The optimal mean variance problem with inflation [1]

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Abstract

The risk of inflation is looming under the current low interest rate environment. Assuming that the investment includes a fixed interest asset and n risky assets under inflation, we consider two scenarios: inflation rate can be observed directly or through a noisy observation. Since the inflation rate is random, all assets become risky. Under this circumstance, we formulate the portfolio selection problem and derive the efficient frontier by solving the associated HJB equation. We find that for a given expected portfolio return, investment at time t is linearly proportional to the price index level. Moreover, the risk for the real value of the portfolio is no longer minimal when all the wealth is put into the fixed interest asset. Finally, for the mutual fund theorem, two funds are needed now instead of the traditional single fund. If an inflation linked bond can be included in the portfolio, the problem is reduced to the traditional mean variance problem with a risk-free and $n + 1$ risky assets with real returns.

Keywords: Mean Variance; Inflation; HJB Equation; Partial Information.

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1 Introduction

The task of developing appropriate quantitative models for long-term asset allocation is challenging. As investors are concerned with asset returns expressed in real terms, uncertainty about inflation is a potentially important source of risk. Inflation risk, also called purchasing power risk, is the chance that the cash flows from an investment won't be worth as much in the future because of changes in purchasing power due to inflation. Inflation causes money to lose value, and any investment that involves cash flows over time is exposed to this risk. The ramifications of this can be serious: The investor earns a lower return that he or she originally expected, in some cases causing the investor to withdraw some of a portfolio's principal if he or she is dependent on it for income. Since substantial fluctuations have been observed in the rate of inflation over a long period of time, it is important to consider inflation effect for long-term asset allocation decisions. The goal of long-term asset allocation is to preserve the real wealth, namely, the purchasing power of terminal wealth. Zhang [16] considered the problem of maximizing the expected utility of terminal real wealth including an inflation-linked bond. Siu [14] added the impact of (macro)-economic conditions to the settings in Zhang [16]. Some other works on using the single period model include Solnik [15], Manaster [12] and Chen and Moore [5], while on maximizing the CRRA utility function include Brennan and Xia [2], Munk et al. [13] for one risky asset and Bensoussan [1] for multiple risky assets and partial information.

Since the Markowitz's celebrated pioneering work [9], mean variance portfolio selection has become the foundation of modern finance theory. Markowitz developed an elegant mathematical framework for the problem and obtained a feasible solution to the problem which is simple and intuitively appealing. He considered a one-period economy and formulated the portfolio selection problem as a static mean-variance optimization problem. There have been continuing efforts in extending the Markowitz model from the static single period model to dynamic multi-period or continuous-time models. Under the mean variance framework, Li and Ng [8] investigated the multi-period problem with an embedding technique. They obtained an explicit mean-variance efficient frontier using the dynamic programming method. Using the same embedding technique, Zhou and Li [17] obtained an explicit expression of the efficient frontier for the continuous-time mean-variance problem, which is formulated as a stochastic linear-quadratic (LQ) optimal control problem. Note that the solution to this problem could also have been obtained (after embedding) via dynamic programming and the associated Hamilton-Jacobi- Bellman (HJB) partial differential equation. Li et al. [10] considered a similar problem but with an additional short-selling constraint. The constraint prevents the Riccati equations approach to be used and the HJB equation has to be employed instead to explicitly derive the efficient frontier. In all these works, the inflation risk has not been considered in the literature for portfolio selection problems under a continuous-time mean-variance setting. Although the portfolio selection problem under inflation was studied within the expected utility maximization (EUM) framework using the CRRA utility function [1, 2, 13] in order to derive analytic solutions, more general utility functions are difficult to be employed. Also, the theory of continuous mean-variance approach has not been utilized and the properties of the efficient frontier under inflation have not been studied.

In this paper, we consider the continuous time mean variance problem for the real wealth, namely the nominal final wealth discounted by the inflation rate. We assume that the investment includes a fixed interest asset and n risky assets. The inflation process [1] driven by a geometric Brownian motion is adopted. All assets become risky due to inflation. We formulate the mean variance problem and derive the efficient frontier by solving the associated Hamilton-Jacobi-Bellman (HJB) equation and Lagrangian multiplier. The results show that the investment is dependent on the dynamics of the inflation, and no risk-free portfolio can be obtained. For the portfolio, the nominal investment is linearly proportional to the current price index level. Also, the variance is not minimal when all money is put into the interest bearing bond.

In addition, we study the Mutual Fund Theorem with inflation. In normal circumstance, one-fund theorem says that any portfolio is a combination of the bank account and a given efficient portfolio known as the tangent fund. If inflation is taken into account, to hedge against the inflation risk, two mutual funds are now needed, namely, the minimum variance portfolio plus another efficient portfolio. To reduce the inflation risk, we introduce an inflation-linked bond (e.g. the iBond issued by the Hong Kong government). If an inflation-linked bond is allowed in the investment portfolio, it acts as a risk-free asset for the real wealth and protects the investor against risk due to unanticipated inflation rate.

The outline of this paper is as follows. In Section 2, we introduce the continuous time mean variance (MV) problem. When inflation is taken into consideration, we treat the MV problem with both full information and partial information in Section 3. When an inflation bond is allowed into the investment, the MV problem is considered in Section 4. The final section gives concluding remarks.

2 Review on Dynamic Mean-Variance Problem

In this section we formulate the mean variance problem. Let $(\Omega, \mathcal{F}, P)$ be a probability space, on which defined an n-dimensional standard Brownian motion. Suppose that the agent is allowed to invest his wealth in a financial market consisting of an interest bearing asset (bond or bank account) and n risky assets (stocks). Specifically, the nominal price process of the interest bearing asset is given by

$$dB(t) = rB(t)dt, \quad r > 0,$$

where r is the nominal interest rate and the price process of the risky asset follows geometric Brownian motion

$$dS(t) = D(S(t))(\boldsymbol{\mu} dt + \boldsymbol{\sigma} dW(t)), \quad S(0) = s_0$$

where the vector $\boldsymbol{\mu} = (\mu_1, ... \mu_n)^\top \in \mathcal{R}^n(\mu_L > r)$ is the appreciation rate, with "$\top$" being the transpose of a matrix or vector; $\boldsymbol{\sigma} \triangleq (\sigma_{i,j})$ is the $n \times n$ volatility, $D(S(t))$ denotes the diagonal matrix $diag[S_1(t), ..., S_n(t)]$. We assume that $\boldsymbol{\sigma}^\top \boldsymbol{\sigma} \geq \delta I$ for some $\delta > 0$, which ensures the market to be arbitrage-free and complete.

Assume that the investment horizon is $[0, T]$. Denote the investor's nominal wealth at time t by $X(t), t \in [0, T]$. The amount put into the i^{th} risky asset is denoted by $\pi_i(t)$, and the rest $x(t) - \sum_i^n \pi_i(t)$ into the interest bearing asset. The portfolio $\boldsymbol{\pi}(t)$ is called *admissible* if

$$E[\int_0^T \boldsymbol{\pi}^\top(t) \boldsymbol{\pi}(t) dt] < \infty.$$

With strategy $\boldsymbol{\pi}$, the dynamics of the nominal wealth evolve as

$$\begin{cases} dX(t) &= rX(t) + \boldsymbol{\pi}^\top(t)(\boldsymbol{\mu} - r\mathbf{1})dt + \boldsymbol{\pi}^\top(t)\boldsymbol{\sigma}dW(t), \\ X(0) &= x_0, \end{cases} \tag{2.1}$$

where $\mathbf{1} = (\underbrace{1, 1, ..., 1}_{n})^\top$. The aim is to find an admissible strategy $\boldsymbol{\pi}$, such that the expected terminal wealth satisfies $E[X(T)] = d$ while the risk measured by the variance of the terminal wealth

$$\text{Var}X(T) = E[X(T) - E[X(T)]]^2 = E[X(T) - d]^2$$

is minimized. Here, we have a condition that $d \geq x_0 e^{rT}$. It states that the investor's expected terminal wealth d can not be less than $x_0 e^{rT}$, which coincides with the amount that the investor would earn if all of the initial wealth is invested in the risk-free asset.

Problem 2.1 *The formal formulation about mean variance (MV) problem is*

$$\min_{\boldsymbol{\pi}} E[X(T) - d]^2$$

$$\textit{subject to } E[X(T)] = d,\ d \geq x_0 e^{rT},\ (X(t), \boldsymbol{\pi}(t)) \textit{ follows } (2.1).$$

The problem is called feasible (with respect to d) if there is at least one admissible portfolio satisfying $E[X(T)] = d$. An optimal portfolio to Problem 2.1, if it exists, is called an *efficient portfolio* with respect to d, and the corresponding point on the diagram is called an *efficient point*. The set of all the efficient points (with different values of d) is called the efficient frontier. If the effect of inflation is neglected, Zhou and Li [17] obtained the analytical solution by stochastic linear-quadratic (LQ) optimal approach. Here we cite their results as the following lemma for comparison.

