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OPTIMAL SHARPE RATIO IN CONTINUOUS-TIME MARKETS WITH AND WITHOUT A RISK-FREE ASSET

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Abstract. In this paper, we investigate a continuous-time mean-variance portfolio selection model with only risky assets and its optimal Sharpe ratio in a new way. We obtain closed-form expressions for the efficient investment strategy, the efficient frontier and the optimal Sharpe ratio. Using these results, we further prove that (i) the efficient frontier with only risky assets is significantly different from the one with inclusion of a risk-free asset and (ii) inclusion of a risk-free asset strictly enhances the optimal Sharpe ratio. Also, we offer an explicit expression for the enhancement of the optimal Sharpe ratio. Finally, we test our theory results using an empirical analysis based on real data of Chinese equity market. Out-of-sample analyses shed light on advantages of our theoretical results established.

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Key words and phrases. Continuous-time mean-variance model; Efficient investment strategy; Efficient frontier; Sharpe ratio; Hamilton-Jacobi-Bellman equation.

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1. **Introduction.** The research of portfolio selection theory can be dated back to Markowitz's seminal work [16]. In his work, he minimized the risk (measured by the variance) for a predetermined expected return, and set up the mean-variance (M-V) model for portfolio selection. Since then, Markowitz's M-V model has become the foundation for modern financial theory. Based on the M-V model and the equilibrium market analysis, Sharpe [20] developed the capital asset pricing model (CAPM). Furthermore, Merton [17] proved that the efficient frontier with both risky and risk-free assets is a straight line tangent to the one with only risky assets, and showed that the tangency portfolio is the market portfolio in the sense of Sharpe's CAPM. This is the well-known one-fund theorem in literature.

Roy [19] proposed a risk-reward ratio to evaluate the performance of an investment strategy. Based on Markowitz's M-V framework and Roy's ideas, Sharpe [21] introduced the well-known Sharpe Ratio to study the portfolio performance. The Sharpe Ratio has become one of the most common measures on the performance of mutual funds handled by managers or individual investors (for example, Bayley and de Prado [1] and Schuster and Auer [18]). Merton [17] showed that the optimal Sharpe ratios are the same for both cases with and without a risk-free asset in the static setting. This means that inclusion of a risk-free asset cannot boost the optimal Sharpe Ratio in the static setting. For more discussions about Sharpe Ratio, one can further read research works in Dowd [12], Zakamouline and Koekebakker [26] and Chow and Lai [6].

The classical Markowitz's M-V model only focuses on the static case, but the financial investment is a long-term dynamic procedure. It naturally requires us to extend the static M-V model to dynamic (multi-period or continuous-time) cases. Researchers did not make any breakthrough until 2000, due to the fact that dynamic M-V models possess non-separability in the sense of dynamic programming. To overcome this fundamental difficulty, Li and Ng [14] and Zhou and Li [28] derived analytical optimal solutions for the multi-period and the continuous-time M-V models by introducing an embedding technique, respectively. Since then, a variety of dynamic M-V portfolio selection problems have been investigated. For example, Li et al. [15] and Cui et al. [8] studied the continuous-time and the multi-period M-V portfolio selection problems with no-shorting constraint, respectively. Zhu et al. [29] and Bielecki et al. [2] considered portfolio selection problems with bankruptcy risk constraint under the multi-period and the continuous-time M-V frameworks, respectively. Wang and Liu [23] investigated the multi-period M-V model with fixed and proportional transaction costs. Chen et al. [3] studied a continuous-time M-V portfolio selection problem with liability and regime switching. For more discussion on the subject of portfolio selection under dynamic M-V model, please refer to [25], [4], [10] and [27]. However, research on Sharpe ratio in dynamic settings is under-explored. To the best of our knowledge, only Cvitanic et al. [9] studied the effect of Sharpe ratio as a performance measure in a dynamic setting.

