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The aggregation of multiple three-way decision spaces

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Abstract

Based on the theory of three-way decisions proposed by Yao, Hu established three-way decision spaces on fuzzy lattices and partially ordered sets. At the same time, multiple three-way decision spaces and its corresponding three-way decisions were also established. How to choose a method for the transformation from multiple three-way decision spaces to a single three-way decision space? This is one of the main problems on multiple three-way decision spaces. In connection with the transformation question on multiple three-way decision spaces, this paper gives out an aggregation method from multiple three-way decision spaces to a single three-way decision space through an axiomatic complement-preserving aggregation function. These aggregation methods in the partially set [0,1] contain the weighted average three-way decisions, max-min average three-way decisions and median three-way decisions etc. These methods are generalized to three-way decisions over two groups of multiple three-way decision spaces. At last we illustrate aggregation methods of multiple three-way decision spaces through a practical example.

Keywords: Partially ordered sets; Fuzzy sets; Rough sets; Three-way decisions; Three-way decision space.

1. Introduction

Since three-way decisions (3WD) were proposed by Yao [37], many authors had studied 3WD [5, 16-17, 22, 38-40]. The existing studies focus mainly on the following four aspects.

- Three-way decisions based on decision-theoretic rough sets are generalized to various fuzzy sets, such as Deng and Yao considered fuzzy sets [5]; Liang and Liu et al. discussed triangular fuzzy sets [18], Liang and Liu looked upon interval-valued fuzzy sets [16] and intuitionistic fuzzy sets [17]; Zhao and Hu also considered interval-valued fuzzy sets [47-48]; Hu analyzed hesitant fuzzy sets and interval-valued hesitant fuzzy sets [8] etc.
- Three-way decisions based on decision-theoretic rough sets are generalized to more patterns, such as Qian and Zhang et al. introduced multigranulation into decision-theoretic rough sets [30]; To reduce boundary regions, Chen and Zhang et al. proposed multi-granular three-way decision based on the multiple-views of granularity [4]; Sang and Liang et al. considered decision-theoretic rough sets under dynamic granulation [31] etc.

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● The theoretical frameworks on three-way decisions are studied, such as the domain of evaluation functions [38], construction and interpretation of evaluation functions [37-39], the mode of three-way decisions [39], the theory of three-way decision spaces [7, 11] and trisecting-and-acting framework of three-way decisions [42] etc.

● The theory of three-way decisions has been applied to incomplete information system [20], risk decision making [15], classification [21] and clustering [43], investment [23], multi-agent [34], group decision making [19], recommender systems [46], face recognition [14] and social networks [26] etc.

For theoretical development of three-way decisions, Hu systematically studied three-way decision models in rough sets and probabilistic rough sets, introduced axiomatic definitions for decision measurement, decision condition and decision evaluation function and established three-way decision spaces based on fuzzy lattices [7, 11] and partially ordered sets [8]. The so-called fuzzy lattice is a complete distributive lattice with an involutive negator (i.e. inverse order and involutive mapping). There are numerous popular fuzzy lattices used in classical logic and fuzzy logic such as crisp sets, fuzzy sets [44], shadowed sets [24-25], intuitionistic fuzzy sets [1-2], interval-valued fuzzy sets [45] and interval sets [35, 36]. A fuzzy lattice is also a partially ordered set. There are many partially ordered sets, which are not fuzzy lattices, such as hesitant fuzzy sets [33], interval-valued hesitant fuzzy sets [3], type-2 fuzzy sets [9] and interval-valued type-2 fuzzy sets [10].

At the same time, based on multi-granulation rough sets [27-31], multiple three-way decision spaces were further discussed in [7]. As a result of the classical single-granulation rough set theory, a multi-granulation rough set model (MGRS) has been developed [28, 29] which is a kind of information fusion strategy through fusing multiple granular structures. The following are some existing multi-granulation fusion strategies.

(1) Pessimistic strategy [27, 30].

(2) Optimistic strategy [28, 29, 30]

(3) Dynamic strategy [31].

In this paper, we consider two problems. The first problem is *are these existing strategies reasonable*? Another one is *are there other reasonable strategies*? This paper answers these problems through considering aggregation methods from multiple three-way decision spaces to a single three-way decision space which is referred to as the aggregation strategy.

From Note 3.1 in [7], we can see that if $E_1, E_2, \cdots, E_n$ are *n* decision evaluation functions, then $\overset{n}{\underset{i=1}{\wedge}} E_i(A)(x)$ and $\overset{n}{\underset{i=1}{\vee}} E_i(A)(x)$ are not necessarily decision evaluation functions because they do not meet the third axiom, Complement Axiom. Are there some methods to construct a decision evaluation function from *n* decision evaluation functions $E_1, E_2, \cdots, E_n$? Although $\overset{n}{\underset{i=1}{\wedge}} E_i(A)(x)$ and $\overset{n}{\underset{i=1}{\vee}} E_i(A)(x)$ are not decision evaluation functions, $\frac{1}{2}\left(\overset{n}{\underset{i=1}{\wedge}} E_i(A)(x) + \overset{n}{\underset{i=1}{\vee}} E_i(A)(x)\right)$ is a decision evaluation function over [0, 1]. And $\frac{1}{n}\sum_{i=1}^{n} E_i(A)(x)$ is also a decision evaluation function in [0, 1]. There are three common properties in these functions, namely regularity, nondecreasing property and complement-preserving property. This is one of our motivations to consider the axiomatic definition on complement-preserving aggregation function. Because general aggregation functions [6] satisfy regularity and nondecreasing property, aggregation functions satisfied complement-preserving property are referred to as a complement-preserving aggregation function in this paper.

And then, through these complement-preserving aggregation functions we can establish transformation methods from multiple three-way decision spaces to a single three-way decision space. These transformation methods in partially ordered set [0, 1] include the weighted average three-way decisions, max-min average three-way decisions and median three-way decisions. These methods are generalized to bi-evaluation functions.

Our method compensates for the defect of the multi-granulation rough sets which only consider two extreme models, the optimistic rough set [29] and the pessimistic rough set [27]. This paper presents more strategies for the aggregation of multi-granulation rough sets. There are the possible applications in the aggregation of the multi-granulation rough sets, the theory of multiple three-way decisions and so on.

The rest of this paper is organized as follows. Section 2, as preliminaries, recalls the decision evaluation function axioms and three-way decision spaces based on partially ordered sets. Section 3 first introduces the axiomatic definition on complement-preserving aggregation function and then gives out methods for the transformation from multiple three-way decision spaces to a single three-way decision spaces based on the axiomatic complement-preserving aggregation function. It also gives an example to illustrate these novel methods. In Section 4, these aggregation methods are generalized to three-way decisions over two groups of multiple three-way decision spaces and a practical example on evaluation of student performance is taken in order to illustrate the thoughts of the aggregation methods over two groups of multiple three-way decision spaces. Finally, Section 5 concludes the paper.

2. Preliminaries

The basic concepts, notations and results of partially ordered sets [8], decision valuation functions [7, 8, 11] and three-way decision spaces [7, 8, 11] are briefly reviewed in this section.

In this paper $(P, \leq_P)$ is a bounded partially ordered set with an involutive negator N_P, the minimum 0_P and maximum 1_P, which is written as $(P, \leq_P, N_P, 0_P, 1_P)$ [7]. In [0, 1], operator $x^c = 1 - x$ ($x \in [0,1]$) is applied.

Let X and Y be two universes. $Map(X,Y)$ is the family of all mappings from X *to* Y, *i.e.* $Map(X,Y) = \{f \mid f : X \to Y\}$. If $A \in Map(U,\{0,1\})$, then A is a subset of U, i.e. $Map(U,\{0,1\})$ is the power set of U, which can also be written as 2^U. If $A \in Map(U,\{0,1,[0,1]\})$, then A is a shadowed set of U [24-25]. If $A \in Map(U,[0,1])$, then A is a fuzzy set of U [44], namely $Map(U,[0,1])$ is the fuzzy power set of U. If $A \in Map(U, I^{(2)})$, then A is an interval-valued fuzzy set of U [45] and an interval-valued fuzzy set A with membership function $[A^-(x), A^+(x)]$ is also denoted as $[A^-, A^+]$. If $A \in Map(U, I_s^{(2)})$, then A is an intuitionistic fuzzy set of U [1-2].

