim_{k\to\infty}\|X^{N_k}\|^{4p-2}_{C(0,T;\mathbb{H}))}\Big]\\
&\leq \liminf_{k\to\infty}\mathbb{E}\Big[\|X^{N_k}\|^{4p-2}_{C(0,T;\mathbb{H}))}\Big]\leq C(X_0,T,p).
\end{aligned}
$$

This implies that $\{X^N\}_{N\in\mathbb{N}^+}$ is also a Cauchy sequence in $L^{2p}(\Omega;C([0,T];\mathbb{H}))$. □

After establishing the well-posedness of Eq. (1.1), in the following, we present the regularity estimates of the exact solution X in both time and space.

COROLLARY 5.1. Let $X_0\in\mathbb{H}^\gamma$, $\gamma\in(0,\frac{3}{2})$, $T>0$ and $p\geq 1$. The unique mild solution X of Eq. (1.1) satisfies

$$
\mathbb{E}\Big[\sup_{t\in[0,T]}\big\|X(t)\big\|^p_{\mathbb{H}^\gamma}\Big]\leq C(X_0,T,p). \tag{5.2}
$$

PROOF. Due to $X^N=Y^N+Z^N$ and $X=Y+Z$, we need show the convergence of Z^N and Y^N in $\mathbb{H}^\gamma,\gamma\in(0,\frac{3}{2})$, respectively. When $\gamma>\frac{1}{2}$, we can apply the Sobolev embedding theorem $\mathbb{H}^\gamma\hookrightarrow E$ and follow the procedures in the proof of Proposition 5.1 to get $X(t)\in L^{2p}(\Omega;C([0,T];\,\mathbb{H}^\gamma))$. When $\gamma\leq\frac{1}{2}$, similar arguments as in the proof of Lemma 3.4 and Proposition 8.1 yield that Z^N is convergent to Z in $L^{2p}(\Omega;C([0,T];\,\mathbb{H}^\gamma))$ for $\gamma\in(0,\frac{3}{2})$. Since $X_0\in\mathbb{H}^\gamma$, to show the convergence of Y^N, it remains to prove that $\int_0^t S(t-s)AP^NF(X^N(s))ds$ is convergent to $\int_0^t S(t-s)AF(X(s))ds$ in $L^{2p}(\Omega;C([0,T];\mathbb{H}^\gamma)),\gamma\in(0,\frac{1}{2}]$. By applying (2.3), Hölder's inequality, Theorem 4.1 and Proposition 8.1, we can get for $\tau<\gamma$, $(\frac{1}{2}+\frac{\gamma}{4})\frac{l}{l-1}<1,2l<8$, $\epsilon>0$ small enough and$(\frac{1}{2}+\frac{\gamma}{4}+\epsilon)\frac{l_1}{l_1-1}<1,3l_1<12$,

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\Big\|\int_0^t S(t-s)AP^NF(X^N(s))ds-\int_0^t S(t-s)AF(X(s))ds\Big\|^{2p}_{\mathbb{H}^\gamma}\Big]\\
&\leq C\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\int_0^t(t-s)^{-\frac{1}{2}-\frac{\gamma}{4}}(1+\|X^N(s)\|_E^2+\|X(s)\|_E^2)ds\Big)^{2p}\sup_{s\in[0,T]}\|X(s)-X^N(s)\|^{2p}\Big]\\
&+C\mathbb{E}\Big[\sup_{t\in[0,T]}\Big\|\int_0^t S(t-s)A(I-P^N)F(X(s))ds\Big\|^{2p}_{\mathbb{H}^\gamma}\Big]\\
&\leq C\lambda_N^{-\tau p}\sup_{t\in[0,T]}\Big(\int_0^t(t-s)^{-(\frac{1}{2}+\frac{\gamma}{4})\frac{l}{l-1}}ds\Big)^{2p\frac{l-1}{l}}\Big(\mathbb{E}\Big[\Big(\int_0^T(1+\|X^N(s)\|_E^{2l}+\|X(s)\|_E^{2l})ds\Big)^{\frac{4p}{l}}\Big]\Big)^{\frac{1}{2}}
\end{aligned}
$$