Chiu and Zhou [5] revealed that, in a continuous-time setting and when there exists a risk-free asset, any efficient portfolio must involve allocation to the risk-free asset at any time. This implies that in a continuous-time setting the efficient frontier with inclusion of a risk-free asset is not tangent to the one with only risky assets. This in turn suggests that a risk-free asset strictly enhance the optimal Sharpe ratio in a continuous-time setting. Cui et al. [7] further characterized the gap between these two efficient frontiers with and without a risk-free asset by constructing a continuous-time M-V model with finite transactions between the risk-free asset and the pool of risky assets. This issue improves the meaningfulness of CAPMs based on continuous-time M-V models. However, neither Chiu and Zhou [5] nor Cui et al. [7] strictly and directly proved the aforementioned results according to the expressions of these two efficient frontiers with and without a risk-free asset.

Although the expressions of continuous-time M-V efficient frontiers for both cases with and without a risk-free asset have been derived by Zhou and Li [28] and Yao et al. [24] [2](or Cui et al. [7]), respectively. However, we cannot strictly prove there exists a gap between these two efficient frontiers by directly comparing the expressions obtained in the existing literature (See Section 2 for more details). In this paper, we re-construct the continuous-time M-V portfolio selection model with only risky assets in a new way where we retain in our model the equality constraint (budget constraint) that the total amount invested in each risky asset equals the wealth at time t. Following the similar approach in Yao et al. [24], we obtain an explicit closed-form expression for the efficient frontier, and then derive the optimal Sharpe ratio with only risky assets. Using these expressions, compared with the existing expression of the efficient frontier with both risky and risk-free assets (see Zhou and Li [28]), we strictly and exactly prove that (i) the efficient frontier with only risky assets is not tangent to the one with both risk-free and risky assets; (ii) inclusion of a risk-free asset strictly enhances the optimal Sharpe ratio. In addition, we provide an explicit expression for the enhancement of the optimal Sharpe ratio, which is referred to as the *premium of dynamic trading* in Chiu and Zhou [5], where they did not present its computational formula.

The remainder of this paper is organized as follows. Continuous-time M-V portfolio selection problems with and without a risk-free asset are formulated and some existing results are introduced in Section 2. Closed form expressions for the efficient investment strategy, the efficient frontier and the optimal Sharpe ratio with only risky assets are obtained in Section 3. In Section 4, we prove that the efficient frontier generated by both risky and risk-free assets is strictly separated from the one generated by only risky assets, and that inclusion of a risk-free asset strictly enhances the optimal Sharpe ratio. Moreover, an analytical expression for the enhancement of the optimal Sharpe ratio is provided. Based on the real data of Chinese stock market, an empirical analysis is presented in Section 5 to test the theory results established, and the conclusion is in Section 6.

2. **Model formulation and existing results review.** Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t\geq 0})$, where $\mathscr{F}_t = \sigma\{W(s); 0 \leq s \leq t\}$ is the natural filtration representing the information available up to time t, and $W(t) = (W_1(t), W_2(t), \cdots, W_m(t))'$ is an m-dimensional standard Brownian motion. Suppose that there are one risk-free asset and n risky assets in the market. The price $P_0(t)$ of the risk-free asset satisfies the following dynamics

$$dP_0(t) = P_0(t)r(t)dt, \quad P_0(0) = p_0, \tag{1}$$

and the prices $P_1(\cdot), P_2(\cdot), \cdots, P_n(\cdot)$ of the n risky assets satisfy the following stochastic differential equations

$$dP_i(t) = P_i(t)\left[b_i(t)dt + \sum_{j=1}^{m}\sigma_{ij}(t)dW_j(t)\right],\ P_i(0) = p_i,\ i = 1, 2, \cdots n, \tag{2}$$

[2]The main reasons of considering market with only risky assets has been discussed in the previous work of Yao et al. [24].

where $r(t)$ is the deterministic interest rate, $b_i(t)$ and $\sigma_i(t) = (\sigma_{i1}(t), \sigma_{i2}(t), \cdots, \sigma_{i\,m}(t))$ are the appreciation rate and the volatility rate of risky asset i, respectively, which are assumed to be deterministic functions of time t.

