Noname manuscript No.
(will be inserted by the editor)

Maximum-norm stability of the finite element Ritz projection under mixed boundary conditions

Dmitriy Leykekhman · Buyang Li

Received: date / Accepted: date

Abstract As a model of the second order elliptic equation with non-trivial boundary conditions, we consider the Laplace equation with mixed Dirichlet and Neumann boundary conditions on convex polygonal domains. Our goal is to establish that finite element discrete harmonic functions with mixed Dirichlet and Neumann boundary conditions satisfy a weak (Agmon-Miranda) discrete maximum principle, and then prove the stability of the Ritz projection with mixed boundary conditions in L^∞ norm. Such results have a number of applications, but are not available in the literature. Our proof of the maximum-norm stability of the Ritz projection is based on converting the mixed boundary value problem to a pure Neumann problem, which is of independent interest.

Keywords stability · maximum norm · finite element method

Mathematics Subject Classification (2000) MSC code1 · MSC code2 · more

1 Introduction

We consider a finite element approximation of the mixed boundary value problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ u &= u_\Gamma, \quad \text{on } \Gamma, \\ \partial_n u &= 0, \quad \text{on } \partial\Omega\backslash\Gamma, \end{aligned} \tag{1.1}$$

where Ω is a convex polygon in $\mathbb{R}^2$, Γ is a union of some edges of Ω ($\Gamma \neq \emptyset$) and u_Γ is a given function defined on Γ.

Let S_h denote a space of continuous piecewise polynomials of degree $r \geq 1$ on a quasi-uniform triangulation of Ω of size h and let $S_h^\Gamma = S_h \cap H^1_\Gamma(\Omega)$, with $H^1_\Gamma(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}$. We define a finite element approximation $u_h \in S_h$ of (1.1) as

$$(\nabla u_h, \nabla v_h) = (f, v_h), \quad \forall\, v_h \in S_h^\Gamma, \tag{1.2}$$

D. Leykekhman
Department of Mathematics, University of Connecticut, USA. E-mail: dmitriy.leykekhman@uconn.edu

B. Li
Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong. E-mail: buyang.li@polyu.edu.hk

with $u_h = I_h u_\Gamma$ on Γ, where I_h denotes the Lagrange interpolation operator. It is well known (cf. [31, Sec. 4.4]) that the finite element solution u_h defined this way is stable in $H^1(\Omega)$ norm and as a result satisfies the following approximation property

$$\|u - u_h\|_{H^1(\Omega)} \le C\|u - I_h u\|_{H^1(\Omega)}. \tag{1.3}$$

The main goal of this paper is to establish the analogous approximation (stability) result in $L^\infty(\Omega)$ norm, i.e.

$$\|u - u_h\|_{L^\infty(\Omega)} \le C\ell_h\|u - I_h u\|_{L^\infty(\Omega)}, \tag{1.4}$$

with a constant C independent of the solution u and the mesh size h, and

$$\ell_h = \begin{cases} 1 + |\ln h| & \text{if } r = 1, \\ 1 & \text{if } r \ge 2. \end{cases} \tag{1.5}$$

The logarithmic factor in the piecewise linear case cannot be removed in general [17] (see also [10]). The estimate (1.4) by the triangle inequality and the stability of the interpolant implies that the Ritz projection $R_h : H^1(\Omega) \cap C(\overline{\Omega}) \to S_h$ defined by

$$(\nabla R_h u, \nabla v_h) = (\nabla u, \nabla v_h), \quad \forall\, v_h \in S_h^\Gamma, \tag{1.6}$$

with $R_h u = I_h u$ on Γ, extends to a bounded linear operator $R_h : C(\overline{\Omega}) \to S_h$, and satisfies

$$\|R_h u\|_{L^\infty(\Omega)} \le C\ell_h\|u\|_{L^\infty(\Omega)}. \tag{1.7}$$

Such result has important applications in maximum-norm error estimates of finite element methods for elliptic and parabolic equations [19–21,33], discrete resolvent estimates of elliptic operators [3,6], and maximal L^p regularity of finite element solutions of parabolic equations [11,12,22,23]. However, all papers that establish (1.7) only deal with Dirichlet or Neumann boundary conditions. Thus, (1.7) has been established for the Dirichlet problem on polygonal domains [30], convex polyhedral domains [18], and smooth domains in $\mathbb{R}^N$ [29]. The mixed boundary conditions are not covered in the literature so far, however such problems have a number of applications [9].

A similar result in $W^{1,\infty}$ norm, namely

$$\|u - u_h\|_{W^{1,\infty}(\Omega)} \le C\|u - I_h u\|_{W^{1,\infty}(\Omega)} \tag{1.8}$$

has been established for Dirichlet problems on convex polygonal domains [24], [27] and convex polyhedral domains with quasi-uniform [16] and mildly graded meshes [8] without any logarithmic factor even for the piecewise linear case. The corresponding $W^{1,\infty}$ estimates were also established for the Stokes problem in convex polyhedra [15] with Dirichlet boundary conditions.

Our work was motivated by [30], where the error estimate (1.4) and as a result (1.7) were established for Dirichlet problems on polygonal domains. The proof is based on a weak discrete maximum principle (Agnom-Miranda maximum principle), namely

$$\|u_h\|_{L^\infty(\Omega)} \le C\|u_h\|_{L^\infty(\Gamma)}, \tag{1.9}$$

for any discrete harmonic function u_h satisfying

$$(\nabla u_h, \nabla v_h) = 0, \quad \forall\, v_h \in S_h^\Gamma, \tag{1.10}$$

with $\Gamma = \partial\Omega$. Using (1.9) and the knowledge of the finite element solution on the boundary, the Dirichlet problem in a polygon can be converted to a problem on a larger domain by extending the exact solution out of the polygon. Then (1.4) can be proved by applying an interior maximum-norm estimates of [28]. Unfortunately, this argument cannot be extended to the Neumann problem or a problem with mixed boundary conditions since the values of the finite element solutions on the boundary in these cases are unknown.

In this paper, we prove the weak discrete maximum principle for mixed boundary value problems on convex polygons, where Γ is only a proper subset of the boundary $\partial\Omega$ on which the Dirichlet bounder condition is specified. Then we apply this weak discrete maximum principle to prove the maximum-norm error estimate (1.4) by converting the mixed boundary conditions to a pure Neumann problem. Our main results are presented in the following two theorems.

The first theorem establishes a weak discrete maximum principle for the problem with mixed boundary conditions.

Theorem 1 (Weak discrete maximum principle) *Let Ω be a convex polygon, Γ be the union of some edges of Ω, and $u_h \in S_h$ be a discrete harmonic function satisfying* (1.10)*. Then there exists a constant C, independent of h and u_h, such that*

$$\|u_h\|_{L^\infty(\Omega)} \le C\|u_h\|_{L^\infty(\Gamma)}.$$

The second result establishes the best approximation property of the error in L^∞ norm.

Theorem 2 (Best approximation property) *Let Ω be a convex polygon, Γ be the union of some edges of Ω, and $u_h \in S_h$ be the finite element solution of* (1.1) *defined by* (1.2) *with $u_h = I_h u_\Gamma$ on Γ. Then there exists a constant C independent of u, u_h and h such that*

$$\|u - u_h\|_{L^\infty(\Omega)} \le C\ell_h\|u - I_h u\|_{L^\infty(\Omega)},$$

where ℓ_h is given in (1.5).

Corollary 1 (Maximum-norm stability) *By the triangle inequality one can immediately deduce that Theorem 2 is equivalent to the stability of the Ritz projection in L^∞ norm, i.e.*

$$\|u_h\|_{L^\infty(\Omega)} := \|R_h u\|_{L^\infty(\Omega)} \le C\ell_h\|u\|_{L^\infty(\Omega)}.$$

Remark 1 Since the regularity of the solution of a mixed boundary value problem on convex polygonal domains is similar to the regularity of a Dirichlet problem on nonconvex polygonal domains, our results for mixed boundary value problems on convex polygonal domains are as sharp as the results for Dirichlet problems on nonconvex polygonal domains.

As we mentioned above, the current paper is inspired by [30] and some elements of the analysis are similar. However, there are some significant technical differences. The paper [30] deals with Dirichlet problem on non-convex domains Ω and the analysis in [30] used an assumption that the original (non-convex) domain Ω can be extended to a larger convex domain $\widetilde{\Omega} \supset \Omega$ and each quasi-uniform triangulation T_h of Ω can be extended to a quasi-uniform triangulation $\widetilde{T}_h$ of $\widetilde{\Omega}$. In our paper, we already dealing with convex domains, but with mixed boundary conditions. Although the regularity of the solution in our case is very similar to the regularity of the solution in [30], new technical difficulties need to be overcome due to the mixed boundary conditions. The key argument in our analysis is essentially reducing the problem to showing the stability of the Ritz projection with only Neumann boundary conditions (cf. Lemma 9).

The rest of the paper is organized as follows. In the next section we provide some important local and global regularity results for the continuous problem (1.1) to be used later in the proofs. In Section 3 we provide proofs of Theorems 1 and 2. Finally, Section 4 is devoted to a proof of the stability of the Ritz projection for a pure inhomogeneous Neumann problem which plays a central role in the proofs of the main results Theorem 1 and Theorem 2.

2 Preliminaries

Throughout the paper we use the usual notation for Lebesgue and Sobolev spaces. We denote by $(\cdot,\cdot)$ the $L^2(\Omega)$ inner-product and we will specify a subdomain in the case it is not the whole Ω. Throughout this paper we assume that Ω is a convex polygonal domain. The boundary segments of $\partial\Omega$ are denoted by Γ_j, $j = 1,\dots,M$ with the corresponding angles α_j, $j = 1,\dots,M$. Without loss of generality we assume $\alpha_M < \pi$ to be the largest interior angle of Ω.