$$
+ C\lambda_N^{-4p\epsilon}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\int_0^t (t-s)^{(-\frac{1}{2}-\frac{\gamma}{4}-\epsilon)\frac{l_1}{l_1-1}}\Big)^{2p\frac{l_1-1}{l_1}}\int_0^t(1+\|X(s)\|_{L^6}^{3l_1})ds\Big)^{\frac{2p}{l_1}}\Big]
\leq C(\lambda_N^{-\tau p}+\lambda_N^{-4p\epsilon}),
$$

which completes the proof. □

PROPOSITION 5.2. Let $X_0 \in E$, $T > 0$ and $p \geq 1$. The unique mild solution X of Eq. (1.1) satisfies

$$
\sup_{t\in[0,T]}\mathbb{E}\Big[\big\|X(t)\big\|_E^p\Big] \leq C(X_0,T,p).
$$

PROOF. The proof is similar to that of Proposition 3.2. □

Under the condition of Proposition 5.2, one can prove that the solution X has almost surely continuous trajectories in E. Assume that X_0 is β-Hölder continuous with $\beta \in (0,1)$. By using the fact that $S(\cdot)$ is an analytical semigroup in E (see, e.g., [**22**]) and similar arguments as in the proof of Proposition 5.3, we have that X is almost surely β-continuous in space and $\frac{\beta}{4}$-continuous in time.

PROPOSITION 5.3. Let $X_0 \in \mathbb{H}^\gamma$, $\gamma \in (0,\frac{3}{2})$, $p \geq 1$. Then the unique mild solution X of Eq. (1.1) satisfies

$$
\|X(t)-X(s)\|_{L^p(\Omega;\mathbb{H})} \leq C(X_0,T,p)(t-s)^{\frac{\gamma}{4}} \tag{5.3}
$$

for a positive constant $C(X_0,T,p)$ and $0 \leq s \leq t \leq T$.

PROOF. From the mild form of X, it follows that

$$
\begin{aligned}
\|X(t)-X(s))\| \leq{}& \|(S(t)-S(s))X_0\| \\
&+\int_0^s \Big\|(S(t-r)-S(s-r))AF(X(r))\Big\|dr \\
&+\int_s^t \Big\|S(t-r)AF(X(r))\Big\|dr \\
&+\Big\|\int_0^s (S(t-r)-S(s-r))G(X(r))dW(r)\Big\| \\
&+\Big\|\int_s^t S(t-r)G(X(r))dW(r)\Big\|.
\end{aligned}
$$

When $\gamma \geq \frac{1}{3}$, taking pth moment, applying (2.3) and the continuity estimate of $S(t)$, which can be obtained by using the similar arguments as in the proof of [**17**, Appendix, Lemma B.9]),

$$
\|(S(t)-I)A^{-\frac{\gamma}{2}}\| \leq Ct^{\frac{\gamma}{4}}, \gamma \in (0,4), \tag{5.4}
$$

and using the Sobolev embedding $\mathbb{H}^\gamma \hookrightarrow L^6$ and (5.2), we get

$$
\begin{aligned}
&\big\|(S(t)-S(s))X_0\big\| \leq C(T,X_0,p,\gamma)(t-s)^{\frac{\gamma}{4}}, \\
&\mathbb{E}\Big[\int_0^s \big\|(S(t-r)-S(s-r))AF(X(r))\big\|^p dr\Big] \\
&\leq C(T,p)\mathbb{E}\Big[\Big(\int_0^s (s-r)^{-\frac{1}{2}-\frac{\gamma}{4}}\big\|(S(t-s)-I)A^{-\frac{\gamma}{2}}\big\|\big\|F(X(r))\big\|dr\Big)^p\Big] \\
&\leq C(T,X_0,p,\gamma)(t-s)^{\frac{\gamma p}{4}},
\end{aligned}
$$

and

$$\mathbb{E}\Big[\Big(\int_s^t \Big\|S(t-r)AF(X(r))\Big\|dr\Big)^p\Big] \leq C(T,p)\mathbb{E}\Big[\Big(\int_s^t (t-r)^{-\frac{1}{2}}\|F(X(r))\|dr\Big)^p\Big]$$
$$\leq C(T,X_0,p)(t-s)^{\frac{p}{2}}.$$

