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MAXIMAL L^p ANALYSIS OF FINITE ELEMENT SOLUTIONS FOR PARABOLIC EQUATIONS WITH NONSMOOTH COEFFICIENTS IN CONVEX POLYHEDRA

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ABSTRACT. The paper is concerned with Galerkin finite element solutions of parabolic equations in a convex polygon or polyhedron with a diffusion coefficient in $W^{1,N+\alpha}$ for some $\alpha > 0$, where N denotes the dimension of the domain. We prove the analyticity of the semigroup generated by the discrete elliptic operator, the discrete maximal L^p regularity and the optimal L^p error estimate of the finite element solution for the parabolic equation.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ (with $N = 2$ or $N = 3$), and let S_h be a finite element subspace of $H_0^1(\Omega)$ consisting of continuous piecewise polynomials of degree $r \geq 1$ subject to certain quasi-uniform triangulation of the domain Ω. We consider the parabolic equation

$$\begin{cases} \partial_t u - \nabla \cdot (a\nabla u) = f & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u^0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

and its finite element approximation

$$\begin{cases} \big(\partial_t u_h, v_h\big) + (a\nabla u_h, \nabla v_h) = (f, v_h), \ \forall\, v_h \in S_h, \\ u_h(0) = u_h^0, \end{cases} \tag{1.2}$$

where f is a given function, and $a = (a_{ij}(x))_{N\times N}$ is an $N \times N$ symmetric matrix which satisfies the ellipticity condition

$$\Lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \text{for } x \in \Omega, \tag{1.3}$$

for some positive constant Λ.

If we define the elliptic operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ and its finite element approximation $A_h : S_h \to S_h$ by

$$(Aw, v) := (a\nabla w, \nabla v), \qquad \forall\, w, v \in H_0^1(\Omega), \tag{1.4}$$

$$(A_h w_h, v_h) := (a\nabla w_h, \nabla v_h), \qquad \forall\, w_h, v_h \in S_h, \tag{1.5}$$

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then the solutions of (1.1) and (1.2) can be expressed by

$$u(t) = E(t)u^0 + \int_0^t E(t-s)f(s)\mathrm{d}s, \tag{1.6}$$

$$u_h(t) = E_h(t)u_h^0 + \int_0^t E_h(t-s)f(s)\mathrm{d}s, \tag{1.7}$$

where $\{E(t) = e^{-tA}\}_{t>0}$ and $\{E_h(t) = e^{-tA_h}\}_{t>0}$ denote the semigroups generated by the operators $-A$ and $-A_h$, respectively. By the theory of parabolic equations and [33], it is well known that $\{E(t)\}_{t>0}$ is an analytic semigroup on $C_0(\overline{\Omega})$ satisfying

$$\|E(t)v\|_{L^\infty} + t\|\partial_t E(t)v\|_{L^\infty} \leq C\|v\|_{L^\infty}, \quad \forall\, v \in C_0(\overline{\Omega}),\ \forall\, t > 0, \tag{1.8}$$

which is equivalent to the resolvent estimate

$$\|(\lambda + A)^{-1}v\|_{L^\infty} \leq C\lambda^{-1}\|v\|_{L^\infty}, \quad \forall\, v \in C_0(\overline{\Omega}),\ \forall\, \lambda \in \Sigma_{\theta+\pi/2},$$

where $\Sigma_{\theta+\pi/2} := \{z \in \mathbb{C} : |\arg(z)| < \theta + \pi/2\}$. The counterparts of these two inequalities above for the discrete finite element operator A_h are the analyticity of the semigroup $\{E_h(t)\}_{t>0}$ on $L^\infty \cap S_h$:

$$\|E_h(t)v_h\|_{L^\infty} + t\|\partial_t E_h(t)v_h\|_{L^\infty} \leq C\|v_h\|_{L^\infty}, \quad \forall\, v_h \in S_h, \forall\, t > 0, \tag{1.9}$$

and the resolvent estimate

$$\|(\lambda + A_h)^{-1}v_h\|_{L^\infty} \leq C\lambda^{-1}\|v_h\|_{L^\infty}, \quad \forall\, v_h \in S_h,\ \forall\, \lambda \in \Sigma_{\varphi+\pi/2}.$$

The estimates of the discrete semigroup have attracted much attention in the past several decades. With these estimates, one may reach more precise analyses of finite element solutions, such as maximum-norm analysis of FEMs [31, 45, 46, 48], error estimates of fully discrete FEMs [30, 34, 45] and the discrete maximal L^p regularity for parabolic finite element equations [14, 15, 22, 25, 27].

The proof of (1.9) dates back to Schatz et. al. [38], who proved (1.9) with a logarithmic factor for the heat equation in a two-dimensional smooth convex domain with the linear finite element method. The logarithmic factor was removed in the case $r \geq 4$ for $N = 1, 2, 3$ in [32], and the analysis was further extended to the case $1 \leq N \leq 5$ in [4]. Later, a unified approach was presented in [39] by Schatz et. al., where they proved (1.9) with the Neumann boundary condition for all $r \geq 1$ and $N \geq 1$. The result was extended to the Dirichlet boundary condition in [47] for the linear finite element method. Some other maximum-norm error estimates can be found in [7, 8, 11, 20, 24, 28], and the resolvent estimates can be found in [1, 2].

A related topic is the discrete maximal L^p regularity (when $u^0 = 0$ and $1 < p, q < \infty$)

$$\|\partial_t u_h\|_{L^p((0,T);L^q)} + \|A_h u_h\|_{L^p((0,T);L^q)} \leq C_{p,q}\|f\|_{L^p((0,T);L^q)}, \tag{1.10}$$

which resembles the maximal L^p regularity of the continuous parabolic problem and was proved by Geissert [14, 15]. A straightforward application of (1.10) is the L^p-norm error estimate

$$\|P_h u - u_h\|_{L^p((0,T);L^q)} \leq C_{p,q}(\|P_h u^0 - u_h^0\|_{L^q} + \|P_h u - R_h u\|_{L^p((0,T);L^q)}), \tag{1.11}$$

where R_h is the Ritz projection associated with the operator A and P_h is the L^2 projection onto the finite element space.

All these estimates were established under the assumption that the coefficients a_{ij} and the domain Ω are smooth enough so that the parabolic Green's function satisfies

(1.12)
$$|\partial_t^\gamma \partial_x^\beta G(t,x,y)| \leq C(t^{1/2}+|x-y|)^{-(N+2\gamma+|\beta|)}e^{-\frac{|x-y|^2}{Ct}}, \ \forall\ 0\leq\gamma\leq 2,\ 0\leq|\beta|\leq 2\,.$$

Although the condition on the coefficients was relaxed to $a_{ij}\in C^{2+\alpha}(\overline{\Omega})$ in [14], this assumption is still too strong for many physical applications. One of the examples is an incompressible miscible flow in porous media [9, 26], where the diffusion-dispersion tensor $[a_{ij}]_{i,j=1}^N$ is only a Lipschitz continuous function of the velocity field. In a recent work [25], the first author proved (1.9) in a smooth domain under the assumption $a_{ij}\in W^{1,\infty}(\Omega)$, together with the estimate (when $u^0=0$ and $1<p,q<\infty$)

$$\|u_h\|_{L^p((0,T);W^{1,q})}\leq C_{p,q}\|f\|_{L^p((0,T);W^{-1,q})}, \tag{1.13}$$

which were then applied to the incompressible miscible flow in porous media [27]. Moreover, the problem in a polygon or a polyhedron is of high interest in practical cases, while the inequality (1.12) does not hold in arbitrary convex polygons or polyhedra, and all the analyses of (1.10)-(1.13) are limited to smooth domains so far. For the problem in two-dimensional polygons with constant coefficients, the inequality (1.9) with an extra logarithmic factor was proved in [3, 35, 45] by using the following estimate of the discrete Green's function Γ_h:

$$\int_\Omega |\Gamma_h(t,x,x_0)|dx\leq C|\ln h|\,.$$

The corresponding results in three-dimensional polyhedra are unknown. More interested is whether these stability estimates hold with the natural regularity $a_{ij}\in W^{1,p}(\Omega)$ for some $1<p<\infty$, since such estimates are important for the extension of the analysis to a general nonlinear model.

This paper focuses on (1.9)-(1.10) and (1.13) in a convex polygon or polyhedron with a weaker regularity of the diffusion coefficient. Instead of estimating Γ_h directly, we present a more precise estimate for the error function $F:=\Gamma_h-\Gamma$ (see Lemma 2.2) with which the logarithmic factor can be removed (this idea was used in [39]), where Γ is a regularized Green's function. To compensate the lack of pointwise estimate of the second-order derivatives of the Green's function, we use local $W^{1,\infty}$ estimate and local energy estimates of the second-order derivatives (see Lemma 4.1). Our main result is the following theorem.

Theorem 1.1. *Assume that $a_{ij}\in W^{1,N+\alpha}(\Omega)$ for some $\alpha>0$, satisfying the condition* (1.3)*, and assume that Ω is either a convex polygon in $\mathbb{R}^2$ or a convex polyhedron in $\mathbb{R}^3$. Then*

(1) the semigroup estimate (1.9) *holds,*
(2) the solution of (1.2) *satisfies* (1.10) *when $f\in L^p((0,T);L^q)$ and $u^0=0$,*
(3) the solution of (1.2) *satisfies* (1.13) *when $f\in L^p((0,T);W^{-1,q})$ and $u^0=0$.*

Under the assumptions in Theorem 1.1 and assuming that the solution of (1.1) satisfies $u\in C(\overline{\Omega}\times[0,T])$, (1.11) follows immediately from (1.10).

The rest of this paper is organized as follows. In section 2, we introduce some notations and present a key lemma based on which our main theorem can be proved. In section 3, we present superapproximation results for smoothly truncated finite element functions and present several estimates for the parabolic Green's functions under the assumed regularity of the coefficients and the domain. Based on these estimates, we prove our key lemma in section 4.

2. Notations, assumptions and sketch of the proof

2.1. **Notations.** For any nonnegative integer k and $1 \le p \le \infty$, we let $W^{k,p}(\Omega)$ be the conventional Sobolev space of functions defined in Ω, and let $W_0^{1,p}(\Omega)$ be the subspace of $W^{1,p}(\Omega)$ consisting of functions whose traces vanish on $\partial\Omega$. As conventions, we denote the dual space of $W_0^{1,p}(\Omega)$ by $W^{-1,p'}(\Omega)$ for $1 \le p < \infty$, and denote $H^k(\Omega) := W^{k,2}(\Omega)$ and $L^p(\Omega) := W^{0,p}(\Omega)$ for any integer k and $1 \le p \le \infty$.

Let $Q_T := \Omega \times (0,T)$. For any Banach space X and a given $T > 0$, we let $L^p((0,T);X)$ be the Bochner spaces equipped with the norm

$$\|f\|_{L^p((0,T);X)} = \begin{cases} \left(\int_0^T \|f(t)\|_X^p dt\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname*{ess\ sup}_{t\in(0,T)} \|f(t)\|_X. & p = \infty, \end{cases}$$

To simplify notations, in the following sections, we write L^p, H^k and $W^{k,p}$ as the abbreviations of $L^p(\Omega)$, $H^k(\Omega)$ and $W^{k,p}(\Omega)$, respectively, and denote by $(\cdot\,,\cdot)$ the inner product in $L^2(\Omega)$. For any subdomain $Q \subset Q_T$, we define

$$Q^t := \{x \in \Omega :\ (x,t) \in Q\},$$

$$\|f\|_{L^{\infty,2}(Q)} := \operatorname*{ess\ sup}_{t\in(0,T)} \|f(\cdot,t)\|_{L^2(Q^t)},$$

$$\|f\|_{L^p(Q)} := \left(\iint_Q |f(x,t)|^p \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}, \quad \forall\, 1 \le p < \infty,$$

and denote $w(t) = w(\cdot,t)$ for any function w defined on Q_T.

We assume that Ω is partitioned into quasi-uniform triangular elements τ_l^h, $l = 1,\cdots,L$, with $h = \max_l\{\operatorname{diam}\tau_l^h\}$, and let S_h be a finite element subspace of $H_0^1(\Omega)$ consisting of continuous piecewise polynomials of degree $r \ge 1$ subject to the triangulation. Let $a(x) = (a_{ij}(x))_{N\times N}$ be the coefficient matrix and define the operators

$$A : H_0^1 \to H^{-1},\ \ A_h : S_h \to S_h,$$
$$R_h : H_0^1 \to S_h,\ \ P_h : L^2 \to S_h,$$

by

$$\begin{aligned} &\big(A\phi, v\big) = \big(a\nabla\phi, \nabla v\big) && \text{for all } \phi, v \in H_0^1, \\ &\big(A_h\phi_h, v\big) = \big(a\nabla\phi_h, \nabla v\big) && \text{for all } \phi_h \in S_h,\ v \in S_h, \\ &\big(A_hR_hw, v\big) = \big(Aw, v\big) && \text{for all } w \in H_0^1 \text{ and } v \in S_h, \\ &\big(P_h\phi, v\big) = \big(\phi, v\big) && \text{for all } \phi \in L^2 \text{ and } v \in S_h. \end{aligned}$$

Clearly, R_h is the Ritz projection operator associated to the elliptic operator A and P_h is the L^2 projection operator onto the finite element space. The following estimates are useful in this paper.

Lemma 2.1. *If Ω is a bounded convex domain and $a_{ij} \in W^{1,N+\alpha}(\Omega)$, $N \geq 2$, then we have*

$$\|w\|_{H^2} \leq C\|\nabla \cdot (a\nabla w)\|_{L^2}, \qquad \forall\, w \in H_0^1, \tag{2.1}$$

$$\|\nabla w\|_{L^\infty} \leq C_p\|\nabla \cdot (a\nabla w)\|_{L^p}, \qquad \text{for any given } p > N, \qquad \forall\, w \in H_0^1, \tag{2.2}$$

and the solution of (1.1) with $u^0 = 0$ satisfies

$$\|\partial_t u\|_{L^p((0,T);L^q)} + \|Au\|_{L^p((0,T);L^q)} \leq C_{p,q}\|f\|_{L^p((0,T);L^q)}, \tag{2.3}$$

$$\|\partial_t u\|_{L^p((0,T);W^{-1,q})} + \|u\|_{L^p((0,T);W^{1,q})} \leq C_{p,q}\|f\|_{L^p((0,T);W^{-1,q})}, \tag{2.4}$$

for all $1 < p, q < \infty$.

In the Lemma above, (2.1) is the standard H^2-regularity estimate in convex domains and (2.2) is a simple consequence of the Green's function estimates given in Theorem 3.3–3.4 of [18], and (2.3)-(2.4) are consequences of the maximal L^p regularity (see Appendix for details).

2.2. **Properties of the finite element space and Green's functions.** For any subdomain $D \subset \Omega$, we denote by $S_h(D)$ the space of functions restricted to the domain D, and denote by $S_h^0(D)$ the subspace of $S_h(D)$ consisting of functions which equal zero outside D. For any given subset $D \subset \Omega$, we denote $B_d(D) = \{x \in \Omega : \mathrm{dist}(x, D) \leq d\}$ for $d > 0$. Then there exist positive constants K and κ such that the triangulation and the corresponding finite element space S_h possess the following properties (K and κ are independent of the subset D and h).