Suppose that an investor endowed with an initial wealth x_0 ($x_0 > 0$) enters the market at time 0 and exits the market at time T. Let $X(t)$ and $\pi_i(t)$ denote the wealth held by the investor and the amount invested in the ith risky asset ($i = 1, 2, \cdots, n$) at time t, respectively. Then, $X(t) - \sum_{i=1}^{n} \pi_i(t)$ is the amount invested in the risk-free asset. Then, the dynamics of wealth process under an investment strategy π is

$$\begin{aligned} dX^\pi(t) &= \left(\left(X^\pi(t) - \sum_{i=1}^{n} \pi_i(t) \right) \frac{dP_0(t)}{P_0(t)} + \sum_{i=1}^{n} \pi_i(t) \frac{dP_i(t)}{P_i(t)} \right) \\ &= \left(r(t)X^\pi(t) + \pi(t)' \left(b(t) - \mathbf{1}_n r(t) \right) \right) dt + \pi(t)'\sigma(t) dW(t). \end{aligned} \tag{3}$$

where

$$\begin{cases} \pi(t) = (\pi_1(t), \pi_2(t), \cdots, \pi_n(t))', & \mathbf{1}_n = (1, 1, \cdots, 1)' \in \mathbb{R}^n, \\ b(t) = (b_1(t), b_2(t), \cdots, b_n(t))', & \sigma(t) = (\sigma_1(t)', \sigma_2(t)', \cdots, \sigma_n(t)')'. \end{cases} \tag{4}$$

Let

$$\mathcal{N} = \Big\{ t \in [0, T] \mid \text{There does not exist a real number } \phi, \text{ such that } b(t) \neq \phi \mathbf{1}_n \Big\}. \tag{5}$$

Similar to most of the existing literature, we have the following technical assumptions throughout this paper.

Assumption 2.1. $\sigma(t)\sigma(t)' \geq \varepsilon I$, $\forall t \in [0, T]$ for some $\varepsilon > 0$, where I is an identity matrix of order n.

Assumption 2.2. Set $\mathcal{N}$ is Lebesgue measurable and $\Im(\mathcal{N}) > 0$, where $\Im(\mathcal{N})$ denotes the Lebesgue measure of set $\mathcal{N}$.

Assumption 2.3. $b(t)$ and $\sigma(t)$ are essentially bounded and measurable on $[0, T]$.

Denote by $L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued, measurable stochastic processes $f(t)$ adapted to $\mathscr{F}_t$, such that $\mathrm{E}[\int_0^T |f(t)|^2 dt] < +\infty$. An investment strategy $\pi = \{\pi(t); t \in [0, T]\}$ is said to be admissible if $\pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ and $(X^\pi(\cdot), \pi(\cdot))$ satisfies (17) and (18) below. The continuous-time M-V portfolio problem refers to the problem of finding the optimal admissible investment strategy such that the variance of the terminal wealth is minimized for a given expected terminal wealth $\mathrm{E}[X^\pi(T)] = u$.

Assume that the trading is continuous, and the transaction cost is not considered. Then, the continuous-time M-V portfolio selection problem with both risk-free and risky assets can be formulated as below

$$\begin{cases} \min\limits_{\pi(\cdot)} & \left\{ \mathrm{Var}[X^\pi(T)] = \mathrm{E}[X^\pi(T) - u]^2 \right\} \\ \text{s.t.} & \mathrm{E}[X^\pi(T)] = u \\ & \pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n), (x(\cdot), \pi(\cdot)) \text{ satisfies (3)}. \end{cases} \tag{6}$$

Problem (6) is called *feasible* if there is at least one admissible investment strategy π, such that $\mathrm{E}[X^\pi(T)] = u$. An admissible investment strategy π is called an *efficient investment strategy* if there exists no other admissible investment strategy $\tilde{\pi}$ such that $\mathrm{Var}[X^{\tilde{\pi}}(T)] \leq \mathrm{Var}[X^\pi(T)]$, $\mathrm{E}[X^{\tilde{\pi}}(T)] \geq \mathrm{E}[X^\pi(T)]$, and at least one of the inequalities holds strictly. Point $(\mathrm{Var}[X^\pi(T)],\ \mathrm{E}[X^\pi(T)])$ corresponding to an efficient investment strategy π on the variance-mean plane is called an efficient point. The set of all efficient points is called the *efficient frontier*.

