The Kalman Decomposition for Linear Quantum Stochastic Systems

Symeon Grivopoulos[1]   Guofeng Zhang[2]   Ian R. Petersen[3]   John Gough[4]

Abstract— The Kalman decomposition for Linear Quantum Stochastic Systems in the real quadrature operator representation, that was derived indirectly in [1] by the authors, is derived here directly, using the "one-sided symplectic" SVD-like factorization of [2] on the observability matrix of the system.

I. INTRODUCTION

Linear Quantum Stochastic Systems (LQSSs) are a class of models used in linear quantum optics [3], [4], [5], circuit QED systems [6], [7], quantum opto-mechanical systems [8], [9], [10], [11], and elsewhere. The mathematical framework for these models is provided by the theory of quantum Wiener processes, and the associated Quantum Stochastic Differential Equations [12], [13], [14]. Potential applications of LQSSs include quantum information processing, and quantum measurement and control. In particular, an important application of LQSSs is as coherent quantum feedback controllers for other quantum systems, i.e. controllers that do not perform any measurement on the controlled quantum system, and thus, have the potential to outperform classical controllers, see e.g. [15], [16], [17], [18], [19], [20], [21], [10], [22].

Controllability (stabilizability) and observability (detectability) of a classical linear system are prerequisites for various control design methods. These notions, and the related mathematical concepts and techniques, can be transferred essentially unchanged to LQSSs, see e.g. [17], [18], [23]. There is, however, an important difference from the classical case: The allowed state transformations in LQSSs (for the purpose of state-space decompositions) cannot be arbitrary, but are fundamentally restricted by the laws of quantum mechanics. More specifically, in the so called *real quadrature operator representation* of an LQSS that is used in this work, the only transformations that preserve its structure (see Subsection II-B) are real symplectic ones. Recently, various investigations of controllability and observability for

[1] Symeon Grivopoulos was with the School of Engineering and Information Technology, UNSW Canberra, Canberra BC 2610, Australia. Current address: 11 Zafeiriou Rapesi St., Mesolonghi 30200, Greece symeon.grivopoulos@gmail.com
[2] Guofeng Zhang is with the Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong Guofeng.Zhang@polyu.edu.hk
[3] Ian R. Petersen is with the Research School of Engineering, The Australian National University, Canberra ACT 2601, Australia i.r.petersen@gmail.com
[4] John Gough is with the Department of Physics, Aberystwyth University, Wales, cSY23 2BZ, Aberystwyth, UK jug@aber.ac.uk

LQSSs have appeared in the literature, see e.g. [24], [25], [1]. In [1], the authors of the present work showed that, a Kalman decomposition of a LQSS is always possible with a real orthogonal and symplectic transformation. Moreover, they uncovered the following interesting structure in the decomposition: The controllable/observable (co), and uncontrollable/unobservable subsystems ($\bar{c}\bar{o}$) are LQSSs in their own right, as is to be expected from a physics perspective. Furthermore, the states of the controllable/unobservable ($c\bar{o}$) subsystem are conjugate variables of the states of the uncontrollable/observable ($\bar{c}o$) subsystem. An immediate consequence of this is that, a $c\bar{o}$ subsystem exists if and only if a $\bar{c}o$ subsystem does, and they always have the same dimension. This is a consequence of the special structure of LQSSs.

The construction of the Kalman decomposition in [1], is performed first in the so called *creation-annihilation operator representation* of a LQSS, where special bases for the co, $\bar{c}\bar{o}$, $c\bar{o}$, and $\bar{c}o$ subspaces are constructed, and the result is then translated in the real quadrature representation. We should point out that the Kalman decomposition of a LQSS in the real quadrature representation offers an advantage over the corresponding decomposition in the creation-annihilation representation of the LQSS: In the former, the $c\bar{o}$ and $\bar{c}o$ subsystems are separate, as usual, while in the latter, the two subsystems are merged, due to the grouping of states imposed by that representation. In this work, we present a derivation of the Kalman decomposition of a LQSS, directly in the real quadrature operator representation. This derivation uses the "one-sided symplectic" SVD-like factorization of [2] on the observability matrix of the LQSS, and leads directly to the desired decomposition. Its value lies in its brevity and directness in uncovering the structure of the Kalman decomposition of LQSSs.

II. BACKGROUND MATERIAL

A. Notation and terminology

1) x^* denotes the complex conjugate of a complex number x or the adjoint of an operator x, respectively. For a matrix $X = [x_{ij}]$ with number or operator entries, $X^\# = [x_{ij}^*]$, $X^\top = [x_{ji}]$ is the usual transpose, and $X^\dagger = (X^\#)^\top$. The commutator of two operators X and Y is defined as $[X, Y] = XY - YX$.