The Burkholder inequality, (5.4), the properties that

$$\sup_{k\in\mathbb{N}^+}\|e_k\|_E \leq C, \text{ and } \sum_{k\in\mathbb{N}^+} e^{-\lambda_k^2 t} \leq Ct^{-\frac{1}{4}},$$

and (5.2) yield that for $\gamma < \frac{3}{2}$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|\int_0^s (S(t-r)-S(s-r))G(X(r))dW(r)\Big\|^p\Big] \\
&\leq C(T,p)\mathbb{E}\Big[\Big(\int_0^s \sum_{i\in\mathbb{N}^+}\|S(s-r)(S(t-s)-I)G(X(r))e_i\|^2 ds\Big)^{\frac{p}{2}}\Big] \\
&\leq C(T,p)\mathbb{E}\Big[\Big(\int_0^s (s-r)^{-\frac{1}{4}-\frac{\gamma}{2}}\|(1+\|X(s)\|^2)ds\Big)^{\frac{p}{2}}\Big] \\
&\leq C(T,X_0,p,\gamma)(t-s)^{\frac{\gamma p}{4}}.
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|\int_s^t S(t-r)G(X(r))dW(r)\Big\|^p\Big] \\
&\leq C(T,p)\mathbb{E}\Big[\Big(\int_s^t \sum_{i\in\mathbb{N}^+}\|S(t-r)G(X(r))e_i\|^2 ds\Big)^{\frac{p}{2}}\Big] \\
&\leq C(T,p)\mathbb{E}\Big[\Big(\int_s^t (t-r)^{-\frac{1}{4}}(1+\|X(r)\|^2)ds\Big)^{\frac{p}{2}}\Big] \leq C(T,p,X_0)(t-s)^{\frac{3p}{8}}.
\end{aligned}$$

Combining all the above estimates, we complete the proof when $\gamma \geq \frac{1}{3}$. When $\gamma < \frac{1}{3}$, making use of Proposition 8.1 and similar arguments, we could also obtain (5.3). □

As a result of Proposition 5.1, we have the following strong convergence rate result of the spectral Galerkin method.

COROLLARY 5.2. Let $X_0 \in \mathbb{H}^\gamma$, $\gamma \in (0,\frac{3}{2})$, $T>0$, $p\geq 1$ and $\sup_{N\in\mathbb{N}^+}\|X_0^N\|_E \leq C(X_0)$. Then for $\alpha\in(0,\gamma)$, there exists $C(X_0,T,p)>0$ such that

$$\big\|X^N - X\big\|_{L^p(\Omega;C([0,T];\mathbb{H})}\leq C(X_0,T,p)\lambda_N^{-\frac{\alpha}{2}}. \tag{5.5}$$

As a consequence of the strong convergence of the spectral Galerkin method, the exponential integrability property of the mild solution also holds (see Corollary 8.1 in Appendix). We would like to mention that the exponential integrability property has many applications in non-global SPDEs and their numerical approximations (see e.g. [**6, 9, 10**]).

6. Conclusion

In this paper, we use the spectral Galerkin method to study the global existence and regularity estimate of the solution process for stochastic Cahn–Hilliard equation driven by multiplicative space-time white noise. We present the explicit convergence rate of finite dimensional approximation of the stochastic Cahn–Hilliard equation with unbounded diffusion. Then we show that the limit of the finite dimensional approximation is the unique mild solution of the stochastic Cahn–Hilliard equation. As a consequence, the optimal regularity estimates and the exponential integrability of the mild solution are shown. One main application of the regularity result is to proceed to the numerical approximation and density functions of stochastic Cahn–Hilliard equation driven by multiplicative space-time white noise [**7**].