2.1 Global Regularity Results

In this section we state the regularity results for the Poisson equation with the Neumann boundary conditions

$$\begin{cases} -\Delta u = f, \text{ in } \ \Omega, \\ \partial_n u = g, \quad \text{on } \ \partial\Omega, \end{cases} \tag{2.1}$$

with the compatibility condition

$$\int_\Omega f + \int_{\partial\Omega} g = 0.$$

The first result is $H^{2+s}(\Omega)$ estimate of the solution to (2.1).

Lemma 1 *Let $s \in [0, 1 - \frac{\alpha_M}{\pi})$ and $u \in H^1(\Omega)$ be a solution of (2.1) with $f \in H^s(\Omega)$ and $g \in H^{1/2+s}_{\text{piecewise}}(\partial\Omega)$, where*

$$H^{1/2+s}_{\text{piecewise}}(\partial\Omega) = \{q \in L^2(\partial\Omega) : \ q \in H^{1/2+s}(\Gamma_i), \ i = 1,\dots,M\},$$

with $\partial\Omega = \cup_{i=1}^M \Gamma_i$. Then there exists a constant C such that

$$\|u\|_{H^{2+s}(\Omega)} \le C(\|f\|_{H^s(\Omega)} + \|g\|_{H^{1/2+s}_{\text{piecewise}}(\partial\Omega)}).$$

The case $s = 0$ can be found in [2]. For $s \in (0, 1 - \frac{\alpha_M}{\pi})$, the H^{2+s} regularity is a consequence of the following two lemmas.

Lemma 2 *For $s \in (0, 1 - \frac{\alpha_M}{\pi})$, suppose that $u \in H^1(\Omega)$ is a solution of (2.1) with $g = 0$ and $f \in H^s(\Omega)$. Then we have*

$$\|u\|_{H^{2+s}(\Omega)} \le C\|f\|_{H^s(\Omega)}. \tag{2.2}$$

Lemma 3 *Let $s \in (0, 1 - \frac{\alpha_M}{\pi})$. Then there exists a bounded linear lifting operator $\mathcal{A} : H^{1/2+s}_{\text{piecewise}}(\partial\Omega) \to H^{2+s}(\Omega)$ such that*

$$\partial_n \mathcal{A} g = g.$$

Lemma 2 is a special case of (23.3) in [7] and Lemma 3 is a consequence of Theorem 5.2 and Corollary 5.3 in Chapter 1 of [4].

2.2 Local Regularity Results

For the Neumann problem (2.1) with $g = 0$, by using Lemma 1 we can show the following local regularity result.

Lemma 4 (Local H^{2+s} regularity in balls) *Let u be a solution to* (2.1) *with $g = 0$ and $s \in [0, 1 - \frac{\alpha_M}{\pi})$. Let $B_d(z) \subset B_{2d}(z)$ be balls of radius d and radius $2d$ centered at the point $z \in \Omega$, respectively, and set $\Omega_d = B_d(z) \cap \Omega$ and $\Omega'_d = B_{2d}(z) \cap \Omega$. Then the following estimate holds:*

$$\|u\|_{H^{2+s}(\Omega_d)} \le Cd^{-s}\left(\|f\|_{L^2(\Omega'_d)} + d\|\nabla f\|_{L^2(\Omega'_d)} + \|\nabla u\|_{L^\infty(\Omega'_d)}\right).$$

Proof Let z be any point in Ω'_d and set $\tilde{u} = (u - u(z))\omega$. We consider a smooth cut-off function ω with the following properties

$$\omega(x) \equiv 1, \quad x \in B_d(z) \tag{2.3a}$$
$$\omega(x) \equiv 0, \quad x \in \Omega \backslash B_{2d}(z) \tag{2.3b}$$
$$|\nabla^k \omega| \le C_k d^{-k}, \quad k = 0, 1, 2, \ldots \tag{2.3c}$$

There holds:

$$\begin{cases} -\Delta\tilde{u} = -\nabla\cdot((u-u(z))\nabla\omega) - \nabla u\cdot\nabla\omega + f\omega, & \text{in } \Omega, \\ \partial_n\tilde{u} = (u-u(z))\nabla\omega\cdot\vec{n} & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

and therefore $\tilde{u}$ satisfies the following equation

$$\begin{cases} -\Delta\tilde{u} = \tilde{f}, & \text{in } \Omega, \\ \partial_n\tilde{u} = \tilde{g}\cdot\vec{n} & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where

$$\tilde{f} = f\omega - 2\nabla u\cdot\nabla\omega - (u-u(z))\Delta\omega \quad \text{and} \quad \tilde{g} = (u-u(z))\nabla\omega,$$

and the compatibility condition is satisfied, i.e. $\int_\Omega \tilde{f} + \int_{\partial\Omega}\tilde{g}\cdot\vec{n} = 0$.

Note that

$$\|f\omega\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega'_d)}$$

and

$$\|f\omega\|_{H^1(\Omega)} \le Cd^{-1}\|f\|_{L^2(\Omega'_d)} + C\|\nabla f\|_{L^2(\Omega'_d)},$$

which imply by the interpolation and Young's inequalities,

$$\begin{aligned} \|f\omega\|_{H^s(\Omega)} &\le \|f\omega\|^{1-s}_{L^2(\Omega)}\|f\omega\|^s_{H^1(\Omega)} \\ &\le Cd^{-s}\|f\|_{L^2(\Omega'_d)} + C\|f\|^{1-s}_{L^2(\Omega'_d)}\|\nabla f\|^s_{L^2(\Omega'_d)} \\ &\le Cd^{-s}\|f\|_{L^2(\Omega'_d)} + Cd^{1-s}\|\nabla f\|_{L^2(\Omega'_d)}. \end{aligned} \tag{2.6}$$

Since the domain Ω'_d is convex, any two points $z_1, z_2 \in \Omega'_d$ can be connected by a line of length at most $4d$ in Ω'_d, by the properties of ω and the Hölder inequality, we have

$$\begin{aligned} \|(u-u(z))\Delta\omega\|_{L^2(\Omega)} &\le Cd^{-2}\|u-u(z)\|_{L^2(\Omega'_d)} \\ &\le Cd^{-1}\|u-u(z)\|_{L^\infty(\Omega'_d)} \le C\|\nabla u\|_{L^\infty(\Omega'_d)} \end{aligned}$$

and

$$\begin{aligned}\|(u-u(z))\Delta\omega\|_{H^1(\Omega)} \leq&\|(u-u(z))\Delta\omega\|_{L^2(\Omega)}+\|\nabla(u-u(z))\Delta\omega\|_{L^2(\Omega)}\\ &+\|(u-u(z))\nabla\Delta\omega\|_{L^2(\Omega)}\\ \leq& Cd^{-2}\|u-u(z)\|_{L^2(\Omega_d')}+Cd^{-2}\|\nabla(u-u(z))\|_{L^2(\Omega_d')}\\ &+Cd^{-3}\|u-u(z)\|_{L^2(\Omega_d')}\\ \leq& Cd^{-1}\|\nabla u\|_{L^\infty(\Omega_d')},\end{aligned}$$

which imply

$$\begin{aligned}\|(u-u(z))\Delta\omega\|_{H^s(\Omega)} &\leq \|(u-u(z))\Delta\omega\|_{L^2(\Omega)}^{1-s}\|(u-u(z))\Delta\omega\|_{H^1(\Omega)}^{s}\\ &\leq Cd^{-s}\|\nabla u\|_{L^\infty(\Omega_d')}.\end{aligned} \tag{2.7}$$

Similarly, we have

$$\|\nabla u\cdot\nabla\omega\|_{L^2(\Omega)}\leq Cd^{-1}\|\nabla u\|_{L^2(\Omega_d')}$$

and

$$\|\nabla u\cdot\nabla\omega\|_{H^1(\Omega)}\leq Cd^{-2}\|\nabla u\|_{L^2(\Omega_d')}+Cd^{-1}\|u\|_{H^2(\Omega_d')},$$

which imply by the interpolation and Young's inequalities,

$$\begin{aligned}\|\nabla u\cdot\nabla\omega\|_{H^s(\Omega)} &\leq \|\nabla u\cdot\nabla\omega\|_{L^2(\Omega)}^{1-s}\|\nabla u\cdot\nabla\omega\|_{H^1(\Omega)}^{s}\\ &\leq Cd^{-(1-s)}\|\nabla u\|_{L^2(\Omega_d')}^{1-s}(Cd^{-2}\|\nabla u\|_{L^2(\Omega_d')}+Cd^{-1}\|u\|_{H^2(\Omega_d')})^s\\ &\leq Cd^{-1-s}\|\nabla u\|_{L^2(\Omega_d')}+Cd^{-1}\|\nabla u\|_{L^2(\Omega_d')}^{1-s}\|u\|_{H^2(\Omega_d')}^{s}.\end{aligned} \tag{2.8}$$