(P0) Quasi-uniformity:

For all triangles (or tetrahedron) τ_l^h in the partition, the diameter h_l of τ_l^h and the radius ρ_l of its inscribed ball satisfy

$$K^{-1}h \leq \rho_l \leq h_l \leq Kh.$$

(P1) Inverse inequality:

If D is a union of elements in the partition, then

$$\|\chi_h\|_{W^{l,p}(D)} \leq Kh^{-(l-k)-(N/q-N/p)}\|\chi_h\|_{W^{k,q}(D)}, \quad \forall\ \chi_h \in S_h,$$

for $0 \leq k \leq l \leq 1$ and $1 \leq q \leq p \leq \infty$.

(P2) Local approximation and superapproximation:

(1) There exists a linear operator $I_h : H_0^1(\Omega) \to S_h$ such that if $d \geq \kappa h$, then

$$\|v - I_h v\|_{L^2(D)} \leq K\sum_{l=0}^{k} h^k d^{-l}\|v\|_{H^{k-l}(B_d(D))}, \quad \forall\ v \in H^k \cap H_0^1,\ 1 \leq k \leq 2.$$

Moreover, if $\mathrm{supp}(v) \subset \overline{D}$, then $I_h v \in S_h^0(B_d(D))$. For example, the Clément interpolation operator defined in [5] has these properties. Also, the Lagrange interpolation operator Π_h satisfies

$$\|v - \Pi_h v\|_{L^2(D)} + h\|\nabla(v - \Pi_h v)\|_{L^2(D)} \leq Kh^2\|\nabla^2 v\|_{L^2(B_d(D))}, \quad \forall\ v \in H^2 \cap H_0^1.$$

(2) If $d \geq \kappa h$, $\omega = 0$ outside $B_{2d}(D)$ and $|\partial^\beta \omega| \leq Cd^{-|\beta|}$ for all multi-index β, then for any $\psi_h \in S_h(B_{3d}(D))$ there exists $\eta_h \in S_h^0(B_{3d}(D))$ such that

$$\|\omega\psi_h - \eta_h\|_{H^k(B_{3d}(D))} \leq Kh^{1-k}d^{-1}\|\psi_h\|_{L^2(B_{3d}(D))}, \quad k = 0, 1.$$

Furthermore, if $\omega \equiv 1$ on $B_d(D)$, then $\eta_h = \psi_h$ on D and

$$\|\omega\psi_h - \eta_h\|_{H^k(B_{3d}(D))} \leq Kh^{1-k}d^{-1}\|\psi_h\|_{L^2(B_{3d}(D)\backslash D)}, \quad k = 0, 1.$$

For example, $\eta_h = \Pi_h(\omega\psi_h)$ has these properties.

(P3) Regularized Delta function:
For any $x_0 \in \overline{\tau}_j^h$, there exists a function $\widetilde{\delta}_{x_0} \in C^3(\overline{\Omega})$ with support in τ_j^h such that

$$\chi_h(x_0) = \int_{\tau_j^h} \chi_h \widetilde{\delta}_{x_0} \mathrm{d}x, \quad \forall\, \chi_h \in S_h,$$
$$\|\widetilde{\delta}_{x_0}\|_{W^{l,p}} \le K h^{-l-N(1-1/p)} \quad \text{for } 1 \le p \le \infty,\ \ l = 0,1,2,3.$$

(P4) Discrete Delta function
Let δ_{x_0} denote the Dirac Delta function centered at x_0, i.e. $\int_\Omega \delta_{x_0}(y)\varphi(y)\mathrm{d}y = \varphi(x_0)$ for any $\varphi \in C(\overline{\Omega})$. The discrete Delta function $P_h\widetilde{\delta}_{x_0}$ satisfies that

$$P_h\widetilde{\delta}_{x_0}(x) \le K h^{-N} e^{-\frac{|x-x_0|}{Kh}}, \quad \forall\, x, x_0 \in \Omega.$$

The properties (P0)-(P4) hold for any quasi-uniform partition with those standard finite element spaces and also, have been used in many previous works such as [25, 39, 41, 47]. The proof can be found in the appendix of [41].

For an element τ_l^h and a point $x_0 \in \overline{\tau}_l^h$, we let $G(t,x,x_0)$ be the Green's function of the parabolic equation, defined by

$$\partial_t G(t,\cdot,x_0) + AG(t,\cdot,x_0) = 0 \quad \text{for } t > 0 \text{ with } G(0,x,x_0) = \delta_{x_0}(x), \tag{2.5}$$

The regularized Green's function $\Gamma(t,x,x_0)$ is defined by

$$\partial_t \Gamma(\cdot,\cdot,x_0) + A\Gamma(\cdot,\cdot,x_0) = 0 \quad \text{for } t > 0 \text{ with } \Gamma(0,\cdot,x_0) = \widetilde{\delta}_{x_0}, \tag{2.6}$$

where $\widetilde{\delta}_{x_0}$ is given in (P2), and the discrete Green's function $\Gamma_h(\cdot,\cdot,x_0)$ is defined by

$$\partial_t \Gamma_h(\cdot,\cdot,x_0) + A_h\Gamma_h(\cdot,\cdot,x_0) = 0 \quad \text{for } t > 0 \text{ with } \Gamma_h(0,\cdot,x_0) = P_h\widetilde{\delta}_{x_0}. \tag{2.7}$$

The functions $G(t,x,x_0)$ and $\Gamma_h(t,x,x_0)$ are symmetric with respect to x and x_0.
By the fundamental estimates of parabolic equations, there exists a positive constant C such that ([12], Theorem 1.6; note that the Green's function in the domain Ω is less than the Green's function in $\mathbb{R}^N$)

$$|G(t,x,y)| \le C(t^{1/2} + |x-y|)^{-N} e^{-\frac{|x-y|^2}{Ct}}. \tag{2.8}$$

By estimating $\Gamma(t,x,x_0) = \int_\Omega G(t,x,y)\widetilde{\delta}_{x_0}(y)\mathrm{d}y$, it is easy to see that (2.8) also holds when G is replaced by Γ and when $\max(t^{1/2}, |x-y|) \ge 2h$.

2.3. **Decomposition of the domain $\Omega \times (0,T)$.** Here we present some further notations on a dyadic decomposition of the domain $\Omega\times(0,T)$, which were introduced in [39] and also used in many other articles [14, 24, 25, 47]. Let R_0 be the smallest distance between a corner and a closed face which does not contained this corner.

For the given polygon/polyhedron Ω, there exists a positive constant $K_0 \ge \max(1, R_0)$ (which depends on the interior angle of the edges/corners of Ω) such that

(1) if z_0 is a point in the interior of Ω and $B_\rho(z_0)$ intersects a face of Ω, then $B_\rho(z_0) \subset B_{2\rho}(z_1)$ for some z_1 which is on a face of Ω;

(2) if z_1 is on a face of Ω and $B_\rho(z_1)$ intersects another face, then $B_\rho(z_1) \subset B_{\rho K_0}(z_2)$ for some z_2 which is on an edge of Ω;

(3) if z_2 is on an edge of Ω and $B_\rho(z_2)$ intersects another face which does not contain this edge, then $B_\rho(z_2) \subset B_{\rho K_0}(z_3)$ for some z_3 which is a corner of Ω.

For any integer $j \geq 1$, we define $d_j = 2^{-j-3}R_0K_0^{-2}$. For a given $x_0 \in \Omega$, we let J_* be an integer satisfying $d_{J_*} = 2^{-J_*-3}R_0K_0^{-2} = C_*h$ with $C_* \geq \max(10, 10\kappa, R_0K_0^{-2}/8)$ to be determined later. Thus, $J_* = \log_2[R_0K_0^{-2}/(8C_*h)] \leq \log_2(2+1/h)$ and $J_* > 1$ when $h < R_0K_0^{-2}/(16C_*)$. Let

$$\begin{aligned}
&Q_*(x_0) = \{(x,t) \in \Omega_T : \max(|x-x_0|, t^{1/2}) \leq d_{J_*}\},\\
&\Omega_*(x_0) = \{x \in \Omega : |x-x_0| \leq d_{J_*}\},\\
&Q_j(x_0) = \{(x,t) \in \Omega_T : d_j \leq \max(|x-x_0|, t^{1/2}) \leq 2d_j\},\\
&\Omega_j(x_0) = \{x \in \Omega : d_j \leq |x-x_0| \leq 2d_j\},\\
&D_j(x_0) = \{x \in \Omega : |x-x_0| \leq 2d_j\}
\end{aligned}$$

for $j \geq 1$; see Figure 1.

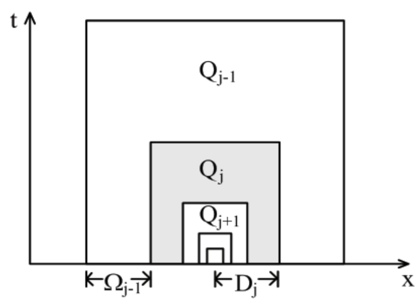

FIGURE 1. Illustration of the subdomains Q_j, Ω_j and D_j.

For $j = 0$ we define $Q_0(x_0) = Q_T\backslash Q_1(x_0)$ and $\Omega_0(x_0) = \Omega\backslash\Omega_1(x_0)$, and for $j < 0$ we simplify define $Q_j(x_0) = \Omega_j(x_0) = \emptyset$. For all $j \geq 1$ we define

$$\begin{aligned}
&\Omega_j'(x_0) = \Omega_{j-1}(x_0) \cup \Omega_j(x_0) \cup \Omega_{j+1}(x_0),\\
&\Omega_j''(x_0) = \Omega_{j-2}(x_0) \cup \Omega_j'(x_0) \cup \Omega_{j+2}(x_0),\\
&\Omega_j'''(x_0) = \Omega_{j-2}(x_0) \cup \Omega_j''(x_0) \cup \Omega_{j+2}(x_0),\\
&Q_j'(x_0) = Q_{j-1}(x_0) \cup Q_j(x_0) \cup Q_{j+1}(x_0),\\
&Q_j''(x_0) = Q_{j-2}(x_0) \cup Q_j'(x_0) \cup Q_{j+2}(x_0),\\
&Q_j'''(x_0) = Q_{j-2}(x_0) \cup Q_j''(x_0) \cup Q_{j+2}(x_0),\\
&D_j'(x_0) = D_{j-1}(x_0) \cup D_j(x_0),\\
&D_j''(x_0) = D_{j-2}(x_0) \cup D_j'(x_0),\\
&D_j'''(x_0) = D_{j-3}(x_0) \cup D_j''(x_0).
\end{aligned}$$

Then we have

$$Q_T = \bigcup_{j=0}^{J_*} Q_j(x_0) \cup Q_*(x_0) \quad \text{and} \quad \Omega = \bigcup_{j=0}^{J_*} \Omega_j(x_0) \cup \Omega_*(x_0),$$

We refer to $Q_*(x_0)$ as the "innermost" set. We shall write $\sum_{*,j}$ when the innermost set is included and $\sum_j$ when it is not. When x_0 is fixed, if there is no ambiguity, we simply write $Q_j = Q_j(x_0)$, $Q'_j = Q'_j(x_0)$, $Q''_j = Q''_j(x_0)$, $\Omega_j = \Omega_j(x_0)$, $\Omega'_j = \Omega'_j(x_0)$ and $\Omega''_j = \Omega''_j(x_0)$.

In the rest of this paper, we denote by C a generic positive constant, which will be independent of h, x_0, and the undetermined constant C_* until it is determined at the end of section 4.2.

2.4. **Proof of Theorem 1.1.** The keys to the proof of Theorem 1.1 are several more precise estimates of the Green's functions. Let $F(t) = \Gamma_h(t) - \Gamma(t)$. Then for any $x_0 \in \Omega$, we have

$$\begin{aligned}(E_h(t)v_h)(x_0) &= (F(t), v_h) + (\Gamma(t), v_h)\\ &= \int_0^t (\partial_t F(s), v_h)\mathrm{d}s + (F(0), v_h) + (\Gamma(t), v_h),\end{aligned} \tag{2.9}$$

$$\begin{aligned}(t\partial_t E_h(t)v_h)(x_0) &= (t\partial_t F(t), v_h) + (t\partial_t\Gamma(t), v_h)\\ &= \int_0^t (s\partial_{ss}F(s) + \partial_s F(s), v_h)\mathrm{d}s + (t\partial_t\Gamma(t), v_h),\end{aligned} \tag{2.10}$$

with $\|F(0)\|_{L^1} = \|\widetilde{\delta}_{x_0} - P_h\widetilde{\delta}_{x_0}\|_{L^1} \le C$ (according to (P3) and (P4)). Moreover, by the analyticity of the continuous parabolic semigroup on $L^1(\Omega)$, we have

$$\|\Gamma(t)\|_{L^1} + t\|\partial_t\Gamma(t)\|_{L^1} \le C\|\Gamma(0)\|_{L^1} = C\|\widetilde{\delta}_{x_0}\|_{L^1} \le C.$$

We present some estimates of these Green's functions in the following lemma. The proof of the lemma is the major work of this paper and will be given in the next two sections.

Lemma 2.2. *Under the assumptions of Theorem 1.1, we have*

$$\int_0^\infty \int_\Omega \big(|\partial_t F(t,x,x_0)| + |t\partial_{tt}F(t,x,x_0)|\big)\mathrm{d}x\mathrm{d}t \le C, \tag{2.11}$$

$$|\nabla\partial_t G(t,x,x_0)| \le C\max(t^{1/2}, |x-x_0|)^{-3-N} \quad \textit{for} \;\; (x,t) \in \Omega\times(0,1). \tag{2.12}$$

The estimates in Lemma 2.2 were proved in [39] for parabolic equations with the Neumann boundary condition and in [47] for the Dirichlet boundary condition. However, their proofs are only valid for smooth coefficients and smooth domains (as clearly mentioned in their papers). Later, these estimates were proved in [25] for parabolic equations in smooth domains of arbitrary dimensions under the Neumann boundary condition with Lipschitz continuous coefficients. Here we are concerned with the problem in a convex polyhedron in two or three dimensional spaces under the Dirichlet boundary condition with $a_{ij} \in W^{1,N+\alpha}$.

Proof of Theorem 1.1: Firstly, from (2.9)-(2.10) we see that (1.9) is a consequence of (2.11).