Lemma 2.1 *If the effect of inflation is neglected, the efficient frontier is*

$$\text{Var}X^*(T) = \frac{\left(EX^*(T) - x_0 e^{rT}\right)^2}{e^{AT} - 1} = \frac{\left(d - x_0 e^{rT}\right)^2}{e^{AT} - 1}, \tag{2.2}$$

where $A = (\boldsymbol{\mu} - r\mathbf{1})^\top \Sigma^{-1} (\boldsymbol{\mu} - r\mathbf{1})$, and the optimal strategy is

$$\boldsymbol{\pi}^*(t) = -\Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{1})(\nu e^{r(T-t)} - x), \tag{2.3}$$

with

$$\begin{aligned}
\Sigma &= \boldsymbol{\sigma}^\top \boldsymbol{\sigma} \triangleq (a_{i,j}), i, j = 1, \cdots, n, \\
\nu &= \frac{EX^*(T) - e^{(r-A)T} x_0}{1 - e^{-AT}}.
\end{aligned} \tag{2.4}$$

This result can be reproduced by considering the HJB equation instead, which is described in the following sections for the problem under inflation.

3 The optimal investment problem under inflation

In the sequel, we present the dynamics of a price index and the price dynamics of the two primitive assets. The price index is a proxy, or an indicator, of inflation. Let $W_L \triangleq \{W_L(t), t \in \mathcal{T}\}$ be another standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$. We denote the process of the price index by $P \triangleq \{P(t), t \in \mathcal{T}\}$ on (Ω, F, P). Then under $\mathcal{P}$, the evolution of the price index P over time is governed by the following geometric Brownian motion (GBM):

$$\begin{cases} dP(t) &= \mu_L P(t) dt + \sigma_L P(t) dW_L(t), \\ P(0) &= 1, \end{cases} \tag{3.1}$$

where σ_L the constant volatility of the price index. Or denote

$$P(t) \triangleq e^{L(t)},$$

then

$$\begin{cases} dL(t) &= (\mu_L - \frac{1}{2}\sigma_L^2) dt + \sigma_L dW_L(t), \\ L(0) &= 0. \end{cases} \tag{3.2}$$

We express the returns from the $n+1$ assets in real terms; that is, we regard the price index as a numeraire and divide the nominal values of the assets by the price index level.

3.1 Full information

In this section we assume that the appreciation rate of inflation $\{P(t) : t \in \mathcal{T}\}$ and the standard Brownian motion W_L is observable. For the full information case, the notation $\tilde{}$ is used to denote the variables or parameters discounted by the price index numeraire. Denote $\tilde{B}(t) = \frac{B(t)}{P(t)}$ and $\tilde{S}(t) = \frac{S(t)}{P(t)}$, then their dynamics follow

$$\begin{aligned} d\tilde{B}(t) &= \tilde{B}(t)(\tilde{r}dt - \sigma_L dW_L(t)), \\ d\tilde{S}(t) &= D(\tilde{S}(t))(\tilde{\boldsymbol{\mu}}dt + \boldsymbol{\sigma}dW(t) - D(\sigma_L)dW_L(t)), \end{aligned}$$

respectively, where

$$\tilde{r} = r - \mu_L + \sigma_L^2,\ D(\tilde{S}(t)) = diag(\underbrace{\tilde{S}(t), \cdots, \tilde{S}(t)}_{n}),\ D(\sigma_L) = diag(\underbrace{\sigma_L, \cdots, \sigma_L}_{n}),$$

$$\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1, \cdots, \tilde{\mu}_n)^\top, \quad \tilde{\mu}_i = \mu_i - \mu_L + \sigma_L^2 - (\boldsymbol{\sigma\rho})_i\sigma_L$$

$$\boldsymbol{\rho} = (\rho_1, \cdots, \rho_n)^\top,\ E(dW_i(t)W_L(t)) = \rho_i dt.$$

A strategy $\boldsymbol{\pi} := \boldsymbol{\pi}(t)$ is described by an n-dimensional stochastic process, which denotes the nominal allocation into the risky assets. Let

$$\tilde{\mathcal{F}}_t \triangleq \sigma\{W(s), W_L(s), s \le t\},$$

be the filtration on which $W(t)$ and $W_L(t)$ are adapted. The strategy $\boldsymbol{\pi}$ is said to be admissible if $\{\boldsymbol{\pi}(t), t \in \mathcal{T}\}$ is $\tilde{\mathcal{F}}_t$ adapted and satisfies

$$E[\int_0^t \boldsymbol{\pi}^\top(s)\boldsymbol{\pi}(s)ds] < \infty \tag{3.3}$$

for all $t \ge 0$. Denote the space of all admissible strategies by $\tilde{\Pi}$. Write

$$\tilde{\boldsymbol{\pi}}(t) = \frac{\boldsymbol{\pi}(t)}{P(t)},$$

which is the real strategy. The real wealth dynamics, denoted by $\tilde{X}(t) \triangleq \frac{X(t)}{P(t)}$, evolves as

$$\begin{aligned} d\tilde{X}(t) &= [\tilde{\boldsymbol{\pi}}^\top(t)\tilde{\boldsymbol{\mu}} + (\tilde{X}(t) - \tilde{\boldsymbol{\pi}}^\top(t)\mathbf{1})\tilde{r}]dt + \tilde{\boldsymbol{\pi}}^\top(t)[\boldsymbol{\sigma}dW(t) - \sigma_L\mathbf{1}dW_L(t)] - (\tilde{X}(t) \\ &\quad -\tilde{\boldsymbol{\pi}}^\top(t)\mathbf{1})\sigma_L dW_L(t) \\ &= [\tilde{\boldsymbol{\pi}}^\top(t)(\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1}) + \tilde{X}(t)\tilde{r}]dt + \tilde{\boldsymbol{\pi}}^\top(t)\boldsymbol{\sigma}dW(t) - \tilde{X}(t)\sigma_L dW_L(t). \end{aligned} \tag{3.4}$$

Then the MV problem becomes

Problem 3.1

$$\min_{\tilde{\pi}} \qquad Var[\tilde{X}(T)] \triangleq E(\tilde{X}(T)-d)^2 \tag{3.5}$$

$$\text{subject to} \quad \begin{cases} E[\tilde{X}(T)] = d, \\ \tilde{\boldsymbol{\pi}}(\cdot) \in \tilde{\Pi}, \\ (\tilde{X}(\cdot), \tilde{\boldsymbol{\pi}}(\cdot)) \text{ satisfy (3.4)}, \end{cases} \tag{3.6}$$

where $d > d_0$ is a fixed value to be decided.

Note that all the investment assets becomes risky, we cannot apply the results of Li et al. [10] directly. Instead, we will use the dynamic programming principle here for solving Problem 3.1. The problem is a convex optimization problem, and the equality constraint $E[\tilde{X}(T)] = d$ can be dealt with by introducing a Lagrange multiplier $\beta \in R$. In this way, Problem 3.1 can be solved via the following optimal stochastic control problem (for every fixed β):

$$\begin{aligned} &\min_{\tilde{\boldsymbol{\pi}}} \qquad E\{[\tilde{X}(T)-d]^2 + 2\beta[E[\tilde{X}(T)]-d]\}, \\ &\text{subject to} \qquad \begin{cases} \tilde{\boldsymbol{\pi}}(\cdot) \in \tilde{\Pi}, \\ (\tilde{X}(\cdot), \tilde{\boldsymbol{\pi}}(\cdot)) \text{ satisfy (3.4)}, \end{cases} \end{aligned} \tag{3.7}$$

where the factor 2 in front of the multiplier β is introduced in the objective function just for convenience. Define

$$J(\tilde{\boldsymbol{\pi}}, \beta) = E\{[\tilde{X}(T)-d]^2 + 2\beta[E[\tilde{X}(T)]-d]\},$$

then by the Lagrange duality theorem, (see [3]),

$$\min_{\tilde{\pi}} Var[\tilde{X}(T)] = \max_{\beta} \min_{\tilde{\boldsymbol{\pi}}} J(\tilde{\boldsymbol{\pi}}, \beta) = \min_{\tilde{\boldsymbol{\pi}}} \max_{\beta} J(\tilde{\boldsymbol{\pi}}, \beta). \tag{3.8}$$

Letting $b = d - \beta$, problem (3.4) is equivalent to

$$\begin{aligned} &\min_{\tilde{\boldsymbol{\pi}}} \qquad E[\tilde{X}(T)-b]^2 \\ &\text{subject to} \qquad \begin{cases} \tilde{\boldsymbol{\pi}}(\cdot) \in \tilde{\Pi}, \\ (\tilde{X}(\cdot), \tilde{\pi}(\cdot)) \text{ satisfy (3.4)}. \end{cases} \end{aligned} \tag{3.9}$$

Define

$$\tilde{J}(\tilde{\boldsymbol{\pi}}, \beta) = E[\tilde{X}(T)-b]^2,$$

(3.8) becomes

$$\min_{\tilde{\boldsymbol{\pi}}} Var[\tilde{X}(T)] = \max_{\beta}(\min_{\tilde{\boldsymbol{\pi}}} \tilde{J}(\tilde{\boldsymbol{\pi}},\beta)-\beta^2) = \min_{\tilde{\boldsymbol{\pi}}} \max_{\beta}(\tilde{J}(\tilde{\boldsymbol{\pi}},\beta)-\beta^2). \tag{3.10}$$

The value function associated with $\tilde{J}(\tilde{\pi},\beta)$ is defined by

$$V(t,\tilde{x}) = \min_{\tilde{\boldsymbol{\pi}}\in\tilde{\Pi}} E\Big[(\tilde{X}(T)-b)^2|\tilde{X}(t)=\tilde{x}\Big]. \tag{3.11}$$