By using Lemma 1, (2.6), (2.7) and (2.8), we obtain

$$\begin{aligned}\|u\|_{H^2(\Omega_d)}=\|\tilde u\|_{H^2(\Omega_d)} &\leq \|\tilde u\|_{H^2(\Omega)}\\ &\leq C\|\tilde f\|_{L^2(\Omega)}+C\|\tilde g\cdot\vec n\|_{H^{1/2}_{\rm piecewise}(\partial\Omega)}\\ &\leq C\|\tilde f\|_{L^2(\Omega)}+C\|\tilde g\|_{H^1(\Omega)}\\ &\leq C\|f\|_{L^2(\Omega_d')}+Cd^{-1}\|\nabla u\|_{L^2(\Omega_d')}+C\|\nabla u\|_{L^\infty(\Omega_d')}\\ &\leq C\|f\|_{L^2(\Omega_d')}+C\|\nabla u\|_{L^\infty(\Omega_d')},\end{aligned}$$

and for $s\in(0,1-\frac{\alpha_M}{\pi})$

$$\begin{aligned}\|u\|_{H^{2+s}(\Omega_d)}&=\|\tilde u\|_{H^{2+s}(\Omega_d)}\\ &\leq \|\tilde u\|_{H^{2+s}(\Omega)}\\ &\leq C\|\tilde f\|_{H^s(\Omega)}+C\|\tilde g\cdot\vec n\|_{H^{1/2+s}_{\rm piecewise}(\partial\Omega)}\\ &\leq C\|\tilde f\|_{H^s(\Omega)}+C\|\tilde g\|_{H^{1+s}(\Omega)}\\ &\leq Cd^{-s}\|f\|_{L^2(\Omega_d')}+C\|f\|_{L^2(\Omega_d')}^{1-s}\|\nabla f\|_{L^2(\Omega_d')}^{s}+Cd^{-1-s}\|\nabla u\|_{L^2(\Omega_d')}\\ &\quad+Cd^{-1}\|\nabla u\|_{L^2(\Omega_d')}^{1-s}\|u\|_{H^2(\Omega_d')}^{s}+Cd^{-s}\|\nabla u\|_{L^\infty(\Omega_d')}\\ &\leq Cd^{-s}\|f\|_{L^2(\Omega_d')}+Cd^{1-s}\|\nabla f\|_{L^2(\Omega_d')}\\ &\quad+Cd^{-1-s}\|\nabla u\|_{L^2(\Omega_d')}+Cd^{-s}\|u\|_{H^2(\Omega_d')}+Cd^{-s}\|\nabla u\|_{L^\infty(\Omega_d')}\\ &\leq Cd^{-s}\|f\|_{L^2(\Omega_d')}+Cd^{1-s}\|\nabla f\|_{L^2(\Omega_d')}\\ &\quad+Cd^{-s}\|u\|_{H^2(\Omega_d')}+Cd^{-s}\|\nabla u\|_{L^\infty(\Omega_d')}.\end{aligned}$$

By substituting the H^2 estimate into the H^{2+s} estimate, we complete the proof.

Since any dyadic annulus $R_d(x_0) = \{x \in \mathbb{R}^2 : d \le |x - x_0| < 2d\}$ can be covered by a finite number of balls $B_{2/d}(z_j)$, $j = 1, \dots, M$, where $z_j \in R_d$ and M is bounded by constant. Lemma 4 immediately implies the following result.

Lemma 5 (Local H^{2+s} regularity in annulus) *Let u be a solution to* (2.1) *with $g = 0$ and $s \in [0, 1 - \frac{\alpha_M}{\pi})$. Let $\Omega_d = \{x \in \Omega : d \le |x - x_0| < 2d\}$ and $\Omega_d' = \{x \in \Omega : d/2 \le |x - x_0| < 4d\}$. Then the following estimate holds:*

$$\|u\|_{H^{2+s}(\Omega_d)} \le Cd^{-s} \left(\|f\|_{L^2(\Omega_d')} + d\|\nabla f\|_{L^2(\Omega_d')} + \|\nabla u\|_{L^\infty(\Omega_d')} \right).$$

2.3 Discretization and local energy estimate

For $h \in (0, h_0]$, $h_0 > 0$, let $\mathcal{T}$ denote a quasi-uniform triangulation of Ω with a mesh size h, i.e., $\mathcal{T} = \{\tau\}$ is a partition of Ω into cells (triangles) τ of diameter h_τ such that for $h = \max_\tau h_\tau$,

$$\operatorname{diam}(\tau) \le h \le C|\tau|^{\frac{1}{2}}, \quad \forall \tau \in \mathcal{T},$$

hold.

Then the usual nodewise interpolant $I_h \colon C(\overline{\Omega}) \to S_h$ has the following approximation properties (cf., e. g., [5, Theorem 3.1.5])

$$\|u - I_h u\|_{L^q(\Omega)} \le Ch^{2+2(\frac{1}{q}-\frac{1}{p})}\|u\|_{W^{2,p}(\Omega)}, \quad \text{for } q \ge p > 1, \tag{2.9a}$$

$$\|u - I_h u\|_{L^\infty(\Omega)} \le Ch^{1+\alpha}\|u\|_{C^{1+\alpha}(\overline{\Omega})}, \quad \text{for } 0 \le \alpha \le 1, \tag{2.9b}$$

where $C^{1+\alpha}(\overline{\Omega})$ is the space of Hölder continuous functions.

We will also require the following superapproximation result [25].

Lemma 6 *Assume $\Omega_0 \subset \Omega_1 \subset \Omega$ with $\{x \in \Omega : \operatorname{dist}(x, \Omega_0) < d\} \subset \Omega_1$ and let ω be a smooth cut-off function satisfying*

$$\omega(x) \equiv 1, \quad x \in \Omega_0 \tag{2.10a}$$

$$\omega(x) \equiv 0, \quad x \in \Omega \backslash \Omega_1 \tag{2.10b}$$

$$|\nabla^k \omega| \le C_k d^{-k}, \quad k = 0, 1, 2, \dots \tag{2.10c}$$

Then there exists a constant C independent of h and d such that for any $\chi \in S_h$

$$\|\nabla(\omega^2\chi - I_h(\omega^2\chi))\|_{L^2(\Omega)} \le Ch \left(d^{-1}\|\nabla\chi\|_{L^2(\Omega_1)} + d^{-2}\|\chi\|_{L^2(\Omega_1)} \right).$$

We will also require a local energy estimate for a problem with mixed boundary conditions that is valid up to the boundary.

Lemma 7 *Suppose that $u \in H^1_\Gamma(\Omega)$ and $u_h \in S_h^\Gamma$ satisfy*

$$(\nabla(u - u_h), \nabla v_h) = 0 \quad \forall\, v_h \in S_h^\Gamma,$$

and $\Omega_0 \subset \Omega_1 \subset \Omega$ with $\{x \in \Omega : \operatorname{dist}(x, \Omega_0) < d\} \subset \Omega_1$, then

$$\|u - u_h\|_{H^1(\Omega_0)} \le C \left(\|u - I_h u\|_{H^1(\Omega_1)} + d^{-1}\|u - I_h u\|_{L^2(\Omega_1)} + d^{-1}\|u - u_h\|_{L^2(\Omega_1)} \right).$$

Proof Let ω be the cut-off function satisfying (2.10). Then inserting $I_h u$ and using that $I_h(\omega^2(I_h u - u_h)) \in S_h^\Gamma$ and the superapproximation Lemma 6, we have

$$\begin{aligned}
&\|\nabla(u-u_h)\|_{L^2(\Omega_0)}^2 \le \|\omega\nabla(u-u_h)\|_{L^2(\Omega_0)}^2 = (\omega^2\nabla(u-u_h), \nabla(u-u_h)) \\
&= (\omega^2\nabla(u-u_h), \nabla(u-I_h u)) + (\omega^2\nabla(u-u_h), \nabla(I_h u - u_h)) \\
&\le \|\omega\nabla(u-u_h)\|_{L^2(\Omega)}\|\omega\nabla(u-I_h u)\|_{L^2(\Omega)} \\
&\quad + (\nabla(u-u_h), \nabla[\omega^2(I_h u - u_h) - I_h(\omega^2(I_h u - u_h))]) \\
&\quad - (2\omega\nabla(u-u_h), (I_h u - u_h)\nabla\omega]) \\
&\le \frac{1}{2}\|\omega\nabla(u-u_h)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\omega\nabla(u-I_h u)\|_{L^2(\Omega)}^2 \\
&\quad + Chd^{-1}\|\nabla(u-u_h)\|_{L^2(\Omega_1)}(\|\nabla(I_h u - u_h)\|_{L^2(\Omega_1)} + d^{-1}\|I_h u - u_h\|_{L^2(\Omega_1)}) \\
&\quad + Cd^{-1}\|\omega\nabla(u-u_h)\|_{L^2(\Omega)}\|I_h u - u_h\|_{L^2(\Omega_1)} \\
&\le \frac{1}{2}\|\omega\nabla(u-u_h)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\omega\nabla(u-I_h u)\|_{L^2(\Omega)}^2 \\
&\quad + Chd^{-1}\|\nabla(u-I_h u)\|_{L^2(\Omega_1)}^2 + Cd^{-2}\|u-I_h u\|_{L^2(\Omega_1)}^2 \\
&\quad + Chd^{-1}\|\nabla(u-u_h)\|_{L^2(\Omega_1)}^2 + Cd^{-2}\|u-u_h\|_{L^2(\Omega_1)}^2.
\end{aligned}$$

Canceling both sides by $\frac{1}{2}\|\omega\nabla(u-u_h)\|_{L^2(\Omega)}^2$ and taking the square root, the above estimate implies

$$\begin{aligned}
\|u-u_h\|_{H^1(\Omega_0)} &\le C\|u-I_h u\|_{H^1(\Omega_1)} + Cd^{-1}\|u-I_h u\|_{L^2(\Omega_1)} \\
&\quad + Ch^{\frac{1}{2}}d^{-\frac{1}{2}}\|u-u_h\|_{H^1(\Omega_1)} + Cd^{-1}\|u-u_h\|_{L^2(\Omega_1)}.
\end{aligned}$$

Iterating the argument and using the inverse estimate, we obtain

$$\begin{aligned}
\|u-u_h\|_{H^1(\Omega_0)} &\le C\|u-I_h u\|_{H^1(\Omega_1)} + Cd^{-1}\|u-I_h u\|_{L^2(\Omega_1)} \\
&\quad + Chd^{-1}\|u-u_h\|_{H^1(\Omega_1)} + Cd^{-1}\|u-u_h\|_{L^2(\Omega_1)} \\
&\le C\|u-I_h u\|_{H^1(\Omega_1)} + Cd^{-1}\|u-I_h u\|_{L^2(\Omega_1)} \\
&\quad + Chd^{-1}\|I_h u-u_h\|_{H^1(\Omega_1)} + Cd^{-1}\|u-u_h\|_{L^2(\Omega_1)} \\
&\le C\|u-I_h u\|_{H^1(\Omega_1)} + Cd^{-1}\|u-I_h u\|_{L^2(\Omega_1)} \\
&\quad + Cd^{-1}\|I_h u-u_h\|_{L^2(\Omega_1)} + Cd^{-1}\|u-u_h\|_{L^2(\Omega_1)} \\
&\le C\|u-I_h u\|_{H^1(\Omega_1)} + Cd^{-1}\|u-I_h u\|_{L^2(\Omega_1)} \\
&\quad + Cd^{-1}\|u-u_h\|_{L^2(\Omega_1)}.
\end{aligned}$$

The proof of Lemma 7 is complete.