Let $(P, \leq_P, N_P, 0_P, 1_P)$ be a bounded partially ordered set. If $A \in Map(U,P)$, then the complement of A is defined pointwise by the following formula

$$N_P(A)(x) = N_P(A(x)).$$

Then $(Map(U,P), \subseteq_P, N_P, \varnothing, U)$ is a bounded partially ordered set, where $\varnothing(x) = 0_P, \forall x \in U$ and

$U(x)=1_P, \forall x \in U$, and for $A, B \in Map(U,P)$, $A \subseteq_P B$ iff $A(x) \le_P B(x)$, $\forall x \in U$.

Let $(P_C, \le_{P_C}, N_{P_C}, 0_{P_C}, 1_{P_C})$ and $(P_D, \le_{P_D}, N_{P_D}, 0_{P_D}, 1_{P_D})$ be two bounded partially ordered sets in the following. Let *U* be a nonempty universe, on which a decision is to make. *U* is called a decision universe. Similarly, let *V* be a nonempty universe where a condition function is defined. *V* is named condition universe.

Definition 2.1. [8]. Let *U* be a decision universe and *V* be a condition universe. Then a mapping $E: Map(V, P_C) \to Map(U, P_D)$ is called a *decision evaluation function* of *U*, if it satisfies the following three axioms.

(E1) Minimum element axiom

$$E(\varnothing)=\varnothing, \text{ i.e., } E(\varnothing)(x)=0_{P_D}, \forall x \in U .$$

(E2) Monotonicity axiom

$$\forall A, B \in Map(V, P_C), A \subseteq_{P_C} B \Rightarrow E(A) \subseteq_{P_D} E(B), \text{ i.e., } E(A)(x) \le_{P_D} E(B)(x), \forall x \in U .$$

(E3) Complement axiom

$$N_{P_D}(E(A)) = E(N_{P_C}(A)), \forall A \in Map(V, P_C), \text{ i.e., } N_{P_D}(E(A))(x) = E(N_{P_C}(A))(x), \forall x \in U .$$

$E(A)$ is called a decision evaluation function of *U* (for $A \in Map(V, P_C)$).

Given universe *U*, the decision condition domain $Map(V, P_C)$, decision measurement domain P_D and decision evaluation function *E*, then $(U, Map(V, P_C), P_D, E)$ is called a *three-way decision space*.

In multiple three-way decision spaces, two extreme transformation methods are discussed in [7], i.e., optimistic and pessimistic three-way decisions of multiple three-way decision spaces.

Definition 2.2. Let $(U, Map(V, P_C), P_D, E_i)$ ($i=1,2,\cdots,n$) be *n* three-way decision spaces, $A \in Map(V, P_C)$, $\alpha, \beta \in P_D$ and $0 \le \beta < \alpha \le 1$. Then the optimistic three-way decisions of multiple three-way decision spaces are defined as follows.

(1) Acceptance region:

$$ACP^{op}_{(\alpha,\beta)}(E_{1\sim n}, A) = \bigcup_{i=1}^{n} ACP_{(\alpha,\beta)}(E_i, A) = \bigcup_{i=1}^{n} \{x \in U \mid E_i(A)(x) \ge \alpha\} ,$$

(2) Rejection region:

$$REJ^{op}_{(\alpha,\beta)}(E_{1\sim n}, A) = \bigcap_{i=1}^{n} REJ_{(\alpha,\beta)}(E_i, A) = \bigcap_{i=1}^{n} \{x \in U \mid E_i(A)(x) \le \beta\} ,$$

(3) Uncertain region:

$$UNC^{op}_{(\alpha,\beta)}(E_{1\sim n}, A) = \left(ACP^{op}_{(\alpha,\beta)}(E_{1\sim n}, A) \cup REJ^{op}_{(\alpha,\beta)}(E_{1\sim n}, A)\right)^c .$$

The pessimistic three-way decisions of multiple three-way decision spaces are defined as

(1) Acceptance region:

$$ACP^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \bigcap_{i=1}^{n} ACP_{(\alpha,\beta)}(E_i, A) = \bigcap_{i=1}^{n} \{x \in U \mid E_i(A)(x) \ge \alpha\} ,$$

(2) Rejection region:

$$REJ^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \bigcup_{i=1}^{n} REJ_{(\alpha,\beta)}(E_i, A) = \bigcup_{i=1}^{n} \{x \in U \mid E_i(A)(x) \le \beta\} ,$$

(3) Uncertain region：

$$UNC^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \left(ACP^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) \cup REJ^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A)\right)^c .$$

In rough set theory, three-way decisions are given by the lower and upper approximations. However, in a three-way decision space, three-way decisions are directly induced by decision evaluation function and the lower and upper approximations are induced by the three-way decisions. Fig 2.1 is a diagram showing that three-way decisions induced by the lower and upper approximations of rough sets and the lower and upper approximations are induced by the three-way decisions in 3WD spaces.

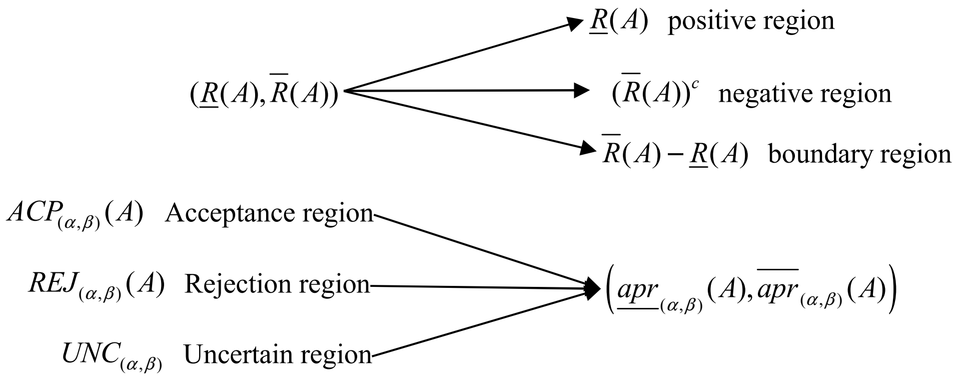

Fig. 2.1. 3WDs induced by the lower and upper approximations of rough sets in rough set theory, and the lower and upper approximations are induced by the 3WDs in 3WD space.

One of the important characteristics of three-way decision space is that the lower approximation and upper approximation of an object can be induced through three-way decisions. Similarly the optimistic and pessimistic three-way decisions over three-way decision spaces can also induce the corresponding lower approximation and upper approximation of an object.

Defintion 2.3. If $A \in Map(V, P_C)$, then

$\underline{apr}^{op}_{(\alpha,\beta)}(E_{1\sim n}, A) = ACP^{op}_{(\alpha,\beta)}(E_{1\sim n}, A)$ and

$\overline{apr}^{op}_{(\alpha,\beta)}(E_{1\sim n}, A) = \left(REJ^{op}_{(\alpha,\beta)}(E_{1\sim n}, A)\right)^c$

are referred to as *the lower approximation and upper approximation* of *A* with regard to optimistic three-way decisions over multiple three-way decision spaces respectively.

$\underline{apr}^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = ACP^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A)$ and

$\overline{apr}^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \left(REJ^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A)\right)^c$

are referred to as *the lower approximation and upper approximation* of *A* with regard to pessimistic three-way decisions over multiple three-way decision spaces respectively.

In [7], the author discussed some properties of the optimistic / pessimistic three-way decisions over multiple three-way decision spaces and the lower / upper approximation with regard to the optimistic / pessimistic three-way decisions over multiple three-way decision spaces.

3. The aggregation methods of multiple three-way decision spaces

In this section, we give a transformation method from multiple three-way decision spaces to a single three-way decision space with complement-preserving aggregation, whose special cases in [0, 1] contain the weighted average, the max-min average and median average.

3.1. Complement-preserving aggregation functions

The transformation method from multiple three-way decision spaces to a single three-way decision space relies on a complement-preserving aggregation function, such as

$$\frac{1}{2}\left(\bigwedge_{i=1}^{n} x_i + \bigvee_{i=1}^{n} x_i\right) \text{ and } \frac{1}{n}\sum_{i=1}^{n} x_i .$$

In these functions, there are some common characteristics, e.g., regularity, nondecreasing property and complement-preserving property. This paper presents some complement-preserving aggregation functions through an axiomatic definition. The axiomatic definition on complement-preserving aggregation function is defined as follows.