In the sequence, we will show how to solve (3.11) with the help of the HJB equation. Then

$$\min_{\tilde{\boldsymbol{\pi}}} Var[\tilde{X}(T)] = \max_{\beta} V(0,\tilde{x}_0). \tag{3.12}$$

As described in III.7 of [7], by the dynamic programming principle, problem (3.9) is reduced to solving the following second-order differential equation

$$V_t + \min_{\tilde{\boldsymbol{\pi}}\in\tilde{\Pi}}\{[\tilde{r}\tilde{x}+\tilde{\boldsymbol{\pi}}^\top(\tilde{\boldsymbol{\mu}}-\tilde{r}\mathbf{1})]V_{\tilde{x}} + [\tfrac{1}{2}\tilde{\boldsymbol{\pi}}^\top\Sigma\tilde{\pi}+\tfrac{1}{2}\sigma_L^2\tilde{x}^2-\tilde{\boldsymbol{\pi}}^\top\boldsymbol{\sigma}\boldsymbol{\rho}\sigma_L\tilde{x}]V_{\tilde{x}\tilde{x}}\} = 0, \tag{3.13}$$

with boundary condition

$$V(T,\tilde{x}) = (\tilde{x}-b)^2, \tag{3.14}$$

where $\Sigma = \boldsymbol{\sigma}^\top\boldsymbol{\sigma}$. For the theory on the dynamic programming principle, we refer it to III.7 of [7]. In the following, we construct a solution of (3.13).

Theorem 3.1 *The function*

$$W_1(t,\tilde{x}) = P_1(t)\tilde{x}^2 + Q_1(t)\tilde{x} + R_1(t), \tag{3.15}$$

is a continuously differentiable solution of (3.13)-(3.14). Here

$$\begin{aligned} P_1(t) &= e^{f_1(t)} \triangleq e^{(M+\sigma_L^2)(T-t)}, \quad Q_1(t) = -2be^{g_1(t)} \triangleq -2be^{N(T-t)}, \\ R_1(t) &= b^2k_1(t)e^{h_1(t)} \triangleq b^2k_1(t)e^{U(T-t)}, \end{aligned} \tag{3.16}$$

with

$$\begin{aligned} M &= 2\tilde{r} - (\boldsymbol{\sigma}\boldsymbol{\rho}\sigma_L-(\tilde{\boldsymbol{\mu}}-\tilde{r}\mathbf{1}))^\top\Sigma^{-1}(\boldsymbol{\sigma}\boldsymbol{\rho}\sigma_L-(\tilde{\boldsymbol{\mu}}-\tilde{r}\mathbf{1})), \\ N &= \tilde{r} + (\boldsymbol{\sigma}\boldsymbol{\rho}\sigma_L-(\tilde{\boldsymbol{\mu}}-\tilde{r}\mathbf{1}))^\top\Sigma^{-1}(\tilde{\boldsymbol{\mu}}-\tilde{r}\mathbf{1}), \\ U &= (2N-M)-\sigma_L^2, \\ k_1(t) &= (1+\frac{[(\tilde{\boldsymbol{\mu}}-\tilde{r}\mathbf{1})]^\top\Sigma^{-1}[(\tilde{\boldsymbol{\mu}}-\tilde{r}\mathbf{1})]}{2N-M-\sigma_L^2})e^{\int_t^T -(2N-M-\sigma_L^2)ds} - \frac{[(\tilde{\boldsymbol{\mu}}-\tilde{r}\mathbf{1})]^\top\Sigma^{-1}[(\tilde{\boldsymbol{\mu}}-\tilde{r}\mathbf{1})]}{2N-M-\sigma_L^2}. \end{aligned} \tag{3.17}$$

Proof. Suppose that (3.13)-(3.14) has a solution with the following form:

$$W_1(t,\tilde{x}) = P_1(t)\tilde{x}^2 + Q_1(t)\tilde{x} + R(t), \tag{3.18}$$

where $P_1(\cdot) > 0, Q_1(\cdot)$ and $R_1(\cdot)$ are three suitable functions. The boundary condition (3.14) implies that

$$P_1(T) = 1, Q_1(T) = -2b, R_1(T) = b^2. \tag{3.19}$$

Then (3.13) attains its minimum at

$$\begin{aligned}
\tilde{\boldsymbol{\pi}}^*(\tilde{x},t) &= -\Sigma^{-1}(\tilde{\boldsymbol{\mu}} - \tilde{r}I)\frac{\partial W_1(t,\tilde{x})/\partial\tilde{x}}{\partial W_1(t,\tilde{x})/\partial\tilde{x}^2} + \Sigma^{-1}\boldsymbol{\sigma}\boldsymbol{\rho}\sigma_L\tilde{x}, \\
&= \Sigma^{-1}(\boldsymbol{\sigma}\boldsymbol{\rho}\sigma_L - (\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1}))\tilde{x} - \Sigma^{-1}(\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1})\frac{Q_1(t)}{2P_1(t)} \\
&= -\Sigma^{-1}(\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1})(\tilde{x} + \frac{Q_1(t)}{2P_1(t)}) + \Sigma^{-1}\boldsymbol{\sigma}\boldsymbol{\rho}\sigma_L\tilde{x}.
\end{aligned} \tag{3.20}$$

Inserting (3.15) and (3.20) into (3.13) and rearranging, we have

$$\begin{aligned}
&\Big(P_1'(t) + [2\tilde{r} + \sigma_L^2 - (\boldsymbol{\sigma}\boldsymbol{\rho}\sigma_L - (\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1}))^\top\Sigma^{-1}(\boldsymbol{\sigma}\boldsymbol{\rho}\sigma_L - (\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1}))]P_1(t)\Big)\tilde{x}^2 \\
&+\Big(Q_1'(t) + 2[\tilde{r} + (\boldsymbol{\sigma}\boldsymbol{\rho}\sigma_L - (\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1}))^\top\Sigma^{-1}(\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1})]Q_1(t)b\Big)\tilde{x} \\
&+R_1'(t) - \frac{1}{4}[(\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1})^\top\Sigma^{-1}(\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1})]\frac{Q_1^2(t)}{P_1(t)}b^2 = 0.
\end{aligned} \tag{3.21}$$

Solving (3.21) with (3.19), we have

$$W_1(t,\tilde{x}) = P_1(t)\tilde{x}^2 + Q_1(t)\tilde{x} + R_1(t), \tag{3.22}$$

where $P_1(t), Q_1(t)$ and $R_1(t)$ are given by (3.16). This completes the proof. □

The following verification theorem guarantees $W_1(t,\tilde{x}) = V(t,\tilde{x})$.

Theorem 3.2 *Let $W_1(t,\tilde{x})$ be defined by Proposition 3.1, then $V(t,x)$ of (3.11) is equal to $W_1(t,\tilde{x})$. Furthermore, the strategy $\boldsymbol{\pi}^*$ defined by (3.20) is optimal.*

The proof is a standard one, which can be done by repeating the steps of Theorem 8.1 of Chapter III in [7]. We therefore omit it here.

3.1.1 Efficient strategy and efficient frontier

In this subsection, we apply the results established in the previous section to the mean-variance problem. In another word, we need to translate the results obtained from (3.11) back to (3.6). By Theorem 3.1, W_1 is equal to the value function of (3.11). Let $t = 0$ and $\tilde{x} = \tilde{x}_0$ in (3.11). Then (3.10) implies

$$\tilde{J}(\boldsymbol{\pi}^*, \beta) - \beta^2 = \tilde{x}_0^2 e^{f_1(0)} - 2b\tilde{x}_0 e^{g_1(0)} + k_1(0) b^2 e^{h_1(0)} - \beta^2, \tag{3.23}$$

where

$$b = d - \beta. \tag{3.24}$$

The above value depends on the Lagrange multiplier β. According to (3.12), we need to maximize the value in (3.23) over $\beta \in R$. A simple calculation, which is given in Appendix, shows that (3.23) attains its maximum value

$$\tilde{x}_0^2 e^{f_1(0)} - 2d\tilde{x}_0 e^{g_1(0)} + k_1(0) d^2 e^{h_1(0)} + \frac{(k_1(0) d e^{h_1(0)} - \tilde{x}_0 e^{g_1(0)})^2}{k_1(0) e^{h_1(0)} - 1} \tag{3.25}$$

at

$$\beta^* = \frac{k_1(0) d e^{h_1(0)} - \tilde{x}_0 e^{g_1(0)}}{k_1(0) e^{h_1(0)} - 1}. \tag{3.26}$$

Therefore,

$$\begin{aligned} VarX^*(T) &= \frac{-k_1(0) e^{h_1(0)} (d - \frac{\tilde{x}_0}{k_1(0)} e^{g_1(0) - h_1(0)})^2}{k_1(0) e^{h_1(0)} - 1} + C_1 \tilde{x}_0^2 \\ &= \frac{-k_1(0) e^{h_1(0)} (E\tilde{X}^*(T) - \frac{\tilde{x}_0}{k_1(0)} e^{g_1(0) - h_1(0)})^2}{k_1(0) e^{h_1(0)} - 1} + C_1 \tilde{x}_0^2, \end{aligned} \tag{3.27}$$

where $C_1 = \frac{e^{2g_1(0)}(1 - k_1(0)) + e^{f_1(0)}(1 - \frac{1}{k_1(0)})}{1 - k_1(0) e^{h_1(0)}} > 0$, $f_1(\cdot), g_1(\cdot), h_1(\cdot)$ and $k_1(\cdot)$ are defined by (3.16). Rewrite (3.27) as

$$E\tilde{X}^*(T) = \frac{\tilde{x}_0}{k_1(0)} e^{g_1(0) - h_1(0)} \pm \sqrt{\frac{1 - k_1(0) e^{h_1(0)}}{k_1(0) e^{h_1(0)}} \mathrm{Var}\tilde{X}^*(T) - C_1 \tilde{x}_0^2}. \tag{3.28}$$