Secondly, from [49, Theorem 4.2] and [50, Lemma 4.c] (with a duality argument for the case $q \ge 2$) we know that the maximal L^p regularity (1.10) holds if the following maximal ergodic estimate holds:

$$\Big\|\sup_{t>0}\frac{1}{t}\int_0^t |E_h(s)|\, v\,\mathrm{d}s\Big\|_{L^q} \le C\|v\|_{L^q}, \quad \forall\, v \in L^q(\Omega), \tag{2.13}$$

where

$$(|E_h(s)|v)(x_0) := \int_\Omega |\Gamma_h(t,x,x_0)|v(x)\mathrm{d}x.$$

Let $G_{\rm tr}(t,x,x_0)$ be a truncated Green's function which is symmetric with respect to x and x_0 and satisfies $G_{\rm tr}(t,x,x_0) = G(t,x,x_0)$ when (x,t) is outside $Q_*(x_0)$ (see [25, Section 4.2] on its construction). Then we have (assuming that τ_0^h is the triangle/tetrahedron which contains x_0)

$$\begin{aligned}
&\iint_{[\Omega\times(0,\infty)]\backslash Q_*(x_0)} |\partial_t\Gamma(t,x,x_0) - \partial_t G_{\rm tr}(t,x,x_0)|\mathrm{d}x\mathrm{d}t \\
&= \iint_{[\Omega\times(0,1)]\backslash Q_*(x_0)} \Big|\int_\Omega \partial_t G(t,x,y)\widetilde{\delta}_{x_0}(y)\mathrm{d}y - \partial_t G(t,x,x_0)\Big|\mathrm{d}x\mathrm{d}t \\
&\quad + \iint_{\Omega\times(1,\infty)} |\partial_t\Gamma(t,x,x_0) - \partial_t G(t,x,x_0)|\mathrm{d}x\mathrm{d}t \\
&\le Ch \iint_{[\Omega\times(0,1)]\backslash Q_*(x_0)} \sup_{y\in\tau_0^h} \big|\nabla_y\partial_t G(t,x,y)\big|\mathrm{d}x\mathrm{d}t \\
&\quad + C\int_1^\infty t^{-1}(\|\Gamma(t/2,\cdot,x_0)\|_{L^1} + \|G(t/2,\cdot,x_0)\|_{L^1})\mathrm{d}t \quad \text{[by semigroup estimate]} \\
&= Ch\sum_j \iint_{Q_j(x_0)} \sup_{(y,t)\in Q_j'(x)} \big|\nabla_y\partial_t G(t,x,y)\big|\mathrm{d}x\mathrm{d}t + C\int_1^\infty t^{-1-N/2}\mathrm{d}t \quad \text{[see (2.8)]} \\
&\le C\sum_j \frac{h}{d_j} + C \quad \text{[see (2.12)]} \\
&\le C.
\end{aligned}$$

By using energy estimates, it is easy to see

$$\begin{aligned}
&\iint_{Q_*(x_0)} (|\partial_t\Gamma(t,x,x_0)| + |\partial_t G_{\rm tr}(t,x,x_0)|)\mathrm{d}x\mathrm{d}t \\
&\le d_{J_*}^{N/2+1}(\|\partial_t\Gamma(\cdot,\cdot,x_0)\|_{L^2(\Omega\times(0,1))} + \|\partial_t G_{\rm tr}(\cdot,\cdot,x_0)\|_{L^2(\Omega\times(0,1))}) \le C_*^{N/2+1},
\end{aligned}$$

where the constant C_* will be determined at the end of Section 4. Then (2.11) and the last two inequalities imply

$$\int_0^\infty \int_\Omega |\partial_t\Gamma_h(t,x,x_0) - \partial_t G_{\rm tr}(t,x,x_0)|\mathrm{d}x\mathrm{d}t \le C.$$

In other words, the symmetric kernel $K(x,y) := \int_0^\infty |\partial_t\Gamma_h(t,x,y) - \partial_t G_{\rm tr}^*(t,x,y)|\mathrm{d}t$ satisfies

$$\sup_{y\in\Omega}\int_\Omega K(x,y)\mathrm{d}x + \sup_{x\in\Omega}\int_\Omega K(x,y)\mathrm{d}y \le C,$$

and therefore, Schur's lemma implies that the corresponding operator M_K, defined by $M_K v(x) = \int_\Omega K(x,y)v(y)\mathrm{d}y$, is bounded on $L^q(\Omega)$ for all $1 \le q \le \infty$. Let

$E^*_{\rm tr}(t)v(x) = \int_\Omega G^*_{\rm tr}(t,x,y)v(y){\rm d}y$ and note that $E^*_{\rm tr}(t)v(x) \le E(t)|v|(x)$ (because $G^*_{\rm tr}(t,x,y) \le G(t,x,y)$). We have

$$\begin{aligned}
&\sup_{t>0}(|E_h(t)|v)(x)\\
&\le \sup_{t>0}\big(|E_h(t)-E^*_{\rm tr}(t)|\,|v|\big)(x) + \sup_{t>0}|(E^*_{\rm tr}(t)|v|)(x)|\\
&= \sup_{t>0}\bigg|(|P_h\widetilde\delta_x|,|v|) + \int_0^t\int_\Omega |\partial_t\Gamma_h(s,x,y)-\partial_t G^*_{\rm tr}(s,x,y)|\,|v(y)|{\rm d}y{\rm d}s\bigg|\\
&\quad + \sup_{t>0}|(E^*_{\rm tr}(t)|v|)(x)|\\
&\le (|P_h\widetilde\delta_x|,|v|) + (M_K|v|)(x) + \sup_{t>0}|(E(t)|v|)(x)|
\end{aligned}$$

where

$$\|\sup_{t>0}E(t)|v|\|_{L^q} \le C_q\|v\|_{L^q}, \quad \forall\, 1<q<\infty,$$

is a simple consequence of the Gaussian estimate (2.8) (Corollary 2.1.12 and Theorem 2.1.6 of [16]). This proves a stronger estimate than (2.13). The proof of (1.10) is completed.

Finally, (1.1)-(1.2) imply that the error $e_h = P_hu - u_h$ satisfies the equation (when $u^0 = u^0_h = 0$)

$$\partial_t(A_h^{-1}e_h) + A_h(A_h^{-1}e_h) = P_hu - R_hu. \tag{2.14}$$

By applying (1.10) to the equation above, we obtain

$$\|e_h\|_{L^p((0,T);L^q)} \le C_{p,q}\|P_hu-R_hu\|_{L^p((0,T);L^q)} \le C_{p,q}h\|u\|_{L^p((0,T);W^{1,q})} \tag{2.15}$$

for $1<p<\infty$ and $2\le q<\infty$, where we have used the inequality $\|P_hu-R_hu\|_{L^q} \le C_qh\|u\|_{W^{1,q}}$, which only holds for $2\le q<\infty$ in convex polygons/polyhedra. Then, by using an inverse inequality and (2.4), we have

$$\begin{aligned}
\|e_h\|_{L^p((0,T);W^{1,q})} &\le Ch^{-1}\|e_h\|_{L^p((0,T);L^q)}\\
&\le C_{p,q}\|u\|_{L^p((0,T);W^{1,q})}\\
&\le C_{p,q}\|f\|_{L^p((0,T);W^{-1,q})},
\end{aligned}$$

which implies (1.13) for the case $1<p<\infty$ and $2\le q<\infty$.

In the case $1<p<\infty$ and $1<q\le 2$, we define $\vec g = \nabla\Delta^{-1}P_hf$ and express the solution of (1.2) by (when $u^0_h = 0$)

$$\nabla u_h = \mathcal{L}_h\vec g := \int_0^t \nabla A_h^{-1/2}A_hE_h(t-s)A_h^{-1/2}\nabla\cdot\vec g(s){\rm d}s.$$

In order to prove the boundedness of the operator $\mathcal{L}_h$ on $L^p((0,T);(L^q)^N)$, we only need to prove the boundedness of its dual operator $\mathcal{L}'_h$ on $L^{p'}((0,T);(L^{q'})^N)$. It is easy to see that

$$\int_0^T(\mathcal{L}_h\vec g,\vec\eta){\rm d}t = \int_0^T\bigg(\vec g, \int_s^T \nabla A_h^{-1/2}A_hE_h(t-s)A_h^{-1/2}\nabla\cdot\vec\eta(t){\rm d}t\bigg){\rm d}s,$$

which gives

$$\mathcal{L}'_h\vec\eta := \int_s^T \nabla A_h^{-1/2}A_hE_h(t-s)A_h^{-1/2}\nabla\cdot\vec\eta(s){\rm d}s.$$

If we define the backward finite element problem

$$\begin{cases} -\big(\partial_t w_h, v_h\big) + (a\nabla w_h, \nabla v_h) = (\nabla\cdot\vec{\eta}, v_h), & \forall\, v_h \in S_h, \\ w_h(T) = 0, \end{cases} \tag{2.16}$$

then $\mathcal{L}_h'\vec{\eta} = \nabla w_h$. By a time reversal we obtain, as shown in the last paragraph,

$$\|\nabla w_h\|_{L^{p'}((0,T);L^{q'})} \le C_{p,q}\|\nabla\cdot\vec{\eta}\|_{L^{p'}((0,T);W^{-1,q'})} \le C_{p,q}\|\vec{\eta}\|_{L^{p'}((0,T);L^{q'})},$$

for $1 < p' < \infty$ and $2 \le q' < \infty$, which implies the boundedness of $\mathcal{L}_h'$ on $L^{p'}((0,T);(L^{q'})^N)$. By duality, we derive the boundedness of $\mathcal{L}_h$ on $L^p((0,T);(L^q)^N)$ and therefore,

$$\begin{aligned} \|\nabla u_h\|_{L^p((0,T);L^q)} &\le C_{p,q}\|\vec{g}\|_{L^p((0,T);L^q)} \\ &\le C_{p,q}\|P_h f\|_{L^p(0,T);W^{-1,q})} \\ &\le C_{p,q}\|f\|_{L^p(0,T);W^{-1,q})}. \end{aligned}$$

This proves (1.13) in the case $1 < p < \infty$ and $1 < q \le 2$.

The proof of Theorem 1.1 is completed (based on Lemma 2.2). ∎

Remark 2.1 In the proof of (1.13), we have used an L^q error estimate of the Ritz projection for $2 \le q < \infty$, which can be proved in the same way as used in [36, Corollary] by using the $W^{1,q}$-stability of the Ritz projection. This $W^{1,q}$-stability is based on an interpolation between these two cases $q = 2$ and $q = \infty$. The case $q = 2$ is trivial and the case $q = \infty$ was studied by several authors, such as [36] for $r = 1$ and 2D convex polygons (which requires H^2 regularity of the elliptic problem), [37] for $r \ge 2$ and 2D arbitrary polygons (as a consequence of the L^∞ stability proved therein, which only requires $H^{3/2+\varepsilon}$ regularity of the elliptic problem), and [19] for $r \ge 1$ in 3D convex polyhedra (which requires H^2 and $C^{1+\alpha}$ regularity of the elliptic problem). These essential properties used by [19, 36, 37] are all possessed by the elliptic problem when the domain is convex polygonal/polyhedral and the coefficients a_{ij} are $W^{1,N+\alpha}$.

In the rest of this paper, we focus on the proof of Lemma 2.2.

3. Preliminary analysis

In this section, we present two propositions.

3.1. Superapproximation of smoothly truncated finite element functions.

In this subsection, we prove the following proposition, which is needed in proving Lemma 2.2.

Proposition 3.1. *If $0 \le \omega \le 1$ is a smooth cut-off function which equals zero in $\Omega\backslash D$, satisfying $|\partial^\beta\omega| \le Cd^{-|\beta|}$ for all multi-index β such that $|\beta| = 0, 1, \cdots, r+1$ and $d \ge 10\kappa h$, then for any $\psi_h \in S_h$ there exists $\chi_h \in S_h^0(B_d(D))$ such that*

$$d^2\|R_h(\omega\psi_h) - \chi_h\|_{H^1} + d\|\omega\psi_h - \chi_h\|_{L^2} \le Ch\|\psi_h\|_{L^2(B_d(D))}. \tag{3.1}$$

Proof. Define $0 \le \widetilde{\omega} \le 1$ as a smooth cut-off function which is zero outside $B_{0.8d}(D)$, satisfying that $\widetilde{\omega} = 1$ on $B_{0.7d}(D)$ and $|\partial^\beta\widetilde{\omega}| \le Cd^{-|\beta|}$ for $|\beta| = 0, 1, \cdots, r+1$.

First we prove the following inequality

$$\|\omega\psi_h - R_h(\omega\psi_h)\|_{L^2} \le Chd^{-1}\|\psi_h\|_{L^2(B_{0.3d}(D))} \tag{3.2}$$

by a duality argument. We define ϕ as the solution of the elliptic PDE

$$\begin{cases} -\nabla\cdot(a\nabla\phi) = v & \text{in } \ \Omega, \\ \phi = 0 & \text{on } \ \partial\Omega. \end{cases}$$

We see that

$$\begin{aligned} \big(v, \omega\psi_h - R_h(\omega\psi_h)\big) &= \big(a\nabla\phi, \nabla(\omega\psi_h - R_h(\omega\psi_h))\big) \\ &= \big(a\nabla(\phi - R_h\phi), \nabla(\omega\psi_h - R_h(\omega\psi_h))\big) \\ &= \big(a\nabla(\phi - R_h\phi), \nabla(\omega\psi_h - \Pi_h(\omega\psi_h))\big) \\ &\le C\|\phi - R_h\phi\|_{H^1}\|\omega\psi_h - \Pi_h(\omega\psi_h)\|_{H^1} \\ &\le Chd^{-1}\|v\|_{L^2}\|\psi_h\|_{L^2(B_{0.3d}(D))}. \end{aligned}$$

where we have used the superapproximation property (P2) in section 2.2 and the H^1 error estimate:

$$\|\phi - R_h\phi\|_{H^1} \le Ch\|\phi\|_{H^2} \le Ch\|v\|_{L^2}.$$

(3.2) follows these inequalities.

Secondly, it is noted that the following inequality

$$\|R_h(\omega\psi_h)\|_{H^1(B_d(D)\backslash B_{0.5d}(D))} \le Cd^{-1}\|R_h(\omega\psi_h)\|_{L^2(B_d(D)\backslash B_{0.3d}(D))} \tag{3.3}$$

was proved in Lemma 4.4 of [40] (also see Page 1374 of [39]) as a consequence of the discrete elliptic equation

$$(a\nabla R_h(\omega\psi_h), \nabla\eta) = 0, \quad \text{ for } \quad \eta \in S_h^0(B_d(D)\backslash D)\,.$$

Let $\chi_h = \Pi_h[\widetilde{\omega}R_h(\omega\psi_h)]$ and note that the support of χ_h is contained in $B_{0.8d}(D)$. By using the superapproximation property (P2), we have

$$\begin{aligned} &d^{-1}\|R_h(\omega\psi_h) - \chi_h\|_{L^2} + \|R_h(\omega\psi_h) - \chi_h\|_{H^1} \\ &\le d^{-1}\|R_h(\omega\psi_h) - \widetilde{\omega}R_h(\omega\psi_h)\|_{L^2} + d^{-1}\|\widetilde{\omega}R_h(\omega\psi_h) - \Pi_h[\widetilde{\omega}R_h(\omega\psi_h)]\|_{L^2} \\ &\quad + \|R_h(\omega\psi_h) - \widetilde{\omega}R_h(\omega\psi_h)\|_{H^1} + \|\widetilde{\omega}R_h(\omega\psi_h) - \Pi_h[\widetilde{\omega}R_h(\omega\psi_h)]\|_{H^1} \\ &\le Cd^{-1}\|R_h(\omega\psi_h)\|_{L^2(B_d(D)\backslash B_{0.3d}(D))} \\ &= Cd^{-1}\|R_h(\omega\psi_h) - \omega\psi_h\|_{L^2(B_d(D)\backslash B_{0.3d}(D))} \qquad \text{(because } \omega = 0 \text{ on } B_d(D)\backslash B_{0.3d}(D)) \\ &\le Chd^{-2}\|\psi_h\|_{L^2(B_{0.3d}(D))} \qquad \text{(as a consequence of (3.2))} \end{aligned} \tag{3.4}$$

and from (3.2) we see that

$$\begin{aligned} \|\omega\psi_h - \chi_h\|_{L^2} &\le \|\omega\psi_h - R_h(\omega\psi_h)\|_{L^2} + \|R_h(\omega\psi_h) - \chi_h\|_{L^2} \\ &\le Chd^{-1}\|\psi_h\|_{L^2(B_{0.3d}(D))}\,. \end{aligned} \tag{3.5}$$

(3.1) follows immediately and the proof of the Proposition 3.1 is completed. ∎

Remark 3.1 In the proof of Proposition 3.1 we have assumed that $d \ge 10\kappa d$ and used $B_{0.3d}(D)$, $B_{0.7d}(D)$, $B_{0.8d}(D)$ ... to make sure that their radius differ from each other by at least κh so that the superapproximation property (P2) can be used.