Problem (6) has been studied by Zhou and Li [28]. According to their results, the efficient investment strategy is

$$\pi^*(t) = \left(\frac{\left(u - e^{-\int_0^T (\rho(s)-r(s))ds} x_0\right) e^{-\int_t^T r(s)ds}}{1 - e^{-\int_0^T \rho(s)ds}} - X^*(t) \right) (\sigma(t)\sigma(t)')^{-1} (b(t) - r(t)\mathbf{1}_n), \tag{7}$$

the efficient frontier is

$$\mathrm{Var}[X^*(T)] = \frac{e^{-\int_0^T \rho(s)ds}}{1 - e^{-\int_0^T \rho(s)ds}} \left(\mathrm{E}[X^*(T)] - x_0 e^{\int_0^T r(s)ds} \right)^2, \text{ for } \mathrm{E}[X^*(T)] \geq x_0 e^{\int_0^T r(s)ds}, \tag{8}$$

where

$$\rho(t) = (b(t) - \mathbf{1}_n r(t))'(\sigma(t)\sigma(t)')^{-1}(b(t) - \mathbf{1}_n r(t)). \tag{9}$$

When there are only n risky assets in the investment opportunity set. Then, we naturally have the budget constraint $\sum_{i=1}^{n} \pi_i(t) = X(t)$ for $t \in [0, T]$. As Yao et al. [24] and Cui et al. [7] do, one can remove the budget constraint by eliminating one of the decision variables, e.g.

$$\pi_1(t) = X(t) - \sum_{i=2}^{n} \pi_i(t). \tag{10}$$

By (2) and (10), the dynamic of the wealth process in case with only risky assets satisfies

$$\begin{aligned} dX^\pi(t) &= \left(X^\pi(t) - \sum_{i=1}^{n} \pi_i(t) \right) \frac{dP_1(t)}{P_1(t)} + \sum_{i=2}^{n} \pi_i(t) \frac{dP_i(t)}{P_i(t)} \\ &= \left[X^\pi(t) b_1(t) + \pi'_{-1}(t) \left(b_{-1}(t) - \mathbf{1}_{n-1} b_1(t) \right) \right] dt \\ &\quad + \left[X^\pi(t) \sigma_1(t) + \pi'_{-1}(t) \left(\sigma_{-1}(t) - \mathbf{1}_{n-1} \sigma_1(t) \right) \right] dW(t). \end{aligned} \tag{11}$$

where

$$\begin{cases} b_{-1}(t) = (b_2(t), \cdots, b_n(t))', \sigma_{-1}(t) = ({\sigma'}_2(t), \cdots, {\sigma'}_n(t))', \\ \pi_{-1}(t) = (\pi_2(t), \cdots, \pi_n(t)), \mathbf{1}_{n-1} = (1, 1, \cdots, 1)' \in \mathbb{R}^{n-1}. \end{cases} \tag{12}$$

We point out that $b_{-1}(t)$ and $\sigma_{-1}(t)$ in this paper are the same meaning of $b(t)$ and $\sigma(t)$ in Cui et al. [7] and Yao et al. [24]. Because, in Cui et al. [7] and Yao et al. [24], the first risky asset is denoted by asset 0 and the subscript labeling the risky assets begins with 0, while in our model the first risky asset is denoted by asset 1 where the subscript labeling the risky assets starts from 1.

Then, the continuous-time M-V portfolio problem with only risky assets can be formulated as follows

$$\begin{cases} \min_{\pi(\cdot)} \left\{ \mathrm{Var}[X^\pi(T)] := \mathrm{E}[X^\pi(T) - u]^2 \right\}. \\ \text{s.t.} \quad \mathrm{E}[X^\pi(T)] = u, \\ \qquad \pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n), (X^\pi(\cdot), \pi(\cdot)) \text{ satisfies (11)}. \end{cases} \tag{13}$$