2) The identity matrix in n dimensions will be denoted by I_n, and a $r \times s$ matrix of zeros will be denoted by $0_{r \times s}$. δ_{ij} denotes the Kronecker delta symbol, i.e. $I = [\delta_{ij}]$. We define $\mathbb{J}_{2k} = \begin{pmatrix} 0_{k \times k} & I_k \\ -I_k & 0_{k \times k} \end{pmatrix}$. Also, $\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix}$ is the

vertical concatenation of the matrices $X_1, X_2, \ldots, X_k$, of equal column dimension, $(Y_1 \ Y_2 \ \ldots \ Y_k)$ is the horizontal concatenation of the matrices $Y_1, Y_2, \ldots, Y_k$ of equal row dimension, and $\mathrm{diag}(Z_1, Z_2, \ldots, Z_k)$ is the block-diagonal matrix formed by the square matrices $Z_1, Z_2, \ldots, Z_k$. $\mathrm{Im}A$, and $\mathrm{Ker}A$ denote the range and the null space of a matrix, respectively.

3) For a $2r \times 2s$ matrix X, define its $\sharp$-*adjoint* $X^\sharp$, by $X^\sharp = -\mathbb{J}_{2s} X^\dagger \mathbb{J}_{2r}$. The $\sharp$-adjoint satisfies properties similar to the usual adjoint, namely $(x_1 A + x_2 B)^\sharp = x_1^* A^\sharp + x_2^* B^\sharp$, $(AB)^\sharp = B^\sharp A^\sharp$, and $(A^\sharp)^\sharp = A$.

4) A $2k \times 2k$ complex matrix T is called *symplectic*, if it satisfies $TT^\sharp = T^\sharp T = I_{2k}$. Hence, any symplectic matrix is invertible, and its inverse is its $\sharp$-adjoint. The set of these matrices forms a non-compact group known as the symplectic group. Real symplectic matrices constitute a subgroup of the (complex) symplectic group.

B. Linear Quantum Stochastic Systems

The material in this Subsection is fairly standard, and our presentation aims mostly at establishing notation and terminology. For the mathematical background necessary for a precise discussion of LQSSs, some standard references are [12], [13], [14], while for a Physics perspective, see [3], [26]. The references [27], [28], [29], [30], [31] contain a lot of relevant material, as well.

The systems we consider in this work are collections of quantum harmonic oscillators interacting among themselves, as well as with their environment. The i-th harmonic oscillator ($i = 1, \ldots, n$) is described by its position and momentum variables, q_i and p_i, respectively. These are self-adjoint operators satisfying the *Canonical Commutation Relations* (CCRs) $[q_i, q_j] = 0$, $[p_i, p_j] = 0$, and $[q_i, p_j] = \imath \delta_{ij}$, for $i, j = 1, \ldots, n$. As in classical mechanics, the states q_i and p_i, $i = 1, \ldots, n$, are called *conjugate* states. If we define the vectors of operators $q = (q_1, q_2, \ldots, q_n)^\top$, $p = (p_1, p_2, \ldots, p_n)^\top$, and $x = \binom{q}{p}$, the CCRs can be expressed as

$$[x, x^\top] \doteq xx^\top - (xx^\top)^\top = \begin{pmatrix} \mathbf{0} & \imath I_n \\ -\imath I_n & \mathbf{0} \end{pmatrix} = \imath \mathbb{J}_{2n}. \quad (1)$$

The environment is modelled as a collection of zero temperature bosonic quantum fields. The i-th field ($i = 1, \ldots, m$) is described by bosonic *field annihilation and creation operators* $\mathcal{A}_i(t)$ and $\mathcal{A}_i^*(t)$, respectively. The field operators are *adapted quantum stochastic processes* with forward differentials $d\mathcal{A}_i(t) = \mathcal{A}_i(t+dt) - \mathcal{A}_i(t)$, and $d\mathcal{A}_i^*(t) = \mathcal{A}_i^*(t + dt) - \mathcal{A}_i^*(t)$. They satisfy the quantum Itô products $d\mathcal{A}_i(t)d\mathcal{A}_j(t) = 0$, $d\mathcal{A}_i^*(t)d\mathcal{A}_j^*(t) = 0$, $d\mathcal{A}_i^*(t)d\mathcal{A}_j(t) = 0$, and $d\mathcal{A}_i(t)d\mathcal{A}_j^*(t) = \delta_{ij}dt$. If we define the vector of field operators $\mathcal{A}(t) = (\mathcal{A}_1(t), \mathcal{A}_2(t), \ldots, \mathcal{A}_m(t))^\top$, and the vector of self-adjoint *field quadratures*

$$\mathcal{V}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{A}(t) + \mathcal{A}(t)^\# \\ -\imath(\mathcal{A}(t) - \mathcal{A}(t)^\#) \end{pmatrix},$$

the quantum Itô products above can be expressed as

$$d\mathcal{V}(t)d\mathcal{V}(t)^\top = \frac{1}{2} \begin{pmatrix} I_m & \imath I_m \\ -\imath I_m & I_m \end{pmatrix} dt = \frac{1}{2} (I_{2m} + \imath \mathbb{J}_{2m})dt. (2)$$