As the efficient frontier means the portfolio has the largest expected return for a given standard deviation. Therefore, the efficient frontier of $mean - variance$ is

$$E\tilde{X}^*(T) = \frac{\tilde{x}_0}{k_1(0)} e^{g_1(0) - h_1(0)} + \sqrt{\frac{1 - k_1(0) e^{h_1(0)}}{k_1(0) e^{h_1(0)}} \mathrm{Var}\tilde{X}^*(T) - C_1 \tilde{x}_0^2}. \tag{3.29}$$

It can be easily checked that $0 < k_1(0)e^{h_1(0)} < 1$, therefore,

$$Var\tilde{X}^*(T) \geq \frac{(k_1(0)-1)e^{2g_1(0)-h_1(0)}\tilde{x}_0^2}{k_1(0)}$$

and $d \geq \frac{\tilde{x}_0}{k_1(0)}e^{g_1(0)-h_1(0)}$. Then the efficient frontier is the upper curve of a hyperbola with the minimal expectation $\frac{\tilde{x}_0}{k_1(0)}e^{g_1(0)-h_1(0)}$ and minimal variance $\frac{(k_1(0)-1)e^{2g_1(0)-h_1(0)}\tilde{x}_0^2}{k_1(0)}$. Therefore, to pick up the efficient porfoltios on the upper half of the hyperbola, we can set $d_0 = \frac{\tilde{x}_0}{k_1(0)}e^{g_1(0)-h_1(0)}$ in Problem 3.1. The above results are summarized in the following theorem.

Theorem 3.3 *The optimal strategy of Problem 3.1, corresponding to the expected terminal wealth $E\tilde{x}^*(T) = d$, are*

$$\tilde{\boldsymbol{\pi}}^*(t) \quad = \quad \Sigma^{-1}\boldsymbol{\sigma}\boldsymbol{\rho}\sigma_L\tilde{x} + \Sigma^{-1}(\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1})(-\tilde{x} + 2(d-\beta^*)\frac{e^{g_1(t)}}{2e^{f_1(t)}}) \tag{3.30}$$

where β^* is given by (3.26). The efficient frontier is

$$E[\tilde{X}^*(T)] = \frac{\tilde{x}_0}{k_1(0)}e^{g_1(0)-h_1(0)} + \sqrt{\frac{1-k_1(0)e^{h_1(0)}}{k_1(0)e^{h_1(0)}}\mathrm{Var}\tilde{X}^*(T) - C_1\tilde{x}_0^2}, \tag{3.31}$$

with

$$d \geq d_0.$$

Remark 3.1 *When there is no inflation, namely, $\mu_L = \sigma_L = 0$, then (3.30) reduces to the strategy without inflation (2.3).*

From (3.30), we can calculate the nominal investment amount

$$\tilde{\boldsymbol{\pi}}^*(t)P(t) = \Sigma^{-1}\boldsymbol{\sigma}\boldsymbol{\rho}\sigma_L x + \Sigma^{-1}(\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1})(-x + 2(d-\beta^*)\frac{e^{g_1(t)}P(t)}{2e^{f_1(t)}}),$$

which shows that the nominal investment varies linearly with respect to the inflation risk $P(t)$. Different from Zhou and Li [17], the efficient frontier is no longer a perfect square and no risk free portfolio can be obtained.

If all the wealth is put into the fixed interest asset, then the variance for the real return is $e^{2\tilde{r}}\sqrt{e^{2\sigma_L^2}-1}$, which is larger than $\frac{(k_1(0)-1)e^{2g_1(0)}\tilde{x}_0^2}{k_1(0)e^{h_1(0)}}$. Thus, another interesting finding is that, unlike the situation without inflation, the risk is not minimal when one puts all the money in the bond. The intuitive explanation is that real values of all assets become risky due to inflation. Thus the real risk of fixed interest asset can be reduced by holding a diversified portfolio of assets. This is in line with the theory of diversification, which has been started by Markowitz and then reinforced by other economists and mathematicians. Here we present an example which illustrates this findings above. The parameters are given as follows: $\tilde{x}_0 = 1$, $n = 1$, $r = 0.005, \mu = 0.02, \mu_L = 0.01, \sigma = 0.1$, $\sigma_L = 0.05$, $\rho = 0$. In Figure 1, the straight line denotes the efficient frontier without consideration of inflation. The continuous line denotes the efficient frontier under inflation. When all the wealth is put into the interest bearing asset, the points A and A' denote the mean-variance in the two cases respectively. The point B denotes the minimal mean-variance point in the efficient frontier with inflation. Obviously B has a lower variance and higher return than A'.

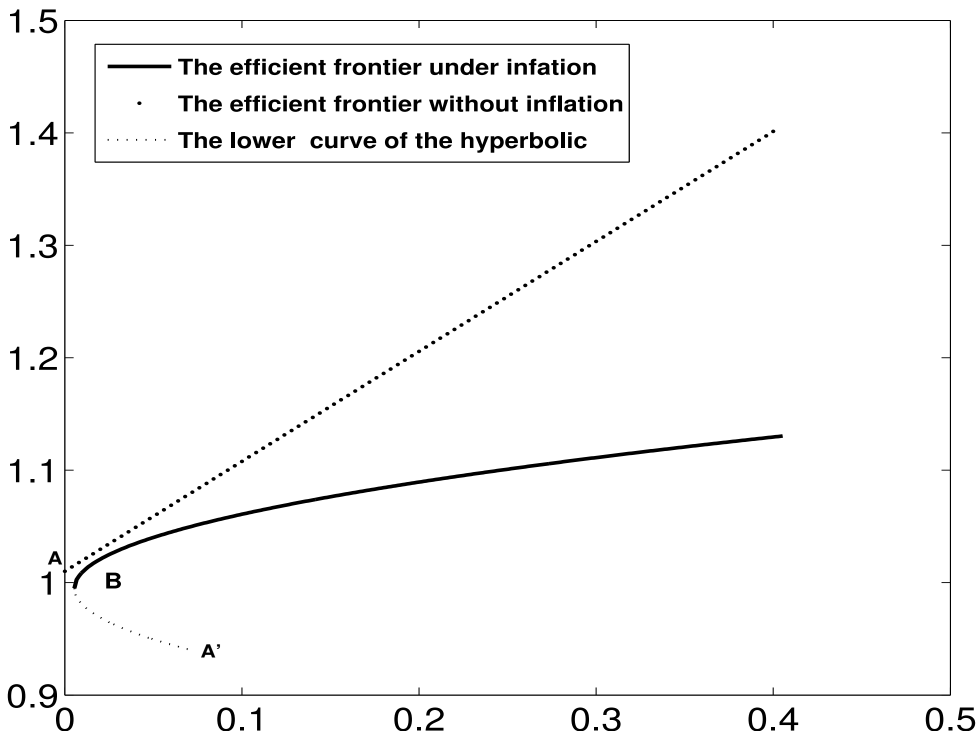

Figure 1: The efficient frontier with and without inflation.

3.2 Partial information basket price

We now turn to a partially observed information model. In reality, there are several observable prices of some consumption goods in basket price which can reflect partially the true value of the basket price. Therefore, the process $L(t)$ is not observable fully, but the investor receives a noisy signal $Z(t)$ on his consumption basket price, which might overestimate or underestimate the true value. Assuming this error is normally distributed, the signal process can be written as follows:

$$\begin{cases} dZ(t) &= L(t)dt + mdW_Z(t), \\ Z(0) &= 0, \end{cases}$$

where m is a small known constant, $L(t)$ is defined as (3.2), $W_Z(t)$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$, which is independent of $W_L(t)$ and $W(t)$. Then the investor makes decision based on knowledge before time t. Let $\mathcal{G}_t = \sigma\{W(s), Z(s), s \leq t\}$. Namely, the investor's strategy $\pi(t)$ should be adapted to $\mathcal{G}_t$, which describes the agent's information. The idea is to reduce the partially observable problem above to an equivalent one with full observation. Now we define the filtered estimate $\hat{L}$ as

$$\hat{L}(t) = E^P[L(t)/\mathcal{G}_t].$$

It is a nonlinear filtration problem to derive the density of $\hat{L}(t)$. In this partial information case, the notation $\hat{}$ is used to denote variables or parameters discounted by $\hat{L}$. From the arguments in Bensoussan et al. [1], the following lemma holds.