Definition 3.1. Let $(P, \leq_P, N_P, 0_P, 1_P)$ be a bounded partially ordered set. A mapping $f: P^n \to P$ is called an *n*-ary complement-preserving aggregation function, if it satisfies the following conditions:

(AF1) Regularity:

$$f(x, x, \cdots, x) = x, \ \forall x_i \in P;$$

(AF2) Nondecreasing Property:

f is a nondecreasing function for each variable over P, i.e. $x_i^{(1)} \leq_P x_i^{(2)}$ $(i = 1, 2, \cdots, n)$ implies $f(x_1, \cdots, x_{i-1}, x_i^{(1)}, x_{i+1}, \cdots, x_n) \leq_P f(x_1, \cdots, x_{i-1}, x_i^{(2)}, x_{i+1}, \cdots, x_n)$, $\forall x_i, y_i \in P$;

(AF3) Complement-preserving Property:

$$f\left(N_P(x_1), N_P(x_2), \cdots, N_P(x_n)\right) = N_P\left(f(x_1, x_2, \cdots, x_n)\right), \ \forall x_i \in P.$$

The family of all *n*-ary complement-preserving aggregation functions over *P* is denoted by $AF_n(P)$.

General aggregation function satisfies nondecreasing property and boundary conditions, $f(0, 0, \cdots, 0) = 0$ and $f(1, 1, \cdots, 1) = 1$. Detailed information about aggregation functions can be found in [6].

The following are examples of complement-preserving aggregation functions over [0, 1].

Example 3.1. Take $P = [0,1]$ and $N_P(x) = 1 - x$. Then the following functions are *n*-ary complement-preserving aggregation functions over [0, 1].

(1) $f^{wa}(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} a_i x_i$, $x_1, x_2, \cdots, x_n \in [0,1]$, where $a_1, a_2, \cdots, a_n \in [0,1]$ and $\sum_{i=1}^{n} a_i = 1$. f^{wa} is called a weighted average complement-preserving aggregation function.

Specially, $\frac{x_1 + x_2 + \cdots + x_n}{n}$, written as $f^{aa}(x_1, x_2, \cdots, x_n)$, is called an absolute average complement-preserving aggregation function. $\frac{x_1 + x_2}{2^{n-1}} + \frac{x_3}{2^{n-2}} + \cdots + \frac{x_n}{2}$, written as $f^{sa}(x_1, x_2, \cdots, x_n)$, is called a stepwise average complement-preserving aggregation function.

(2) $f^{ma}(x_1, x_2, \cdots, x_n) = \dfrac{\max_i\{a_i \mathrm{T} x_i\} + \min_i\{(1 - a_i) \perp x_i\}}{2}$, $x_1, x_2, \cdots, x_n \in [0,1]$, where $a_1, a_2, \cdots, a_n \in [0,1]$, $\max_i a_i = 1$, T and $\perp$ are t-norm and t-conorm over [0,1] respectively and are dual w.r.t $N_P(x) = 1 - x$. f^{ma} is called a max-min average complement-preserving aggregation

function.

We verify that the function f^{ma} satisfies complement-preserving axiom as follows.

$$f^{ma}(1-x_1,1-x_2,\cdots,1-x_n)=\frac{\max_i\{a_i\mathrm{T}(1-x_i)\}+\min_i\{(1-a_i)\perp(1-x_i)\}}{2}$$

$$=\frac{\max_i\{1-\left((1-a_i)\perp x_i\right)\}+\min_i\{1-a_i\mathrm{T}x_i)\}}{2}$$

$$=\frac{1-\min_i\{(1-a_i)\perp x_i\}+1-\max_i\{a_i\mathrm{T}x_i)\}}{2}$$

$$=1-\frac{\max_i\{a_i\mathrm{T}x_i)\}+\min_i\{(1-a_i)\perp x_i\}}{2}$$

$$=1-f^{ma}(x_1,x_2,\cdots,x_n).$$

Specially, take $a_i=1$, $i=1,2,\cdots,n$, then

$$f^{ma}(x_1,x_2,\cdots,x_n)=\frac{\max_i\{x_i\}+\min_i\{x_i\}}{2}.$$

(3) $f^{me}(x_1,x_2,\cdots,x_n)=Med\{x_i\}$

$$=\begin{cases} x'_{[\frac{n}{2}]+1}, & n \textit{ is an odd number} \\ \dfrac{x'_{\frac{n}{2}}+x'_{\frac{n}{2}+1}}{2}, & n \textit{ is an even number}\end{cases},\quad x_1,x_2,\cdots,x_n\in[0,1]$$

and $\{x'_i\}$ is an ordering of $\{x_i\}$ from smallest to largest or from largest to smallest. f^{me} is called a median complement-preserving aggregation function.

It follows from the Example 3.1. (1) that if we consider $AF_2([0,1])$, then $f^{aa}(x_1,x_2)=f^{sa}(x_1,x_2)=f^{ma}(x_1,x_2)=\dfrac{x_1+x_2}{2}$.

In practical application, $f^{wa}(x_1,x_2,\cdots,x_n)$ is used when different data are considered different significance; the equal weighted function $f^{aa}(x_1,x_2,\cdots,x_n)$ is used when all data are of the same significance; $f^{sa}(x_1,x_2,\cdots,x_n)$ is used when more recent data are considered more important than older data; $f^{ma}(x_1,x_2,\cdots,x_n)$ is used when some datum x_i is considered particularly significance; $f^{me}(x_1,x_2,\cdots,x_n)$ is used when we consider the median of data.

In Example 3.1, all examples satisfy the following property.

(AF4) $x_i <_P y_i$, $\forall i\in\{1,2,\cdots,n\}$ implies $f(x_1,x_2\cdots,x_n)<_P f(y_1,y_2,\cdots,y_n)$, $\forall x_i,y_i\in P$.

The following are some properties and constructions of complement-preserving aggregation functions.

Theorem 3.1. *Let* $(L,\le_L,N_L,0_L,1_L)$ *be a fuzzy lattice and* $f\in AF_n(L)$. *Then* $\overset{n}{\underset{i=1}{\wedge}} x_i \le_L f(x_1,x_2,\cdots,x_n)\le_L \overset{n}{\underset{i=1}{\vee}} x_i$.

Proof. Because $\overset{n}{\underset{i=1}{\wedge}} x_i \le_L x_i \le_L \overset{n}{\underset{i=1}{\vee}} x_i$ for any $x_1,x_2,\cdots,x_n\in L$ and the nondecreasing property of *f*, we have

$$f\left(\overset{n}{\underset{i=1}{\wedge}} x_i, \overset{n}{\underset{i=1}{\wedge}} x_i, \cdots, \overset{n}{\underset{i=1}{\wedge}} x_i\right) \leq_L f(x_1, x_2, \cdots, x_n) \leq_L f\left(\overset{n}{\underset{i=1}{\vee}} x_i, \overset{n}{\underset{i=1}{\vee}} x_i, \cdots, \overset{n}{\underset{i=1}{\vee}} x_i\right).$$

It follows from the regularity of *f* that $\overset{n}{\underset{i=1}{\wedge}} x_i \leq_L f(x_1, x_2, \cdots, x_n) \leq_L \overset{n}{\underset{i=1}{\vee}} x_i$. □

The following theorems are simple and proofs are omitted.

Theorem 3.2. *Let* $f^0 \in AF_2(P)$ *and*

$$f(x_1, x_2, \cdots, x_n) = f^0(f^0(\cdots f^0(f^0(x_1, x_2), x_3)\cdots), x_n)$$

Then $f \in AF_n(P)$.

Note in Example 3.1(1), stepwise average complement-preserving aggregation function $f^{sa}(x_1, x_2, \cdots, x_n) = \frac{x_1 + x_2}{2^{n-1}} + \frac{x_3}{2^{n-2}} + \cdots + \frac{x_n}{2}$ are also thought to be a result of Theorem 3.2 as $f^0(x_1, x_2) = \frac{x_1 + x_2}{2}$.