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8. Appendix

Proof of (2.4)**.**

Proof. Thanks to the series expansion $S(t)v = \sum_{k=1}^\infty e^{-\lambda_k^2 t}\langle v, e_k\rangle e_k$, using the fact that $|\int_{\mathcal{O}} e_k(x)dx| \le \frac{C}{k}$, and $\int_{c_0}^\infty e^{-x^4 t}x^{-1}dx \le C(c_0) < \infty$ for any $c_0 > 0$, we have that $\|S(t)v\|_{L^\infty} \le C\|v\|_{L^\infty}$ and $\|S(t)v\|_{L^1} \le C\|v\|_{L^1}$. By using the Riesz–Thorin interpolation theorem (see e.g. [**21**]), it follows that for $q \ge 1$,

$$\|S(t)v\|_{L^q} \le C\|v\|_{L^q}. \tag{8.1}$$

We first show that the contractivity property (2.4) holds for the case $1 \le p \le 2$. When $p = 1$, by using Minkowski's inequality and Hölder's inequality,

$$\|S(t)v\|_{L^\infty} \le \sum_{k=1}^\infty e^{-\lambda_k^2 t}|\langle v, e_k\rangle|\|e_k\|_{L^\infty} \le Ct^{-\frac{1}{4}}\|v\|_{L^1},$$

where we use the fact that $\sum_{k=1}^\infty e^{-\lambda_k^2 t} \le Ct^{-\frac{1}{4}}$. By (8.1), the above estimate and an interpolation argument, it follows that (2.4) holds for $p = 1, q \ge 1$. Similarly, by using Hölder's inequality, for $p = 2$, we obtain that

$$\begin{aligned}\|S(t)v\|_{L^\infty} &\le \sum_{k=1}^\infty e^{-\lambda_k^2 t}|\langle v, e_k\rangle|\|e_k\|_{L^\infty}\\ &\le C\Big(\sum_{k=1}^\infty e^{-2\lambda_k^2 t}\Big)^{\frac{1}{2}}\sum_{k=1}^\infty |\langle v, e_k\rangle|^2 \le Ct^{-\frac{1}{8}}\|v\|.\end{aligned}$$

Thus, by (8.1) and interpolation arguments, we also have (2.4) in the case of $p = 2, q \ge 2$. By using the Riesz–Thorin interpolation theorem (see e.g. [**21**]), we could get (2.4) for $1 \le p \le 2, q \ge p$. To show the case that $q \ge p > 2$, we use the dual form of L^q norm. In fact, according to the self-adjoint property of $S(t)$, it follows that $\|S(t)v\|_{L^q} = \sup\limits_{\|w\|_{L^{q'}} \le 1}\Big|\langle S(t)v, w\rangle\Big|$, where $\frac{1}{q} + \frac{1}{q'} = 1$. Moreover, using Hölder's inequality and (2.4) for $1 \le p' \le 2, q' \le p'$, we obtain that for $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\begin{aligned}\|S(t)v\|_{L^q} &= \sup_{\|w\|_{L^{q'}} \le 1} \Big|\langle S(t)v, w\rangle\Big| \\ &\le \sup_{\|w\|_{L^{q'}} \le 1} \|v\|_{L^p} \|S(t)w\|_{L^{p'}} \\ &\le C \sup_{\|w\|_{L^{q'}} \le 1} \|v\|_{L^p} t^{\frac{1}{4}(\frac{1}{q'} - \frac{1}{p'})} \|w\|_{L^{q'}} \\ &\le C t^{\frac{1}{4}(\frac{1}{p} - \frac{1}{q})} \|v\|_{L^p},\end{aligned}$$

which completes the proof for any $1 \le p \le q \le \infty$. Since for any $t > 0$, $v \in L^p$, $S(t)v \in E$, we also have

$$\|S(t)v\|_E \le C t^{-\frac{1}{4p}} \|v\|_{L^p}.$$

□

The proof of Proposition 5.1 in the case that $\gamma \le \frac{1}{2}$:
Following the steps in the proof of Proposition 5.1, it suffices to show that $Y = X - Z$ satisfies

$$Y(t) = S(t)X(0) - \int_0^t S(t-s)AF(X(s))ds.$$

More precisely, we will show that $S(t)X^N(0)$ converges to $S(t)X(0)$ and that $-A\int_0^t S(t-s)P^N F(X^N(s))ds$ converges to $-A\int_0^t S(t-s)F(X(s))ds$. To this end, we need the following convergence result.