Lemma 3.1 *The conditional probability law of $L(t)$ given $\mathcal{G}_t$ is Gaussian with mean $\hat{L}(t)$ and variance*

$$\hat{\sigma}_z(t) = \begin{cases} m\Lambda_1 \frac{\Lambda_2 exp(a\Lambda_1 t/m) - 1}{\Lambda_2 exp(2\Lambda_1 t/m) + 1}, & \text{if } \hat{\sigma}_z(0) < m\Lambda_1, \\ m\lambda_1, & \text{if } \hat{\sigma}_z(0) = m\Lambda_1, \\ m\Lambda_1 \frac{\Lambda_2 exp(a\Lambda_1 t/m) + 1}{\Lambda_2 exp(2\Lambda_1 t/m) - 1}, & \text{if } \hat{\sigma}_z(0) > m\Lambda_1. \end{cases} \tag{3.32}$$

The process follows

$$\begin{cases} d\hat{L}(t) = (\mu_L - \frac{1}{2}\sigma_L^2)dt + \sigma_L \rho^\top dW(t) + \hat{\sigma}_z(t) d\tilde{W}_Z(t), \\ \hat{L}(0) = 0, \end{cases}$$

where $\Lambda_1 = \sigma_L\sqrt{|\rho|-1}, |\rho| = \sqrt{\rho^T\rho}, \Lambda_2 = |\frac{m\Lambda_1+\hat{\sigma}_z(0)}{m\Lambda_1+\hat{\sigma}_z(0)}|$, *and the innovation process* $\tilde{W}_Z(t)$ *is defined by*

$$\begin{cases} d\tilde{W}_Z(t) &= \frac{1}{m}(dZ(t)-\hat{L}(t)dt), \\ \tilde{W}_Z(0) &= 0. \end{cases} \tag{3.33}$$

Here $\tilde{W}_Z(t)$ *and* $W(t)$ *form an* $(n+1)$*-dimensional* $(P,\mathcal{G}_t)$ *Wiener process.*

3.2.1 The optimal problem with partial information

Let $\hat{\boldsymbol{\pi}}$ be the investment in the nominal risk assets. Recall that the investment strategy $\hat{\boldsymbol{\pi}}$ is admissible if it is $\mathcal{G}_t$ adapted and satisfies that

$$E[\int_0^T \boldsymbol{\pi}^\top(t)\boldsymbol{\pi}(t)dt] < \infty.$$

Let $\hat{\Pi}$ denote the space of admissible strategies. With strategy $\hat{\boldsymbol{\pi}}$, the dynamics of the wealth, denoted by $\hat{X}(t)$, evolves as

$$\begin{cases} dX(t) &= rX(t)+\boldsymbol{\pi}^\top(t)(\boldsymbol{\mu}-r\mathbf{1})dt+\boldsymbol{\pi}^\top(t)\boldsymbol{\sigma}dW(t), \\ X(0) &= x_0. \end{cases} \tag{3.34}$$

Now the optimal portfolio problem is to choose $\boldsymbol{\pi} \in \hat{\Pi}$ to minimize the variance with the expected real return d. Notice that $L(t)$ is not observable and the signals $Z(t)$ give information of the price index. However, the conditional probability law $L(t)$ given $\mathcal{G}_t$ is known by Lemma 3.1. Therefore, the optimal portfolio problem can be transformed into

$$\begin{aligned} &\min_{\boldsymbol{\pi}} E\Big(E[(X(T)e^{-L(T)}-d)^2|\mathcal{G}_T]\Big) \\ &\text{subject to} \quad \begin{cases} E\Big(E[X(T)e^{-L(T)}|\mathcal{G}_T]\Big) = d, \\ \boldsymbol{\pi}(\cdot) \in \hat{\Pi}, \\ (X(\cdot),\boldsymbol{\pi}(\cdot)) \text{ satisfy (3.34)}, \end{cases} \end{aligned} \tag{3.35}$$

where $d \geq \hat{d}_0$ is a fixed value to be decided. Again, introducing Lagrange multiplier $\beta \in R$, this problem is equivalent to (letting $b = d - \beta$),

$$\begin{aligned} \min_{\tilde{\boldsymbol{\pi}}} \quad & E\Big(E[(X(T)e^{-L(T)} - b)^2 | \mathcal{G}_T]\Big) && (3.36) \\ \text{subject to} \quad & \begin{cases} \boldsymbol{\pi}(\cdot) \in \hat{\Pi}, \\ (X(\cdot), \pi(\cdot)) \text{ satisfy (3.34).} \end{cases} && (3.37) \end{aligned}$$

Denote $\hat{X}(t) \triangleq X(t)e^{-\hat{L}(t)}$ and

$$V(t, \hat{x}) \triangleq \min_{\hat{\pi} \in \hat{\Pi}} E\Big[E((X(T)e^{-L(T)} - b)^2 | \mathcal{G}_T) | \mathcal{G}_t, \hat{X}(t) = \hat{x}\Big].$$

From Lemma 3.1,

$$V(t, \hat{x}) = \min_{\hat{\pi} \in \hat{\Pi}} E[\hat{X}^2(T)e^{\hat{\sigma}_z^2(T)} - 2b\hat{X}(T)e^{\frac{1}{2}\hat{\sigma}_z^2(T)} + b^2]. \quad (3.38)$$

If we know the dynamics of $\hat{X}(t)$, the problem can be solved. This dynamic will be derived in the following section.

3.2.2 The real wealth dynamics with filtration

From Ito's lemma, $e^{\hat{L}(t)}$ satisfies

$$\begin{aligned} \frac{de^{\hat{L}(t)}}{e^{\hat{L}(t)}} &= [(\mu_L - \frac{1}{2}\sigma_L^2) + \frac{1}{2}(\boldsymbol{\rho}^\top \boldsymbol{\rho}\sigma_i^2 + \hat{\sigma}_z^2)]dt + \sigma_L \boldsymbol{\rho} dW(t) + \hat{\sigma}_z dW_Z(t) \\ &= \hat{\mu}_I(t)dt + \sigma_L \boldsymbol{\rho}^\top dW(t) + \hat{\sigma}_z(t)dW_Z(t), \end{aligned} \quad (3.39)$$

where $\hat{\mu}_I(t) = (\mu_L - \frac{1}{2}\sigma_L^2) + \frac{1}{2}(\boldsymbol{\rho}^\top \boldsymbol{\rho}\sigma_L^2 + \hat{\sigma}_z^2(t))$.

Denote $\hat{B}(t) = \frac{B(t)}{e^{\hat{L}(t)}}$ and $\hat{S}(t) = \frac{S(t)}{e^{\hat{L}(t)}}$, then we have

$$\begin{cases} d\hat{B}(t) = \hat{B}(t)(\hat{r}(t)dt - \sigma_L \boldsymbol{\rho}^\top dW(t) - \hat{\sigma}_z(t)dW_Z(t)), \\ d\hat{S}(t) = D(\hat{S}(t))(\hat{\boldsymbol{\mu}}dt + \boldsymbol{\sigma}dW(t) - \sigma_L \mathbf{1}\boldsymbol{\rho}^\top dW(t) - \hat{\sigma}_z(t)\mathbf{1}dW_Z(t)), \end{cases}$$

respectively, where

$$\begin{aligned} \hat{r}(t) &= r - \hat{\mu}_I(t) + \boldsymbol{\rho}^\top \boldsymbol{\rho}\sigma_L^2 + \hat{\sigma}_z^2(t), && (3.40) \\ \hat{\mu}_i(t) &= \mu_L - \hat{\mu}_I(t) + \boldsymbol{\rho}^\top \boldsymbol{\rho}\sigma_L^2 + \hat{\sigma}_z^2(t) - (\boldsymbol{\sigma}\boldsymbol{\rho})_i \sigma_L, \\ D(\tilde{S}(t)) &= diag(\underbrace{\tilde{S}(t), \cdots, \tilde{S}(t)}_{n}) && (3.41) \end{aligned}$$

for each $i = 1, \cdots, n$. Write $\hat{\boldsymbol{\pi}} := \hat{\boldsymbol{\pi}} e^{-\hat{L}(t)}$, then the dynamics of $\hat{X}(t)$ follows

$$\begin{aligned} d\hat{X}(t) &= [\hat{\boldsymbol{\pi}}^\top(t)\hat{\boldsymbol{\mu}} + (\hat{X}(t) - \hat{\boldsymbol{\pi}}^\top(t)\mathbf{1})\hat{r}]dt + \hat{\boldsymbol{\pi}}^\top(t)[\boldsymbol{\sigma} dW(t) - \sigma_L \boldsymbol{\rho}^\top dW(t) - \hat{\sigma}_z \mathbf{1} dW_z(t)] \\ &\quad -(\hat{X}(t) - \hat{\boldsymbol{\pi}}^\top(t)\mathbf{1})(-\sigma_L \boldsymbol{\rho}^\top dW(t) - \hat{\sigma}_z(t) dW_z(t)) \\ &= [\hat{r}\hat{X}(t) + \hat{\boldsymbol{\pi}}^\top(t)(\hat{\mu} - \hat{r}\mathbf{1})]dt + \hat{X}(t)(-\sigma_L \boldsymbol{\rho}^\top dW(t) - \hat{\sigma}_z(t) dW_z(t)). \end{aligned} \tag{3.42}$$

Now we are ready to solve (3.38).