Theorem 3.3. *Let* $f^0 \in AF_m(P)$, $f^{(k)} \in AF_n(P)$ $(k = 1, 2, \cdots, m)$ *and* $\forall x_i \in P$, $i \in \{1, 2, \cdots, n\}$,

$$f(x_1, x_2, \cdots, x_n) = f^0(f^{(1)}(x_1, x_2, \cdots, x_n), f^{(2)}(x_1, x_2, \cdots, x_n), \cdots, f^{(m)}(x_1, x_2, \cdots, x_n))$$

Then $f \in AF_n(P)$.

If we consider

$$f^{(1)}(x_1, x_2, x_3) = f^{aa}(x_1, x_2, x_3) = (x_1 + x_2 + x_3)/3,$$

$$f^{(2)}(x_1, x_2, x_3) = f^{sa}(x_1, x_2, x_3) = (x_1 + x_2)/4 + x_3/2 \quad \text{and}$$

$$f^{(0)}(x_1, x_2) = f^{aa}(x_1, x_2) = (x_1 + x_2)/2,$$

then it follows from Theorem 3.3 that

$$f(x_1, x_2, x_3) = f^0(f^{(1)}(x_1, x_2, x_3), f^{(2)}(x_1, x_2, x_3)) = \frac{7}{24}(x_1 + x_2) + \frac{5}{12}x_3$$

is a 3-ary complement-preserving aggregation function over [0, 1].

It is easy to derive from Theorem 3.3 and Example 3.1 the following two statements.

Let $f^{(k)} \in AF_n([0,1])$ $(k = 1, 2, \cdots, m)$. Then the following functions are *n*-ary complement-preserving aggregation functions over [0, 1].

(1) $f(x_1, x_2, \cdots, x_n) = \sum_{k=1}^{m} a_k f^{(k)}(x_1, x_2, \cdots, x_n)$, $x_1, x_2, \cdots, x_n \in [0,1]$, where $a_1, a_2, \cdots, a_n \in [0,1]$ and $\sum_{i=1}^{n} a_i = 1$.

(2) $$f(x_1, x_2, \cdots, x_n) = \frac{\max_k\{a_k \mathrm{T} f^{(k)}(x_1, x_2, \cdots, x_n)\} + \min_k\{(1 - a_k) \perp f^{(k)}(x_1, x_2, \cdots, x_n)\}}{2},$$

$x_1, x_2, \cdots, x_n \in [0,1]$, where $a_1, a_2, \cdots, a_n \in [0,1]$, $\max_i a_i = 1$, T and $\perp$ are t-norm and t-conorm over [0,1] respectively and are dual w.r.t $N_P(x) = 1 - x$.

The following discussions are some relationships between complement-preserving aggregation functions under different partially ordered sets.

Theorem 3.4. *Let* $f \in AF_n([0,1])$ *and*

$$f^{(2)}(x_1,x_2,\cdots,x_n)=\left[f(x_1^-,x_2^-,\cdots,x_n^-),f(x_1^+,x_2^+,\cdots,x_n^+)\right],\quad x_i=\left[x_i^-,x_i^+\right]\in I^{(2)}.$$

Then $f^{(2)} \in AF_2(I^{(2)})$.

Proof. Obviously $f^{(2)}(x_1,x_2,\cdots,x_n)$ satisfies regularity and nondecreasing property. In the following we only verify that it is complement-preserving. For any $x_i=\left[x_i^-,x_i^+\right]\in I^{(2)}$,

$$\begin{aligned}
f^{(2)}\left(\overline{1}-x_1,\overline{1}-x_2,\cdots,\overline{1}-x_n\right) &= f^{(2)}\left([1-x_1^+,1-x_1^-],[1-x_2^+,1-x_2^-],\cdots,[1-x_n^+,1-x_n^-]\right)\\
&=\left[f\left(1-x_1^+,1-x_2^+,\cdots,1-x_n^+\right),f\left(1-x_1^-,1-x_2^-,\cdots,1-x_n^-\right)\right]\\
&=\left[1-f\left(x_1^+,x_2^+,\cdots,x_n^+\right),1-f\left(x_1^-,x_2^-,\cdots,x_n^-\right)\right]\\
&=\overline{1}-\left[f\left(x_1^-,x_2^-,\cdots,x_n^-\right),f\left(x_1^+,x_2^+,\cdots,x_n^+\right)\right]\\
&=\overline{1}-f^{(2)}\left(x_1,x_2\cdots,x_n\right). \qquad \square
\end{aligned}$$

Theorem 3.4 tells us complement-preserving aggregation functions under $I^{(2)}$ can be constructed through complement-preserving aggregation functions under [0, 1].

On the other hand, in the following theorem, complement-preserving aggregation functions under [0, 1] can be also constructed through complement-preserving aggregation functions under $I^{(2)}$.

Theorem 3.5. *Let* $f^{(2)} \in AF_n(I^{(2)})$ *and*

$$f(x_1,x_2,\cdots,x_n)=\frac{1}{2}\left(\left(f^{(2)}([x_1,x_1],[x_2,x_2],\cdots,[x_n,x_n])\right)^-+\left(f^{(2)}([x_1,x_1],[x_2,x_2],\cdots,[x_n,x_n])\right)^+\right),$$
$$x_i\in[0,1].$$

Then $f \in AF_n([0,1])$.

Proof. Obviously *f* satisfies regularity and nondecreasing property. In the following, we only verify that it is complement-preserving. For any $x_i \in [0,1]$,

$$\begin{aligned}
&f(1-x_1,1-x_2,\cdots,1-x_n)\\
&=\frac{1}{2}\Big(\left(f^{(2)}([1-x_1,1-x_1],[1-x_2,1-x_2],\cdots,[1-x_n,1-x_n])\right)^-\\
&\qquad\qquad+\left(f^{(2)}([1-x_1,1-x_1],[1-x_2,1-x_2],\cdots,[1-x_n,1-x_n])\right)^+\Big)\\
&=\frac{1}{2}\left(\left(f^{(2)}(\overline{1}-[x_1,x_1],\overline{1}-[x_2,x_2],\cdots,\overline{1}-[x_n,x_n])\right)^-+\left(f^{(2)}(\overline{1}-[x_1,x_1],\overline{1}-[x_2,x_2],\cdots,\overline{1}-[x_n,x_n])\right)^+\right)\\
&=\frac{1}{2}\left(\left(\overline{1}-f^{(2)}([x_1,x_1],[x_2,x_2],\cdots,[x_n,x_n])\right)^-+\left(\overline{1}-f^{(2)}([x_1,x_1],[x_2,x_2],\cdots,[x_n,x_n])\right)^+\right)\\
&=\frac{1}{2}\left(1-\left(f^{(2)}([x_1,x_1],[x_2,x_2],\cdots,[x_n,x_n])\right)^++1-\left(f^{(2)}([x_1,x_1],[x_2,x_2],\cdots,[x_n,x_n])\right)^-\right)\\
&=1-\frac{1}{2}\left(\left(f^{(2)}([x_1,x_1],[x_2,x_2],\cdots,[x_n,x_n])\right)^-+\left(f^{(2)}([x_1,x_1],[x_2,x_2],\cdots,[x_n,x_n])\right)^+\right)\\
&=1-f(x_1,x_2,\cdots,x_n). \qquad \square
\end{aligned}$$

Example 3.2. Let

$$f^{(2)}\left([x_1^-,x_1^+],[x_2^-,x_2^+],\cdots,[x_n^-,x_n^+]\right)=\left[\min\{x_1^-,x_2^-,\cdots,x_n^-\},\max\{x_1^+,x_2^+,\cdots,x_n^+\}\right],\ \left[x_i^-,x_i^+\right]\in I^{(2)}.$$

Then $f^{(2)}\in AF_n(I^{(2)})$. In fact we only verify it is complement-preserving. For any $\left[x_i^-,x_i^+\right]\in I^{(2)}$,

$$\begin{aligned}
&f^{(2)}\left(\overline{1}-[x_1^-,x_1^+],\overline{1}-[x_2^-,x_2^+],\cdots,\overline{1}-[x_n^-,x_n^+]\right)\\
&=f^{(2)}\left([1-x_1^+,1-x_1^-],[1-x_2^+,1-x_2^-],\cdots,[1-x_n^+,1-x_n^-]\right)\\
&=\left[\min\left\{1-x_1^+,1-x_2^+,\cdots,1-x_n^+\right\},\max\left\{1-x_1^-,1-x_2^-,\cdots,1-x_n^-\right\}\right]\\
&=\left[1-\max\left\{x_1^+,x_2^+,\cdots,x_n^+\right\},1-\min\left\{x_1^-,x_2^-,\cdots,x_n^-\right\}\right]\\
&=\overline{1}-\left[\min\left\{x_1^-,x_2^-,\cdots,x_n^-\right\},\max\left\{x_1^+,x_2^+,\cdots,x_n^+\right\}\right]\\
&=\overline{1}-f^{(2)}\left([x_1^-,x_1^+],[x_2^-,x_2^+],\cdots,[x_n^-,x_n^+]\right).
\end{aligned}$$

3.2. Aggregation three-way decisions

The aggregation of multiple decision evaluation functions is a decision evaluation function. The following theorem confirms this.