To describe the dynamics of the harmonic oscillators and the quantum fields, we introduce certain operators. We begin with the Hamiltonian operator $H = \frac{1}{2}x^\top R x$, which specifies the dynamics of the harmonic oscillators in the absence of any environmental influence. R is a $2n \times 2n$ real symmetric matrix referred to as the Hamiltonian matrix. Next, we have the coupling operator L (vector of operators) that specifies the interaction of the harmonic oscillators with the quantum fields. L depends linearly on the position and momentum operators of the oscillators, and can be expressed as $L = L_q q + L_p p$. We construct the real coupling matrix $C^{2m \times 2n}$ from $L_q^{m \times n}$ and $L_p^{m \times n}$, as $C = \frac{1}{\sqrt{2}} \begin{pmatrix} L_q + L_q^\# & L_p + L_p^\# \\ -\imath(L_q - L_q^\#) & -\imath(L_p - L_p^\#) \end{pmatrix}$. Finally, we have the unitary scattering matrix $S^{m \times m}$, that describes the mixing $\mathcal{A}(t) \to S\mathcal{A}(t)$ effected on the quantum field operators by a linear static passive network (made up of phase shifters and beam splitters) preceding the inputs of the dynamical elements of the LQSS (such as cavities, degenerate parametric amplifiers, etc).

In the *Heisenberg picture* of quantum mechanics, the joint evolution of the harmonic oscillators and the quantum fields is described by the following system of *Quantum Stochastic Differential Equations* (QSDEs):

$$\begin{aligned} dx &= (\mathbb{J}R - \frac{1}{2}C^\sharp C)xdt - C^\sharp \Sigma \, d\mathcal{V}, \\ d\mathcal{V}_{out} &= Cxdt + \Sigma \, d\mathcal{V}, \end{aligned} \quad (3)$$

where

$$\Sigma = \frac{1}{2} \begin{pmatrix} S + S^\# & \imath(S - S^\#) \\ -\imath(S - S^\#) & S + S^\# \end{pmatrix},$$

is, by construction, $2m \times 2m$ real orthogonal and symplectic. The field quadrature operators $\mathcal{V}_{i\,out}(t)$ describe the outputs of the system. (3) is a description of the dynamics of the LQSS in the real quadrature operator representation, where the states, inputs, and outputs are all self-adjoint operators. We are going to use a version of (3) generalized in two ways: First, we replace the real orthogonal symplectic transformation Σ, with a more general real symplectic transformation Σ, see e.g. [31] for a discussion in the creation-annihilation representation. Second, in the context of coherent quantum systems in particular, the output of a quantum system may be fed into other quantum system, so we substitute the more general input and output notations $\mathcal{U}$ and $\mathcal{Y}$, for $\mathcal{V}$ and $\mathcal{V}_{out}$, respectively. The resulting QSDEs are the following:

$$\begin{aligned} dx &= (\mathbb{J}R - \frac{1}{2}C^\sharp C)xdt - C^\sharp \Sigma \, d\mathcal{U}, \\ d\mathcal{Y} &= Cxdt + \Sigma \, d\mathcal{U}, \end{aligned} \quad (4)$$

The forward differentials $d\mathcal{U}$ and $d\mathcal{Y}$ of inputs and outputs, respectively (or, more precisely, of their quadratures), contain "quantum noises", as well as a "signal part" (linear combinations of variables of other systems). One can prove that, the structure of (4) is preserved under linear transformations

of the state $\bar{x} = Tx$, if and only if T is real symplectic (with $\bar{R} = T^{-\top}RT^{-1}$, and $\bar{C} = CT^{-1} = CT^{\#}$). From the point of view of quantum mechanics, T must be real symplectic so that the transformed position and momentum operators are also self-adjoint and satisfy the same CCRs, as one can verify from (1). It is exactly this additional constraint on the allowed state transformations of LQSSs that complicates the construction of the Kalman decomposition for these systems.

III. THE KALMAN DECOMPOSITION FOR LINEAR QUANTUM STOCHASTIC SYSTEMS

System (4) has the standard form of a linear, time-invariant, system with $A = \mathbb{J}_{2n}R - \frac{1}{2}C^{\sharp}C$, $B = -C^{\sharp}\Sigma$, and $D = \Sigma$. In the following, we prove that there exists a real symplectic transformation of the state that puts (4) in a Kalman-like canonical form. Before we state and prove this result, we introduce the conventions used in this work regarding the uncontrollable and observable subspaces. Let

$$\mathcal{C} = \begin{pmatrix} B & AB & \cdots & A^{2n-1}B \end{pmatrix}, \text{ and}$$
$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{2n-1} \end{pmatrix}, \quad (5)$$

be the controllability and observability matrices of the system (4). As usual, $\mathrm{Im}\,\mathcal{C}$, and $\mathrm{Ker}\,\mathcal{O}$ define the controllable and unobservable subspaces. The uncontrollable and observable subspaces are defined as the *orthogonal complements* of $\mathrm{Im}\,\mathcal{C}$, and $\mathrm{Ker}\,\mathcal{O}$ in $\mathbb{R}^{2n}$, respectively. With this convention, we have the following theorem:

Theorem 1: Given the LQSS (4), there exists a real symplectic transformation V such that the following hold:

1) The transformed states $\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} = \hat{x} = Vx = V\begin{pmatrix} q \\ p \end{pmatrix}$, can be partitioned as follows:

$$\hat{q} = \begin{pmatrix} \hat{q}_a \\ \hat{q}_b \\ \hat{q}_c \end{pmatrix}, \hat{p} = \begin{pmatrix} \hat{p}_a \\ \hat{p}_b \\ \hat{p}_c \end{pmatrix}, \quad (6)$$

   where

   a) The states $\hat{q}_a$ and $\hat{p}_a$ are both controllable and observable.
   b) The states $\hat{p}_b$ are controllable but unobservable.
   c) The states $\hat{q}_b$ are uncontrollable but observable.
   d) The states $\hat{q}_c$ and $\hat{p}_c$ are both uncontrollable and unobservable.

2) In the transformed states, (4) takes the form

$$d\hat{x} = \hat{A}\hat{x}dt + \hat{B}d\mathcal{U},$$
$$d\mathcal{Y} = \hat{C}\hat{x}dt + Dd\mathcal{U}, \quad (7)$$

   where

$$\hat{A} = \begin{pmatrix} A_{co,11} & A_{13,1} & 0 & A_{co,12} & 0 & 0 \\ 0 & A_{\bar{c}o} & 0 & 0 & 0 & 0 \\ 0 & A_{43,1} & A_{\bar{c}\bar{o},11} & 0 & 0 & A_{\bar{c}\bar{o},12} \\ A_{co,21} & A_{13,2} & 0 & A_{co,22} & 0 & 0 \\ A_{21,1} & A_{23} & A_{24,1} & A_{21,2} & A_{c\bar{o}} & A_{24,2} \\ 0 & A_{43,2} & A_{\bar{c}\bar{o},21} & 0 & 0 & A_{\bar{c}\bar{o},22} \end{pmatrix},$$

and

$$\hat{B} = \begin{pmatrix} B_{co,1} \\ 0 \\ 0 \\ B_{co,2} \\ B_{c\bar{o}} \\ 0 \end{pmatrix}, \hat{C} = \begin{pmatrix} C_{co,1} & C_{\bar{c}o} & 0 & C_{co,2} & 0 & 0 \end{pmatrix}. \square \quad (8)$$

To prove Theorem 1, we shall need the following lemmas:

Lemma 1: Let

$$\tilde{\mathcal{C}} = \begin{pmatrix} B & (\mathbb{J}R)B & \cdots & (\mathbb{J}R)^{2n-1}B \end{pmatrix}, \text{ and}$$
$$\tilde{\mathcal{O}} = \begin{pmatrix} C \\ C(\mathbb{J}R) \\ \vdots \\ C(\mathbb{J}R)^{2n-1} \end{pmatrix}. \quad (9)$$

Then, $\mathrm{Im}\,\tilde{\mathcal{C}} = \mathrm{Im}\,\mathcal{C}$, and $\mathrm{Ker}\,\tilde{\mathcal{O}} = \mathrm{Ker}\,\mathcal{O}$. $\square$

This follows from standard results of linear systems theory, since the system (4) can be constructed from a system with $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (\mathbb{J}R, B, C, D)$, with state feedback with gain $\frac{1}{2}D^{-1}C$, or from a system with $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (\mathbb{J}R, \frac{1}{2}B, C, D)$ with output injection with gain $-\frac{1}{2}C^{\sharp}$. Hence, in all of the constructions above, we may use $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{O}}$ in place of $\mathcal{C}$ and $\mathcal{O}$. From now on, we shall refer to $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{O}}$ simply as the controllability and observability matrices of the system (4).

The second result we shall make use of, is the following:

Lemma 2: There exists a symplectic matrix T_0, such that $\tilde{\mathcal{O}} = T_0\tilde{\mathcal{C}}^{\sharp}$, or, equivalently, $\tilde{\mathcal{C}} = \tilde{\mathcal{O}}^{\sharp}T_0$. $\square$