3.2.3 HJB equation and its solution

By the dynamic programming principle, the problem (3.38) is reduced to solving

$$V_t + \min_{\hat{\boldsymbol{\pi}}}\{[\hat{r}\hat{x} + \hat{\boldsymbol{\pi}}^\top(\hat{\boldsymbol{\mu}} - \hat{r}\mathbf{1})]V_{\hat{x}} + [\tfrac{1}{2}\hat{\boldsymbol{\pi}}^\top \Sigma \tilde{\boldsymbol{\pi}} + \tfrac{1}{2}\hat{x}^2 \hat{\sigma}_L^2 \boldsymbol{\rho}^\top \boldsymbol{\rho} - \hat{\boldsymbol{\pi}}^\top \boldsymbol{\sigma}\boldsymbol{\rho}\hat{\sigma_L}\hat{x}]V_{\hat{x}\hat{x}}\} = 0, \tag{3.43}$$

with boundary condition

$$V(T, \hat{x}) = \hat{x}_0^2 e^{\hat{\sigma}_z^2(T)} - 2b\hat{x}_0 e^{\frac{1}{2}\hat{\sigma}_z^2(T)} + b^2. \tag{3.44}$$

The solution of (3.43) is given by the following proposition.

Proposition 3.1 *The function*

$$W_2(t, \tilde{x}) = P_2(t)\tilde{x}^2 + Q_2(t)\tilde{x} + R_2(t) \tag{3.45}$$

is a continuously differentiable solution to (3.43)-(3.44), where

$$\begin{aligned} P_2(t) &= e^{f_2(t)} \triangleq e^{\int_t^T (M_2(s) + \hat{\sigma}_z^2(s))ds + \hat{\sigma}_z^2(T)}, \\ Q_2(t) &= e^{g_2(t)} \triangleq -2be^{\int_t^T N_2(s)ds + \frac{1}{2}\hat{\sigma}_z^2(T)}, \\ R_2(t) &= b^2 k_2(t) e^{-h_2(t-T)} \triangleq b^2 k_2^2(0) e^{\int_t^T (2N_2(s) - M_2(s) - \sigma_z^2(s))ds}, \end{aligned} \tag{3.46}$$

in which

$$\begin{aligned} M_2(t) &= 2\hat{r}(t) - (\sigma\rho\hat{\sigma}_z(t) - (\hat{\mu} - \hat{r}(t)\mathbf{1}))\Sigma^{-1}(\sigma\rho\hat{\sigma}_z(t) - (\hat{\mu}(t) - \hat{r}(t)\mathbf{1})), \\ N_2(t) &= \hat{r}(t) + (\sigma\rho\hat{\sigma}_z(t) - (\hat{\mu}(t) - \hat{r}(t)\mathbf{1}))\Sigma^{-1}(\hat{\mu}(t) - \hat{r}(t)\mathbf{1}), \\ U_2(t) &= 2N_2(t) - M_2(t) - \hat{\sigma}_z^2(t), \end{aligned} \tag{3.47}$$

with

$$k_2(t) = e^{-\int_t^T U_2(s)ds} - \int_t^T e^{\int_s^t U_2(u)du}[(\hat{\mu}(s) - \hat{r}(s)\mathbf{1})]^\top \Sigma^{-1}[(\hat{\mu}(s) - \hat{r}(s)\mathbf{1})]ds. \tag{3.48}$$

We omit the proof here as it can be carried out by repeating the procedure of Theorem 3.1.

3.2.4 Efficient strategy and efficient frontier

Similar to Theorem 3.2, the solution W_2 is exactly the value function $V(t, \hat{x})$ defined by (3.38). Let $t = 0$ and $\hat{x} = \hat{x}_0$ in (3.45). Then (3.45) is translated into the following function with respect to β:

$$V(t, \hat{x}) - \beta^2 = \hat{x}_0^2 e^{f_2(0)} - 2b\hat{x}_0 e^{g_2(0)} + k_2(t) b^2 e^{h_2(0)} - \beta^2, \tag{3.49}$$

where

$$b = d - \beta. \tag{3.50}$$

Again we can show that (3.49) attains its maximum value

$$\hat{x}_0^2 e^{f_2(0)} - 2d\hat{x}_0 e^{g_2(0)} + d^2 k_2(0) e^{h_2(0)} + \frac{(k_2(0) d e^{h_2(0)} - \hat{x}_0 e^{g_2(0)})^2}{k_2(0) e^{h_2(0)} - 1} \tag{3.51}$$

at

$$\beta^* = \frac{k_2(0) d e^{h_2(0)} - \hat{x}_0 e^{g_2(0)}}{k_2(0) e^{h_2(0)} - 1}. \tag{3.52}$$

As $k_2(0) e^{h_2(0)} < 1$, just as the full information case, we have $d \geq \frac{\hat{x}_0 e^{g_2(0)}}{k_2(0) e^{h_2(0)}}$, which is denoted as $\hat{d}_0$.

The results are summarized in the following theorem.

Theorem 3.4 *The efficient strategy of the problem (3.35) corresponding to the expected terminal wealth* $E\Big(E[X^*(T) e^{-L(T)} | \mathcal{G}_T]\Big) = d$ *is*

$$\hat{\boldsymbol{\pi}}^*(t) = \Sigma^{-1}(\boldsymbol{\sigma}\boldsymbol{\rho}\hat{\sigma}_z(t) - (\hat{\boldsymbol{\mu}}(t) - \hat{r}(t)\mathbf{1}))\hat{x} - \Sigma^{-1}(\hat{\boldsymbol{\mu}}(t) - \hat{r}(t)\mathbf{1})(d - \beta^*) e^{g_2(t) - h_2(t)}, \tag{3.53}$$

and the efficient frontier is

$$\mathrm{Var}\hat{X}^*(T) = \frac{-k_2(0) e^{h_2(0)} (d - \frac{\hat{x}_0}{k_2(0)} e^{g_2(0) - h_2(0)})^2}{k_2(0) e^{h_2(0)} - 1} + C_2 \hat{x}_0^2, \tag{3.54}$$

where $C_2 = [e^{2g_2(0)}(k_2(0)-1) + e^{f_2(0)}(\frac{1}{k_2^2(0)} - 1)]$*,* β^* *is given by (3.52),* $f_2(\cdot), g_2(\cdot), h_2(\cdot)$ *and* $k_2(\cdot)$ *are defined by (3.46).*

3.3 Mutual fund theorem

When inflation is introduced into the portfolio, the original mutual fund theorem no longer holds due to the lack of a totally risk free asset. We revisit the case with full information and derive the following modified mutual fund theorem.

Theorem 3.5 *Suppose* $\tilde{\boldsymbol{\pi}}_1^*$ *and* $\tilde{\boldsymbol{\pi}}_2^*$ *be the optimal investment strategy corresponding to the expected returns* d_1 *and* d_2 *respectively, where* $\tilde{\boldsymbol{\pi}}_1^*$ *is the minimal mean-variance portfolio, then for any feasible* d_3*, the optimal portfolio* $\tilde{\boldsymbol{\pi}}_3^*$ *can be written as a weighted average of* $\tilde{\boldsymbol{\pi}}_1^*$ *and* $\tilde{\boldsymbol{\pi}}_2^*$*, that is*

$$\tilde{\boldsymbol{\pi}}_3^* = \alpha \tilde{\boldsymbol{\pi}}_1^* + (1-\alpha)\tilde{\boldsymbol{\pi}}_2^*. \tag{3.55}$$

Proof. Denote $\boldsymbol{\xi} = -\Sigma^{-1}(\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1})$, $\boldsymbol{\eta}(t) = \frac{\Sigma^{-1}(\tilde{\boldsymbol{\mu}} - \tilde{r}\mathbf{1})}{1-k_1(0)e^{h_1(0)}} \frac{e^{g_1(t)}}{2e^{f_1(t)}}$, $c = x_0 e^{g_1(0)}$. Then

$$\tilde{\boldsymbol{\pi}}_i^*(t) = \boldsymbol{\xi} x + (d_i - c)\boldsymbol{\eta}(t), i = 1, 2, 3.$$

Let

$$\alpha = \frac{d_1 - d_3}{d_1 - d_2},$$

thus (3.55) holds. This completes the proof.

For the case without inflation and all the market parameters are deterministic, the corresponding investment reduces to the case described by the one-fund theorem, which says that any efficient portfolio is the combination of the risk free asset and the tangential portfolio with the highest Sharpe ratio. With inflation, now the theorem shows that an investor needs to invest in the minimum variance portfolio and another pre-specified efficient portfolio, which are able to generate all the other efficient portfolios. For the case with partially observed information, Theorem 3.5 still holds but with the projected $\hat{\boldsymbol{\mu}}(t), \hat{r}(t)$ instead of $\tilde{\boldsymbol{\mu}}, \tilde{r}$, respectively.

4 Dynamic Mean-Variance Problem with an inflation-linked bond

In the previous section, the MV problem under inflation is converted into a MV problem with risky assets only. In this section, we study the situation when an inflation-linked bond is available for investment, which acts as a risk-free asset. We also compare the maximum portfolio Sharpe ratio with and without the addition of the inflation-linked bond. Indeed, in the literature, it was suggested there is a positive premium, in terms of the Sharpe ratio, arising from the dynamic trading, when a risk-free asset is included in the portfolio of risky assets only [6]. However, the value of the premium has not been derived explicitly. Here, under inflation, we derive an explicit expression for the highest Sharpe ratio and study the premium for dynamic trading under inflation.

4.1 Portfolio with an inflation-linked bond

Assume that the agent can also invest in an inflation-linked bond, and the price index process is observable with full information [5].