Proposition 8.1. Under the condition of Proposition 5.1, $\{X^N\}_{N\in\mathbb{N}^+}$ is a Cauchy sequence in $L^{2p}(\Omega; L^{\kappa_1}([0,T];E)) \cap L^{2p}(\Omega; L^{\kappa_2}([0,T];L^6))$, where $2 \le \kappa_1 < 8$ and $2 \le \kappa_2 < 12$. Furthermore, there exists $\widetilde{\tau} \in (0, \max(\frac{\gamma}{2}, \frac{3}{4}))$ such that

$$\|X^N - X^M\|_{L^{2p}(\Omega;L^{\kappa_1}([0,T];E))} + \|X^N - X^M\|_{L^{2p}(\Omega;L^{\kappa_2}([0,T];L^6))} \le C\lambda_N^{-\widetilde{\tau}},$$

where $M \ge N$.

Proof. The proof of the convergence in $L^{2p}(\Omega; L^{\kappa_1}([0,T];E))$ and $L^{2p}(\Omega; L^{\kappa_2}([0,T];L^6))$ are similar. We only present the details on the convergence in $L^{2p}(\Omega; L^{\kappa_1}([0,T];E))$. Since $X^N = Y^N + Z^N$, it suffices to show the convergence of Y^N and Z^N respectively. Let $M \ge N$. Then by using the arguments in the proof of Lemma 3.2, we have that for $\frac{3}{8} - \frac{\alpha}{4} > \alpha_1 > \frac{1}{8} + \frac{1}{p}$ with a large enough $p \ge 1$, and a small enough $\epsilon > 0$,

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]} \|Z^N(t) - Z^M(t)\|_E^{2p}\Big] \\ &\le C_p\mathbb{E}\Big[\sup_{t\in[0,T]} \|Z^N(t) - P^N Z^M(t)\|_E^{2p}\Big] + C_p\mathbb{E}\Big[\sup_{t\in[0,T]} \|(I-P^N)Z^M(t)\|_E^{2p}\Big] \\ &\le C(p,T)\int_0^T \mathbb{E}\Big[\Big(\int_0^t (t-r)^{-2\alpha_1 - \frac{1}{4}} \|X^N(r) - X^M(r)\|^2 dr\Big)^p\Big] dt \\ &+ C(p,T)\int_0^T \mathbb{E}\Big[\Big(\int_0^t (t-r)^{-2\alpha_1 - \frac{1}{4} - \epsilon} \lambda_N^{-2\epsilon} (1 + \|X^M(r)\|^2) dr\Big)^p\Big] dt,\end{aligned}$$

where we have used the following estimate

$$\|S(\frac{t-r}{2})(I-P^N)u\| \le \|S(\frac{t-r}{2})A^{\epsilon}\|\|(I-P^N)A^{-\epsilon}u\| \\ \le C(t-r)^{-\frac{\epsilon}{2}}\lambda_N^{-\epsilon}\|u\|.$$

Lemma 3.3 and Theorem 4.1 lead to

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|Z^N(t)-Z^M(t)\|_E^{2p}\Big] \le C(T,X_0,p)(\lambda_N^{-\tau p}+\lambda_N^{-2\epsilon p}) \tag{8.2}$$

where $\tau<\gamma$. Since $C([0,T];E)\subset L^{\kappa_1}([0,T];E)$, it remains to prove the convergence Y^N. The mild form Y^N-Y^M, together with (2.4), (2.3) and (4.2), yields that for any $\gamma_2<\frac{3}{4}$,

$$\begin{aligned}
&\|Y^N(t)-Y^M(t)\|_E \\
&\le \|S(t)(I-P^N)X^M(0)\|_E + \Big\|\int_0^t S(t-s)P^M A(I-P^N)F(X^M(s))ds\Big\|_E \\
&+\Big\|\int_0^t S(t-s)P^N A(F(X^M(s))-F(X^N(s)))ds\Big\|_E \\
&\le Ct^{-\frac{1}{8}}\lambda_N^{-\frac{\gamma}{2}}\|X^M(0)\|_{\mathbb{H}^\gamma} \\
&+C\int_0^t (t-s)^{-\frac{5}{8}-\frac{\gamma_2}{2}}\lambda_N^{-\gamma_2}(1+\|X^M(s)\|_{L^6}^3)ds \\
&+C\int_0^t (t-s)^{-\frac{5}{8}}\big(1+\|X^N(s)\|_E^2+\|X^M(s)\|_E^2\big)ds \sup_{s\in[0,t]}\|X^N(s)-X^M(s)\|.
\end{aligned}$$