Theorem 3.6. *Let* $(U,Map(V,P_C),P_D,E_i)$ *(* $i=1,2,\cdots,n$ *) be n three-way decision spaces,* $f\in AF_n(P_D)$ *and* $E^f(A)(x)=f\left(E_1(A)(x),E_2(A)(x),\cdots,E_n(A)(x)\right)$ *for* $A\in Map(V,P_C)$ *and* $x\in U$. *Then* $(U,Map(V,P_C),P_D,E^f)$ *is a three-way decision space.*

Proof. Axioms E1 and E2 are easy to verify. And for $A\in Map(V,P_C)$,

$$\begin{aligned}
E^f(N_{P_C}(A))(x)&=f\left(E_1(N_{P_C}(A))(x),E_2(N_{P_C}(A))(x),\cdots,E_n(N_{P_C}(A))(x)\right)\\
&=f\left(N_{P_D}(E_1(A))(x),N_{P_D}(E_2(A))(x),\cdots,N_{P_D}(E_n(A))(x)\right)\\
&=N_{P_D}\left(f\left(E_1(A)(x),E_2(A)(x),\cdots,E_n(A)(x)\right)\right)\\
&=N_{P_D}\left(E^f(A)\right)(x).
\end{aligned}$$

Namely $E^f(N_{P_C}(A))=N_{P_D}\left(E^f(A)\right)$. □

In $P_D=[0,1]$, E^f is written as E^{wa}, E^{sa}, E^{ma} and E^{me} if $f=f^{wa},f^{sa},f^{ma},f^{me}$ respectively, i.e. for $A\in Map(V,P_C)$ and $x\in[0,1]$,

$$E^{wa}(A)(x)=\sum_{i=1}^{n}a_iE_i(A)(x),\text{ where } a_1,a_2,\cdots,a_n\in[0,1] \text{ and } \sum_{i=1}^{n}a_i=1,$$

$$E^{sa}(A)(x)=\frac{E_1(A)(x)+E_2(A)(x)}{2^{n-1}}+\frac{E_3(A)(x)}{2^{n-2}}+\cdots+\frac{E_n(A)(x)}{2},$$

$$E^{ma}(A)(x)=\frac{\max_i\left(a_i\top E_i(A)(x)\right)+\min_i\left((1-a_i)\perp E_i(A)(x)\right)}{2}\text{ and }\max_i a_i=1,$$

$$E^{me}(A)(x)=Median\left\{E_i(A)(x)\right\}.$$

E^{wa}, E^{sa}, E^{ma} and E^{me} are called the weighted average, stepwise average, max-min average and median decision evaluation function respectively.

Through aggregation of multiple three-way decision spaces, it follows aggregation three-way decisions and the lower and upper approximations of the aggregation three-way decisions based on complement-preserving aggregation functions.

Let $(U, Map(V, P_C), P_D, E_i)$ $(i = 1, 2, \cdots, n)$ be n 3WD spaces, $f \in AF_n(P_D)$, $A \in Map(V, P_C)$ and $0 \le \beta < \alpha \le 1$. Then the aggregation three-way decisions over n three-way decision spaces are

(1) Acceptance region:

$$ACP^f_{(\alpha,\beta)}(E_{1\sim n}, A) = ACP_{(\alpha,\beta)}(E^f, A) = \left\{x \in U \mid E^f(A)(x) \ge \alpha\right\},$$

(2) Rejection region:

$$REJ^f_{(\alpha,\beta)}(E_{1\sim n}, A) = REJ_{(\alpha,\beta)}(E^f, A) = \left\{x \in U \mid E^f(A)(x) \le \beta\right\},$$

(3) Uncertain region:

$$UNC^f_{(\alpha,\beta)}(E_{1\sim n}, A) = U - ACP^f_{(\alpha,\beta)}(E_{1\sim n}, A) \cup REJ^f_{(\alpha,\beta)}(E_{1\sim n}, A).$$

If P_D is a linear order, then $UNC^f_{(\alpha,\beta)}(E_{1\sim n}, A) = \left\{x \in U \mid \beta < E^f(A)(x) < \alpha\right\}$.

The lower and upper approximations of the aggregation three-way decisions over n three-way decision spaces are respectively

$$\underline{apr}^f_{(\alpha,\beta)}(E_{1\sim n}, A) = ACP^f_{(\alpha,\beta)}(E_{1\sim n}, A) = \left\{x \in U \mid E^f(A)(x) \ge \alpha\right\}$$

and

$$\overline{apr}^f_{(\alpha,\beta)}(E_{1\sim n}, A) = \left(REJ^f_{(\alpha,\beta)}(E_{1\sim n}, A)\right)^c = \left\{x \in U \mid E^f(A)(x) > \beta\right\}.$$

3.3. Relationships among aggregation, the optimistic and pessimistic three-way decisions

In the following, we discuss properties of the aggregation three-way decisions and the lower and upper approximations, and relationships among the aggregation three-way decisions, the optimistic three-way decisions and the pessimistic three-way decisions.

Theorem 3.7. *Let $(U, Map(V, P_C), P_D, E_i)$ ($i = 1, 2, \cdots, n$) be n three-way decision spaces, $f \in AF_n(P_D)$ and satisfy (AF4), $A \in Map(V, P_C)$ and $0 \le \beta < \alpha \le 1$. Then the following statements hold.*

(1) $ACP^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq ACP^f_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq ACP^{op}_{(\alpha,\beta)}(E_{1\sim n}, A)$,

(2) $REJ^{op}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq REJ^f_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq REJ^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A)$.

Proof. Let $x \in ACP^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \bigcap_{i=1}^{n} ACP_{(\alpha,\beta)}(E_i, A)$. Then $\forall i$, $x \in ACP_{(\alpha,\beta)}(E_i, A)$, i.e. $E_i(A)(x) \ge \alpha$. So

$$\begin{aligned} E^f(A)(x) &= f(E_1(A)(x), E_2(A)(x), \cdots, E_n(A)(x)) \\ &\ge f(\alpha, \alpha, \cdots, \alpha) = \alpha, \end{aligned}$$

i.e. $x \in ACP^f_{(\alpha,\beta)}(E_{1\sim n}, A)$. If $x \in ACP^f_{(\alpha,\beta)}(E_{1\sim n}, A)$, then

$$E^f(A)(x) = f(E_1(A)(x), E_2(A)(x), \cdots, E_n(A)(x)) \ge \alpha.$$

Hence there is an i such that $E_i(A)(x) \ge \alpha$ due to the condition (AF4) of *f*, i.e. $x \in \bigcup_{i=1}^{n} \underline{apr}_{(\alpha,\beta)}(E_i, A) = ACP^{op}_{(\alpha,\beta)}(E_{1\sim n}, A)$.