3 Proof of Theorems 1–2

3.1 Proof of Theorem 1

To prove Theorem 1, we choose an arbitrary fixed $x_0 \in \Omega$ and prove that $|u_h(x_0)| \le C\|u_h\|_{L^\infty(\Gamma)}$, where the constant C is independent of x_0. Let $B_d(x_0)$ denote the ball of radius d centered at x_0 and let $S_d(x_0) := B_d(x_0) \cap \Omega$. Then we have the following lemma, which is a consequence of the maximum norm estimates proved in [28] (also see [30]).

Lemma 8 *Let $\rho = \max(\mathrm{dist}(x_0, \partial\Omega), 2h)$. Then we have*

$$|u_h(x_0)| \le C\rho^{-1}\|u_h\|_{L^2(S_\rho(x_0))}.$$

In view of Lemma 8, it suffices to estimate $\|u_h\|_{L^2(S_\rho(x_0))}$ in order to prove Theorem 1. By a duality argument, it suffices to estimate $|(u_h, \varphi)|$ for any given function $\varphi \in L^2(S_\rho(x_0))$ which is supported in $S_\rho(x_0)$ and $\|\varphi\|_{L^2(S_\rho(x_0))} = 1$. For this purpose, we define $v \in H^1_\Gamma(\Omega)$ as the solution of

$$\begin{cases} -\Delta v = \varphi \text{ in } \Omega, \\ v = 0 \quad \text{on } \Gamma, \\ \partial_n v = 0 \text{ on } \partial\Omega\backslash\Gamma, \end{cases} \tag{3.1}$$

and $v_h \in S_h^\Gamma$ as the solution of

$$(\nabla v_h, \nabla \psi_h) = (\varphi, \psi_h), \quad \forall\, \psi_h \in S_h^\Gamma. \tag{3.2}$$

Then $v \in W^{2,p}(\Omega) \cap H^1_\Gamma(\Omega)$ for any $p \in \big(1, \frac{2}{2-\pi/(2\alpha_M)}\big)$, where α_M is the maximal interior angle of the polygon Ω (Theorem 4.4.3.7 of [14]), and $v_h \in S_h^\Gamma$ is the finite element approximation of v. Since $\frac{2}{2-\pi/(2\alpha_M)} > \frac{4}{3}$, there exists some $p_0 > \frac{4}{3}$ such that

$$\|v\|_{W^{2,p_0}(\Omega)} \leq C\|\varphi\|_{L^{p_0}(\Omega)}. \tag{3.3}$$

Let w be the solution of

$$\begin{cases} \Delta w = 0 \text{ in } \Omega, \\ w = u_h \text{ on } \Gamma, \\ \partial_n w = 0 \text{ on } \partial\Omega\backslash\Gamma, \end{cases} \tag{3.4}$$

which satisfies the maximum principle (cf. [26])

$$\|w\|_{L^\infty(\Omega)} \leq \|w\|_{L^\infty(\Gamma)} = \|u_h\|_{L^\infty(\Gamma)}. \tag{3.5}$$

Then we have

$$\begin{aligned} |(u_h, \varphi)| &\leq |(u_h - w, \varphi)| + |(w, \varphi)| \\ &= |(\nabla(u_h - w), \nabla v)| + |(w, \varphi)| \\ &= |(\nabla u_h, \nabla v)| + |(w, \varphi)| \\ &= |(\nabla u_h, \nabla(v - v_h))| + |(w, \varphi)| \\ &= |(\nabla(u_h - \widetilde{u}_h), \nabla(v - v_h))| + |(w, \varphi)| \\ &\leq |(\nabla(u_h - \widetilde{u}_h), \nabla(v - v_h))| + \|w\|_{L^\infty(\Omega)}\|\varphi\|_{L^1(S_\rho(x_0))} \\ &\leq |(\nabla(u_h - \widetilde{u}_h), \nabla(v - v_h))| + \rho\|u_h\|_{L^\infty(\Gamma)}\|\varphi\|_{L^2(S_\rho(x_0))}, \end{aligned} \tag{3.6}$$

where $\widetilde{u}_h$ can be any function in S_h^Γ. We simply choose $\widetilde{u}_h$ to be equal u_h at the nodes of $\Omega\backslash\overline{\Gamma}$ and equal zero on $\overline{\Gamma}$. If we define $D_h = \{x \in \overline{\Omega} : \text{dist}(x, \Gamma) \leq h\}$, then we have

$$\begin{aligned} |(u_h, \varphi)| &\leq \|\nabla(u_h - \widetilde{u}_h)\|_{L^\infty(D_h)}\|\nabla(v - v_h)\|_{L^1(D_h)} + \rho\|u_h\|_{L^\infty(\Gamma)} \\ &\leq (Ch^{-1}\|\nabla(v - v_h)\|_{L^1(D_h)} + \rho)\|u_h\|_{L^\infty(\Gamma)}. \end{aligned} \tag{3.7}$$

Thus, in order to prove the theorem we need to show

$$\|\nabla(v - v_h)\|_{L^1(D_h)} \leq C\rho h. \tag{3.8}$$

To establish (3.8), we divide the domain D_h into $S_{8\rho}(x_0) \cap D_h$ and $O_j \cap D_h$, $j = 0, 1, 2, \cdots, J = \log_2(R_0/8\rho)$, with

$$O_j = \{x \in \Omega : d_{j+1} \leq |x - x_0| < d_j\}$$

and $d_j = 2^{-j}\mathrm{diam}(\Omega)$. Then we have $|O_j \cap D_h| \le Chd_j$ and

$$\begin{aligned}\|\nabla(v-v_h)\|_{L^1(D_h)} &\le \sum_{j=0}^{J} \|\nabla(v-v_h)\|_{L^1(O_j\cap D_h)} + \|\nabla(v-v_h)\|_{L^1(S_{8\rho}(x_0)\cap D_h)} \\ &\le C\sum_{j=0}^{J} h^{\frac12} d_j^{\frac12} \|\nabla(v-v_h)\|_{L^2(O_j\cap D_h)} + Ch^{\frac12}\rho^{\frac12}\|\nabla(v-v_h)\|_{L^2(S_{8\rho}(x_0)\cap D_h)}.\end{aligned} \tag{3.9}$$

The second term on the right-hand side of (3.9) can be estimated as

$$\begin{aligned}\|\nabla(v-v_h)\|_{L^2(S_{8\rho}(x_0)\cap D_h)} &\le \|\nabla(v-v_h)\|_{L^2(\Omega)} \\ &\le \|\nabla(v-I_h v)\|_{L^2(\Omega)} \\ &\le Ch^{2-\frac{2}{p_0}}\|v\|_{H^{3-2/p_0}(\Omega)} \\ &\le Ch^{2-\frac{2}{p_0}}\|v\|_{W^{2,p_0}(\Omega)} \\ &\le Ch^{2-\frac{2}{p_0}}\|\varphi\|_{L^{p_0}(S_\rho(x_0))} \\ &\le Ch^{2-\frac{2}{p_0}}\rho^{\frac{2}{p_0}-1}\|\varphi\|_{L^2(S_\rho(x_0))}.\end{aligned}$$

As a result

$$h^{\frac12}\rho^{\frac12}\|\nabla(v-v_h)\|_{L^2(S_{8\rho}(x_0)\cap D_h)} \le Ch^{\frac52-\frac{2}{p_0}}\rho^{\frac{2}{p_0}-\frac12} = Ch\rho\left(\frac{h}{\rho}\right)^{\frac32-\frac{2}{p_0}} \le Ch\rho, \tag{3.10}$$

where in last step we used that $h \le \rho$ and $\frac32 > \frac{2}{p_0}$.