Using Proposition 3.2 and Theorem 4.1, we obtain that for $\gamma_2<\frac{3}{4}$, $\tau<\gamma$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^T \|Y^N(t)-Y^M(t)\|_E^{\kappa_1}dt\Big)^{\frac{2p}{\kappa_1}}\Big] \\
&\le C(\lambda_N^{-2\gamma_2 p}+\lambda_N^{-\tau p})+C\lambda_N^{-\gamma p}\|X^M(0)\|_{\mathbb{H}^\gamma}^{2p}\mathbb{E}\Big[\Big(\int_0^T t^{-\frac{1}{8}\kappa_1}dt\Big)^{2p}\Big] \\
&\le C(\lambda_N^{-2\gamma_2 p}+\lambda_N^{-\tau p}+\lambda_N^{-\gamma p}).
\end{aligned}$$

Thus, $\{Y^N\}_{N\in\mathbb{N}^+}$ is a Cauchy sequence in $L^{2p}(\Omega;L^{\kappa_1}([0,T];E))$ and $\widetilde{\tau}\le\max(\epsilon,\frac{\tau}{2},\frac{3}{4})$ This, together with the convergence of Z^N in $L^{2p}(\Omega;L^{\kappa_1}([0,T];E))$, implies that $\{X^N\}_{N\in\mathbb{N}^+}$ forms a Cauchy sequence in $L^{2p}(\Omega;L^{\kappa_1}([0,T];E))$. □

Since $S(t)X^N(0)$ is convergent to $S(t)X(0)$, we only need to estimate

$$Err_1(t):=\Big\|\int_0^t S(t-s)A(F(X(s))-P^NF(X^N(s)))ds\Big\|.$$

Applying (2.3) and Theorem 4.1, we obtain that for $\gamma_3\in(0,\frac{1}{2})$,

$$\begin{aligned}
Err_1(t) &\le \Big\|\int_0^t S(t-s)A(I-P^N)F(X(s)ds\Big\| \\
&+\Big\|\int_0^t S(t-s)AP^N(F(X^N(s))-F(X(s)))ds\Big\| \\
&\le C\int_0^t (t-s)^{-\frac{1}{2}-\gamma_3}\lambda_N^{-2\gamma_3}(1+\|X(s)\|_{L^6}^3)ds
\end{aligned}$$

$$+ C\int_0^t (t-s)^{-\frac12}(1+\|X^N(s)\|_E^2+\|X(s)\|_E^2)\|X^N(s)-X(s)\|ds.$$

Hölder's inequality, together with Proposition 8.1, yields that for $\tau<\gamma$, $\gamma_3\in(0,\frac12)$, $3l<12$, $(\frac12+\gamma_3)\frac{l}{l-1}<1$ and $2l_1<8$, $\frac{l_1}{2(l_1-1)}<1$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\|Err_1(t)\|^{2p}\Big]\\
&\le C\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\int_0^t (t-s)^{-\frac12-\gamma_3}\lambda_N^{-2\gamma_3}(1+\|X(s)\|_{L^6}^3)ds\Big)^{2p}\Big]\\
&+C\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\int_0^t (t-s)^{-\frac12}(1+\|X^N(s)\|_E^2+\|X(s)\|_E^2)ds\Big)^{2p}\sup_{t\in[0,T]}\|X^N(s)-X(s)\|^{2p}\Big]\\
&\le C\lambda_N^{-4\gamma_3 p}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\int_0^t (t-s)^{-\frac12-\gamma_3}(1+\|X(s)\|_{L^6}^3)ds\Big)^{2p}\Big]\\
&+C\lambda_N^{-\tau p}\Big(\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\int_0^t (t-s)^{-\frac12}(1+\|X^N(s)\|_E^2+\|X(s)\|_E^2)ds\Big)^{4p}\Big]\Big)^{\frac12}\\
&\le C\lambda_N^{-4\gamma_3 p}\Big(\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\int_0^t (t-s)^{(-\frac12-\gamma_3)\frac{l}{l-1}}\Big)^{4p\frac{l-1}{l}}\Big(\int_0^T(1+\|X(s)\|_{L^6}^{3l})ds\Big)^{\frac{4p}{l}}\Big]\Big)^{\frac12}\\
&+C\lambda_N^{-\tau p}\sup_{t\in[0,T]}\Big(\int_0^t (t-s)^{-\frac{l_1}{2(l_1-1)}}ds\Big)^{2p\frac{l_1-1}{l_1}}\Big(\mathbb{E}\Big[\Big(\int_0^T(1+\|X^N(s)\|_E^{2l_1}+\|X(s)\|_E^{2l_1})ds\Big)^{\frac{4p}{l_1}}\Big]\Big)^{\frac12}\\
&\le C(\lambda_N^{-4\gamma_3 p}+\lambda_N^{-\tau p}),
\end{aligned}$$