The second relation of inclusion can be proved in a similar way. □

Theorem 3.8. *Let* $(U, Map(V, P_C), P_D, E_i)$ *(* $i=1,2,\cdots,n$ *) be* n *three-way decision spaces,* $f \in AF_n(P_D)$, $A, B \in Map(V, P_C)$ *and* $0 \le \beta < \alpha \le 1$. *Then*

$$\underline{apr}^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \bigcap_{i=1}^{n} \underline{apr}_{(\alpha,\beta)}(E_i, A) \subseteq \underline{apr}^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq \bigcup_{i=1}^{n} \underline{apr}_{(\alpha,\beta)}(E_i, A) = \underline{apr}^{op}_{(\alpha,\beta)}(E_{1\sim n}, A)$$

and

$$\overline{apr}^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \bigcap_{i=1}^{n} \overline{apr}_{(\alpha,\beta)}(E_i, A) \subseteq \overline{apr}^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq \bigcup_{i=1}^{n} \overline{apr}_{(\alpha,\beta)}(E_i, A) = \overline{apr}^{op}_{(\alpha,\beta)}(E_{1\sim n}, A).$$

Proof. It is straightforward from the definition of the lower and upper approximations of the aggregation three-way decisions and Theorem 3.7. □

Theorem 3.9. *Let* $(U, Map(V, P_C), P_D, E_i)$ *(* $i=1,2,\cdots,n$ *) be* n *three-way decision spaces,* $f \in AF_n(P_D)$, $A \in Map(V, P_C)$. *If* $0 \le \beta \le \beta' < \alpha' \le \alpha \le 1$, *then*

$$\underline{apr}^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq \underline{apr}^{f}_{(\alpha',\beta')}(E_{1\sim n}, A) \quad and$$

$$\overline{apr}^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \supseteq \overline{apr}^{f}_{(\alpha',\beta')}(E_{1\sim n}, A).$$

Proof. It is straightforward from the definition of the lower and upper approximations of the aggregation three-way decisions. □

3.4. Illustrative example

Here, we use an example to illustrate some notions on the aggregation of multiple three-way decision spaces.

Example 3.3. Let $U = \{x_1, x_2, \cdots, x_8\}$, $P_C = I^{(2)}$, $P_D = [0,1]$, an interval-valued fuzzy set over U is

$$A = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{[0.5, 0.6]}{x_4} + \frac{[0.4, 0.6]}{x_5} + \frac{[0.6, 0.8]}{x_6} + \frac{[0.8, 1]}{x_7} + \frac{[0.2, 0.4]}{x_8} \text{ and}$$

$U/R = \{\{x_1, x_8\}, \{x_2\}, \{x_3\}, \{x_4, x_5\}, \{x_6\}, \{x_7\}\}$ is a classification of U based on an equivalence relation (or an attribute) R.

If follows from $E_1(A)(x) = \frac{|A^- \cap [x]_R|}{|[x]_R|}$, $E_2(A)(x) = \frac{|A^+ \cap [x]_R|}{|[x]_R|}$, $E_3(A)(x) = A^-(x)$ and $E_4(A)(x) = A^+(x)$ that

$$E_1(A) = \frac{0.6}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{0.45}{x_4} + \frac{0.45}{x_5} + \frac{0.6}{x_6} + \frac{0.8}{x_7} + \frac{0.6}{x_8},$$

$$E_2(A) = \frac{0.7}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5} + \frac{0.8}{x_6} + \frac{1}{x_7} + \frac{0.7}{x_8},$$

$$E_3(A) = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{0.5}{x_4} + \frac{0.4}{x_5} + \frac{0.6}{x_6} + \frac{0.8}{x_7} + \frac{0.2}{x_8},$$

$$E_4(A) = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5} + \frac{0.8}{x_6} + \frac{1}{x_7} + \frac{0.4}{x_8}.$$

Then

$$E^{wa}(A) = \frac{0.825}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{0.5375}{x_4} + \frac{0.5125}{x_5} + \frac{0.7}{x_6} + \frac{0.9}{x_7} + \frac{0.475}{x_8},$$

where the weight vector $(a_1, a_2, a_3, a_4) = (0.25, 0.25, 0.25, 0.25)$,

$$E^{sa}(A) = \frac{0.9125}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{0.55625}{x_4} + \frac{0.53125}{x_5} + \frac{0.725}{x_6} + \frac{0.925}{x_7} + \frac{0.4125}{x_8},$$

$$E^{ma}(A) = \frac{0.8}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{0.525}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6} + \frac{0.9}{x_7} + \frac{0.45}{x_8}, \text{ (take } a_1 = a_2 = a_3 = a_4 = 1\text{)},$$

$$E^{me}(A) = \frac{0.85}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{0.55}{x_4} + \frac{0.525}{x_5} + \frac{0.7}{x_6} + \frac{0.9}{x_7} + \frac{0.5}{x_8}.$$

Consider $\beta = 0.6$ and $\alpha = 0.9$, then weighted average three-way decisions over three-way decision spaces are given as follows:

Acceptance region: $ACP^{wa}_{(0.9,0.6)}(E_{1\sim4}, A) = \{x \in U \mid E^{wa}(A)(x) \geq 0.9\} = \{x_2, x_3, x_7\}$,

Rejection region: $REJ^{wa}_{(0.9,0.6)}(E_{1\sim4}, A) = \{x \in U \mid E^{wa}(A)(x) \leq 0.6\} = \{x_4, x_5, x_8\}$;

Uncertain region: $UNC^{wa}_{(0.9,0.6)}(E_{1\sim4}, A) = \left(ACP^{wa}_{(0.9,0.6)}(E_{1\sim4}, A) \cup REJ^{wa}_{(0.9,0.6)}(E_{1\sim4}, A)\right)^c$

$= \left\{x \in U \mid 0.6 < E^{wa}(A)(x) < 0.9\right\} = \{x_1, x_6\}$.

The lower and upper approximations of A are

$\underline{apr}^{wa}_{(0.9,0.6)}(E_{1\sim4}, A) = ACP^{wa}_{(0.9,0.6)}(E_{1\sim4}, A) = \{x_2, x_3, x_7\}$ and

$\overline{apr}^{wa}_{(0.9,0.6)}(E_{1\sim4}, A) = \left(REJ^{wa}_{(0.9,0.6)}(E_{1\sim4}, A)\right)^c = \{x_1, x_2, x_3, x_6, x_7\}$.

In the following 4 three-way decision spaces, if we consider three groups of different parameters α, β, the aggregation three-way decisions over four three-way decision spaces, the lower and upper approximations of A are listed in Table 3.1.

Table 3.1

The aggregation three-way decisions, the lower and upper approximations of A for different α, β.

	$\beta = 0.6, \alpha = 0.9$	$\beta = 0.7, \alpha = 0.8$	$\beta = 0.55, \alpha = 0.7$
$ACP^{wa}_{(\alpha,\beta)}(E_{1\sim4}, A)$	$\{x_2, x_3, x_7\}$	$\{x_1, x_2, x_3, x_7\}$	$\{x_1, x_2, x_3, x_6, x_7\}$
$ACP^{sa}_{(\alpha,\beta)}(E_{1\sim4}, A)$	$\{x_1, x_2, x_3, x_7\}$	$\{x_1, x_2, x_3, x_7\}$	$\{x_1, x_2, x_3, x_6, x_7\}$
$ACP^{ma}_{(\alpha,\beta)}(E_{1\sim4}, A)$	$\{x_2, x_3, x_7\}$	$\{x_1, x_2, x_3, x_7\}$	$\{x_1, x_2, x_3, x_6, x_7\}$
$ACP^{me}_{(\alpha,\beta)}(E_{1\sim4}, A)$	$\{x_2, x_3, x_7\}$	$\{x_1, x_2, x_3, x_7\}$	$\{x_1, x_2, x_3, x_6, x_7\}$
$REJ^{wa}_{(\alpha,\beta)}(E_{1\sim4}, A)$	$\{x_4, x_5, x_8\}$	$\{x_4, x_5, x_6, x_8\}$	$\{x_4, x_5, x_8\}$
$REJ^{sa}_{(\alpha,\beta)}(E_{1\sim4}, A)$	$\{x_4, x_5, x_8\}$	$\{x_4, x_5, x_8\}$	$\{x_5, x_8\}$
$REJ^{ma}_{(\alpha,\beta)}(E_{1\sim4}, A)$	$\{x_4, x_5, x_8\}$	$\{x_4, x_5, x_6, x_8\}$	$\{x_4, x_5, x_8\}$
$REJ^{me}_{(\alpha,\beta)}(E_{1\sim4}, A)$	$\{x_4, x_5, x_8\}$	$\{x_4, x_5, x_6, x_8\}$	$\{x_4, x_5, x_8\}$
$UNC^{wa}_{(\alpha,\beta)}(E_{1\sim4}, A)$	$\{x_1, x_6\}$	$\varnothing$	$\varnothing$
$UNC^{sa}_{(\alpha,\beta)}(E_{1\sim4}, A)$	$\{x_6\}$	$\{x_6\}$	$\{x_4\}$
$UNC^{ma}_{(\alpha,\beta)}(E_{1\sim4}, A)$	$\{x_1, x_6\}$	$\varnothing$	$\varnothing$
$UNC^{me}_{(\alpha,\beta)}(E_{1\sim4}, A)$	$\{x_6\}$	$\{x_6\}$	$\varnothing$
$\underline{apr}^{wa}_{(\alpha,\beta)}(E_{1\sim4}, A)$	$\{x_2, x_3, x_7\}$	$\{x_1, x_2, x_3, x_7\}$	$\{x_1, x_2, x_3, x_6, x_7\}$