To estimate the first term on the right-hand side of (3.9), it suffices to consider the sets O_j such that $O_j \cap D_h \ne \emptyset$. For such a set O_j we have $O_j''' \subset \{x \in \Omega : \mathrm{dist}(x,\Gamma) < 8d_j\}$, and by Lemma 7 we have

$$\begin{aligned}\|\nabla(v-v_h)\|_{L^2(O_j\cap D_h)} &\le C\big(\|\nabla(v-I_h v)\|_{L^2(O_j')} + d_j^{-1}\|v-I_h v\|_{L^2(O_j')} \\ &\quad + d_j^{-1}\|v-v_h\|_{L^2(O_j')}\big) \\ &\le Ch^{2-\frac{2}{p_0}}\|v\|_{W^{2,p_0}(O_j'')} + Cd_j^{-1}\|v-v_h\|_{L^2(O_j')}.\end{aligned}$$

Since v is harmonic in O_j''', a classical interior estimate for elliptic equations gives

$$\|v\|_{W^{2,p_0}(O_j'')} \le Cd_j^{\frac{2}{p_0}-2}\|\nabla v\|_{L^2(O_j''')} + Cd_j^{\frac{2}{p_0}-3}\|v\|_{L^2(O_j''')}, \tag{3.11}$$

and a standard energy estimate shows

$$\|\nabla v\|_{L^2(\Omega)} \le C\rho\|\varphi\|_{L^2(S_\rho(x_0))}. \tag{3.12}$$

Moreover, since $v = 0$ on Γ and $O_j''' \subset \{x \in \Omega : \mathrm{dist}(x,\Gamma) < 8d_j\}$, it follows from Lemma 1.1 of [30] that

$$\|v\|_{L^2(O_j''')} \le Cd_j\|\nabla v\|_{L^2(\Omega)}. \tag{3.13}$$

The last four inequalities imply

$$\begin{aligned}\sum_{j=0}^{J} h^{\frac12}d_j^{\frac12}\|\nabla(v-v_h)\|_{L^2(O_j\cap D_h)} &\le C\sum_{j=0}^{J}\left(\frac{h}{d_j}\right)^{\frac32-\frac{2}{p_0}}\rho h + h^{\frac12}d_j^{-\frac12}\|v-v_h\|_{L^2(O_j')} \\ &\le C\rho h + C\sum_{j=0}^{J} h^{\frac12}d_j^{-\frac12}\|v-v_h\|_{L^2(O_j')},\end{aligned} \tag{3.14}$$

where in the last step we used $h \le d_j$ and $\frac{3}{2} > \frac{2}{p_0}$.

The second term on the right-hand side above can be estimated by a duality argument. We consider $|(v - v_h, \phi)|$ for $\phi \in L^2(O'_j)$. For this purpose, we define $w \in H^1_\Gamma(\Omega)$ and $w_h \in S_h^\Gamma$ by

$$\begin{cases} -\Delta w = \phi \text{ in } & \Omega, \\ w = 0 \quad \text{ on } & \Gamma, \\ \partial_n w = 0 \text{ on } & \partial\Omega \backslash \Gamma, \end{cases} \tag{3.15}$$

and

$$(\nabla w_h, \nabla \psi_h) = (\phi, \psi_h), \quad \forall \psi_h \in S_h^\Gamma, \tag{3.16}$$

respectively. Then we have

$$\begin{aligned} (v - v_h, \phi) &= (\nabla(v - v_h), \nabla w) = (\nabla(v - v_h), \nabla(w - w_h)) \\ &\le \|\nabla(v - v_h)\|_{L^2(\Omega)} \|\nabla(w - w_h)\|_{L^2(\Omega)} \\ &\le \|\nabla(v - I_h v)\|_{L^2(\Omega)} \|\nabla(w - I_h w)\|_{L^2(\Omega)} \\ &\le C h^{4-\frac{4}{p_0}} \|v\|_{H^{3-\frac{2}{p_0}}(\Omega)} \|w\|_{H^{3-\frac{2}{p_0}}(\Omega)} \\ &\le C h^{4-\frac{4}{p_0}} \|v\|_{W^{2,p_0}(\Omega)} \|w\|_{W^{2,p_0}(\Omega)} \\ &\le C h^{4-\frac{4}{p_0}} \|\varphi\|_{L^{p_0}(S_\rho(x_0))} \|\phi\|_{L^{p_0}(O'_j)} \\ &\le C h^{4-\frac{4}{p_0}} \rho^{\frac{2}{p_0}-1} d_j^{\frac{2}{p_0}-1} \|\varphi\|_{L^2(S_\rho(x_0))} \|\phi\|_{L^2(O'_j)}, \end{aligned} \tag{3.17}$$

that implies

$$h^{\frac{1}{2}} d_j^{-\frac{1}{2}} \|v - v_h\|_{L^2(O'_j)} \le C\rho h \left(\frac{h}{\rho}\right)^{2-\frac{2}{p_0}} \left(\frac{h}{d_j}\right)^{\frac{3}{2}-\frac{2}{p_0}}. \tag{3.18}$$

Since $h \le d_j$, $h \le \rho$ and $\frac{3}{2} > \frac{2}{p_0}$, we have

$$\sum_{j=0}^{J} h^{\frac{1}{2}} d_j^{-\frac{1}{2}} \|v - v_h\|_{L^2(O'_j)} \le C\rho h,$$

which together with (3.10) establishes (3.8). The inequalities (3.7) and (3.8) show

$$\|u_h\|_{L^2(S_\rho(x_0))} \le C\rho \|u_h\|_{L^\infty(\Gamma)}, \tag{3.19}$$

which together with Lemma 8 implies

$$|u_h(x_0)| \le C\|u_h\|_{L^\infty(\Gamma)}. \tag{3.20}$$

This completes the proof of Theorem 1. □

3.2 Proof of Theorem 2

For u, the solution of (1.1), we define $\widetilde{u}_h$ as the finite element solution of

$$(\nabla(u - \widetilde{u}_h), \nabla \psi_h) = 0, \quad \forall \psi_h \in S_h. \tag{3.21}$$

with the condition $\int_\Omega (u - \widetilde{u}_h)\, dx = 0$ for the uniqueness of $\widetilde{u}_h$. In other words, $\widetilde{u}_h$ is the finite element approximation of u with certain "inhomogeneous" Neumann boundary condition. We shall use the following lemma, whose proof is deferred to the next section.

Lemma 9 *In a convex polygon Ω, we have*

$$\|\widetilde{u}_h\|_{L^\infty(\Omega)} \le C\ell_h \|u\|_{L^\infty(\Omega)}. \tag{3.22}$$

Now we have

$$\|u_h\|_{L^\infty(\Omega)} \le \|u_h - \widetilde{u}_h\|_{L^\infty(\Omega)} + \|\widetilde{u}_h\|_{L^\infty(\Omega)} \le \|u_h - \widetilde{u}_h\|_{L^\infty(\Omega)} + C\ell_h\|u\|_{L^\infty(\Omega)}. \tag{3.23}$$

From (1.1), (1.2) and (3.21) we see that

$$(\nabla(u_h - \widetilde{u}_h), \nabla\psi_h) = 0, \quad \forall\, \psi_h \in S_h^\Gamma,$$

which together with Theorem 1 implies

$$\|u_h - \widetilde{u}_h\|_{L^\infty(\Omega)} \le C\|u_h - \widetilde{u}_h\|_{L^\infty(\Gamma)} = C\|I_h u - \widetilde{u}_h\|_{L^\infty(\Gamma)} \le C\ell_h\|u\|_{L^\infty(\Omega)}. \tag{3.24}$$

Overall, (3.23) and (3.24) imply Theorem 2. □

4 Proof of Lemma 9

To establish the result we require pointwise estimates of the Green's function for the pure Neumann problem. Such estimates on convex polygonal domains are establish for example, in [1].

Lemma 10 *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain and $\Gamma(x,\xi)$ be the Green's function defined by*

$$\begin{cases} -\Delta\Gamma(\cdot,\xi) = \delta(\cdot - \xi) - \frac{1}{|\Omega|} & \text{in } \Omega, \\ \partial_n\Gamma(\cdot,\xi) = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\delta(x-\xi)$ is the delta function satisfying $\int_\Omega \delta(x-\xi)\phi(x)\, dx = \phi(\xi)$ for any $\phi \in C(\overline{\Omega})$. We also impose the normalization condition $\int_\Omega \Gamma(x,\xi)\, dx = 0$ for the uniqueness of the solution. Then there exists a constant C such that

$$|\nabla_x\Gamma(x,\xi)| \le C|x-\xi|^{-1}. \tag{4.2}$$

The following local energy estimate can be proved in the same way as Lemma 7.

Lemma 11 *Suppose that $u \in H^1(\Omega)$ and $\widetilde{u}_h \in S_h$ satisfy*

$$(\nabla(u - \widetilde{u}_h), \nabla v_h) = 0 \quad \forall\, v_h \in S_h,$$

with $\int_\Omega (u - \widetilde{u}_h)\, dx = 0$, and $\Omega_0 \subset \Omega_1 \subset \Omega$ with $\{x \in \Omega : \mathrm{dist}(x,\Omega_0) < d\} \subset \Omega_1$, then

$$\begin{aligned}\|u - &\widetilde{u}_h\|_{H^1(\Omega_0)} \\ &\le C\left(\|u - I_h u\|_{H^1(\Omega_1)} + d^{-1}\|u - I_h u\|_{L^2(\Omega_1)} + d^{-1}\|u - \widetilde{u}_h\|_{L^2(\Omega_1)}\right),\end{aligned}$$

where I_h is the Lagrange interpolant.

For a fixed $x_0 \in \Omega$, we shall prove that $|\widetilde{u}_h(x_0)| \le C\ell_h\|u\|_{L^\infty(\Omega)}$, where the constant C does not depend on x_0. Without loss of generality, we suppose that x_0 is contained in a triangle τ_0. We introduce a smooth Delta function [32, Lemma 2.2], which we will denote by $\tilde{\delta}_{x_0}^h := \tilde{\delta} \in C_0^2(\tau_0)$ such that

$$(\tilde{\delta}, \chi)_{\tau_0} = \chi(x_0), \quad \forall \chi \in S_h, \tag{4.3a}$$

$$\int_\Omega \tilde{\delta}(x)\, dx = 1, \tag{4.3b}$$

$$\|\tilde{\delta}\|_{W^{l,p}(\Omega)} \le Ch^{-l-2(1-\frac{1}{p})} \quad \text{for } 1 \le p \le \infty,\ l = 0,1,2. \tag{4.3c}$$

Thus in particular $\|\tilde{\delta}\|_{L^1(\Omega)} \le C$, $\|\tilde{\delta}\|_{L^2(\Omega)} \le Ch^{-1}$, and $\|\tilde{\delta}\|_{L^\infty(\Omega)} \le Ch^{-2}$.