Proof: Let $X_1, X_2, \ldots, X_k$ be complex matrices of corresponding dimensions $2r \times 2s_1, \ldots, 2r \times 2s_k$. Then,

$$\begin{pmatrix} X_1 & \cdots & X_k \end{pmatrix}^{\sharp} = -\mathbb{J}_{2(s_1+\ldots+s_k)}\begin{pmatrix} X_1 & \cdots & X_k \end{pmatrix}^{\dagger}\mathbb{J}_{2r}$$
$$= -\mathbb{J}_{2(s_1+\ldots+s_k)}\begin{pmatrix} X_1^{\dagger}\mathbb{J}_{2r} \\ \vdots \\ X_k^{\dagger}\mathbb{J}_{2r} \end{pmatrix}$$
$$= -\mathbb{J}_{2(s_1+\ldots+s_k)}\begin{pmatrix} \mathbb{J}_{2s_1} & & 0 \\ & \ddots & \\ 0 & & \mathbb{J}_{2s_k} \end{pmatrix}\begin{pmatrix} -\mathbb{J}_{2s_1}X_1^{\dagger}\mathbb{J}_{2r} \\ \vdots \\ -\mathbb{J}_{2s_1}X_k^{\dagger}\mathbb{J}_{2r} \end{pmatrix}$$
$$= -\mathbb{J}_{2(s_1+\ldots+s_k)}\,\mathrm{diag}(\mathbb{J}_{2s_1}, \ldots, \mathbb{J}_{2s_k})\begin{pmatrix} X_1^{\sharp} \\ \vdots \\ X_k^{\sharp} \end{pmatrix}.$$

Applying the above result to $\tilde{\mathcal{C}}$, we have that

$$\tilde{\mathcal{C}}^{\sharp} = -\mathbb{J}_{4nm}\,\mathrm{diag}(\underbrace{\mathbb{J}_{2m}, \ldots, \mathbb{J}_{2m}}_{2n})\begin{pmatrix} B^{\sharp} \\ ((\mathbb{J}R)B)^{\sharp} \\ \vdots \\ ((\mathbb{J}R)^{2n-1}B)^{\sharp} \end{pmatrix}$$
$$= -\mathbb{J}_{4nm}\,\mathrm{diag}(\mathbb{J}_{2m}, \ldots, \mathbb{J}_{2m})\begin{pmatrix} B^{\sharp} \\ B^{\sharp}(\mathbb{J}R)^{\sharp} \\ \vdots \\ B^{\sharp}((\mathbb{J}R)^{\sharp})^{2n-1} \end{pmatrix}.$$

However, $B^\sharp = (-C^\sharp D)^\sharp = -D^\sharp C = -D^{-1}C$, since $T^\sharp = T^{-1}$ for a symplectic T, and $(\mathbb{J}R)^\sharp = R^\sharp \mathbb{J}^\sharp = (-\mathbb{J}R^\dagger \mathbb{J})(-\mathbb{J}) = -\mathbb{J}R^\dagger = -\mathbb{J}R$, due to the fact that R is real symmetric. Putting everything together, we have that $\tilde{\mathcal{C}}^\sharp = T_0^{-1}\tilde{\mathcal{O}}$, with

$$T_0^{-1} = \mathbb{J}_{4nm}\,\mathrm{diag}(\mathbb{J}_{2m}, -\mathbb{J}_{2m}, \ldots, \mathbb{J}_{2m}, -\mathbb{J}_{2m})$$
$$\times\ \mathrm{diag}(D^{-1}, \ldots, D^{-1}).$$

Since each of the matrices $\mathbb{J}_{4nm}$, $\mathrm{diag}(\mathbb{J}_{2m}, -\mathbb{J}_{2m}, \ldots, \mathbb{J}_{2m}, -\mathbb{J}_{2m})$, and $\mathrm{diag}(D^{-1}, \ldots, D^{-1})$ is real symplectic, the conclusion of the lemma follows with

$$T_0 = \mathrm{diag}(D, \ldots, D)$$
$$\times\ \mathrm{diag}(\mathbb{J}_{2m}, -\mathbb{J}_{2m}, \ldots, \mathbb{J}_{2m}, -\mathbb{J}_{2m})\,\mathbb{J}_{4nm}. \blacksquare$$

The final result we need is the following "one-sided symplectic" SVD from [2]:

Lemma 3: [2, Theorem 3] For any matrix $F \in \mathbb{R}^{s \times 2r}$, there exist an orthogonal matrix $Q^{s \times s}$, a real symplectic matrix $Z^{2r \times 2r}$, and integers $0 \le k \le r$, and $0 \le l \le \min\{r-k, s-2k\}$ such that $F = Q\,E\,Z^{-1}$, where

$$E^{s \times 2r} = \begin{pmatrix} \Xi_k & 0 & 0 & 0 & 0 & 0 \\ 0 & I_l & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Xi_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} k \\ l \\ k \\ l' \end{matrix}$$

with column groupings k, l, $r-k-l$, k, l, $r-k-l$,

with $l' = s - 2k - l$, and $\Xi_k = \mathrm{diag}(\xi_1, \ldots, \xi_k) > 0$. $\square$