$\underline{apr}^{sa}_{(\alpha,\beta)}(E_{1\sim4},A)$	$\{x_1,x_2,x_3,x_7\}$	$\{x_1,x_2,x_3,x_7\}$	$\{x_1,x_2,x_3,x_6,x_7\}$
$\underline{apr}^{ma}_{(\alpha,\beta)}(E_{1\sim4},A)$	$\{x_2,x_3,x_7\}$	$\{x_1,x_2,x_3,x_7\}$	$\{x_1,x_2,x_3,x_6,x_7\}$
$\underline{apr}^{me}_{(\alpha,\beta)}(E_{1\sim4},A)$	$\{x_2,x_3,x_7\}$	$\{x_1,x_2,x_3,x_7\}$	$\{x_1,x_2,x_3,x_6,x_7\}$
$\overline{apr}^{wa}_{(\alpha,\beta)}(E_{1\sim4},A)$	$\{x_1,x_2,x_3,x_6,x_7\}$	$\{x_1,x_2,x_3,x_7\}$	$\{x_1,x_2,x_3,x_6,x_7\}$
$\overline{apr}^{sa}_{(\alpha,\beta)}(E_{1\sim4},A)$	$\{x_1,x_2,x_3,x_6,x_7\}$	$\{x_1,x_2,x_3,x_6,x_7\}$	$\{x_1,x_2,x_3,x_4,x_6,x_7\}$
$\overline{apr}^{ma}_{(\alpha,\beta)}(E_{1\sim4},A)$	$\{x_1,x_2,x_3,x_6,x_7\}$	$\{x_1,x_2,x_3,x_7\}$	$\{x_1,x_2,x_3,x_6,x_7\}$
$\overline{apr}^{me}_{(\alpha,\beta)}(E_{1\sim4},A)$	$\{x_1,x_2,x_3,x_6,x_7\}$	$\{x_1,x_2,x_3,x_7\}$	$\{x_1,x_2,x_3,x_6,x_7\}$

From Table 3.1, we have the following observations.

(1) Acceptance regions contain decision objects x_2,x_3 and x_7 for all different parameters and transformation methods.

(2) Rejection regions contain decision objects x_4,x_5 and x_8 for almost all different parameters and transformation methods.

(3) Decision object x_6 appears most frequently in the uncertain region for all different parameters and transformation methods.

4. The aggregations of multiple three-way decisions with a pair of evaluation functions

4.1 Three-way decisions with a pair of evaluation functions

Depending upon the number of evaluation functions, Yao gave two modes of three-way decisions, which are the single evaluation function and dual evaluation functions [37]. Hu discussed three-way decisions with a pair of evaluation functions through two three-way decision spaces [7]. In the following, we discuss the aggregation of three-way decisions over two groups of multiple three-way decision spaces.

Definition 4.1. Let $(U,Map(V,P_C),P_D,E_i)$ ($i=1,2,\cdots,m$) and $(U,Map(V,P_C),P_D,F_j)$ ($j=1,2,\cdots,n$) be two groups of multiple three-way decision spaces, $f\in AF_m(P_D)$, $g\in AF_n(P_D)$, $A,B\in Map(V,P_C)$ and $\alpha,\beta\in P_D$. Then *three-way decisions over two groups of multiple three-way decision spaces* are defined as follows.

(1) Acceptance region：

$$ACP^{f,g}_{(\alpha,\beta)}((E_{1\sim m},F_{1\sim n}),(A,B))=\left\{x\in U\mid E^f(A)(x)\geq\alpha\right\}\cap\left\{x\in U\mid F^g(B)(x)<\beta\right\}$$
$$=ACP_{(\alpha,\beta)}(E^f,A)\cap REJ_{(\alpha,\beta)}(F^g,B).$$

(2) Rejection region：

$$REJ^{f,g}_{(\alpha,\beta)}((E_{1\sim m},F_{1\sim n}),(A,B))=\left\{x\in U\mid E^f(A)(x)<\alpha\right\}\cap\left\{x\in U\mid F^g(B)(x)\geq\beta\right\}$$
$$=REJ_{(\alpha,\beta)}(E^f,A)\cap ACP_{(\alpha,\beta)}(F^g,B).$$

(3) Uncertain region：

$$UNC^{f,g}_{(\alpha,\beta)}((E_{1\sim m},F_{1\sim n}),(A,B))=\left(ACP^{f,g}_{(\alpha,\beta)}((E_{1\sim m},F_{1\sim n}),(A,B))\cup REJ^{f,g}_{(\alpha,\beta)}((E_{1\sim m},F_{1\sim n}),(A,B))\right)^c.$$

Definition 4.2. Let $(U, Map(V, P_C), P_D, E_i)$ ($i = 1, 2, \cdots, m$) and $(U, Map(V, P_C), P_D, F_j)$ ($j = 1, 2, \cdots, n$) be two groups of multiple 3WD spaces, $f \in AF_m(P_D)$, $g \in AF_n(P_D)$, $A, B \in Map(V, P_C)$ and $\alpha, \beta \in P_D$. Then

$$\underline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (A, B)) = ACP^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (A, B))$$

and

$$\begin{aligned}\overline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (A, B))) &= \left(REJ^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (A, B))\right)^c \\ &= \left\{x \in U \mid E^{f,g}(A)(x) \geq \alpha\right\} \cup \left\{x \in U \mid F^{f,g}(B)(x) < \beta\right\}\end{aligned}$$

are referred to as *the lower approximation and upper approximation* of (*A*, *B*) with regard to three-way decisions over two groups of multiple three-way decision spaces respectively.

Theorem 4.1. *Let* $(U, Map(V, P_C), P_D, E_i)$ *(* $i = 1, 2, \cdots, m$ *) and* $(U, Map(V, P_C), P_D, F_j)$ *(* $j = 1, 2, \cdots, n$ *) be two groups of multiple three-way decision spaces,* $f \in AF_m(P_D)$ *,* $g \in AF_n(P_D)$ *,* $A, B \in Map(V, P_C)$ *and* $\alpha, \beta \in P_D$ *. Then the following hold.*

(1) $\underline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (A, B)) \subseteq \overline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (A, B))$.

(2) $\underline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (U, \varnothing)) = U$, $\overline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (\varnothing, U)) = \varnothing$.

(3) $\left(\underline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (A, B))\right)^c = \overline{apr}^{f,g}_{(\beta,\alpha)}((F_{1\sim n}, E_{1\sim m}), (B, A))$.

(4) *If* $A \subseteq C, B \supseteq D$ *, then*

$\underline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (A, B)) \subseteq \underline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (C, D))$ *, and*

$\overline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (A, B)) \subseteq \overline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (C, D))$.

Proof. Propositions (2) and (4) are immediate from Definition 4.1 and Definition 4.2. The proofs of Propositions (1) and (3) are given as follows.