3.2. **Local error estimate.** The following proposition is concerned with a local energy error estimate of parabolic equations.

Proposition 3.2. *Suppose that* $\phi, \partial_t\phi \in L^2((0,T); H_0^1)$ *and* $\phi_h \in H^1((0,T); S_h)$, *and* $e = \phi_h - \phi$ *satisfies the equation*

$$(e_t, \chi) + (a\nabla e, \nabla\chi) = 0, \quad \forall\, \chi \in S_h,\ t > 0, \tag{3.6}$$

with $\phi(0) = 0$ *in* Ω_j''''. *Then for any* $m > 0$, *there exists a constant* $C_m > 0$, *independent of* h *and* d_j, *such that*

$$\begin{aligned} &\|e_t\|_{L^2(Q_j)} + d_j^{-1}\|\nabla e\|_{L^2(Q_j)} \\ &\le C_m\big(I_j(\phi_h(0)) + X_j(\Pi_h\phi - \phi) + H_j(e) + d_j^{-2}\|e\|_{L^2(Q_j''')}\big), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} I_j(\phi_h(0)) &= d_j^{-1}\|\phi_h(0)\|_{L^2(\Omega_j''')} + \|\phi_h(0)\|_{H^1(\Omega_j''')}, \\ X_j(\Pi_h\phi - \phi) &= d_j\|\nabla\partial_t(\Pi_h\phi - \phi)\|_{L^2(Q_j''')} + \|\partial_t(\Pi_h\phi - \phi)\|_{L^2(Q_j''')} \\ &\qquad + d_j^{-1}\|\nabla(\Pi_h\phi - \phi)\|_{L^2(Q_j''')} + d_j^{-2}\|\Pi_h\phi - \phi\|_{L^2(Q_j''')}, \\ H_j(e) &= (h/d_j)^m\big(\|e_t\|_{L^2(Q_j''')} + d_j^{-1}\|\nabla e\|_{L^2(Q_j''')}\big). \end{aligned}$$

Before we prove Proposition 3.2, we present a local energy estimate for finite element solutions of parabolic equations.

Lemma 3.3. *Suppose that* $\phi_h(t) \in S_h$ *satisfies*

$$\begin{aligned} &\big(\partial_t\phi_h, \chi\big) + \big(a\nabla\phi_h, \nabla\chi\big) = 0, \qquad \text{for } \chi \in S_h^0(\Omega_j''),\ t \in (0, d_j^2], \\ &\big(\partial_t\phi_h, \chi\big) + \big(a\nabla\phi_h, \nabla\chi\big) = 0, \qquad \text{for } \chi \in S_h^0(D_j''),\ t \in (d_j^2/4, 4d_j^2). \end{aligned}$$

Then for any $m > 0$ *there exists* $C_m > 0$, *independent of* h *and* d_j, *such that*

$$\begin{aligned} &\|\partial_t\phi_h\|_{L^2(Q_j)} + d_j^{-1}\|\nabla\phi_h\|_{L^2(Q_j)} \\ &\le C_m\left(\|\nabla\phi_h(0)\|_{L^2(\Omega_j'')} + d_j^{-1}\|\phi_h(0)\|_{L^2(\Omega_j'')}\right) \\ &\quad + C_m\left(\frac{h}{d_j}\right)^m\left(\|\partial_t\phi_h\|_{L^2(Q_j'')} + d_j^{-1}\|\nabla\phi_h\|_{L^2(Q_j'')}\right) + C_m d_j^{-2}\|\phi_h\|_{L^2(Q_j'')}. \end{aligned} \tag{3.8}$$

Proof. Note that $Q_j = [\Omega_j \times (0, d_j^2)] \cup [D_j \times (d_j^2, 4d_j^2)]$. We first present estimates in the domain $\Omega_j \times (0, d_j^2)$ and then present estimates in the domain $D_j \times (d_j^2, 4d_j^2)$.

Let ω be a smooth cut-off function which equals 1 in Ω_j and vanishes outside Ω_j', and let $\widetilde{\omega}$ be a smooth cut-off function which equals 1 in Ω_j''' and vanishes outside Ω_j'''' with

$$|\partial^\beta\omega| \le Cd_j^{-|\beta|} \qquad \text{and} \qquad |\partial^\beta\widetilde{\omega}| \le Cd_j^{-|\beta|}. \tag{3.9}$$

Let $v_h = \Pi_h(\widetilde{\omega}\phi_h) \in S_h^0(\Omega_j''''')$ so that $v_h = \phi_h$ in Ω_j'', satisfying (due to the superapproximation property (P2))

$$\begin{aligned} &\|v_h\|_{L^2} \le C\|\phi_h\|_{L^2(\Omega_j'''')}, \\ &\|\nabla v_h\|_{L^2} \le C\|\nabla\phi_h\|_{L^2(\Omega_j'''')} + Cd_j^{-1}\|\phi_h\|_{L^2(\Omega_j'''')} \end{aligned}$$

and

$$\big(\partial_t v_h, \chi\big) + \big(a\nabla v_h, \nabla\chi\big) = 0, \quad \forall\, \chi \in S_h^0(\Omega_j'')\,.$$

It follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega v_h\|^2 + (\omega^2 a\nabla v_h, \nabla v_h) \\
&= \big[\big(\partial_t v_h, \omega^2 v_h - \chi_h\big) + \big(a\nabla v_h, \nabla(R_h(\omega^2 v_h) - \chi_h)\big)\big] \\
&\quad + \big[\big(\omega^2 a\nabla v_h, \nabla v_h\big) - \big(a\nabla v_h, \nabla(\omega^2 v_h)\big)\big] \\
&\leq \big[C\|\partial_t v_h\|_{L^2}\|v_h\|_{L^2} h d_j^{-1} + C\|\nabla v_h\|_{L^2}\|v_h\|_{L^2} h d_j^{-2}\big] + C\big(\omega a\nabla v_h, 2v_h\nabla\omega\big),
\end{aligned}
$$

where we have used (P2) and Proposition 3.1, and from (3.9) we see that

$$
\big(\omega a\nabla v_h, 2v_h\nabla\omega\big) \leq \big(|\omega a\nabla v_h|,\, 2|v_h|\big)d_j^{-1} \leq C\|\omega a\nabla v_h\|_{L^2}\|v_h\|_{L^2} d_j^{-1}\,.
$$

The last two inequalities imply

$$
\begin{aligned}
&\|\phi_h\|_{L^\infty((0,d_j^2)\times L^2(\Omega_j))} + \|\nabla\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j))} \\
&\leq C\|\phi_h(0)\|_{L^2(\Omega_j'''')} + C\|\partial_t\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j''''))} h \\
&\quad + C\|\nabla\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j''''))} h d_j^{-1} + C\|\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j''''))} d_j^{-1}.
\end{aligned} \tag{3.10}
$$

By using Proposition 3.1 again, we derive that

$$
\begin{aligned}
&\|\omega^2\partial_t v_h\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\omega^4 a\nabla v_h, \nabla v_h) \\
&= \big[(\partial_t v_h, \omega^4\partial_t v_h - \chi_h) + (a\nabla v_h, \nabla[R_h(\omega^4\partial_t v_h) - \chi_h])\big] + (4\omega^3 a\nabla v_h, \partial_t v_h\nabla\omega) \\
&\leq C\|\partial_t v_h\|_{L^2}^2 h d_j^{-1} + C\|\nabla v_h\|_{L^2}\|\partial_t v_h\|_{L^2} h d_j^{-2} + C\|\omega\nabla v_h\|_{L^2}\|\omega^2\partial_t v_h\|_{L^2} d_j^{-1} \\
&\leq C\|\partial_t v_h\|_{L^2}^2 h d_j^{-1} + C\|\nabla v_h\|_{L^2}^2 d_j^{-2} + \frac{1}{2}\|\omega^2\partial_t v_h\|_{L^2}^2,
\end{aligned}
$$

which reduces to

$$
\begin{aligned}
&\|\omega^2\partial_t v_h\|_{L^2((0,d_j^2);L^2(\Omega))}^2 \\
&\leq C\|\nabla v_h(0)\|_{L^2(\Omega)}^2 + C\|\partial_t v_h\|_{L^2((0,d_j^2);L^2(\Omega))}^2 h d_j^{-1} + C\|\nabla v_h\|_{L^2((0,d_j^2);L^2(\Omega))}^2 d_j^{-2}.
\end{aligned}
$$

The inequality above further implies

$$
\begin{aligned}
\|\partial_t\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j))} &\leq C(\|\nabla\phi_h(0)\|_{L^2(\Omega_j'''')} + d_j^{-1}\|\phi_h(0)\|_{L^2(\Omega_j'''')}) \\
&\quad + C\|\partial_t\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j''''))} h^{1/2} d_j^{-1/2} \\
&\quad + C d_j^{-1}(\|\phi_h\|_{L^\infty((0,d_j^2)\times L^2(\Omega_j''''))} + \|\nabla\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j''''))}).
\end{aligned} \tag{3.11}
$$

With an obvious change of indices (from Ω'''' to Ω''' on the right-hand side, and from Ω to Ω' on the left-hand side), (3.10)-(3.11) imply

$$
\begin{aligned}
&\|\phi_h\|_{L^\infty((0,d_j^2);L^2(\Omega_j'))} + \|\nabla\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j'))} \\
&\leq C\|\phi_h(0)\|_{L^2(\Omega_j''')} + Ch\|\partial_t\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j'''))} \\
&\quad + Ch d_j^{-1}\|\nabla\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j'''))} + C d_j^{-1}\|\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j'''))}.
\end{aligned} \tag{3.12}
$$

and

$$
\begin{aligned}
\|\partial_t\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j))} &\leq C(\|\nabla\phi_h(0)\|_{L^2(\Omega_j')} + d_j^{-1}\|\phi_h(0)\|_{L^2(\Omega_j')}) \\
&\quad + Ch^{1/2} d_j^{-1/2}\|\partial_t\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j'))} \\
&\quad + C d_j^{-1}(\|\phi_h\|_{L^\infty((0,d_j^2);L^2(\Omega_j'))} + \|\nabla\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j'))}).
\end{aligned} \tag{3.13}
$$

In the same way as we derive (3.12)-(3.13), by choosing $\overline{\omega}(x,t) = \omega_1(x)\omega_2(t)$ with $\omega_1 = 1$ in D_j', $\omega_1 = 0$ outside D_j'', $\omega_2 = 1$ for $t \in (d_j^2, 4d_j^2)$ and $\omega_2 = 0$ for $t \in (0, d_j^2/2)$, we can derive that

$$\begin{aligned}
&\|\overline{\omega}\phi_h\|_{L^\infty((0,4d_j^2);L^2(\Omega))} + \|\overline{\omega}\nabla\phi_h\|_{L^2((0,4d_j^2);L^2(\Omega))} \\
&\leq Ch\|\partial_t\phi_h\|_{L^2((d_j^2/4,4d_j^2);L^2(D_j'''))} \\
&\quad + Chd_j^{-1}\|\nabla\phi_h\|_{L^2((d_j^2/4,4d_j^2);L^2(D_j'''))} \\
&\quad + Cd_j^{-1}\|\phi_h\|_{L^2((d_j^2/4,4d_j^2);L^2(D_j'''))}
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
\|\partial_t\phi_h\|_{L^2((d_j^2,4d_j^2);L^2(D_j))} &\leq Ch^{1/2}d_j^{-1/2}\|\partial_t\phi_h\|_{L^2((d_j^2/4,4d_j^2);L^2(D_j'''))} \\
&\quad + Cd_j^{-1}\big(\|\overline{\omega}\phi_h\|_{L^\infty((0,4d_j^2);L^2(\Omega))} + \|\overline{\omega}\nabla\phi_h\|_{L^2((0,4d_j^2);L^2(\Omega))}\big).
\end{aligned} \tag{3.15}$$

By noting the definition of ω and $\overline{\omega}$, we have

$$\begin{aligned}
&\|\partial_t\phi_h\|_{L^2(Q_j)} \leq C\big(\|\partial_t\phi_h\|_{L^2((0,d_j^2);L^2(\Omega_j))} + \|\partial_t\phi_h\|_{L^2((d_j^2,4d_j^2);L^2(D_j))}\big) \\
&\|\nabla\phi_h\|_{L^2(Q_j)} \leq C\big(\|\omega\nabla\phi_h\|_{L^2((0,d_j^2);L^2(\Omega))} + \|\overline{\omega}\nabla\phi_h\|_{L^2((0,4d_j^2);L^2(\Omega))}\big).
\end{aligned}$$

With the last two inequalities, combining (3.10)-(3.15) gives

$$\begin{aligned}
\|\partial_t\phi_h\|_{L^2(Q_j)} + d_j^{-1}\|\nabla\phi_h\|_{L^2(Q_j)} &\leq C(\|\nabla\phi_h(0)\|_{L^2(\Omega_j''')} + d_j^{-1}\|\phi_h(0)\|_{L^2(\Omega_j''')}) \\
&\quad + C\left(\frac{h}{d_j}\right)^{\frac{1}{2}}\big(\|\partial_t\phi_h\|_{L^2(Q_j''')} + d_j^{-1}\|\nabla\phi_h\|_{L^2(Q_j''')}\big) \\
&\quad + Cd_j^{-2}\|\phi_h\|_{L^2(Q_j''')}.
\end{aligned}$$

Iterating the inequality above and changing the indices, we derive (3.8). ∎

Now we are ready to prove Proposition 3.2.