Proof of Theorem 1: We begin by applying Lemma 3 to the observability matrix $\tilde{\mathcal{O}}$ of system (4). Then, $\tilde{\mathcal{O}} = Q\,E\,Z^{-1}$ as above, with $s = 4nm$ and $r = n$, while the integers k and l are determined by the lemma. Using Lemma 2, we have that $\tilde{\mathcal{C}} = \tilde{\mathcal{O}}^\sharp T_0 = (Q\,E\,Z^{-1})^\sharp T_0 = (Z^{-1})^\sharp E^\sharp Q^\sharp T_0 = Z\,E^\sharp Q^\sharp T_0$. Now, we perform the state transformation $\left(\begin{smallmatrix}\hat{q}\\\hat{p}\end{smallmatrix}\right) = Z^{-1}\left(\begin{smallmatrix}q\\p\end{smallmatrix}\right)$. Since Z and Z^{-1} are real symplectic, the transformed system is also of the form (4). From standard results in systems theory, the controllability and observability matrices of the transformed system are given by

$$\hat{\tilde{\mathcal{C}}} = Z^{-1}\tilde{\mathcal{C}} = Z^{-1}Z\,E^\sharp Q^\sharp T_0 = E^\sharp Q^\sharp T_0, \quad (10)$$
$$\hat{\tilde{\mathcal{O}}} = \tilde{\mathcal{O}}(Z^{-1})^{-1} = Q\,E\,Z^{-1}Z = Q\,E. \quad (11)$$

Since Q is of full rank, (11) implies that $\mathrm{Ker}\hat{\tilde{\mathcal{O}}} = \mathrm{Ker}E$. Let e_i denote the i-th vector of the standard basis of $\mathbb{R}^{2n}$. From Lemma 3, we conclude that

$$\mathrm{Ker}\hat{\tilde{\mathcal{O}}} = \mathrm{span}\{e_{k+l+1}, \ldots, e_n, e_{n+k+1}, \ldots, e_{2n}\}.$$

From (10), we have that

$$\mathrm{Im}\hat{\tilde{\mathcal{C}}} = \mathrm{Im}E^\sharp Q^\sharp T_0 = \mathrm{Im}E^\sharp = \mathrm{Im}(-\mathbb{J}E^\top \mathbb{J}) = \mathrm{Im}\mathbb{J}E^\top$$

$$= \mathrm{Im}\begin{pmatrix} 0 & 0 & \Xi_k & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\Xi_k & 0 & 0 & 0 \\ 0 & -I_l & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \mathrm{span}\{e_1, \ldots, e_k, e_{n+1}, \ldots, e_{n+k}, e_{n+k+1}, \ldots, e_{n+k+l}\}.$$

The fact that Q and T_0 are of full rank was used in the above derivation. If we partition the states as

$$\hat{q} = \begin{pmatrix} \hat{q}_a^{k\times 1} \\ \hat{q}_b^{l\times 1} \\ \hat{q}_c^{(n-k-l)\times 1} \end{pmatrix}, \text{ and } \hat{p} = \begin{pmatrix} \hat{p}_a^{k\times 1} \\ \hat{p}_b^{l\times 1} \\ \hat{p}_c^{(n-k-l)\times 1} \end{pmatrix},$$

the calculations of the controllable and unobservable subspaces above, along with our convention for the uncontrollable and observable subspaces, lead to the following picture:

1) The states $\hat{q}_a$, $\hat{p}_a$, and $\hat{p}_b$ are controllable, and the states $\hat{q}_b$, $\hat{q}_c$, and $\hat{p}_c$ are uncontrollable.
2) The states $\hat{q}_c$, $\hat{p}_b$, and $\hat{p}_c$ are unobservable, and the states $\hat{q}_a$, $\hat{q}_b$, and $\hat{p}_a$ are observable.

Combining the above controllability and observability results, we end up with the classification of states announced in the statement of the theorem.

Hence, the state transformation $\left(\begin{smallmatrix}\hat{q}\\\hat{p}\end{smallmatrix}\right) = V\left(\begin{smallmatrix}q\\p\end{smallmatrix}\right)$, with $V = Z^{-1}$, essentially puts the system in the Kalman canonical form. The qualification has to do with the fact that, the usual grouping of states in the Kalman canonical form, $(x_{co}, x_{c\bar{o}}, x_{\bar{c}o}, x_{\bar{c}\bar{o}})$, is incompatible with the grouping of the states of (4) in conjugate pairs of position and momentum coordinates, $(\hat{q}, \hat{p})$, that is necessary for the structure of (4) to be preserved. The resolution of this issue is, to modify the usual Kalman canonical form. To do this, we start from the usual Kalman canonical form [32], [33]

$$d\begin{pmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{pmatrix} = \begin{pmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{pmatrix}\begin{pmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{pmatrix} dt$$
$$+ \begin{pmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{pmatrix} d\mathcal{U},$$

$$d\mathcal{Y} = \begin{pmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{pmatrix}\begin{pmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{pmatrix} dt + D\,d\mathcal{U}, \quad (12)$$