Assume that the dynamics of the bond evolve over time as

$$\begin{cases} dI(t) &= (\mu_L + r)dt + \sigma_L I(t) dW_L(t), \\ I(0) &= 1. \end{cases} \tag{4.1}$$

The appreciation rate $\{I(t) : t \in \mathcal{T}\}$ is estimated by the observed price index process (3.1). Write $\tilde{I}(t) = \frac{I(t)}{P(t)}$, by Ito's Lemma,we have

$$\begin{cases} \frac{d\tilde{I}_t}{\tilde{I}_t} &= rdt, \\ \tilde{I}_0 &= 1. \end{cases} \tag{4.2}$$

Let $(\boldsymbol{\pi}, \pi_0)$ denote the nominal investment into nominal risky asset and interest bearing asset. Write

$$\breve{\boldsymbol{\pi}} = \frac{(\boldsymbol{\pi}, \pi_0)}{P(t)}. \tag{4.3}$$

[5] If the price index process is only partially observable, we can repeat the same procedure in this section together with the projected values defined in Section 3.2.

The resulting surplus process evolves as

$$d\tilde{X}(t) = r\tilde{X}(t) + \breve{\boldsymbol{\pi}}^\top(t)(\tilde{\boldsymbol{\mu}} - r\tilde{\mathbf{1}})dt + \breve{\boldsymbol{\pi}}^\top(t)\tilde{\boldsymbol{\sigma}}d\boldsymbol{W}(t), \tag{4.4}$$

where $\tilde{\mathbf{1}} = (\underbrace{1, 1, ..., 1}_{n+1})$, $\tilde{\boldsymbol{\mu}} = (\mu, r)/\mathrm{P(t)}$,

$$\tilde{\boldsymbol{\sigma}} = \begin{pmatrix} \boldsymbol{\sigma} - \boldsymbol{\sigma}_\rho & -\sqrt{1 - \boldsymbol{\rho}^\top\boldsymbol{\rho}}\sigma_L\mathbf{1} \\ -\boldsymbol{\rho}\sigma_L & -\sqrt{1 - \boldsymbol{\rho}^\top\boldsymbol{\rho}}\sigma_L \end{pmatrix}, \tag{4.5}$$

in which $(\boldsymbol{\sigma}_\rho)_{i,j} = \rho_i\sigma_L, i, j = 1, \cdots, n$, $\tilde{\boldsymbol{W}} = (W, W_I)^\top$, $W_I(t) = \boldsymbol{\rho}^\top W(t) + \sqrt{1 - \boldsymbol{\rho}^\top\boldsymbol{\rho}}W_L(t)$, and $W_L(t)$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$, which is independent of $W(t)$.

Let $\breve{\mathbf{\Pi}}$ denote the space of all the investment strategy $\breve{\boldsymbol{\pi}}(t)$ which is $\tilde{\mathcal{F}}_t$-measurable and satisfies that

$$E[\int_0^T \breve{\boldsymbol{\pi}}^\top(t)\breve{\boldsymbol{\pi}}(t)dt] < \infty.$$

The MV problem can be formulated as the following optimization problem parameterized with $d \geq \tilde{x}_0 e^{rT}$.

Problem 4.1

$$\min_{\breve{\boldsymbol{\pi}}} \quad \mathrm{Var}\tilde{\mathrm{X}}(\mathrm{T}) = E[\tilde{X}(T) - d]^2,$$

$$\text{subject to} \quad \begin{cases} E[\tilde{X}(T)] = d, \\ \breve{\boldsymbol{\pi}}(\cdot) \in \breve{\mathbf{\Pi}}, \\ (\tilde{X}(\cdot), \breve{\boldsymbol{\pi}}(\cdot)) \text{ satisfy (4.4)}, \\ d \in [\tilde{x}_0 e^{rT}, +\infty). \end{cases} \tag{4.6}$$

4.1.1 Efficient strategy and efficient frontier

Theorem 4.1 *The efficient strategy of the problem (4.6) corresponding to the expected terminal wealth* $E\tilde{X}^*(T) = d$ *is*

$$\breve{\boldsymbol{\pi}}^*(t) = -\mathbf{\Sigma}^{-1}(\tilde{\boldsymbol{\mu}} - r\tilde{\mathbf{1}})(\tilde{X}^*(t) - (d - \beta^*)e^{r(t-T)}), \tag{4.7}$$

where

$$\tilde{\mathbf{1}} = (\underbrace{1, 1, ..., 1}_{n+1}), \ \beta^* = \frac{d - \tilde{x}_0 e^{rT}}{1 - e^{\mathbf{A}T}}. \tag{4.8}$$

Moreover, the efficient frontier is

$$\mathrm{Var}\tilde{X}^*(T) = \frac{\left(d - \tilde{x}_0 e^{rT}\right)^2}{e^{\mathbf{A}T} - 1} = \frac{\left(E[\tilde{X}^*(T)] - \tilde{x}_0 e^{rT}\right)^2}{e^{\mathbf{A}T} - 1}, \tilde{\mathbf{\Sigma}} = \boldsymbol{\sigma}^\top \boldsymbol{\sigma}, \tag{4.9}$$

where

$$\mathbf{A} = (\tilde{\boldsymbol{\mu}} - r\tilde{\mathbf{1}})^\top \tilde{\mathbf{\Sigma}}^{-1} (\tilde{\boldsymbol{\mu}} - r\tilde{\mathbf{1}}). \tag{4.10}$$

The proof is given in Appendix.

The form of the efficient strategy (4.9) implies that the inflation-linked bond acts as the risk-free asset for the real return. The bond protects the investor against risk arising from the unanticipated inflation. In particular, if one wants to eliminate any risk for real return, i.e., $Var\tilde{X}^*(T) = 0$, then one has to put all the money in the inflation bond $E\tilde{X}^*(T) = \tilde{x}_0 e^{rT}$. Also, mathematically the inflation-linked bond makes the efficient frontier a perfect square.

Let $\sigma(\tilde{X}(T))$ denote the standard deviation of the real terminal wealth, then the efficient frontier in the mean-standard-deviation diagram (capital market line) is

$$E[\tilde{X}(T)] = \tilde{x}_0 e^{rT} + \sqrt{e^{\mathbf{A}T} - 1}\ \sigma(\tilde{X}(T)). \tag{4.11}$$

Hence the efficient frontier in the mean-standard-deviation diagram is still a straight line, similar to the classical result in the mean-variance portfolio selection theory. Let

$$\breve{\boldsymbol{\pi}}^*_{tan}(t) = -\tilde{\mathbf{\Sigma}}^{-1} (\tilde{\boldsymbol{\mu}} - r\tilde{\mathbf{1}}) \left(\tilde{x} - b e^{r(t-T)}\right),$$

be the tangency portfolio. It behaves like the mutual fund in the classical theory. Then for any feasible required return d', the efficient portfolio can be composed of this tangency portfolio and the risk free asset, and the portion α invested in the mutual fund is

$$\alpha = \frac{\tilde{x}_0 e^{rT} - e^{\mathbf{A}T} d'}{\tilde{x}_0 e^{rT} - e^{\mathbf{A}T} d}.$$

4.1.2 The premium for dynamic trading under inflation

With the inflation-linked bond, using (4.10), the Sharpe ratio (denoted by SP_f) is

$$SP_f = \frac{1}{\sqrt{e^{\mathbf{A}T} - 1}},$$

where $\mathbf{A}$ is defined in (4.10). We denote the highest Sharpe ratio by SP_r when the inflation-linked bond is not included. It can be calculated as follows:

1. If $\frac{x_0}{k_1(0)} e^{g_1(0)-h_1(0)} \le e^{rT}$, then

$$SP_r = \sqrt{\frac{1 - k_1(0)e^{h_1(0)}}{k_1(0)e^{h_1(0)}}}.$$

2. If $\frac{x_0}{k_1(0)} e^{g_1(0)-h_1(0)} > e^{rT}$, then

$$SP_r = \sqrt{\frac{\left(r - \frac{x_0}{k_1(0)} e^{g_1(0)-h_1(0)}\right) + \frac{1-k_1(0)e^{h_1(0)}}{k_1(0)e^{h_1(0)}}}{x_0^2}} \sqrt{\frac{1 - k_1(0)e^{h_1(0)}}{k_1(0) - 1}}.$$

While the inflation-linked bond makes the efficient frontier a line, the Sharpe ratio gap between the line and the region is given $SP_f - SP_r$.

5 Discussion and concluding remarks

In this work, we have considered the continuous-time mean variance problem with inflation for both full and partial information scenarios. As inflation makes the fixed interest asset risky, we have derived the efficient frontier and investment strategy when all the assets are discounted by the real interest rate. As a result, no risk-free portfolio can be obtained and the shape of the mean-variance frontier becomes the upper part of hyperbolic instead of the straight line, which is different from the classical result of mean-variance portfolio selection. Moreover, when all the wealth is put into the nominal risk free asset, the variance is not minimal , in contrast to the case without inflation.

For the situation without inflation, one mutual fund and the risk-free set are needed in the Mutual Fund Theorem. When taking inflation into consideration, the result shows that a minimum of two efficient portfolios are needed in order to generate all the other efficient portfolios. Finally, when an inflation-linked bond is included, all the efficient portfolios combining the mutual fund and the bond is still lying on a straight line, which is similar to the situation without inflation. For the real return, the inflation-linked bond acts as a protection for the investor from inflation and therefore improves the overall Sharpe ratio.

6 Appendix

Appendix 1 The derivation of (3.25) and (3.26).