Proof of Proposition 3.2 Let $\widetilde{\omega}(x,t)$ be a smooth cut-off function which equals 1 in Q_j'' and vanishes outside Q_j''', and let $\widetilde{\phi} = \widetilde{\omega}\phi$. Then $\widetilde{\phi}(0) = 0$ and we have

$$\begin{aligned}
&\big(\partial_t(\widetilde{\phi} - \phi_h), \chi\big) + \big(a\nabla(\widetilde{\phi} - \phi_h), \nabla\chi\big) = 0, \qquad \text{for } \chi \in S_h^0(\Omega_j'),\ t \in (0, d_j^2], \\
&\big(\partial_t(\widetilde{\phi} - \phi_h), \chi\big) + \big(a\nabla(\widetilde{\phi} - \phi_h), \nabla\chi\big) = 0, \qquad \text{for } \chi \in S_h^0(D_j'),\ t \in (d_j^2/4, 4d_j^2).
\end{aligned}$$

Let $\widetilde{\phi}_h \in S_h$ be the solution of

$$\big(\partial_t(\widetilde{\phi} - \widetilde{\phi}_h), \chi_h\big) + \big(a\nabla(\widetilde{\phi} - \widetilde{\phi}_h), \nabla\chi_h\big) = 0, \qquad \text{for } \chi_h \in S_h \tag{3.16}$$

with $\widetilde{\phi}_h(0) = \Pi_h\widetilde{\phi}(0) = 0$ so that

$$\big(\partial_t(\widetilde{\phi}_h - \phi_h), \chi_h\big) + \big(a\nabla(\widetilde{\phi}_h - \phi_h), \nabla\chi_h\big) = 0, \quad \forall\ \chi_h \in S_h^0(\Omega_j'),\ t \in (0, d_j^2], \tag{3.17}$$

$$\big(\partial_t(\widetilde{\phi}_h - \phi_h), \chi_h\big) + \big(a\nabla(\widetilde{\phi}_h - \phi_h), \nabla\chi_h\big) = 0, \quad \forall\ \chi_h \in S_h^0(D_j'),\ t \in (d_j^2/4, 4d_j^2). \tag{3.18}$$

Substituting $\chi_h = P_h\widetilde{\phi} - \widetilde{\phi}_h$ into (3.16) we obtain

$$\|\widetilde{\phi} - \widetilde{\phi}_h\|_{L^\infty((0,T);L^2)}^2 + \int_0^T \big(a\nabla(\widetilde{\phi} - \widetilde{\phi}_h), \nabla(\widetilde{\phi} - \widetilde{\phi}_h)\big)\mathrm{d}t$$

$$= \int_0^T \big(\partial_t(\widetilde{\phi}-\widetilde{\phi}_h), \widetilde{\phi}-P_h\widetilde{\phi}\big)\mathrm{d}t + \int_0^T \big(a\nabla(\widetilde{\phi}-\widetilde{\phi}_h), \nabla(\widetilde{\phi}-P_h\widetilde{\phi})\big)\mathrm{d}t,$$

which implies

$$\begin{aligned} &\|\widetilde{\phi}-\widetilde{\phi}_h\|_{L^\infty((0,T);L^2(Q_T))}^2 + \|\nabla(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_T)}^2 \\ &\le C\|\partial_t(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_T)}\|\widetilde{\phi}\|_{L^2(Q_T)} + C\|\nabla\widetilde{\phi}\|_{L^2(Q_T)}^2. \end{aligned} \tag{3.19}$$

Substituting $\chi_h = \partial_t(P_h\widetilde{\phi}-\widetilde{\phi}_h)$ into (3.16) we obtain

$$\begin{aligned} &\|\partial_t(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_T)}^2 + \sup_{0\le t\le T}\big(a\nabla(\widetilde{\phi}-\widetilde{\phi}_h), \nabla(\widetilde{\phi}-\widetilde{\phi}_h)\big) \\ &= \int_0^T \big(\partial_t(\widetilde{\phi}-\widetilde{\phi}_h), \partial_t(\widetilde{\phi}-P_h\widetilde{\phi})\big)\mathrm{d}t + \int_0^T \big(a\nabla(\widetilde{\phi}-\widetilde{\phi}_h), \nabla\partial_t(\widetilde{\phi}-P_h\widetilde{\phi})\big)\mathrm{d}t \\ &\le \|\partial_t(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_T)}\|\partial_t\widetilde{\phi}\|_{L^2(Q_T)} + \|\nabla(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_T)}\|\nabla\partial_t\widetilde{\phi}\|_{L^2(Q_T)}, \end{aligned}$$

which implies

$$\|\partial_t(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_T)}^2 \le C\|\partial_t\widetilde{\phi}\|_{L^2(Q_T)}^2 + C\|\nabla(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_T)}\|\nabla\partial_t\widetilde{\phi}\|_{L^2(Q_T)}, \tag{3.20}$$

It follows that

$$\begin{aligned} &\|\partial_t(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_T)}^2 + d_j^{-2}\|\nabla(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_T)}^2 + d_j^{-2}\|\widetilde{\phi}-\widetilde{\phi}_h\|_{L^\infty((0,T);L^2)}^2 \\ &\le \frac{1}{2}\|\partial_t(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_T)}^2 + Cd_j^{-4}\|\widetilde{\phi}\|_{L^2(Q_T)}^2 + Cd_j^{-2}\|\nabla\widetilde{\phi}\|_{L^2(Q_T)}^2, \\ &\quad + C\|\partial_t\widetilde{\phi}\|_{L^2(Q_T)}^2 + \frac{1}{2}d_j^{-2}\|\nabla(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_T)}^2 + Cd_j^2\|\nabla\partial_t\widetilde{\phi}\|_{L^2(Q_T)}^2. \end{aligned}$$

which in turn produces

$$\begin{aligned} &\|\partial_t(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_T)} + d_j^{-1}\|\nabla(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_T)} + d_j^{-1}\|\widetilde{\phi}-\widetilde{\phi}_h\|_{L^\infty((0,T);L^2)} \\ &\le Cd_j^{-2}\|\widetilde{\phi}\|_{L^2(Q_T)} + Cd_j^{-1}\|\nabla\widetilde{\phi}\|_{L^2(Q_T)} + C\|\partial_t\widetilde{\phi}\|_{L^2(Q_T)} + Cd_j\|\nabla\partial_t\widetilde{\phi}\|_{L^2(Q_T)}. \end{aligned}$$

By applying Lemma 3.3 to (3.17)-(3.18) and using the inequality above, we derive that

$$\begin{aligned} &\|\partial_t(\widetilde{\phi}_h-\phi_h)\|_{L^2(Q_j)} + d_j^{-1}\|\nabla(\widetilde{\phi}_h-\phi_h)\|_{L^2(Q_j)} \\ &\le C_m\left(\|\nabla(\widetilde{\phi}_h-\phi_h)(0)\|_{L^2(\Omega_j'')} + d_j^{-1}\|(\widetilde{\phi}_h-\phi_h)(0)\|_{L^2(\Omega_j'')}\right) \\ &\quad + C_m\left(\frac{h}{d_j}\right)^m\big(\|\partial_t(\widetilde{\phi}_h-\phi_h)\|_{L^2(Q_j'')} + d_j^{-1}\|\nabla(\widetilde{\phi}_h-\phi_h)\|_{L^2(Q_j'')}\big) \\ &\quad + C_m d_j^{-2}\|\widetilde{\phi}_h-\phi_h\|_{L^2(Q_j'')} \\ &\le C_m\left(\|\nabla\phi_h(0)\|_{L^2(\Omega_j'')} + d_j^{-1}\|\phi_h(0)\|_{L^2(\Omega_j'')}\right) \\ &\quad + C_m\left(\frac{h}{d_j}\right)^m\big(\|\partial_t e\|_{L^2(Q_j'')} + d_j^{-1}\|\nabla e\|_{L^2(Q_j'')}\big) + C_m d_j^{-2}\|e\|_{L^2(Q_j'')} \\ &\quad + C_m\left(\frac{h}{d_j}\right)^m\big(\|\partial_t(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_j'')} + d_j^{-1}\|\nabla(\widetilde{\phi}-\widetilde{\phi}_h)\|_{L^2(Q_j'')}\big) \\ &\quad + C_m d_j^{-1}\|\widetilde{\phi}-\widetilde{\phi}_h\|_{L^\infty L^2(Q_j'')} \\ &\le C_m\left(\|\nabla\phi_h(0)\|_{L^2(\Omega_j'')} + d_j^{-1}\|\phi_h(0)\|_{L^2(\Omega_j'')}\right) \end{aligned}$$

$$+ C_m\left(\frac{h}{d_j}\right)^m \big(\|\partial_t e\|_{L^2(Q_j'')} + d_j^{-1}\|\nabla e\|_{L^2(Q_j'')}\big) + C_m d_j^{-2}\|e\|_{L^2(Q_j'')}$$
$$+ C_m d_j^{-2}\|\widetilde{\phi}\|_{L^2(Q_T)} + C_m d_j^{-1}\|\nabla\widetilde{\phi}\|_{L^2(Q_T)} + C_m\|\partial_t\widetilde{\phi}\|_{L^2(Q_T)}$$
$$+ C_m d_j\|\nabla\partial_t\widetilde{\phi}\|_{L^2(Q_T)}$$

The last two inequalities imply

(3.21)
$$\|e_t\|_{L^2(Q_j)} + d_j^{-1}\|\nabla e\|_{L^2(Q_j)} \le C_m\big(I_j(\phi_h(0)) + X_j(\phi) + H_j(e) + d_j^{-2}\|e\|_{L^2(Q_j''')}\big).$$

We have proved that any $e = \phi_h - \phi$ satisfying (3.6) also satisfies (3.21). Since $e = \phi_h - \Pi_h\phi - (\phi - \Pi_h\phi)$ and $\phi(0) - \Pi_h\phi(0) = 0$ in Ω_j''', we can replace ϕ_h by $\phi_h - \Pi_h\phi$ and ϕ by $\phi - \Pi_h\phi$ in (3.21). Then (3.7) follows immediately. ∎

4. Proof of Lemma 2.2

Now we turn back to the proof of Lemma 2.2.

4.1. **The proof of (2.12).** In this subsection, we present several local energy estimates for the Green's function, the regularized Green's function and the discrete Green's function, which then are used to prove (2.12). These energy estimates will also be used to prove (2.11) in the next subsection. In this subsection we let $T = 1$ and fix $x_0 \in \Omega$. We write G and Γ as abbreviations for the functions $G(\cdot,\cdot,x_0)$ and $\Gamma(\cdot,\cdot,x_0)$, respectively, when there is no ambiguity. We use the decomposition of Section 2.3 for all $j \ge 1$ (not restricted to $j \le J_*$) and so we do not require $h < R_0K_0^{-2}/(16C_*)$ in this subsection.

Lemma 4.1. *For the Green's functions Γ, G and Γ_h defined in (2.5)-(2.7), we have the following estimates:*

(4.1)
$$\sum_{l,k=0}^{2} d_j^{2l+k-1+N/2}(\|\nabla^k\partial_t^l G(\cdot,\cdot,x_0)\|_{L^2(Q_j(x_0))} + \|\nabla^k\partial_t^l\Gamma(\cdot,\cdot,x_0)\|_{L^2(Q_j(x_0))}) \le C,$$

(4.2)
$$\|\nabla^2 G(\cdot,\cdot,x_0)\|_{L^{\infty,2}(\cup_{k\le j}Q_k(x_0))} \le Cd_j^{-N/2-2},$$

(4.3)
$$\|\nabla_{x_0}\partial_t G(\cdot,\cdot,x_0)\|_{L^\infty(Q_j(x_0))} \le Cd_j^{-N-3},$$

(4.4)
$$\|\partial_t G(\cdot,\cdot,x_0)\|_{L^1(\Omega\times(1,\infty))} + \|t\partial_{tt}G(\cdot,\cdot,x_0)\|_{L^1(\Omega\times(1,\infty))}$$
$$+ \|\partial_t\Gamma(\cdot,\cdot,x_0)\|_{L^1(\Omega\times(1,\infty))} + \|t\partial_{tt}\Gamma(\cdot,\cdot,x_0)\|_{L^1(\Omega\times(1,\infty))} \le C,$$

Proof. For the given x_0 and j, we define a coordinate transformation $x - x_0 = d_j\widetilde{x}$ and $t = d_j^2\widetilde{t}$, and define $\widetilde{G}(\widetilde{t},\widetilde{x}) := G(t,x,x_0)$, $\widetilde{a}(\widetilde{x}) := a(x)$. Via the coordinates transformation, we assume that the sets Q_j, Q_j', Ω_j, Ω_j' and Ω are transformed to $\widetilde{Q}_j$, $\widetilde{Q}_j'$, $\widetilde{\Omega}_j$, $\widetilde{\Omega}_j'$ and $\widetilde{\Omega}$, respectively. Let $0 \le \widetilde{\omega}_i(x,t) \le 1$, $i = 0,1,2,3$, be smooth cut-off functions which vanishes outside $\widetilde{Q}_j'$ and equals 1 in $\widetilde{Q}_j$. Moreover, $\widetilde{\omega}_i$ equals 1 at the points where $\widetilde{\omega}_{i+1} \ne 0$, and $|\nabla\widetilde{\omega}_i| \le C$, $|\partial_t\widetilde{\omega}_i| \le C$ for $i = 0,1,2,3$. Since $\cup_{k\ge j}\widetilde{\Omega}_k' \cup \widetilde{\Omega}_*$ is of unit size, there exists a convex domain $\widetilde{D} = B_\rho(z) \cap \widetilde{\Omega} \supset$

$\cup_{k\geq j}\widetilde\Omega'_k\cup\widetilde\Omega_*$, with $4\leq\rho\leq 8K_0^2$, which belongs to one of the following cases (there are only a finite number of shapes for $\widetilde D$):

(i) $z\in\widetilde\Omega$, $\rho=4$, and $B_4(z)$ has no intersection with the boundary of $\widetilde\Omega$, thus $B_\rho(z)\cap\widetilde\Omega=B_\rho(z)$,

(ii) z is on a face of $\widetilde\Omega$, $\rho=8$ and $B_8(z)$ has no intersection with other faces of $\widetilde\Omega$, thus $B_\rho(z)\cap\Omega$ is a half ball,

(iii) z is on an edge of $\widetilde\Omega$, $\rho=8K_0$ and $B_{8K_0}(z)$ has no intersection with any closed faces of $\widetilde\Omega$ which do not contain this edge, thus $B_\rho(z)\cap\Omega$ is the intersection of a ball with a sector spanned by the edge,

(iv) z is a corner of $\widetilde\Omega$ and $\rho=8K_0^2<R_0$, and $B_\rho(z)\cap\widetilde\Omega$ coincides with the intersection of the ball $B_\rho(z)$ with the cone spanned by the corner z.