which implies that Y satisfies $Y(t)=S(t)X(0)-\int_0^t S(t-s)AF(X(s))ds$. This, together with the convergence of Z^N to Z, shows that X is the global mild solution.

Exponential integrability of the mild solution

Corollary 8.1. Let $X_0\in\mathbb{H}^\gamma,\gamma\in(0,\frac32)$. There exist $\beta>0$, $c>0$ such that for $t\in[0,T]$,

(8.3)
$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\frac12 e^{-\beta t}\|X(t)\|_{\mathbb{H}^{-1}}^2+c\int_0^t e^{-\beta s}\|X(s)\|_{L^4}^4ds+c\int_0^t e^{-\beta s}\|\nabla X(s)\|^2ds\Big)\Big]\\
&\le C(X_0,T).
\end{aligned}$$

Proof. From the Gagliardo–Nirenberg inequality

$$\|u\|_{L^4}\le C\|\nabla u\|_{L^2}^{\frac14}\|u\|^{\frac34}+C\|u\|,$$

and Young's inequality, it follows that

$$\begin{aligned}
&\int_0^t \|X^N(s)-X(s)\|_{L^4}^4ds\\
&\le C\int_0^t\|\nabla(X^N(s)-X(s))\|^2ds+C\int_0^t(1+\|X^N(s)-X(s)\|^6)ds.
\end{aligned}$$

We first claim that

$$\lim_{N\to\infty}\|X^N-X\|_{L^2(\Omega;L^2([0,t];\mathbb{H}^1))}+\lim_{N\to\infty}\|X^N-X\|_{C([0,t];L^6(\Omega;\mathbb{H}))}=0. \tag{8.4}$$

The estimate of the second term is similar to that of Proposition 8.1. To show the convergence of $\|X^N-X\|_{L^2(\Omega;L^2([0,t];\mathbb{H}^1))}$, one needs to recall that from the proof of Proposition 4.1 and taking $p = 1$, it holds that

$$\begin{aligned}&\mathbb{E}[\|X^N(t) - P^N X^M(t)\|^2_{\mathbb{H}^{-1}}] + 2\mathbb{E}[\int_0^t \|X^N(s) - P^N X^M(s)\|^2_{\mathbb{H}^1} ds]\\&\le C(T, X_0, p)\lambda_N^{-\gamma}.\end{aligned}$$

For convenience, we assume that $\sup_{N\in\mathbb{N}^+} \|X_0^N\|_E \le C(X_0)$ or $\gamma > \frac{1}{2}$ here. Otherwise, one needs to deal with the singularity appearing in each integral term via a tedious and technical calculus. Next, it remains to estimate $\mathbb{E}[\int_0^t \|(I - P^N)X^M(s)\|^2_{\mathbb{H}^1} ds]$. Indeed, the a priori estimates of X^M and Z^M, $\sum_{i=1}^\infty e^{-\lambda_i^2 t} \le Ct^{-\frac{1}{4}}$, (2.3) and (5.4) yield that