Denote

$$U_\beta(0,\tilde{x}_0) = \tilde{x}_0^2 e^{f_1(0)} - 2b\tilde{x}_0 e^{g_1(0)} + k_1(0)b^2 e^{h_1(0)} - \beta^2,$$

we have

$$\begin{aligned} U_\beta(0,\tilde{x}_0) &= \tilde{x}_0^2 e^{f_1(0)} - 2(d-\beta)\tilde{x}_0 e^{g_1(0)} + k_1(0)(d-\beta)^2 e^{h_1(0)} - \beta^2 \\ &= \tilde{x}_0^2 e^{f_1(0)} - 2d\tilde{x}_0 e^{g_1(0)} + k_1(0)d^2 e^{h_1(0)} + \Big((k_1(0)e^{h_1(0)} - 1)\beta^2 \\ &\quad -2[k_1(0)de^{h_1(0)} - \tilde{x}_0 e^{g_1(0)}]\beta \Big), \end{aligned} \tag{A.1}$$

therefore, $U_\beta(0,\tilde{x}_0)$ *arrives at its maximal value at* $\frac{k_1(0)de^{h_1(0)} - \tilde{x}_0 e^{g_1(0)}}{k_1(0)e^{h_1(0)} - 1}$, *the value is*

$$\begin{aligned} U_\beta(0,\tilde{x}_0) &= \tilde{x}_0^2 e^{f_1(0)} - 2d\tilde{x}_0 e^{g_1(0)} + k_1(0)d^2 e^{h(0)} - \frac{(k_1(0)de^{h_1(0)} - \tilde{x}_0 e^{g_1(0)})^2}{k_1(0)e^{h_1(0)} - 1} \\ &= \frac{\tilde{x}_0^2 e^{f_1(0)}[k_1(0)e^{h_1(0)} - 1] + 2d\tilde{x}_0 e^{g_1(0)} - k_1(0)d^2 e^{h_1(0)} - \tilde{x}_0^2 e^{2g_1(0)}}{k_1(0)e^{h_1(0)} - 1} \\ &= \frac{-k_1(0)e^{h_1(0)}(d - \frac{\tilde{x}_0}{k_1(0)} e^{g_1(0)-h_1(0)})^2 + \tilde{x}_0^2[e^{2g_1(0)}(k_1(0)-1) + e^{f_1(0)}(\frac{1}{k_1(0)} - 1)]}{k_1(0)e^{h_1(0)} - 1} \\ &= \frac{k_1(0)e^{h_1(0)}(d - \frac{\tilde{x}_0}{k_1(0)} e^{g_1(0)-h_1(0)})^2}{1 - k_1(0)e^{h_1(0)}} + C_1\tilde{x}_0^2. \end{aligned} \tag{A.2}$$

Appendix 2 The proof of Theorem 4.1.

First we derive the solution to the auxiliary problem. For the auxiliary problem (4.6), we define the associated optimal value function by

$$V(t,\tilde{x}) = \min_{\boldsymbol{\pi}\in\boldsymbol{\Pi}} E[(\tilde{x}(T)-b)^2|\tilde{x}(t)=\tilde{x}]. \tag{A.3}$$

By the dynamic programming principle, the problem reduces to solve the following HJB equation:

$$V_t + \min_{\tilde{\pi}}\{[r\tilde{x}+\tilde{\boldsymbol{\pi}}^\top(\tilde{\boldsymbol{\mu}}-r\tilde{\mathbf{1}})]V_{\tilde{x}} + \tfrac{1}{2}\tilde{\boldsymbol{\mu}}^\top\tilde{\boldsymbol{\Sigma}}\tilde{\boldsymbol{\mu}}V_{\tilde{x}\tilde{x}}\} = 0, \tag{A.4}$$

with the boundary condition

$$V(T,\tilde{x}) = (\tilde{x}-b)^2. \tag{A.5}$$

The following theorem shows that (A.4)-(A.5) has a continuously differentiable solution.

Theorem 6.1 *The function*

$$W_3(t,\tilde{x}) = P_3(t)\tilde{x}^2 + Q_3(t)\tilde{x} + R_3(t), \tag{A.6}$$

is a continuously differentiable solution of (A.4)-(A.5), where

$$P_3(t) = e^{(\mathbf{A}-2r)(t-T)}, \quad Q_3(t) = -2be^{(\mathbf{A}-r)(t-T)}, \quad R_3(t) = b^2e^{\mathbf{A}(t-T)}. \tag{A.7}$$

Then $\tilde{\pi}^*(t)$ that minimizes the left-hand side of (A.4) is

$$\tilde{\boldsymbol{\pi}}^*(t) = -\tilde{\boldsymbol{\Sigma}}^{-1}(\tilde{\boldsymbol{\mu}}-r\tilde{\mathbf{1}})\left(\tilde{x}-be^{r(t-T)}\right), \tag{A.8}$$

and the nominal value

$$\boldsymbol{\pi}^*(t) = -\tilde{\boldsymbol{\Sigma}}^{-1}(\tilde{\boldsymbol{\mu}}-r\tilde{\mathbf{1}})\left(x-be^{r(t-T)}P(t)\right). \tag{A.9}$$

Proof. *Suppose that (A.4)-(A.5) has a solution with the following form:*

$$W(t,\tilde{x}) = P_3(t)\tilde{x}^2 + Q_3(t)\tilde{x} + R_3(t), \tag{A.10}$$

where $P_3(\cdot) > 0, Q_3(\cdot)$ and $R_3(\cdot)$ are three suitable functions. The boundary condition (3.14) implies that $P_3(T) = 1, Q_3(T) = -2b$ and $R_3(T) = b^2$.

Inserting the solution (A.10) into (A.4) and rearranging result in

$$\begin{aligned}\min_{\tilde{\pi}}\{&\tilde{\boldsymbol{\pi}}^\top\tilde{\boldsymbol{\Sigma}}\boldsymbol{\pi}P_3(t)+\tilde{\boldsymbol{\pi}}^\top(\tilde{\boldsymbol{\mu}}-r\tilde{\mathbf{1}})\,(2P_3(t)\tilde{x}+Q_3(t))\\ &+\left(P_3^{'}(t)+2rP_3(t)\right)\tilde{x}^2+\left(Q_3^{'}(t)+rQ_3(t)\right)\tilde{x}+R_3^{'}(t)\}=0.\end{aligned} \tag{A.11}$$

In view of $P_3(t) > 0$, (A.11) attains its minimum at

$$\tilde{\boldsymbol{\pi}}(t) = -\boldsymbol{\Sigma}^{-1}(\tilde{\boldsymbol{\mu}} - r\mathbf{1})\left(\tilde{x} + \frac{Q_3(t)}{2P_3(t)}\right). \tag{A.12}$$

This allows us to replace (A.11) by

$$[P_3^{'}(t) + (2r- \mathbf{A})P_3(t)]\tilde{X}^2+[Q_3^{'}(t)+(r-\mathbf{A})Q_3(t)]\tilde{X}+R^{'}(t)-\frac{\mathbf{A}Q_3^2(t)}{4P_3(t)} = 0, \tag{A.13}$$

where $\mathbf{A}$ is given by (4.10). Therefore, we require $P_3(\cdot)$, $Q_3(\cdot)$ and $R_3(\cdot)$ to satisfy the following differential equations (the first being a special Riccati equation)

$$\begin{aligned}P_3^{'}(t) + (2r - \mathbf{A})P_3(t) &= 0,\\ Q_3^{'}(t) + (r - \mathbf{A})Q_3(t) &= 0,\\ R_3^{'}(t) - \frac{\mathbf{A}Q_3^2(t)}{4P_3(t)} &= 0,\end{aligned} \tag{A.14}$$

with the boundary conditions

$$P_3(T) = 1,\ Q_3(T) = -2b,\ R_3(T) = b^2. \tag{A.15}$$

Solving (A.14)-(A.15), we obtain

$$W_3(t, \tilde{x}) = P_3(t)\tilde{x}^2 + Q_3(t)\tilde{x} + R_3(t), \tag{A.16}$$

where $P_3(t), Q_3(t)$ and $R_3(t)$ are given by (A.7). □

Now we apply the results established above to the mean-variance problem. First of all, we present the optimal value function for the problem (3.9). By Theorem 6.1, it is equal to W of (A.6). Let $t = 0$ and $\tilde{X} = \tilde{X}_0$ in (A.6). Define

$$U_\beta(0, \tilde{x}_0)= \tilde{x}_0^2e^{-(\mathbf{A}-2r)T}-2g(0)\tilde{x}_0e^{-(\mathbf{A}-r)T}+g^2(0)e^{-\mathbf{A}T} - \beta^2, \tag{A.17}$$

where

$$g(0) = d - \beta. \tag{A.18}$$

Note that the above value still depends on the Lagrange multiplier β*. To obtain the optimal value (i.e., the minimum variance* $\mathrm{Var}\tilde{X}_T$*) and optimal strategy for original problem (4.6), one needs to maximize the value in (A.17) over* $\beta \in R$ *according to the Lagrange duality theorem. A simple calculation shows that (A.17) attains its maximum value*

$$\frac{\left(d - \tilde{x}_0 e^{rT}\right)^2}{e^{\mathbf{A}T} - 1} \quad \text{at} \quad \beta^* = \frac{d - \tilde{x}_0 e^{rT}}{1 - e^{\mathbf{A}T}}. \tag{A.19}$$

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