Note that $\widetilde D\times(0,16)$ contains $\widetilde Q'_j$, and consider $\widetilde\omega_i\widetilde G$, $i=1,2$, which are solutions of

$$\partial_{\widetilde t}(\widetilde\omega_1\widetilde G)-\nabla_{\widetilde x}\cdot(\widetilde a\nabla_{\widetilde x}(\widetilde\omega_1\widetilde G))=\widetilde\omega_0\widetilde G\partial_{\widetilde t}\widetilde\omega_1+\widetilde\omega_0\widetilde G\nabla_{\widetilde x}\cdot\left(\widetilde a\nabla_{\widetilde x}\widetilde\omega_1\right)-\nabla_{\widetilde x}\cdot(2\widetilde a\widetilde\omega_0\widetilde G\nabla_{\widetilde x}\widetilde\omega_1) \tag{4.5}$$

and

$$\partial_{\widetilde t}(\widetilde\omega_2\widetilde G)-\nabla_{\widetilde x}\cdot(\widetilde a\nabla_{\widetilde x}(\widetilde\omega_2\widetilde G))=\widetilde\omega_1\widetilde G\partial_{\widetilde t}\widetilde\omega_2+\widetilde\omega_1\widetilde G\nabla_{\widetilde x}\cdot\left(\widetilde a\nabla_{\widetilde x}\widetilde\omega_2\right)-\nabla_{\widetilde x}\cdot(2\widetilde a\widetilde\omega_1\widetilde G\nabla_{\widetilde x}\widetilde\omega_2) \tag{4.6}$$

in the domain $\widetilde D\times(0,16)$, respectively, both with zero boundary/initial conditions. Since $\widetilde D$ is a convex domain, for $p=N+\alpha$ and $p_0=Np/(N+p)$ so that $W^{1,p}(\widetilde D)\hookrightarrow L^\infty(\widetilde D)$ and $W^{1,p_0}(\widetilde D)\hookrightarrow L^p(\widetilde D)$, the standard $L^p((0,16);W^{1,p}(\widetilde D))$ estimate of (4.5) (the inequality (2.3)-(2.4) with $p=q$, Lemma 2.1) gives

$$\begin{aligned}
&\|\nabla_{\widetilde x}(\widetilde\omega_1\widetilde G)\|_{L^p((0,16);L^p(\widetilde D))}\\
&\leq C\|\widetilde\omega_0\widetilde G\|_{L^p((0,16);L^{p_0}(\widetilde D))}+C\|\widetilde\omega_0\widetilde G\nabla_{\widetilde x}\widetilde a\|_{L^p((0,16);L^{p_0}(\widetilde D))}+C\|\widetilde\omega_0\widetilde G\|_{L^p((0,16);L^p(\widetilde D))}\\
&\leq C\|\widetilde\omega_0\widetilde G\|_{L^p((0,16);L^p(\widetilde D))}(C+C\|\nabla_{\widetilde x}\widetilde a\|_{L^N(\widetilde D)})\\
&\leq C\|\widetilde\omega_0\widetilde G\|_{L^p((0,16);L^p(\widetilde D))},
\end{aligned}$$

and the maximal L^p regularity of (4.6) yields that (see inequality (2.3), Lemma 2.1)

$$\begin{aligned}
&\|\partial_{\widetilde t}(\widetilde\omega_2\widetilde G))\|_{L^p((0,16);L^p(\widetilde D))}+\|\nabla_{\widetilde x}\cdot(\widetilde a\nabla_{\widetilde x}(\widetilde\omega_2\widetilde G))\|_{L^p((0,16);L^p(\widetilde D))}\\
&\leq C\|\widetilde\omega_1\widetilde G\|_{L^p((0,16);L^p(\widetilde D))}+C\|\widetilde\omega_1\widetilde G\nabla_{\widetilde x}\widetilde a\|_{L^p((0,16);L^p(\widetilde D))}\\
&\quad+C\|\nabla_{\widetilde x}(\widetilde\omega_1\widetilde G)\|_{L^p((0,16);L^p(\widetilde D))}\\
&\leq C\|\widetilde\omega_1\widetilde G\|_{L^p((0,16);L^\infty(\widetilde D))}(C+C\|\nabla_{\widetilde x}\widetilde a\|_{L^p(\widetilde D)})+C\|\nabla_{\widetilde x}(\widetilde\omega_1\widetilde G)\|_{L^p((0,16);L^p(\widetilde D))}\\
&\leq C\|\nabla_{\widetilde x}(\widetilde\omega_1\widetilde G)\|_{L^p((0,16);L^p(\widetilde D))}.
\end{aligned}$$

By using (2.1)-(2.2), we have

$$\|\nabla_{\widetilde x}(\widetilde\omega_2\widetilde G)\|_{L^\infty(\widetilde D)}+\sum_{k=0}^{2}\|\nabla^k_{\widetilde x}(\widetilde\omega_2\widetilde G)\|_{L^2(\widetilde D)}\leq C\|\nabla_{\widetilde x}\cdot(\widetilde a\nabla_{\widetilde x}(\widetilde\omega_2\widetilde G))\|_{L^p(\widetilde D)}.$$

The last three inequalities imply that

$$\begin{aligned}&\|\nabla_{\widetilde{x}}(\widetilde{\omega}_2\widetilde{G})\|_{L^2((0,16);L^\infty(\widetilde{D}))}+\sum_{k=0}^{2}\|\nabla_{\widetilde{x}}^k(\widetilde{\omega}_2\widetilde{G})\|_{L^2((0,16);L^2(\widetilde{D}))}\\&\quad+\|\partial_{\widetilde{t}}(\widetilde{\omega}_2\widetilde{G})\|_{L^{N+\alpha}((0,16);L^{N+\alpha}(\widetilde{D}))}\\&\le C\|\widetilde{\omega}_0\widetilde{G}\|_{L^{N+\alpha}((0,16);L^{N+\alpha}(\widetilde{D}))}.\end{aligned}$$

Similarly, replacing $\widetilde{G}$ by $\partial_{\widetilde{t}}\widetilde{G}$ and $\partial_{\widetilde{t}}^2\widetilde{G}$ in the above estimates, respectively, one can derive that

$$\begin{aligned}&\sum_{l=0}^{2}\|\nabla_{\widetilde{x}}(\widetilde{\omega}_3\partial_{\widetilde{t}}^l\widetilde{G})\|_{L^2((0,16);L^\infty(\widetilde{D}))}+\sum_{l,k=0}^{2}\|\nabla_{\widetilde{x}}^k(\widetilde{\omega}_3\partial_{\widetilde{t}}^l\widetilde{G})\|_{L^2((0,16);L^2(\widetilde{D}))}\\&\le C\|\widetilde{\omega}_0\widetilde{G}\|_{L^{N+\alpha}((0,16);L^{N+\alpha}(\widetilde{D}))}\le C\|\widetilde{\omega}_0\widetilde{G}\|_{L^\infty((0,16);L^\infty(\widetilde{D}))}.\end{aligned}$$

Since $\widetilde{\omega}_3\widetilde{G}\equiv 0$ at $t=0$, it follows that

$$\begin{aligned}\|\nabla_{\widetilde{x}}\partial_{\widetilde{t}}(\widetilde{\omega}_3\widetilde{G})\|_{L^\infty((0,16);L^\infty(\widetilde{D}))}&\le C\|\nabla_{\widetilde{x}}\partial_{\widetilde{t}}^2(\widetilde{\omega}_3\widetilde{G})\|_{L^2((0,16);L^\infty(\widetilde{D}))}\\&\le C\|\widetilde{\omega}_0\widetilde{G}\|_{L^\infty((0,16);L^\infty(\widetilde{D})))},\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{2}\|\nabla_{\widetilde{x}}^k(\widetilde{\omega}_3\widetilde{G})\|_{L^\infty((0,16);L^2(\widetilde{D}))}&\le C\sum_{k=0}^{2}\|\partial_{\widetilde{t}}\nabla_{\widetilde{x}}^k(\widetilde{\omega}_3\widetilde{G})\|_{L^2((0,16);L^2(\widetilde{D}))}\\&\le C\|\widetilde{\omega}_0\widetilde{G}\|_{L^\infty((0,16);L^\infty(\widetilde{D}))}.\end{aligned}$$

Moreover, from the last three inequalities, we have

$$\|\nabla_{\widetilde{x}}\partial_{\widetilde{t}}\widetilde{G}\|_{L^\infty(\widetilde{Q}_j)}+\sum_{k=0}^{2}\|\nabla_{\widetilde{x}}^k\widetilde{G}\|_{L^{\infty,2}(\widetilde{Q}_j)}+\sum_{l,k=0}^{2}\|\partial_{\widetilde{t}}^l\nabla_{\widetilde{x}}^k\widetilde{G}\|_{L^2(\widetilde{Q}_j)}\le C\|\widetilde{G}\|_{L^\infty(\widetilde{Q}_j')}.$$

Transforming back to the (x,t) coordinates, we see from the last two inequalities that

$$\begin{aligned}&d_j^3\|\nabla\partial_t G\|_{L^\infty(Q_j)}+\sum_{k=0}^{2}d_j^{k-N/2}\|\nabla^k G\|_{L^{\infty,2}(Q_j)}+\sum_{l,k=0}^{2}d_j^{2l+k-1-N/2}\|\partial_t^l\nabla^k G\|_{L^2(Q_j)}\\&\le C\|G\|_{L^\infty(Q_j')}\le Cd_j^{-N},\end{aligned}$$

where we have used (2.8) in the last inequality. By the symmetry of G with respect to x and x_0 we also get

$$d_j^3\|\nabla_{x_0}\partial_t G(\cdot,\cdot,x_0)\|_{L^\infty(Q_j)}\le C\|G\|_{L^\infty(Q_j')}\le Cd_j^{-N}.$$

This proves (4.1)-(4.3) for G.

By using the expression

$$\Gamma(x,t\,;x_0)=\int_\Omega G(x,t\,;y)\widetilde{\delta}(y,x_0)\mathrm{d}y. \tag{4.7}$$

one can derive the same estimates for Γ:

$$\|\nabla^k\partial_t^l\Gamma(\cdot,\cdot\,;x_0)\|_{L^2(Q_j)}\le\int_\Omega\|\nabla^k\partial_t^l G(\cdot,\cdot\,;y)\|_{L^2(Q_j)}|\widetilde{\delta}(y,x_0)|\mathrm{d}y$$

$$\leq \int_\Omega C d_j^{-2l-k+1-N/2} |\widetilde{\delta}(y,x_0)| \mathrm{d}y \leq C d_j^{-2l-k+1-N/2}.$$

Finally, we note that

$$\begin{aligned}
&\|\partial_t \Gamma(t,\cdot,x_0)\|_{L^\infty} + t\|\partial_{tt}\Gamma(t,\cdot,x_0)\|_{L^\infty} \\
&\leq C\|\partial_t \Gamma(t,\cdot,x_0)\|_{H^2} + Ct\|\partial_{tt}\Gamma(t,\cdot,x_0)\|_{H^2} \\
&\leq C\|\partial_t A\Gamma(t,\cdot,x_0)\|_{L^2} + Ct\|\partial_{tt}A\Gamma(t,\cdot,x_0)\|_{L^2} \quad [H^2 \text{ estimate, Lemma 2.1}] \\
&= C\|\partial_{tt}\Gamma(t,\cdot,x_0)\|_{L^2} + Ct\|\partial_{ttt}\Gamma(t,\cdot,x_0)\|_{L^2} \\
&\leq Ct^{-2}\|\Gamma(t/2,\cdot,x_0)\|_{L^2} \quad [\text{semigroup estimate}] \\
&\leq Ct^{-2-N/2}, \qquad (4.8)
\end{aligned}$$

which implies the first part of (4.4) and the second part of (4.4) (the estimates of G) can be proved in the same way.

The proof of Lemma 4.1 is completed. ∎

When $\max(t^{1/2},|x-x_0|) < d_1$ the inequality (2.12) follows from (4.3). When $\max(t^{1/2},|x-x_0|) \geq d_1$, the estimate $\|\nabla_{x_0}\partial_t G(\cdot,\cdot,x_0)\|_{L^\infty(Q_j(x_0))} \leq C$ can be proved directly (without using the scale transformation) in the same way as above.

4.2. **Proof of (2.11).** The proof is also based on Lemma 4.1.

First we consider the case $h < h_0 := (R_0 K_0^{-2}/(16C_*)$ and let $T = 1$. The basic energy estimates of the equations (2.6)-(2.7) yield

$$\begin{aligned}
\|\partial_t\Gamma_h\|_{L^2(Q_T)} + \|\partial_t\Gamma\|_{L^2(Q_T)} &\leq \|\nabla\Gamma_h(0)\|_{L^2(\Omega)} + \|\nabla\Gamma(0)\|_{L^2(\Omega)} \\
&= \|\nabla P_h\widetilde{\delta}_{x_0}\|_{L^2(\Omega)} + \|\nabla\widetilde{\delta}_{x_0}\|_{L^2(\Omega)} \leq Ch^{-1-N/2}
\end{aligned}$$

and

$$\begin{aligned}
\|\partial_{tt}\Gamma_h\|_{L^2(Q_T)} + \|\partial_{tt}\Gamma\|_{L^2(Q_T)} &\leq \|\nabla\partial_t\Gamma_h(0)\|_{L^2(\Omega)} + \|\nabla\partial_t\Gamma(0)\|_{L^2(\Omega)} \\
&= \|\nabla A_h P_h\widetilde{\delta}_{x_0}\|_{L^2(\Omega)} + \|\nabla A\widetilde{\delta}_{x_0}\|_{L^2(\Omega)} \leq Ch^{-3-N/2},
\end{aligned}$$

which imply

$$\begin{aligned}
\|\partial_t F\|_{L^2(Q_*)} + \|t\partial_{tt}F\|_{L^2(Q_*)} &\leq Ch^{-1-N/2} + Cd_{J_*}^2 h^{-3-N/2} \\
&\leq Ch^{-1-N/2} + CC_*^2 h^{-1-N/2}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\|\partial_t F\|_{L^1(Q_T)} + \|t\partial_{tt}F\|_{L^1(Q_T)} \\
&\leq Cd_{J_*}^{1+N/2}\big(\|\partial_t F\|_{L^2(Q_*)} + \|t\partial_{tt}F\|_{L^2(Q_*)}\big) \\
&\quad + \sum_j Cd_j^{1+N/2}\big(\|\partial_t F\|_{L^2(Q_j)} + \|t\partial_{tt}F\|_{L^2(Q_j)}\big) \\
&\leq CC_*^{3+N/2} + \sum_j Cd_j^{1+N/2}\big(\|\partial_t F\|_{L^2(Q_j)} + \|t\partial_{tt}F\|_{L^2(Q_j)}\big) \\
&\leq CC_*^{3+N/2} + C\mathcal{K}, \qquad (4.9)
\end{aligned}$$

where

$$\mathcal{K} := \sum_j d_j^{1+N/2}(d_j^{-1}\|\nabla F\|_{L^2(Q_j)} + \|\partial_t F\|_{L^2(Q_j)} + d_j^2\|\partial_{tt}F\|_{L^2(Q_j)}). \qquad (4.10)$$