Let $G(x, x_0)$ be the regularized Green's function defined by

$$\begin{cases} -\Delta G(\cdot, x_0) = \tilde{\delta}_{x_0}(\cdot) - \frac{1}{|\Omega|} & \text{in } \Omega, \\ \partial_n G(\cdot, x_0) = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\int_\Omega (\tilde{\delta}(x) - \frac{1}{|\Omega|})\ dx = 0$, the equation above admits a unique solution up to a constant. Let $G_h(\cdot, x_0) \in S_h$ be the discrete Green's function, i.e. finite element solution of

$$(\nabla G_h(\cdot, x_0), \nabla v_h) = v_h(x_0) - \frac{1}{|\Omega|}\int_\Omega v_h(x)\ dx, \quad \forall\, v_h \in S_h.$$

The function $G_h(\cdot, x_0) \in S_h$ is also well defined up to a constant. For the uniqueness, we further impose the condition

$$\int_\Omega G(x, x_0)\ dx = \int_\Omega G_h(x, x_0)\ dx = 0.$$

Then we have

$$\begin{aligned}
&\left| \widetilde{u}_h(x_0) - (I_h u)(x_0) - \frac{1}{|\Omega|}\int_\Omega (\widetilde{u}_h - I_h u)\ dx \right| \\
&= |(\nabla G_h, \nabla(\widetilde{u}_h - I_h u))| \\
&= |(\nabla G_h, \nabla(u - I_h u))| \\
&= |(\nabla(G_h - G), \nabla(u - I_h u)) + (\nabla G, \nabla(u - I_h u))| \\
&= \left| (\nabla(G_h - G), \nabla(u - I_h u)) + u(x_0) - I_h u(x_0) - \frac{1}{|\Omega|}\int_\Omega (u - I_h u)\ dx \right| \\
&= \Bigg| \sum_{\tau\in\mathcal{T}_h} (-\Delta(G_h - G), u - I_h u)_\tau + \sum_{e\in\mathcal{E}_h} ([\partial_n(G_h - G)], u - I_h u)_e \\
&\qquad + u(x_0) - I_h u(x_0) - \frac{1}{|\Omega|}\int_\Omega (u - I_h u)\ dx \Bigg| \\
&\le \left(\sum_{\tau\in\mathcal{T}_h} \|\Delta(G_h - G)\|_{L^1(\tau)} + \sum_{e\in\mathcal{E}_h} \|[\partial_n(G_h - G)]\|_{L^1(e)} + C \right) \|u - I_h u\|_{L^\infty(\Omega)} \\
&\le C\big(\|G_h - G\|_{W_h^{2,1}(\Omega)} + h^{-1}\|\nabla(G_h - G)\|_{L^1(\Omega)} + 1 \big) \|u - I_h u\|_{L^\infty(\Omega)},
\end{aligned} \tag{4.4}$$

where we have used the notation

$$\|v\|_{W_h^{2,1}(\Omega)} := \sum_{\tau\in\mathcal{T}_h} \|v\|_{W^{2,1}(\tau)}$$

and the trace inequality

$$\|\nabla v\|_{L^1(e)} \le C\left(h^{-1}\|v\|_{W^{1,1}(\tau)} + \|v\|_{W^{2,1}(\tau)} \right).$$

We will estimate $\|G_h - G\|_{W_h^{2,1}(\Omega)} + h^{-1}\|\nabla(G_h - G)\|_{L^1(\Omega)}$ in the next two subsections for the two cases $r = 1$ and $r \ge 2$, respectively, via the decomposition

$$\begin{aligned}
&\|G_h - G\|_{W_h^{2,1}(\Omega)} + h^{-1}\|\nabla(G_h - G)\|_{L^1(\Omega)} \\
&\qquad \le \|G_h - G\|_{W_h^{2,1}(O_*)} + h^{-1}\|\nabla(G_h - G)\|_{L^1(O_*)} \\
&\qquad\quad + \sum_{j=0}^{J} \big(\|G_h - G\|_{W_h^{2,1}(O_j)} + h^{-1}\|\nabla(G_h - G)\|_{L^1(O_j)} \big)
\end{aligned}$$

where

$$O_* = \{x \in \Omega : |x - x_0| < d_{J+1}\}, \tag{4.5a}$$
$$O_j = \{x \in \Omega : d_{j+1} \le |x - x_0| < d_j\}, \quad j = 0, 1, \cdots, J, \tag{4.5b}$$

with $d_j = 2^{-j}\mathrm{diam}(\Omega)$ and $J = \log_2\left(\frac{\mathrm{diam}(\Omega)}{\kappa h}\right)$. Here κ is a positive parameter to be determined later, and in the rest part of this paper we shall keep the generic constant C to be independent of κ until the parameter κ is determined.

The following lemma concerns some local estimates of the regularized Green's function, which will be used in the next two subsections.

Lemma 12 *Let $s \in (0, 1 - \frac{\pi}{\alpha_M})$, $x_0 \in \tau_0$ and $\mathrm{dist}(\tau_0, O_j) \ge d_j/C$. Then the regularized Green's function G satisfies the following estimates*

$$\|G\|_{H^1(O_j)} \le C, \tag{4.6a}$$
$$\|\nabla G\|_{L^\infty(O_j)} + \|G\|_{H^2(O_j)} \le C d_j^{-1}, \tag{4.6b}$$
$$\|G\|_{H^{1+s}(O_j)} \le C d_j^{-s}, \tag{4.6c}$$
$$\|G\|_{H^{2+s}(O_j)} \le C d_j^{-1-s}. \tag{4.6d}$$

Proof We use the Green's function representation

$$G(x, x_0) = \int_\Omega \Gamma(x, y)\left(\tilde\delta(y - x_0) - \frac{1}{|\Omega|}\right)\, dy = \int_\Omega \Gamma(x, y)\tilde\delta(y)\, dy,$$

where $\Gamma(x, y)$ is the Green's function of the Neumann problem, which also satisfies the normalization condition $\int_\Omega \Gamma(x, y)\, dy = 0$. For $x \in O_j$, we have

$$|\nabla_x G(x, x_0)| \le \int_{\tau_0} |\nabla_x \Gamma(x, y)||\tilde\delta(y)|\, dy \le C d_j^{-1}\|\tilde\delta\|_{L^1(\tau_0)} \le C d_j^{-1},$$

where we used Lemma 10 and the fact that $dist(x_0, O_j) \ge d_j/C$. Hence

$$\|\nabla G\|_{L^\infty(O_j)} \le C d_j^{-1}, \tag{4.7}$$

and the Hölder inequality also implies

$$\|\nabla G\|_{L^2(O_j)} \le C. \tag{4.8}$$

Clearly, if we define $O_j' := O_{j-1} \cup O_j \cup O_{j+1}$, then we have the similar estimates

$$\|\nabla G\|_{L^\infty(O_j')} \le C d_j^{-1} \quad \text{and} \quad \|\nabla G\|_{L^2(O_j')} \le C. \tag{4.9}$$

To obtain (4.6d), we apply Lemma 5 to $\|G\|_{H^{2+s}(O_j)}$ and using that $\tilde\delta$ is supported outside of O_j, we obtain

$$\|G\|_{H^{2+s}(O_j)} \le C d_j^{-s}\Big\|\frac{1}{|\Omega|}\Big\|_{L^2(O_j')} + C d_j^{-s}\|\nabla G\|_{L^\infty(O_j')} \le C d_j^{-1-s}. \tag{4.10}$$

4.1 Linear elements

For piecewise linear elements (the case $r = 1$), we have $\|G_h - G\|_{W_h^{2,1}(\Omega)} = \|G\|_{W^{2,1}(\Omega)}$. To estimate $\|G\|_{W^{2,1}(\Omega)}$ we use $W^{2,p}$ regularity for convex domains. Thus for any $p > 1$,

$$\begin{aligned}\|G\|_{W^{2,1}(\Omega)} \le \|G\|_{W^{2,p}(\Omega)} &\le C_p \left(\left\| \tilde{\delta} - \frac{1}{|\Omega|} \right\|_{L^p(\Omega)} \right) \\ &\le C_p \left(\|\tilde{\delta}\|_{L^p(\Omega)} + \left\| \frac{1}{|\Omega|} \right\|_{L^p(\Omega)} \right) \le C_p h^{-2(1-1/p)},\end{aligned}$$

where the precise behavior of the constant $C_p \approx \frac{1}{p-1}$ as $p \to 1$ can be traced for example from Theorem 9.9 in [13]. Choosing $p = 1 + 1/\ell_h$ we obtain

$$\|G\|_{W^{2,1}(\Omega)} \le C\ell_h. \tag{4.11}$$

It remains to prove that

$$\|\nabla(G_h - G)\|_{L^1(\Omega)} \le C\ell_h h. \tag{4.12}$$

Since $|O_*| \le C(\kappa h)^2$ using the approximation theory and H^2 regularity, we have

$$\begin{aligned}\|\nabla(G - G_h)\|_{L^1(O_*)} &\le C\kappa h \|\nabla(G - G_h)\|_{L^2(O_*)} \\ &\le C_\kappa h^2 \|G\|_{H^2(\Omega)} \le C_\kappa h^2 \|\tilde{\delta}\|_{L^2(\Omega)} \le C_\kappa h,\end{aligned} \tag{4.13}$$