and let $x_{co} = \left(\begin{smallmatrix}\hat{q}_a\\\hat{p}_a\end{smallmatrix}\right)$, $x_{c\bar{o}} = \hat{p}_b$, $x_{\bar{c}o} = \hat{q}_b$, $x_{\bar{c}\bar{o}} = \left(\begin{smallmatrix}\hat{q}_c\\\hat{p}_c\end{smallmatrix}\right)$. Also, partition $A_{co} = \left(\begin{smallmatrix}A_{co,11} & A_{co,12}\\A_{co,21} & A_{co,22}\end{smallmatrix}\right)$, $A_{13} = \left(\begin{smallmatrix}A_{13,1}\\A_{13,2}\end{smallmatrix}\right)$, $A_{21} = \left(A_{21,1}\ A_{21,2} \right)$, $A_{24} = \left(A_{24,1}\ A_{24,2} \right)$, $A_{43} = \left(\begin{smallmatrix}A_{43,1}\\A_{43,2}\end{smallmatrix}\right)$, $A_{\bar{c}\bar{o}} = \left(\begin{smallmatrix}A_{\bar{c}\bar{o},11} & A_{\bar{c}\bar{o},12}\\A_{\bar{c}\bar{o},21} & A_{\bar{c}\bar{o},22}\end{smallmatrix}\right)$, $B_{co} = \left(\begin{smallmatrix}B_{co,1}\\B_{co,2}\end{smallmatrix}\right)$, and $C_{co} = \left(C_{co,1}\ C_{co,2} \right)$, accordingly. Then, by rearranging the Kalman canonical form (12), we end up with (7), where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are given by (8). $\blacksquare$

Though Theorem 1 constructs one particular Kalman decomposition for a LQSS (or, equivalently, one particular Kalman-like canonical form (7)), it is easy to generate many more by use of the following corollary:

Corollary 1: Let E be the reduced form of the observability matrix $\tilde{\mathcal{O}}$ of system (4), according to Lemma 3. Let $X \in \mathbb{R}^{4nm \times 4nm}$ be invertible, and $Y \in \mathbb{R}^{2n \times 2n}$ symplectic,

such that,

$$
XEY = \begin{pmatrix}
\overset{k}{\Xi'_k} & \overset{l}{0} & \overset{n-k-l}{0} & \overset{k}{0} & \overset{l}{0} & \overset{n-k-l}{0} \\
0 & \Xi'_l & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Xi''_k & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{matrix} k \\ l \\ k \\ l' \end{matrix},
$$

with $l' = 4nm - 2k - l$, and every element of the diagonal matrices Ξ'_k, Ξ'_l, and Ξ''_k, be non-zero. If V is the symplectic transformation to the Kalman-like canonical form in Theorem 1, then the theorem holds for $V' = Y^{-1}V$, as well. $\square$

Proof: We have that

$$
\tilde{\mathcal{O}} = Q\,E\,Z^{-1} = (Q\,X^{-1})\,(X\,E\,Y)\,(Y^{-1}Z^{-1}).
$$

In the proof of Theorem 1, the fact that Q is unitary was used only to guarantee that it is of full rank. Also, the exact values of the elements of the non-zero diagonal blocks of E were unimportant. It is straightforward to see that, the proof of the theorem follows through using the decomposition above, instead of Lemma 3. The conclusion of the corollary follows. ∎

IV. An Example

Consider the following 3-mode, 1 input/output LQSS with Hamiltonian

$$
H = \frac{\omega}{2}(q_3^2 + p_3^2) + \lambda q_1 q_3 + \lambda q_2 q_3,
$$

and coupling operator

$$
L = \frac{\gamma}{\sqrt{2}}(q_3 + \imath p_3).
$$

This LQSS models the linearized dynamics of an optomechanical system where the resonant modes of two optical cavities, with states (q_1, p_1) and (q_2, p_2), respectively, interact with a mechanical mode with states (q_3, p_3), of frequency ω. We assume that the cavities are lossless, and that their interaction strengths with the mechanical oscillator are equal. The only source of damping in the system is mechanical. The system QSDEs (4), take the following form:

$$
\begin{aligned}
dq_1 &= 0, \\
dq_2 &= 0, \\
dq_3 &= \left(-\frac{\gamma^2}{2}q_3 + \omega p_3\right) dt - \gamma d\mathcal{U}_1, \\
dp_1 &= -\lambda q_3 dt, \\
dp_2 &= -\lambda q_3 dt, \\
dp_3 &= -\left(\lambda q_1 + \lambda q_2 + \omega q_3 + \frac{\gamma^2}{2}p_3\right) dt - \gamma d\mathcal{U}_2, \\
d\mathcal{Y}_1 &= \gamma q_3\, dt + d\mathcal{U}_1, \\
d\mathcal{Y}_2 &= \gamma p_3\, dt + d\mathcal{U}_2.
\end{aligned}
$$

Recall that $\mathcal{U}_1$ and $\mathcal{U}_2$ are the two real quadratures of a single input, and similarly for the outputs.