We proceed to estimate $\mathcal{K}$. We set $e = F$ ($\phi_h = \Gamma_h$ and $\phi = \Gamma$) and $e = \partial_t F$ ($\phi_h = \partial_t\Gamma_h$ and $\phi = \partial_t\Gamma$) in (3.6) (Proposition 3.2), respectively, and note that $\Gamma(0) = \partial_t\Gamma(0) = 0$ on Ω'_j. We obtain that

$$\begin{aligned} &d_j^{-1}\|\nabla F\|_{L^2(Q_j)} + \|\partial_t F\|_{L^2(Q_j)} + d_j^2\|\partial_{tt}F\|_{L^2(Q_j)} \\ &\leq C(\widehat{I}_j + \widehat{X}_j + \widehat{H}_j + d_j^{-2}\|F\|_{L^2(Q'_j)}) \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} \widehat{I}_j &= \|\nabla P_h\widetilde{\delta}_{x_0}\|_{L^2(\Omega'_j)} + d_j^{-1}\|P_h\widetilde{\delta}_{x_0}\|_{L^2(\Omega'_j)} \\ &\quad + d_j^2\|\nabla A_hP_h\widetilde{\delta}_{x_0}\|_{L^2(\Omega'_j)} + d_j\|A_hP_h\widetilde{\delta}_{x_0}\|_{L^2(\Omega'_j)}, \\ \widehat{X}_j &= d_j\|\nabla\partial_t(\Pi_h\Gamma-\Gamma)\|_{L^2(Q'_j)} + \|\partial_t(\Pi_h\Gamma-\Gamma)\|_{L^2(Q'_j)} + d_j^{-1}\|\nabla(\Pi_h\Gamma-\Gamma)\|_{L^2(Q'_j)} \\ &\quad + d_j^{-2}\|\Pi_h\Gamma-\Gamma\|_{L^2(Q'_j)} + d_j^3\|\nabla\partial_{tt}(\Pi_h\Gamma-\Gamma)\|_{L^2(Q'_j)} + d_j^2\|\partial_{tt}(\Pi_h\Gamma-\Gamma)\|_{L^2(Q'_j)} \\ \widehat{H}_j &= \big(h/d_j\big)^m\big(\|\partial_t F\|_{L^2(Q'_j)} + d_j^{-1}\|\nabla F\|_{L^2(Q'_j)} + d_j^2\|\partial_{tt}F\|_{L^2(Q'_j)} + d_j\|\nabla\partial_t F\|_{L^2(Q'_j)}\big)\,. \end{aligned}$$

By noting the exponential decay of $P_h\widetilde{\delta}_{x_0}(y)$ (see (P2) in section 2.2) we derive

$$\begin{aligned} \widehat{I}_j &\leq Chd_j^{-2-N/2}, \\ \widehat{X}_j &\leq (d_jh+h^2)\|\nabla^2\partial_t\Gamma\|_{L^2(Q''_j)} + (d_j^{-1}h + d_j^{-2}h^2)\|\nabla^2\Gamma\|_{L^2(Q''_j)} \\ &\quad + (d_j^3h + d_j^2h^2)\|\nabla^2\partial_{tt}\Gamma\|_{L^2(Q''_j)} \\ &\leq Chd_j^{-2-N/2}, \quad \text{[by using Lemma 4.1]} \\ \widehat{H}_j &\leq \big(h/d_j\big)^m\big(\|\partial_t F\|_{L^2(Q_T)} + d_j^{-1}\|\nabla F\|_{L^2(Q_T)} + d_j^2\|\partial_{tt}F\|_{L^2(Q_T)} + d_j\|\nabla\partial_t F\|_{L^2(Q_T)}\big) \\ &\leq C\big(h/d_j\big)^m\big(\|P_h\widetilde{\delta}_{x_0}\|_{H^1} + d_j^{-1}\|P_h\widetilde{\delta}_{x_0}\|_{L^2} + d_j^2\|A_hP_h\widetilde{\delta}_{x_0}\|_{H^1} + d_j\|A_hP_h\widetilde{\delta}_{x_0}\|_{L^2}\big) \\ &\leq C\big(h/d_j\big)^m\big(h^{-1-N/2} + d_j^{-1}h^{-N/2} + d_j^2h^{-3-N/2} + d_jh^{-2-N/2}\big) \quad \text{[by (P3)-(P4)]} \\ &\leq Chd_j^{-2-N/2}, \quad \text{[by choosing } m = 4+N/2\text{]} \end{aligned}$$

Therefore, by (4.10),

$$\begin{aligned} \mathcal{K} &\leq \sum_j Chd_j^{-1} + \sum_j Cd_j^{-1+N/2}\|F\|_{L^2(Q'_j)} \\ &\leq C + C\sum_j d_j^{-1+N/2}\|F\|_{L^2(Q'_j)}. \end{aligned} \tag{4.12}$$

To estimate $\|F\|_{L^2(Q'_j)}$, we apply a duality argument. Let w be the solution of the backward parabolic equation

$$\begin{cases} -\partial_t w + Aw = v & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \\ w(T) = 0 & \text{in } \Omega, \end{cases}$$

where v is a function which is supported on Q'_j and $\|v\|_{L^2(Q'_j)} = 1$. Multiplying the above equation by F, with integration by parts we get

$$\iint_{Q_T} Fv\,dxdt = (F(0), w(0)) + \iint_{Q_T} \partial_t Fw\,dxdt + \sum_{i,j=1}^N \iint_{Q_T} a_{ij}\partial_j F\partial_i w\,dxdt, \tag{4.13}$$

where

$$\begin{aligned}(F(0),w(0)) &= (P_h\widetilde{\delta}_{x_0}-\widetilde{\delta}_{x_0},w(0))\\ &= (P_h\widetilde{\delta}_{x_0}-\widetilde{\delta}_{x_0},w(0)-I_hw(0))\\ &= (P_h\widetilde{\delta}_{x_0},w(0)-I_hw(0))_{\Omega_j''}+(P_h\widetilde{\delta}_{x_0}-\widetilde{\delta}_{x_0},w(0)-I_hw(0))_{(\Omega_j'')^c}\\ &:= \mathcal{I}_1+\mathcal{I}_2.\end{aligned}$$

By using the exponential decay of $P_h\widetilde{\delta}_{x_0}$ (see (P4) of section 2) and the local approximation property (see (P2) of section 2), we derive that

$$\begin{aligned}|\mathcal{I}_1| &\leq Ch\|P_h\widetilde{\delta}_{x_0}\|_{L^2(\Omega_j'')}(d_j^{-1}\|w(0)\|_{L^2(\Omega)}+\|\nabla w(0)\|_{L^2(\Omega)})\\ &\leq Ch^{-N/2+1}e^{-Cd_j/h}(d_j^{-1}\|v\|_{L^{2(N+2)/(N+4)}(Q_j')}+\|v\|_{L^2(Q_j')})\\ &\leq Ch^{-N/2+1}e^{-Cd_j/h}\|v\|_{L^2(Q_j')}\\ &\leq C(d_j/h)^{1+N/2}e^{-Cd_j/h}h^2d_j^{-1-N/2}\\ &\leq Ch^2d_j^{-1-N/2}, \end{aligned} \tag{4.14}$$

$$\begin{aligned}|\mathcal{I}_2| &\leq C\|\widetilde{\delta}_{x_0}\|_{L^2}\|w(0)-I_hw(0)\|_{L^2((\Omega_j'')^c)}\\ &\leq Ch^{2-N/2}\sum_{k=0}^{2}d_j^{k-2}\|\nabla^k w(0)\|_{L^2((\Omega_j'')^c)}.\end{aligned} \tag{4.15}$$

To estimate $\|\nabla^k w(0)\|_{L^2((\Omega_j'')^c)}$, we let W_j be a set containing $(\Omega_j'')^c$ but its distance to Ω_j' is larger than $d_j/8$. Since

$$\nabla_x^k w(x,0)=\int_0^T\int_\Omega \nabla_x^k G(s,x,y)v(y,s)\mathrm{d}y\mathrm{d}s,$$

by noting the fact

$$|x-y|+s^{1/2}\geq d_j/8 \quad \text{for } x\in W_j \text{ and } (y,s)\in Q_j'$$

and using (4.2), we further derive

$$\begin{aligned}&\sum_{k=0}^{2}d_j^{k-2}\|\nabla^k w(\cdot,0)\|_{L^2(W_j)}\\ &\leq C\sum_{k=0}^{2}d_j^{k-2}\sup_{y\in\Omega}\|\nabla^k G(\cdot,\cdot,y)\|_{L^{\infty,2}(\bigcup_{k\leq j-3}Q_k(y))}\|v\|_{L^1(Q_j')}\\ &\leq \sum_{k=0}^{2}d_j^{k-2}d_j^{-N-k+N/2}\|v\|_{L^1(Q_j')}\\ &\leq \sum_{k=0}^{2}d_j^{k-2}d_j^{-N-k+N/2}d_j^{N/2+1}\|v\|_{L^2(Q_j')}=Cd_j^{-1}.\end{aligned} \tag{4.16}$$

From (4.14)-(4.16), we see that the first term on the right-hand side of (4.13) is bounded by

$$|(F(0),w(0))|\leq Ch^2d_j^{-N/2-1}+Ch^{2-N/2}d_j^{-1}\leq Ch^{2-N/2}d_j^{-1}, \tag{4.17}$$

and the rest terms are bounded by

$$
\begin{aligned}
&\iint_{Q_T} \partial_t F w dxdt + \sum_{i,j=1}^N \iint_{Q_T} a_{ij}\partial_j F \partial_i w dxdt \\
&\qquad = \iint_{Q_T} \partial_t F(w-\Pi_h w) dxdt + \sum_{i,j=1}^N \iint_{Q_T} a_{ij}\partial_j F \partial_i (w-\Pi_h w) dxdt \\
&\qquad \le \sum_{*,i} C\|\nabla^2 w\|_{L^2(Q_i')}(h^2\|\partial_t F\|_{L^2(Q_i)} + h\|\nabla F\|_{L^2(Q_i)}). \qquad (4.18)
\end{aligned}
$$

Moreover, to estimate $\|\nabla^2 w\|_{L^2(Q_i')}$ we consider the expression

$$\nabla_x^2 w(x,t) = \int_0^T \int_\Omega \nabla_x^2 G(s-t,x,y)v(y,s)1_{s>t}\, dyds.$$

For $i \le j-3$ (so that $d_i > d_j$), we see that $w(x,t)=0$ for $t > 16d_j^2$ (because v is supported in Q_j'); $d_i/2 \le |x-y| \le 4d_i$ and $s-t < d_i^2$ for $t < 16d_j^2$, $(x,t)\in Q_i$ and $(y,s)\in Q_j'$. Therefore, $(x,t)\in Q_i'(y)$ and we obtain

$$
\begin{aligned}
\|\nabla^2 w\|_{L^2(Q_i')} &\le \sup_y \|\nabla^2 G(\cdot,\cdot,y)\|_{L^2(Q_i'(y))}\|v\|_{L^1(Q_j')} \\
&\le C d_i^{-N/2-1} d_j^{N/2+1}\|v\|_{L^2(Q_i')} \\
&\le C(d_j/d_i)^{N/2+1} \le \frac{Cd_j}{d_i}.
\end{aligned}
$$

For $i \ge j+3$ (so that $d_i < d_j$), $\max(|s-t|^{1/2}, |x-y|) \ge d_{j+1}$ for $(x,t)\in Q_i$ and therefore,

$$
\begin{aligned}
\|\nabla^2 w\|_{L^2(Q_i')} \le &\sup_{y\in\Omega} \|\nabla^2 G(\cdot,\cdot,y)1_{\bigcup_{k\le j+1} Q_k(y)}\|_{L^2(Q_i')}\|v\|_{L^1(Q_j')} \\
&\le C d_i \sup_y \|\nabla^2 G(\cdot,\cdot,y)\|_{L^{\infty,2}(\bigcup_{k\le j+1} Q_k(y))}\|v\|_{L^2(Q_j')} d_j^{N/2+1} \\
&\le C d_i d_j^{-N-2+N/2} d_j^{N/2+1} = \frac{Cd_i}{d_j}.
\end{aligned}
$$

Finally for $|i-j|\le 2$, applying the standard energy estimate leads to

$$\|w\|_{L^2((0,T);H^2)} \le C\|v\|_{L^2(Q_T)} = C.$$

Combining the three cases, we have

$$\|\nabla^2 w\|_{L^2(Q_i')} \le C\min\big(d_i/d_j, d_j/d_i\big) := Cm_{ij}. \qquad (4.19)$$

Substituting (4.17)-(4.19) into (4.13) gives the estimate

$$\|F\|_{L^2(Q_j')} \le Ch^2 d_j^{-N/2-1} + C\sum_{*,i} m_{ij}(h^2\|\partial_t F\|_{L^2(Q_i)} + h\|\nabla F\|_{L^2(Q_i)}), \qquad (4.20)$$

which together with (4.12) implies

$$
\begin{aligned}
\mathcal{K} &\le C + C\sum_j \left(\frac{h}{d_j}\right)^{2-N/2} + C\sum_j d_j^{N/2-1}\sum_{*,i} m_{ij}\big(h^2\|\partial_t F\|_{L^2(Q_i)} + h\|\nabla F\|_{L^2(Q_i)}\big) \\
&\le C + C\sum_{*,i}\big(h^2\|\partial_t F\|_{L^2(Q_i)} + h\|\nabla F\|_{L^2(Q_i)}\big)\sum_j d_j^{N/2-1} m_{ij}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C + C\sum_{*,i}\left(h^2\|\partial_t F\|_{L^2(Q_i)} + h\|\nabla F\|_{L^2(Q_i)}\right) d_i^{N/2-1}\\
&\leq C + C\left(h^2\|\partial_t F\|_{L^2(Q_*)} + h\|\nabla F\|_{L^2(Q_*)}\right)(C_* h)^{N/2-1}\\
&\quad + C\sum_i d_i^{1+N/2}\left(\|\partial_t F\|_{L^2(Q_i)} + d_i^{-1}\|\nabla F\|_{L^2(Q_i)}\right)\left(\frac{h}{d_i}\right)\\
&\leq C + CC_*^{-1+N/2} + C\sum_i d_i^{1+N/2}\left(\|\partial_t F\|_{L^2(Q_i)} + d_i^{-1}\|\nabla F\|_{L^2(Q_i)}\right)\left(\frac{h}{d_i}\right)\\
&\leq C_2 + C_2 C_*^{-1+N/2} + C_2 C_*^{-1}\mathcal{K}
\end{aligned}
$$

for some positive constant C_2. By choosing

$$
C_* = \max(10, 10\kappa, R_0 K_0^{-2}/8) + 2C_2, \tag{4.21}
$$

the above inequality shows that $\mathcal{K} \leq C$.