where we have used the standard energy error estimate and (4.3c) in the last inequality. By applying the Hölder inequality and Lemma 11, we also obtain

$$\begin{aligned}\|\nabla(G - G_h)\|_{L^1(O_j)} &\le C d_j \|\nabla(G - G_h)\|_{L^2(O_j)} \\ &\le C d_j \|G - I_h G\|_{H^1(O_j')} + C\|G - I_h G\|_{L^2(O_j')} + C\|G - G_h\|_{L^2(O_j')} \\ &\le C h d_j |G|_{H^2(O_j')} + C\|G - G_h\|_{L^2(O_j')} \\ &\le Ch + C\|G - G_h\|_{L^2(O_j')},\end{aligned}$$

where we have used Lemma 12 in the last step of the inequality above. We see that

$$\begin{aligned}\|\nabla(G - G_h)\|_{L^1(\Omega)} &\le \|\nabla(G - G_h)\|_{L^1(O_*)} + \sum_{j=0}^{J} \|\nabla(G - G_h)\|_{L^1(O_j)} \\ &\le C\kappa h + \sum_{j=0}^{J} Ch + \sum_{j=0}^{J} C\|G - G_h\|_{L^2(O_j')} \\ &\le C\kappa h + C\ell_h h + \sum_{j=0}^{J} C\|G - G_h\|_{L^2(O_j')} \\ &\le C\kappa h + C\ell_h h + \left(\sum_{j=0}^{J} 1 \right)^{1/2} \left(\sum_{j=0}^{J} \|G - G_h\|_{L^2(O_j')}^2 \right)^{1/2} \\ &\le C\kappa h + C\ell_h h + C\ell_h^{1/2} \|G - G_h\|_{L^2(\Omega)} \\ &\le C\kappa h + C\ell_h h + C\ell_h^{1/2} h^2 \|\tilde{\delta}\|_{L^2} \le C(\kappa + \ell_h) h.\end{aligned} \tag{4.14}$$

This proves (4.12) and completes the proof of Lemma 9 in the case $r = 1$.

4.2 Higher order elements

To estimate $\|G_h - G\|_{W^{2,1}(\Omega)}$ by the triangle inequality and the inverse inequality, we have

$$\begin{aligned}\|G_h - G\|_{W^{2,1}(\Omega)} &\leq \|G_h - I_h G\|_{W^{2,1}(\Omega)} + \|G - I_h G\|_{W^{2,1}(\Omega)} \\ &\leq Ch^{-1}\|G_h - I_h G\|_{W^{1,1}(\Omega)} + \|G - I_h G\|_{W^{2,1}(\Omega)} \\ &\leq Ch^{-1}\left(\|G_h - G\|_{W^{1,1}(\Omega)} + \|G - I_h G\|_{W^{1,1}(\Omega)}\right) + \|G - I_h G\|_{W^{2,1}(\Omega)}.\end{aligned}$$

The last two terms in the inequality above are estimated in the following lemma.

Lemma 13 *For $r \geq 2$, there exists a constant C independent of h such that*

$$h^{-1}\|G - I_h G\|_{W^{1,1}(\Omega)} + \|G - I_h G\|_{W^{2,1}(\Omega)} \leq C.$$

Proof Using the dyadic decomposition we have

$$\begin{aligned}&h^{-1}\|G - I_h G\|_{W^{1,1}(\Omega)} + \|G - I_h G\|_{W^{2,1}(\Omega)} \\ &\leq h^{-1}\|G - I_h G\|_{W^{1,1}(O_*)} + \|G - I_h G\|_{W^{2,1}(O_*)} \\ &\quad + \sum_{j=1}^{J} h^{-1}\|G - I_h G\|_{W^{1,1}(O_j)} + \|G - I_h G\|_{W^{2,1}(O_j)}.\end{aligned} \tag{4.15}$$

Using the Cauchy-Schwarz inequality and H^2 regularity, and the properties of $\tilde{\delta}$, we obtain

$$\begin{aligned}h^{-1}\|G - I_h G\|_{W^{1,1}(O_*)} + \|G - I_h G\|_{W^{2,1}(O_*)} &\\ \leq C\|G - I_h G\|_{H^1(O_*)} + h\|G - I_h G\|_{H^2(O_*)} &\\ \leq Ch\|G\|_{H^2(\Omega)} \leq Ch\|\tilde{\delta}\|_{L^2(\Omega)} \leq C.&\end{aligned} \tag{4.16}$$

To estimate the terms in the sum we use the Cauchy-Schwarz inequality together with the approximation theory and (4.6d) to obtain

$$\begin{aligned}h^{-1}\|G - I_h G\|_{W^{1,1}(O_j)} + \|G - I_h G\|_{W^{2,1}(O_j)} &\\ \leq Cd_j\left(h^{-1}\|G - I_h G\|_{H^1(O_j)} + \|G - I_h G\|_{H^2(O_j)}\right) &\\ = Cd_j h^s\|G\|_{H^{2+s}(O_j)} \leq Cd_j^{-s}h^s,&\end{aligned} \tag{4.17}$$

for some $0 < s < 1 - \frac{\pi}{\alpha_M}$. Combing (4.15)–(4.17), we obtain

$$h^{-1}\|G - I_h G\|_{W^{1,1}(\Omega)} + \|G - I_h G\|_{W^{2,1}(\Omega)} \leq C + \sum_{j=1}^{J} h^s d_j^{-s} \leq C.$$

To finish the proof of of Lemma 9, it remains to prove that

$$\|\nabla(G_h - G)\|_{L^1(\Omega)} \leq Ch. \tag{4.18}$$

Using the dyadic decomposition and the Cauchy-Schwarz inequality again, we obtain

$$\begin{aligned}\|\nabla(G - G_h)\|_{L^1(\Omega)} &\leq \|\nabla(G - G_h)\|_{L^1(\Omega_*)} + \sum_{j=1}^{J}\|\nabla(G - G_h)\|_{L^1(O_j)} \\ &\leq Ch\|\nabla(G - G_h)\|_{L^2(\Omega_*)} + \sum_{j=1}^{J} d_j\|\nabla(G - G_h)\|_{L^2(O_j)}.\end{aligned}$$

Using the global best approximation property of the Galerkin solution in H^1 norm we obtain

$$\|\nabla(G-G_h)\|_{L^2(\Omega_*)} \le \|\nabla(G-G_h)\|_{L^2(\Omega)} \le Ch\|G\|_{H^2(\Omega)} \le Ch\|\tilde{\delta}\|_{L^2(\Omega)} \le C.$$

Thus we have

$$\|\nabla(G-G_h)\|_{L^1(\Omega)} \le Ch + \sum_{j=1}^{J} M_j, \tag{4.19}$$

where

$$M_j := d_j\|\nabla(G-G_h)\|_{L^2(O_j)}. \tag{4.20}$$

By the local energy estimates, Lemma 11, and using Lemma 5 and that $h \le d_j$, we obtain

$$\begin{aligned} d_j\|\nabla(G-G_h)\|_{L^2(O_j)} &\\ &\le C\left(d_j\|\nabla(G-I_hG)\|_{L^2(O_j')} + \|G-I_hG\|_{L^2(O_j')} + \|G-G_h\|_{L^2(O_j')}\right)\\ &\le C\left(d_jh^{1+s}\|G\|_{H^{2+s}(O_j'')} + \|G-G_h\|_{L^2(O_j')}\right)\\ &\le Cd_j^{-s}h^{1+s} + C\|G-G_h\|_{L^2(O_j')}. \end{aligned} \tag{4.21}$$

Since $\sum_j d_j^{-s}h^s \le C$, we only need to estimate $\|G-G_h\|_{L^2(O_j')}$. We will accomplish that by a duality argument. By duality

$$\|G-G_h\|_{L^2(O_j')} = \sup_{\substack{\psi\in C_0^\infty(O_j')\\ \|\psi\|_{L^2(\Omega)}\le 1}} (G-G_h,\psi).$$

For each such ψ, we define w as the solution of

$$\begin{cases} -\Delta w = \psi - \bar{\psi} & \text{in } \ \Omega,\\ \partial_n w = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{4.22}$$

where $\bar{\psi} = \frac{1}{|\Omega|}\int_\Omega \psi(x)dx$. Such a solution w exists and is unique if we impose the condition $\int_\Omega w(x)\ dx = 0$. Thus,

$$\begin{aligned} (G-G_h,\psi) &= (G-G_h,\psi-\bar{\psi}) = (G-G_h,-\Delta w) = (\nabla(G-G_h),\nabla w)\\ &= (\nabla(G-G_h),\nabla(w-I_hw))\\ &= (\nabla(G-G_h),\nabla(w-I_hw))_{O_j''} + (\nabla(G-G_h),\nabla(w-I_hw))_{\Omega\setminus O_j''}\\ &:= I_1 + I_2. \end{aligned}$$

Using the Cauchy-Schwarz inequality and H^2 regularity

$$I_1 \le \|\nabla(G-G_h)\|_{L^2(O_j'')}\|\nabla(w-I_hw)\|_{L^2(\Omega)} \le Ch\|\nabla(G-G_h)\|_{L^2(O_j'')}.$$