In order to derive expression for the efficient frontier, let

$$\begin{cases} \bar{r}(t) = b_1(t) - \sigma_1(t)\left(\sigma_{-1}(t) - \mathbf{1}_{n-1}\sigma_1(t)\right)' \\ \quad \times \left[\left(\sigma_{-1}(t) - \mathbf{1}_{n-1}\sigma_1(t)\right)\left(\sigma_{-1}(t) - \mathbf{1}_{n-1}\sigma_1(t)\right)'\right]^{-1}\left(b_{-1}(t) - \mathbf{1}_{n-1}b_1(t)\right), \\ \bar{\rho}(t) = \left(b_{-1}(t) - \mathbf{1}_{n-1}b_1(t)\right)'\left[\left(\sigma_{-1}(t) - \mathbf{1}_{n-1}\sigma_1(t)\right)\left(\sigma_{-1}(t) - \mathbf{1}_{n-1}\sigma_1(t)\right)'\right]^{-1} \\ \quad \times \left(b_{-1}(t) - \mathbf{1}_{n-1}b_1(t)\right), \\ \gamma(t) = \sigma_1(t)\sigma_1'(t) - \sigma_1(t)\left(\sigma_{-1}(t) - \mathbf{1}_{n-1}\sigma_1(t)\right)' \\ \quad \times \left[\left(\sigma_{-1}(t) - \mathbf{1}_{n-1}\sigma_1(t)\right)\left(\sigma_{-1}(t) - \mathbf{1}_{n-1}\sigma_1(t)\right)'\right]^{-1}\left(\sigma_{-1}(t) - \mathbf{1}_{n-1}\sigma_1(t)\right)\sigma_1'(t). \end{cases} \tag{14}$$

where $\mathbf{1}_{n-1} := (1, 1, \cdots, 1)' \in \mathbb{R}^{n-1}$. Following the results (with some modifications) of Yao et al. [24], the efficient frontier with only risky assets can be given by

$$Var[x(T)] = \frac{M(0)}{1-M(0)}\left(u - \frac{L(0)x_0}{M(0)}\right)^2 + \frac{K(0)M(0) - L^2(0)}{M(0)}x_0^2, \tag{15}$$

where

$$\begin{cases} K(t) := e^{-\int_t^T (\bar{\rho}(s) - 2\bar{r}(s) - \gamma(s))ds}, L(t) := e^{-\int_t^T (\bar{\rho}(s) - \bar{r}(s))ds} \\ M(t) := e^{-\int_t^T (\bar{\rho}(s) + \gamma(s))ds}\left(1 + \int_t^T e^{\int_t^z (\bar{\rho}(s) + \gamma(s))ds}\gamma(z)dz\right). \end{cases} \tag{16}$$

Though the efficient frontiers for both cases with and without a risk-free asset have been obtained by (8) and (15), it is very difficult for us to derive the geometrical relationships between the two efficient frontiers by directly comparing the expressions of these two efficient frontiers. Because it is very difficult to find out the quantitative relationships (e.g., '=', '>' and '<', etc.) between expression symbols $\rho(t)$ in case with inclusion of a risk-free asset and the expression symbols $\bar{r}(t)$, $\bar{\rho}(t)$, and $\gamma(t)$ in case with only risky assets. In the following, we explain this more specifically.

It is known from (9) and (14) that vector $b(t) = (b_1(t), b_{-1}'(t))'$ and matrix $\sigma(t) = (\sigma_1'(t), \sigma_{-1}'(t))'$ are the bridge connecting symbol $\rho(t)$ (the main expression symbol of the efficient frontier in case with both risk-free and risky assets) and symbols $\bar{r}(t)$, $\bar{\rho}(t)$, $\gamma(t)$ (the main expression symbols of the efficient frontier in case with only risky assets). $b(t)$ and $\sigma(t)$ appear in the expressions of $\bar{r}(t)$, $\bar{\rho}(t)$ and $\gamma(t)$, in term of $b_1(t)$, $b_{-1}(t)$, $\sigma_1(t)$ and $\sigma_{-1}'(t)$); while in the expression of $\rho(t)$, both $b(t)$ and $\sigma(t)$ appear in term of themselves as a whole. This leads to the fundamental difficulties for setting up the quantitative (function) relationships between symbols $\bar{r}(t)$, $\bar{\rho}(t)$, $\gamma(t)$ and symbol $\rho(t)$. For example, it is difficult for us to express $\rho(t)$ in terms of $\bar{r}(t)$, $\bar{\rho}(t)$ and $\gamma(t)$.

In order to overcome these difficulties aforementioned, in the next section, we will establish the continuous-time mean-variance model with only risky assets in a new way different from Yao et al. [24] and Cui et al. [7].