Returning to (4.9), the boundedness of $\mathcal{K}$ implies

$$
\|\partial_t F\|_{L^1(Q_T)} + \|t\partial_{tt} F\|_{L^1(Q_T)} \leq C. \tag{4.22}
$$

From (4.10) we also see that, the boundedness of $\mathcal{K}$ implies

$$
\|\partial_t F\|_{L^2(Q_j)} + \|t\partial_{tt} F\|_{L^2(Q_j)} \leq C d_j^{-1-N/2}.
$$

Since $\Omega \times (1/4, 1) \subset \cup_{d_j \geq 1/2} Q_j$, it follows that

$$
\begin{aligned}
&\|\partial_t F\|^2_{L^2(\Omega\times(1/4,1))} + \|t\partial_{tt} F\|^2_{L^2(\Omega\times(1/4,1))}\\
&\leq \sum_{d_j \geq 1/2}\left(\|\partial_t F\|^2_{L^2(Q_j)} + \|t\partial_{tt} F\|^2_{L^2(Q_j)}\right)\\
&\leq \sum_{d_j \geq 1/2} C d_j^{-2-N} \leq C.
\end{aligned}
$$

The above inequality and (4.8) imply

$$
\|\partial_t \Gamma_h\|_{L^2(\Omega\times(1/4,1))} + \|t\partial_{tt}\Gamma_h\|_{L^2(\Omega\times(1/4,1))} \leq C.
$$

Furthermore, differentiating the equation (2.7) with respect to t and multiplying the result by $\partial_t \Gamma_h$ give

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t \Gamma_h(t,\cdot,x_0)\|^2_{L^2} + c_0\|\partial_t \Gamma_h(t,\cdot,x_0)\|^2_{L^2}\\
&\leq \frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t \Gamma_h(t,\cdot,x_0)\|^2_{L^2} + (A_h\partial_t\Gamma_h(t,\cdot,x_0), \partial_t\Gamma_h(t,\cdot,x_0)) = 0, \quad \text{for} \quad t \geq 1,
\end{aligned}
$$

which further shows that

$$
\|\partial_t \Gamma_h(t,\cdot,x_0)\|^2_{L^2} \leq Ce^{-c_0(t-1)}\|\partial_t\Gamma_h(\cdot,\cdot,x_0)\|^2_{L^2((1/4,1);L^2(\Omega))} \leq Ce^{-c_0 t} \quad \text{for } t \geq 1.
$$

In a similar way one can derive $\|\partial_{tt}\Gamma_h(t,\cdot,x_0)\|_{L^2} \leq Ce^{-c_0 t}$ for $t \geq 1$. These inequalities together with (4.4) imply

$$
\|\partial_t F(\cdot,\cdot,x_0)\|_{L^1(\Omega\times(1,\infty))} + \|t\partial_{tt} F(\cdot,\cdot,x_0)\|_{L^1(\Omega\times(1,\infty))} \leq C, \tag{4.23}
$$

which together with (4.22) leads to (2.11) for the case $h < h_0 := R_0 K_0^{-2}/(16C_*)$ with C_* being given by (4.21).

Secondly when $h \geq h_0$, the decomposition in subsection 2.3 is not needed and the energy estimates of (2.6)-(2.7) yield

$$\|\partial_t \Gamma_h\|_{L^2(Q_T)} + \|\partial_t \Gamma\|_{L^2(Q_T)} \leq \|\nabla \Gamma_h(0)\|_{L^2(\Omega)} + \|\nabla \Gamma(0)\|_{L^2(\Omega)} \leq C h_0^{-1-N/2}$$

$$\|\partial_{tt} \Gamma_h\|_{L^2(Q_T)} + \|\partial_{tt} \Gamma\|_{L^2(Q_T)} \leq \|\nabla \partial_t \Gamma_h(0)\|_{L^2(\Omega)} + \|\nabla \partial_t \Gamma(0)\|_{L^2(\Omega)} \leq C h_0^{-3-N/2},$$

which imply

$$\int_0^1 \int_\Omega \left(|\partial_t F(t,x,x_0)| + |t\partial_{tt} F(t,x,x_0)|\right) \mathrm{d}x\mathrm{d}t \leq C. \tag{4.24}$$

Since both $\|\partial_t \Gamma_h(t)\|_{L^2(\Omega)} + \|\partial_{tt} \Gamma_h(t)\|_{L^2(\Omega)}$ and $\|\partial_t \Gamma(t)\|_{L^2(\Omega)} + \|\partial_{tt} \Gamma(t)\|_{L^2(\Omega)}$ decay exponentially as $t \to \infty$, it follows that (2.11) still holds when $h \geq h_0$.

The proof of Lemma 2.2 is completed. ∎

5. Conclusion

In this paper we have proved that the discrete elliptic operator $-A_h$ generates a bounded analytic semigroup and has the maximal L^p regularity, uniformly with respect to h, in arbitrary convex polygons and polyhedra under the regularity assumption $a_{ij} \in W^{1,N+\alpha}$. We have assumed the quasi-uniformity of the triangulation, and analysis of the problem under non-quasi-uniform triangulations remains open. As far as we know, only the analytic semigroup estimate (1.9) and its equivalent resolvent estimate were studied with an extra logarithmic factor for some special cases of non-quasi-uniform triangulations, see [6, 44]. The discrete maximal regularity estimates (1.10) and (1.13) have not been established with more general triangulations even in smooth settings.

Appendix: The proof of Lemma 2.1

Proof of (2.1): The inequality (2.1) is similar to Theorem 3.1.3.1 of [17], which was proved by using the local energy inequality of Lemma 3.1.3.2, and the lemma was proved under the assumption $a_{ij} \in W^{1,\infty}(\Omega)$, where Ω is a convex domain. In the following, we show that this assumption can be relaxed to $a_{ij} \in W^{1,N+\alpha}(\Omega) \hookrightarrow C^\gamma(\overline{\Omega})$, where $\gamma = \alpha/(N+\alpha)$

Lemma A.1. *If Ω is convex and $a_{ij} \in W^{1,N+\alpha}(\Omega)$, then each point $y \in \Omega$ has a neighborhood $B_R(y) \cap \Omega$ such that*

$$\|u\|_{H^2(\Omega)} \leq C\|Au\|_{L^2(\Omega)} + C\|u\|_{H^1(\Omega)}. \tag{A.1}$$

for all $u \in H^2 \cap H^1_0$ such that the support of u is contained in $B_R(y) \cap \overline{\Omega}$. The radius R depends only on the semi-norms $|a_{ij}|_{W^{1,N+\alpha}(\Omega)}$ and $|a_{ij}|_{C^\gamma(\overline{\Omega})}$.

Proof. Following the proof of Lemma 3.1.3.2 in [17] (see page 143, (3.1.3.4) and the equality above (3.1.3.5)), we have (using our notations)

$$\begin{aligned}\|u\|_{H^2(\Omega)} &\leq C\|Au\|_{L^2(\Omega)} + C\sum_{i,j=1}^N \max_{x\in V_y} |a_{ij}(y) - a_{ij}(x)| \|u\|_{H^2(\Omega)} \\ &\quad + C\sum_{i,j=1}^N \|\partial_i a_{ij} \partial_j u\|_{L^2(\Omega)} \\ &\leq C\|Au\|_{L^2(\Omega)} + CR^\beta \|u\|_{H^2(\Omega)}\end{aligned}$$

$$+ C \sum_{i,j=1}^{N} \|\nabla a_{ij}\|_{L^{N+\alpha}(\Omega)} \|\nabla u\|_{L^{2(N+\alpha)/(N-2+\alpha)}(\Omega)}.$$

When R is small enough we have

$$\|u\|_{H^2(\Omega)} \le C\|Au\|_{L^2(\Omega)} + C\|\nabla u\|_{L^{2(N+\alpha)/(N-2+\alpha)}(\Omega)}.$$

Since H^2 is compactly embedded into $W^{1,2(N+\alpha)/(N-2+\alpha)}$ which is again embedded into H^1, there exists $\theta_\alpha \in (0,1)$ such that

$$\|\nabla u\|_{L^{2(N+\alpha)/(N-2+\alpha)}(\Omega)} \le \epsilon\|u\|_{H^2(\Omega)} + C_\epsilon\|u\|_{H^1(\Omega)}, \quad \forall \epsilon \in (0,1).$$

Choosing ϵ small enough, (A.1) follows from the last the last two inequalities This completes the proof of Lemma A.1. ∎

Then (2.1) can be proved by using Lemma A.1 and a perturbation procedure (as mentioned in the proof of [17, Theorem 3.1.3.1]).

Proof of (2.2): Theorem 3.4 of [18] states that if $\Omega \subset \mathbb{R}^N$ ($N \ge 3$) is convex and the coefficients a_{ij} are Hölder continuous (so that (3.1)-(3.3) of [18] hold), the Green's function of the elliptic operator A with the Dirichlet boundary condition satisfies

$$|\nabla_x G_e(x,y)| + |\nabla_y G_e(x,y)| \le \frac{C}{|x-y|^{N-1}}, \tag{A.2}$$

where we have used the symmetry $G_e(x,y) = G_e(y,x)$. Therefore, any H_0^1 solution of the equation $-\nabla\cdot(a\nabla u) = f$ satisfies

$$|\nabla u(x)| = \left|\int_\Omega \nabla_x G_e(x,y) f(y)\mathrm{d}y\right| \le \left\|\frac{C}{|x-y|^{N-1}}\right\|_{L^{p'}(\Omega)} \|f\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}$$

for $p > N$. As pointed out in [13] (page 227, the paragraph below Proposition 1), the inequality (A.2) for $N = 2$ can be proved with some minor modifications on the proof of [18, Theorem 3.3–3.4] since Theorem 3.3–3.4 of [18] only requires a_{ij} being Hölder continuous coefficients.

Proof of (2.3)-(2.4): Since $W^{1,N+\alpha}(\Omega) \hookrightarrow C(\overline{\Omega})$, Theorem 1 of [21] implies that the solution of the elliptic equation

$$\begin{cases} Au = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.3}$$

with continuous coefficients a_{ij} in a convex domain $\Omega \subset \mathbb{R}^N$ satisfies

$$\|u\|_{W^{1,q}(\Omega)} \le C\|f\|_{W^{-1,q}(\Omega)}, \quad \forall\, 1 < q < \infty, \tag{A.4}$$

where we have noted that a continuous function is (δ, R) vanishing, and a convex domain is (δ, σ, R)-quasiconvex [21]. Since the solution of (1.1) with $f = 0$ satisfies (integrating the equation against $|u|^{q-2}u$)

$$\|u\|_{L^q(\Omega)} \le \|u^0\|_{L^q(\Omega)},$$

it follows that the semigroup generated by the elliptic operator A is a contraction semigroup on $L^q(\Omega)$. By Theorem 1, Section 2, Chapter 3 of [43], the semigroup $\{E(t)\}_{t>0}$ generated by the elliptic operator A has an analytic continuation (analyticity of the semigroup $\{E(t)\}_{t>0}$). Moreover, by the maximum principle we have $u^0 \ge 0 \implies u \ge 0$ (positivity of the semigroup $\{E(t)\}_{t>0}$) and then, by Corollary 4.d of [50], the solution of (1.1) with $u^0 = 0$ has the maximal L^p regularity (2.3).

In other words, the map from f to Au given by the formula

$$Au = \int_0^t AE(t-s)f(\cdot,s)\mathrm{d}s$$

is bounded in $L^p((0,T);L^q)$, for all $1<p,q<\infty$. Since

$$A^{1/2}u = \int_0^t A^{1/2}E(t-s)f(\cdot,s)\mathrm{d}s = \int_0^t AE(t-s)A^{-1/2}f(\cdot,s)\mathrm{d}s,$$

it follows that

$$\|A^{1/2}u\|_{L^p((0,T);L^q)} \le C_{p,q}\|A^{-1/2}f\|_{L^p((0,T);L^q)}, \quad \forall\, 1<p,q<\infty. \tag{A.5}$$

It remains to prove the boundedness of the Riesz transform $\nabla A^{-1/2}$:

$$\|\nabla A^{-1/2}f\|_{L^q} \le C_q\|f\|_{L^q}, \quad \forall\, 1<q<\infty. \tag{A.6}$$

Then the last two inequalities imply

$$\begin{aligned}\|\nabla u\|_{L^p((0,T);L^q)} &= \|\nabla A^{-1/2}(A^{1/2}u)\|_{L^p((0,T);L^q)}\\ &\le C_q\|A^{1/2}u\|_{L^p((0,T);L^q)}\\ &\le C_{p,q}\|A^{-1/2}f\|_{L^p((0,T);L^q)}\\ &\le C_{p,q}\|f\|_{W^{-1,q}}, \qquad\qquad \forall\, 1<p,q<\infty,\end{aligned}$$

where the last step of the inequality above is due to the following duality argument ($A^{-1/2}$ is self-adjoint):

$$(A^{-1/2}f,g) = (f,A^{-1/2}g) \le C\|f\|_{W^{-1,q}}\|\nabla A^{-1/2}g\|_{L^{q'}} \le C\|f\|_{W^{-1,q}}\|g\|_{L^{q'}}.$$

It has been proved in [42, Theorem B] that the Riesz transform is bounded on $L^q(\Omega)$ (i.e. the inequality (A.6) holds) if and only if the solution of the homogeneous equation

$$Au = 0 \tag{A.7}$$

satisfies the local estimate

$$\left(\frac{1}{r^N}\int_{\Omega\cap B_r(x_0)}|\nabla u|^q\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\frac{1}{r^N}\int_{\Omega\cap B_{\sigma_0 r}(x_0)}|\nabla u|^2\mathrm{d}x\right)^{\frac{1}{2}} \tag{A.8}$$

for all $x_0\in\Omega$ and $0<r<r_0$, where r_0 and $\sigma_0\ge 2$ are any given small positive constants such that $\Omega\cap B_{\sigma_0 r_0}(x_0)$ can be given by the intersection of $B_{\sigma_0 r_0}(x_0)$ with a Lipschitz graph. It remains to prove (A.8).

Let ω be a smooth cut-off function which equals zero outside $B_{2r} := B_{2r}(x_0)$ and equals 1 on B_r. Extend u to be zero on $B_{2r}\backslash\Omega$ and denote by u_{2r} the average of u over B_{2r}. Then (A.7) implies

$$\begin{aligned}&\sum_{i,j=1}^N \partial_i(a_{ij}\partial_j(\omega(u-u_{2r})))\\ &= \sum_{i,j=1}^N \partial_i(a_{ij}(u-u_{2r})\partial_j\omega) + \sum_{i,j=1}^N a_{ij}\partial_i\omega\partial_j(u-u_{2r}) \quad \text{in } \Omega,\end{aligned} \tag{A.9}$$

and the $W^{1,q}$ estimate (A.4) implies

$$\begin{aligned}\|\omega(u-u_{2r})\|_{W^{1,q}(\Omega)} &\le C\|(u-u_{2r})\partial_j\omega\|_{L^q(\Omega)} + C\|\partial_i\omega\partial_j u\|_{W^{-1,q}(\Omega)}\\ &\le C\|(u-u_{2r})\partial_j\omega\|_{L^q(\Omega)} + C\|\partial_i\omega\partial_j u\|_{L^s(\Omega)}\end{aligned}$$

$$= C\|(u-u_{2r})\partial_j\omega\|_{L^q(B_{2r})} + C\|\partial_i\omega\partial_j u\|_{L^s(B_{2r})}$$
$$\leq Cr^{-1}\|\nabla u\|_{L^s(B_{2r})},$$

where $s = qN/(q+N) < q$ satisfies $L^s(\Omega) \hookrightarrow W^{-1,q}(\Omega)$ and $W^{1,s}(\Omega) \hookrightarrow L^q(\Omega)$. The last inequality implies

$$\|\nabla u\|_{L^q(\Omega\cap B_r)} \leq Cr^{-1}\|\nabla u\|_{L^s((\Omega\cap B_{2r})}. \tag{A.10}$$

If $s \leq 2$ then one can derive

$$\|\nabla u\|_{L^q(\Omega\cap B_r)} \leq Cr^{N/q-N/2}\|\nabla u\|_{L^2(\Omega\cap B_{2r})}.$$

by using one more Hölder's inequality on the right-hand side. Otherwise, one only needs a finite number of iterations of (A.10) to reduce s to be less than 2. This completes the proof of (A.8).

The proof of (2.3)-(2.4) is complete. ∎

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