$$\begin{aligned}&\mathbb{E}\Big[\int_0^t \|(I - P^N)X^M(s)\|^2_{\mathbb{H}^1} ds\Big]\\&\le \mathbb{E}\Big[\int_0^t \|(I - P^N)S(s)X^M(0)\|^2_{\mathbb{H}^1} ds\Big]\\&+ \mathbb{E}\Big[\int_0^t \|(I - P^N)\int_0^s S(s-r)AF(X^M(r))dr\|^2_{\mathbb{H}^1} ds\Big]\\&+ \mathbb{E}\Big[\int_0^t \|(I - P^N)Z^M(s)\|^2_{\mathbb{H}^1} ds\Big]\\&\le C\lambda_N^{-2\epsilon}\Big(\int_0^t s^{-\frac{1}{2}-\epsilon} ds + \int_0^t (\int_0^s (s-r)^{-\frac{3}{4}-\epsilon} dr)^2 ds + \int_0^t \int_0^s (s-r)^{-\frac{3}{4}-\epsilon} dr ds\Big)\\&\le C\lambda_N^{-2\epsilon}.\end{aligned}$$

The above claim (8.4) implies that

$$\lim_{N\to\infty} \|X^N - X\|_{L^4(\Omega;L^4([0,t];L^4))} = 0.$$

Take a subsequence X^{N_k} of X^N such that

$$X^{N_k} \to X \text{ in } C([0,T];\mathbb{H}^{-1}) \cap L^2([0,T];\mathbb{H}^1) \cap L^4([0,T];L^4), \text{a.s.}$$

Thus, to prove (8.3), by using Fatou's lemma, it suffices to show the uniform boundedness of the exponential moment for X^N, i.e.,

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}e^{-\beta t}\|X^N(t)\|^2_{\mathbb{H}^{-1}} + c\int_0^t e^{-\beta s}\|X^N(s)\|^4_{L^4} ds + c\int_0^t e^{-\beta s}\|\nabla X^N(s)\|^2 ds\Big)\Big]$$

since the terms inside the expectation converges to those of X, a.s.

Denote $\mu(x) = -A^2 x - AP^N F(x)$ and $\sigma(x) = P^N G(x) I_{\mathbb{H}}$ and $U(x) = \frac{1}{2}\|x\|^2_{\mathbb{H}^{-1}}$, where $x \in P^N(\mathbb{H})$. Using (2.5), the Lipschitz continuity of G, and applying Hölder's and Young's inequality, we get for a small $\epsilon > 0$,

$$\begin{aligned}&\langle DU(x), \mu(x)\rangle + \frac{1}{2}\mathrm{tr}[D^2 U(x)\sigma(x)\sigma^*(x)] + \frac{1}{2}\|\sigma(x)^* DU(x)\|^2\\&= \langle x, -A^2 x - AF(x)\rangle_{\mathbb{H}^{-1}} + \frac{1}{2}\sum_{i\in\mathbb{N}^+} \|P^N(G(x)e_i)\|^2_{\mathbb{H}^{-1}} + \frac{1}{2}\sum_{i\in\mathbb{N}^+} \langle x, G(x)e_i\rangle^2_{\mathbb{H}^{-1}}\\&\le -(1-\epsilon)\|\nabla x\|^2 - (4c_4 - \epsilon)\|x\|^4_{L^4} + \epsilon\|x\|^2_{\mathbb{H}^{-1}} + C(\epsilon).\end{aligned}$$

Using the exponential integrability lemma in [**8**] and taking $\beta = \epsilon$, we have

$$\mathbb{E}\Big[\exp\Big(e^{-\beta t}\frac{1}{2}\|X^N(t)\|^2_{\mathbb{H}^{-1}} + (4c_4-\epsilon)\int_0^t e^{-\beta s}\|X^N(s)\|^4_{L^4}ds + (1-\epsilon)\int_0^t e^{-\beta s}\|\nabla X^N(s)\|^2 ds\Big)\Big] \le C(X_0,T,\epsilon),$$

which completes the proof. □

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Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong

Email address: jianbo.cui@polyu.edu.hk(Corresponding author)

1. LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China 2. School of Mathematical Science, University of Chinese Academy of Sciences, Beijing, 100049, China

Email address: hjl@lsec.cc.ac.cn