(1)
$$\begin{aligned}\overline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (A, B)) &= \left(REJ^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (A, B))\right)^c \\ &= \left(\{x \in U \mid E^f(A)(x) < \alpha\} \cap \{x \in U \mid F^g(B)(x) \geq \beta\}\right)^c \\ &= \left(\{x \in U \mid E^f(A)(x) < \alpha\}\right)^c \cup \left(\{x \in U \mid F^g(B)(x) \geq \beta\}\right)^c \\ &\supseteq \left\{x \in U \mid E^f(A)(x) \geq \alpha\right\} \cap \left\{x \in U \mid F^g(B)(x) < \beta\right\} \\ &= \underline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (A, B)).\end{aligned}$$

(3)
$$\begin{aligned}\left(\underline{apr}^{f,g}_{(\alpha,\beta)}((E_{1\sim m}, F_{1\sim n}), (A, B))\right)^c &= \left(\left\{x \in U \mid E^f(A)(x) \geq \alpha\right\} \cap \left\{x \in U \mid F^g(B)(x) < \beta\right\}\right)^c \\ &= \left(REJ^f_{(\beta,\alpha)}((F_{1\sim n}, E_{1\sim m}), (B, A))\right)^c \\ &= \overline{apr}^{f,g}_{(\beta,\alpha)}((F_{1\sim n}, E_{1\sim m}), (B, A)). \qquad \square\end{aligned}$$

4.2. Illustrative example

Here, we use an example to illustrate some notions on the aggregation of multiple three-way decisions with a pair of evaluation functions.

Example 4.1. Let $U=\{x_1,x_2,\cdots,x_8\}$ be a universe consisting of 8 students and their classification on examination scores of a course is $U/R=\{\{x_1,x_2,x_3,x_7\},\{x_4,x_5,x_8\},\{x_6\}\}$ where students x_1, x_2, x_3 and x_7 obtain grade A, x_4, x_5 and x_8 obtain grade B and x_6 obtains grade C. Their presentation scores of this course are

$$S_p=\frac{1}{x_1}+\frac{0.7}{x_2}+\frac{0.9}{x_3}+\frac{0.6}{x_4}+\frac{0.6}{x_5}+\frac{0.8}{x_6}+\frac{1}{x_7}+\frac{0.3}{x_8}$$

and their rates of absence for this course are

$$S_a=\frac{0.1}{x_1}+\frac{0.3}{x_2}+\frac{0.2}{x_3}+\frac{0.5}{x_4}+\frac{0.6}{x_5}+\frac{0.1}{x_6}+\frac{0.2}{x_7}+\frac{0.4}{x_8}.$$

In the following, we give students' evaluation on the course by considering decision evaluation functions $E_1(S_p)(x)=\frac{|S_p\cap[x]_R|}{|[x]_R|}$, $E_2(S_p)(x)=S_p(x)$, $F_1(S_a)(x)=\frac{|S_a\cap[x]_R|}{|[x]_R|}$ and $F_2(S_a)(x)=S_a(x)$. By computing, we have

$$E_1(S_p)=\frac{0.9}{x_1}+\frac{0.9}{x_2}+\frac{0.9}{x_3}+\frac{0.5}{x_4}+\frac{0.5}{x_5}+\frac{0.8}{x_6}+\frac{0.9}{x_7}+\frac{0.5}{x_8},$$

$$E_2(S_p)=\frac{1}{x_1}+\frac{0.7}{x_2}+\frac{0.9}{x_3}+\frac{0.6}{x_4}+\frac{0.6}{x_5}+\frac{0.8}{x_6}+\frac{1}{x_7}+\frac{0.3}{x_8},$$

$$F_1(S_a)=\frac{0.2}{x_1}+\frac{0.2}{x_2}+\frac{0.2}{x_3}+\frac{0.5}{x_4}+\frac{0.5}{x_5}+\frac{0.1}{x_6}+\frac{0.2}{x_7}+\frac{0.25}{x_8},$$

$$F_2(S_a)=\frac{0.1}{x_1}+\frac{0.3}{x_2}+\frac{0.2}{x_3}+\frac{0.5}{x_4}+\frac{0.6}{x_5}+\frac{0.1}{x_6}+\frac{0.2}{x_7}+\frac{0.4}{x_8}.$$

Considering that E_1 and E_2 (F_1 and F_2) are equally important, we choose 2-ary complement-preserving aggregation functions $f(x,y)=\frac{x+y}{2}$ and $g(x,y)=\frac{(0.5\wedge x)\vee y+(0.5\vee x)\wedge y}{2}$. Then

$$E^f(S_p)=\frac{0.95}{x_1}+\frac{0.8}{x_2}+\frac{0.9}{x_3}+\frac{0.55}{x_4}+\frac{0.55}{x_5}+\frac{0.8}{x_6}+\frac{0.95}{x_7}+\frac{0.4}{x_8}, \text{ and}$$

$$F^g(S_a)=\frac{0.15}{x_1}+\frac{0.3}{x_2}+\frac{0.2}{x_3}+\frac{0.5}{x_4}+\frac{0.55}{x_5}+\frac{0.1}{x_6}+\frac{0.2}{x_7}+\frac{0.4}{x_8}.$$

Consider $\alpha=0.6$ and $\beta=0.3$, then three-way decisions over two groups of multiple three-way decision spaces are given as follows:

Acceptance region：

$$ACP^{f,g}_{(0.6,0.2)}((E_{1,2},F_{1,2}),(S_p,S_a))=\{x\in U\mid E^f(S_p)(x)\geq 0.6\}\cap\{x\in U\mid F^g(S_a)(x)<0.3\}$$
$$=\{x_1,x_3,x_6,x_7\},$$

Rejection region：

$$REJ^{f,g}_{(0.6,0.2)}((E_{1,2},F_{1,2}),(S_p,S_a))=\{x\in U\mid E^f(S_p)(x)<0.6\}\cap\{x\in U\mid F^g(S_a)(x)\geq 0.3\}$$
$$=\{x_4,x_5,x_8\};$$

Uncertain region：

$$UNC^{f,g}_{(0.6,0.2)}((E_{1,2},F_{1,2}),(S_p,S_a))=\left(ACP^{f,g}_{(0.6,0.2)}((E_{1,2},F_{1,2}),(S_p,S_a))\cup REJ^{f,g}_{(0.6,0.2)}((E_{1,2},F_{1,2}),(S_p,S_a))\right)^c$$
$$=\{x_2\}.$$

The three regions tell us students x_1, x_3, x_6 and x_7 pass the exam of this course; students x_4, x_5, and x_8 no pass; student x_2 cannot be determined to be or not pass, and must be further examined on this course.

The lower and upper approximations of (S_p, S_a) with regard to three-way decisions over two groups of multiple three-way decision spaces are

$$\underline{apr}^{f,g}_{(0.6,0.2)}((E_{1,2},F_{1,2}),(S_p,S_a)) = ACP^{f,g}_{(0.6,0.2)}((E_{1,2},F_{1,2}),(S_p,S_a)) = \{x_1,x_3,x_6,x_7\} \text{ and}$$

$$\overline{apr}^{f,g}_{(0.6,0.2)}((E_{1,2},F_{1,2}),(S_p,S_a)) = \left(REJ^{f,g}_{(0.6,0.2)}((E_{1,2},F_{1,2}),(S_p,S_a))\right)^c = \{x_1,x_2,x_3,x_6,x_7\}.$$

The lower approximation of (S_p, S_a) contains students who are sure to pass the exam. Students in the upper approximation of (S_p, S_a) may pass the exam.

5. Conclusions

This paper presents several transformation methods from multiple three-way decision spaces to single three-way decisions. Aggregated three-way decisions can be made by single three-way decisions. Main conclusions in this paper and future work are listed as follows.

(1) The existing work only considers the two methods in multiple three-way decision spaces, the optimistic method and the pessimistic method. This paper presents several transformation methods from multiple three-way decision spaces to single three-way decision space.

(2) Transformation methods are presented based on axiomatic complement-preserving aggregation functions.

(3) This paper gives some methods of construction of complement-preserving aggregation functions. Especially in [0, 1] this paper demonstrates many examples on complement-preserving aggregation functions such as the weighted average method, max-min average method and median method etc.

(4) These methods are generalized to three-way decisions over two groups of multiple three-way decision spaces.

(5) We may consider single three-way decisions and multiple three-way decisions through Yao's view on the two sides of the theory of rough sets [41]. If so, it makes more sense to transformation methods proposed in this paper.

(6) We may consider potential industrial applications of the theory of aggregated three-way decision spaces, such as data mining [4], prediction [12], attribute reduction [13], pattern recognition [14], social networks [26], granular computing [31, 32], clustering [43] and so on.

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