Applying the "one-sided symplectic" SVD of [2] to the observability matrix of the above LQSS, we obtain the

symplectic transformation V that puts the system in the Kalman-like canonical form (7):

$$
V = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
-1 & -1 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
-\lambda a & -\lambda a & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/2 & -1/2 & \lambda a \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0
\end{pmatrix},
$$

where

$$
a = \omega \frac{\omega^8 + \omega^6 + \omega^4 + \omega^2 + 1}{\omega^{10} + \omega^8 + \omega^6 + \omega^4 + \omega^2 + 1}.
$$

The new states of the system are given by

$$
\begin{pmatrix}
\hat{q}_1 \\ \hat{q}_2 \\ \hat{q}_3 \\ \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3
\end{pmatrix} = \begin{pmatrix}
p_3 \\
-(q_1 + q_2) \\
\frac{1}{\sqrt{2}}(q_1 - q_2) \\
-q_3 - \lambda a(q_1 + q_2) \\
\lambda a\, p_3 - \frac{1}{2}(p_1 + p_2) \\
\frac{1}{\sqrt{2}}(p_1 - p_2)
\end{pmatrix}.
$$

$\hat{q}_1$ and $\hat{p}_1$ are the co states, $\hat{q}_2$ and $\hat{p}_2$ are the $\bar{c}o$ and $c\bar{o}$ states, respectively, and $\hat{q}_3$ and $\hat{p}_3$ are the $\bar{c}\bar{o}$ states. This is confirmed by the system QSDEs in the transformed states, which take the following form:

$$
\begin{aligned}
d\hat{q}_1 &= \left(-\frac{\gamma^2}{2}\hat{q}_1 + \lambda b\,\hat{q}_2 + \omega\hat{p}_1\right) dt - \gamma d\mathcal{U}_2, \\
d\hat{q}_2 &= 0, \\
d\hat{q}_3 &= 0, \\
d\hat{p}_1 &= \left(-\omega\hat{q}_1 + \lambda a\frac{\gamma^2}{2}\hat{q}_2 - \frac{\gamma^2}{2}\hat{p}_1\right) dt + \gamma d\mathcal{U}_1, \\
d\hat{p}_2 &= \left(-\lambda a\frac{\gamma^2}{2}\hat{q}_1 + \lambda^2 a(b+1)\,\hat{q}_2 - \lambda b\,\hat{p}_1\right) dt \\
&\quad - \gamma\lambda a\, d\mathcal{U}_2, \\
d\hat{p}_3 &= 0, \\
d\mathcal{Y}_1 &= \gamma(\lambda a\,\hat{q}_2 - \hat{p}_1)\, dt + d\mathcal{U}_1, \\
d\mathcal{Y}_2 &= \gamma\hat{q}_1\, dt + d\mathcal{U}_2,
\end{aligned}
$$

where $b = 1/(\omega^{10} + \omega^8 + \omega^6 + \omega^4 + \omega^2 + 1)$. We can use Corollary 1, to produce a simpler Kalman decomposition of the system. Indeed, with

$$
Y = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -\sqrt{2}\lambda a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda a & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

and

$$
X = \mathrm{diag}\left(\begin{pmatrix} 0 & -\lambda\gamma a/\sqrt{b} & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, I_9\right),
$$

we obtain the following orthogonal symplectic transformation $V' = Y^{-1}V$, that puts the system in the Kalman-like

canonical form (7):

$$V' = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

The new states of the system are given by

$$\begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \hat{q}_3 \\ \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \end{pmatrix} = \begin{pmatrix} q_3 \\ \frac{1}{\sqrt{2}}(q_1 + q_2) \\ \frac{1}{\sqrt{2}}(q_1 - q_2) \\ p_3 \\ \frac{1}{2}(p_1 + p_2) \\ \frac{1}{\sqrt{2}}(p_1 - p_2) \end{pmatrix}.$$

Again, $\hat{q}_1$ and $\hat{p}_1$ are the co states, $\hat{q}_2$ and $\hat{p}_2$ are the $\bar{c}o$ and $c\bar{o}$ states, respectively, and $\hat{q}_3$ and $\hat{p}_3$ are the $\bar{c}\bar{o}$ states. This is confirmed by the system QSDEs in the transformed states, which take the following form:

$$d\hat{q}_1 = \left(-\frac{\gamma^2}{2}\hat{q}_1 + \omega\hat{p}_1 \right) dt - \gamma d\mathcal{U}_1,$$
$$d\hat{q}_2 = 0,$$
$$d\hat{q}_3 = 0,$$
$$d\hat{p}_1 = \left(-\omega\hat{q}_1 - \sqrt{2}\lambda\hat{q}_2 - \frac{\gamma^2}{2}\hat{p}_1 \right) dt - \gamma d\mathcal{U}_2,$$
$$d\hat{p}_2 = -\sqrt{2}\lambda\hat{q}_1 \, dt,$$
$$d\hat{p}_3 = 0,$$
$$d\mathcal{Y}_1 = \gamma\hat{q}_1 \, dt + d\mathcal{U}_1,$$
$$d\mathcal{Y}_2 = \gamma\hat{p}_1 \, dt + d\mathcal{U}_2.$$

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