To estimate the second term we the Hölder inequality, the approximation theory and embedding $H^{2+s} \hookrightarrow C^{1+s}$, to obtain

$$\begin{aligned} I_2 &= (\nabla(G-G_h),\nabla(w-I_hw))_{\Omega\setminus O_j''}\\ &\le \|\nabla(G-G_h)\|_{L^1(\Omega)}\|\nabla(w-I_hw)\|_{L^\infty(\Omega\setminus O_j'')}\\ &\le \|\nabla(G-G_h)\|_{L^1(\Omega)}Ch^s\|w\|_{C^{1+s}(\Omega\setminus O_j'')}\\ &\le \|\nabla(G-G_h)\|_{L^1(\Omega)}Ch^s\|w\|_{H^{2+s}(\Omega\setminus O_j'')}, \end{aligned}$$

for some $s \in (0, 1 - \frac{\pi}{\alpha_M})$. Using the representation

$$w(x) = \int_\Omega \Gamma(x,y)\left(\psi(y) - \bar{\psi}(y)\right)\, dy = \int_\Omega \Gamma(x,y)\psi(y)\, dy,$$

similarly to the Lemma 12 for $x \in \Omega \backslash O''_j$ and since $\Omega \backslash O''_j$ is separated from O'_j by at least d_j, we obtain

$$\|w\|_{H^{2+s}(\Omega \backslash O''_j)} \le C d_j^{-s}.$$

That implies that

$$(\nabla(G - G_h), \nabla(w - I_h w))_{\Omega \backslash O''_j} \le C h^s d_j^{-s} \|\nabla(G - G_h)\|_{L^1(\Omega)}.$$

Therefore,

$$\|G - G_h\|_{L^2(O'_j)} \le C h^s d_j^{-s} \|\nabla(G - G_h)\|_{L^1(\Omega)} + Ch \|\nabla(G - G_h)\|_{L^2(O''_j)}.$$

To summarize, for some $s \in (0, 1 - \frac{\pi}{\alpha_M})$, we have

$$M_j \le Ch(h/d_j)^s + C(h/d_j)^s \|\nabla(G - G_h)\|_{L^1(\Omega)} + Ch\|\nabla(G - G_h)\|_{L^2(O''_j)}.$$

Summing over j we obtain

$$\sum_{j=0}^{J} M_j \le \frac{Ch}{\kappa^s} + \frac{C}{\kappa^s} \|\nabla(G - G_h)\|_{L^1(\Omega)} + \frac{C}{\kappa} \sum_{j=0}^{J} d_j \|\nabla(G - G_h)\|_{L^2(O''_j)},$$

where we have used that

$$\sum_{j=0}^{J} (h/d_j)^s \le h^s \sum_{j=0}^{J} 2^{js} \le C h^s 2^{sJ} \le C\kappa^{-s} \quad \text{and} \quad d_j^{-1} \le d_J^{-1}.$$

Clearly,

$$\begin{aligned} \sum_{j=0}^{J} d_j \|\nabla(G - G_h)\|_{L^2(O''_j)} &\le C \sum_{j=0}^{J} M_j + C\kappa h \|\nabla(G - G_h)\|_{L^2(O_*)} \\ &\le C \sum_{j=0}^{J} M_j + C\kappa h. \end{aligned}$$

Thus, using that $h/d_J \le \kappa^{-1}$, and taking κ large enough we have

$$\sum_{j=0}^{J} M_j \le Ch + \frac{C}{\kappa^s} \|\nabla(G - G_h)\|_{L^1(\Omega)}.$$

Therefore, if we plug this result into (4.19) we get

$$\|\nabla(G - G_h)\|_{L^1(\Omega)} \le Ch + \frac{C}{\kappa^s} \|\nabla(G - G_h)\|_{L^1(\Omega)}.$$

Again by choosing κ large enough we can conclude

$$\|\nabla(G - G_h)\|_{L^1(\Omega)} \le Ch.$$

This proves (4.18) and completes the proof of Lemma 9 in the case $r \ge 2$.

Acknowledgements We would like to thank the anonymous referee for the valuable suggestions. The research of B. Li was partially supported by the start-up grant of The Hong Kong Polytechnic University, and was partially carried during a research stay at University of Tübingen, funded by the Alexandre von Humboldt foundation and NSFC 11301262.

References

1. Y. Alkhutov and V. G. Maz'ya. $L^{1,p}$-coercitivity and estimates of the Green function of the Neumann problem in a convex domain. *Journal of Mathematical Sciences*, 196(3):245–261, 2014.
2. T. Apel, J. Pfefferer, and A. Rösch. Finite element error estimates for Neumann boundary control problems on graded meshes. *Comput. Optim. Appl.*, 52(1):3–28, 2012.
3. N. Y. Bakaev, V. Thomée, and L. B. Wahlbin. Maximum-norm estimates for resolvents of elliptic finite element operators. *Math. Comp.*, 72(244):1597–1610 (electronic), 2003.
4. C. Bernardi, M. Dauge, and Y. Maday. Polynomials in the Sobolev World. working paper or preprint, September 2007.
5. P. G. Ciarlet. *The Finite Element Method for Elliptic Problems.* North–Holland, Amsterdam, New York, 1978.
6. M. Crouzeix, S. Larsson, and V. Thomée. Resolvent estimates for elliptic finite element operators in one dimension. *Math. Comp.*, 63(207):121–140, 1994.
7. M. Dauge. *Elliptic boundary value problems on corner domains*, volume 1341 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1988. Smoothness and asymptotics of solutions.
8. A. Demlow, D. Leykekhman, A. H. Schatz, and L. B. Wahlbin. Best approximation property in the W^1_∞ norm for finite element methods on graded meshes. *Math. Comp.*, 81(278):743–764, 2012.
9. V. I. Fabrikant. *Mixed boundary value problems of potential theory and their applications in engineering*, volume 68 of *Mathematics and its Applications.* Kluwer Academic Publishers Group, Dordrecht, 1991.
10. I. Fried. On the optimality of the pointwise accuracy of the finite element solution. *Internat. J. Numer. Methods Engrg.*, 15(3):451–456, 1980.
11. M. Geissert. Discrete maximal L_p regularity for finite element operators. *SIAM J. Numer. Anal.*, 44(2):677–698 (electronic), 2006.
12. M. Geissert. Applications of discrete maximal L_p regularity for finite element operators. *Numer. Math.*, 108(1):121–149, 2007.
13. D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin, second (Classics in Mathematics) edition, 2001.
14. P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics.* Pitman (Advanced Publishing Program), Boston, MA, 1985.
15. J. Guzmán and D. Leykekhman. Pointwise error estimates of finite element approximations to the Stokes problem on convex polyhedra. *Math. Comp.*, 81(280):1879–1902, 2012.
16. J. Guzmán, D. Leykekhman, J. Rossmann, and A. H. Schatz. Hölder estimates for Green's functions on convex polyhedral domains and their applications to finite element methods. *Numer. Math.*, 112(2):221–243, 2009.
17. R. Haverkamp. Eine Aussage zur L_∞-Stabilität und zur genauen Konvergenzordnung der H^1_0-Projektionen. *Numer. Math.*, 44(3):393–405, 1984.
18. D. Leykekhman and B. Vexler. Finite Element Pointwise Results on Convex Polyhedral Domains. *SIAM J. Numer. Anal.*, 54(2):561–587, 2016.
19. D. Leykekhman and B. Vexler. Pointwise best approximation results for Galerkin finite element solutions of parabolic problems. *accepted by SIAM J. Numer. Anal.*, 2016.
20. D. Leykekhman and B. Vexler. A priori error estimates for three dimensional parabolic optimal control problems with pointwise control. *submitted*, 2016.
21. B. Li. Maximum-norm stability and maximal L^p regularity of FEMs for parabolic equations with Lipschitz continuous coefficients. *Numer. Math.*, 131(3):489–516, 2015.
22. B. Li and W. Sun. Regularity of the diffusion-dispersion tensor and error analysis of Galerkin FEMs for a porous medium flow. *SIAM J. Numer. Anal.*, 53(3):1418–1437, 2015.
23. B. Li and W. Sun: Maximal L^p analysis of finite element solutions for parabolic equations with non-smooth coefficients in convex polyhedra. *Math. Comp.*, in press. http://arxiv.org/abs/1501.07345
24. J. Nitsche. L_∞-convergence of finite element approximations. In *Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975)*, pages 261–274. Lecture Notes in Math., Vol. 606. Springer, Berlin, 1977.
25. J. A. Nitsche and A. H. Schatz. Interior estimates for Ritz-Galerkin methods. *Math. Comp.*, 28:937–958, 1974.
26. N. Pernavs. On a mixed boundary value problem of harmonic functions. *Proc. Amer. Math. Soc.*, 7:127–130, 1956.
27. R. Rannacher and R. Scott. Some optimal error estimates for piecewise linear finite element approximations. *Math. Comp.*, 38(158):437–445, 1982.
28. A. H. Schatz and L. B. Wahlbin. Interior maximum norm estimates for finite element methods. *Math. Comp.*, 31(138):414–442, 1977.
29. A. H. Schatz and L. B. Wahlbin. On the quasi-optimality in L_∞ of the $\dot{H}^1$-projection into finite element spaces. *Math. Comp.*, 38(157):1–22, 1982.
30. A. H. Schatz. A weak discrete maximum principle and stability of the finite element method in L_∞ on plane polygonal domains. I. *Math. Comp.*, 34(149):77–91, 1980.

31. G. Strang and G. J. Fix. *An Analysis of the Finite Element Method*. Wellesley-Cambridge Press, Wellesley, MA, 1988.
32. V. Thomée and L. B. Wahlbin. Stability and analyticity in maximum-norm for simplicial Lagrange finite element semidiscretizations of parabolic equations with Dirichlet boundary conditions. *Numer. Math.*, 87(2):373–389, 2000.
33. V. Thomée. *Galerkin finite element methods for parabolic problems*, volume 25 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, second edition, 2006.