In the following, we retain the budget constraint $\sum_{i=1}^{n} \pi_i(t) = X(t)$ for $t \in [0, T]$. According to (2), the wealth process $X^\pi(t)$ in case with only risky assets can been rewritten as follows

$$dX^\pi(t) = \sum_{i=1}^{n} \pi_i(t)\frac{dP_i(t)}{P_i(t)} = \pi(t)'\left[b(t)dt + \sigma(t)dW(t)\right], \ X^\pi(0) = x_0 \tag{17}$$

with budge constraint

$$\pi(t)'\mathbf{1}_n = X^\pi(t), \tag{18}$$

And the continuous-time M-V portfolio problem with only risky assets can be reformulated as follows

$$\begin{cases} \min_{\pi(\cdot)} \left\{ \mathrm{Var}[X^\pi(T)] := \mathrm{E}[X^\pi(T) - u]^2 \right\}. \\ \text{s.t.} \quad \mathrm{E}[X^\pi(T)] = u, \\ \qquad \pi(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n), (X^\pi(\cdot), \pi(\cdot)) \text{ satisfies (17) and (18).} \end{cases} \tag{19}$$

Now $b(t)$ and matrix $\sigma(t)$ appears as a whole in the mean-variance model. As will be seen below, this is beneficial to find out the relation between the two efficient frontiers with and without a risk-free asset.

3. **Efficient frontier and optimal Sharpe ratio with only risky assets.**

For later use, we introduce the following notations

$$\begin{cases} A(t) = \mathbf{1}_n' \left(\sigma(t)\sigma(t)'\right)^{-1} \mathbf{1}_n, \qquad C(t) = \mathbf{1}_n' \left(\sigma(t)\sigma(t)'\right)^{-1} b(t), \\ B(t) = b(t)' \left(\sigma(t)\sigma(t)'\right)^{-1} b(t), \quad D(t) = A(t)B(t) - C^2(t). \end{cases} \tag{20}$$

Following the similar approach in Yao et al. [24], we can solve Problem (19) and then have the following theorem.

Theorem 3.1. *For the continuous-time M-V portfolio selection problem (19) with only risky assets, the efficient investment strategy and the efficient frontier are respectively given by*

$$\pi^*(t) = \frac{\left(\sigma(t)\sigma(t)'\right)^{-1}}{A(t)} \left(\left(X^*(t) + \frac{\left(\psi(0)x_0 - u\right)\psi(t)}{\left(1 - \vartheta(0)\right)\varphi(t)} \right) \left(C(t)\mathbf{1}_n - A(t)b(t)\right) + X^*(t)\mathbf{1}_n \right). \tag{21}$$

and

$$\mathrm{Var}[X^*(T)] = \frac{\vartheta(0)}{1-\vartheta(0)} \left(u - \frac{\psi(0)x_0}{\vartheta(0)} \right)^2 + \left(\varphi(0) - \frac{\psi^2(0)}{\vartheta(0)} \right) x_0^2. \tag{22}$$

where $u \geq u_{\sigma_{\min}}$, and

$$\varphi(t) = e^{\int_t^T \frac{2C(s)-D(s)+1}{A(s)} ds}, \quad \psi(t) = e^{\int_t^T \frac{C(s)-D(s)}{A(s)} ds}, \tag{23}$$

$$\vartheta(t) = 1 - \int_t^T e^{-\int_z^T \frac{D(s)+1}{A(s)} ds} \frac{D(z)}{A(z)} dz. \tag{24}$$

Moreover, the global minimum variance of the terminal wealth is strictly larger than zero.

For later use, we give the following proposition first.

Proposition 3.1. *For $t \in [0,T]$, we have $A(t) > 0$, $B(t) \geq 0$ and $D(t) \geq 0$. Furthermore, $B(t) > 0$ and $D(t) > 0$ for all $t \in \mathcal{N}$, where $\mathcal{N}$ is defined in (5).*

Proof. Let $t \in [0,T]$. By Assumption 2.1, $\sigma(t)\sigma(t)'$ is positive definite, and so is $\left(\sigma(t)\sigma(t)'\right)^{-1}$. Therefore, it follows that $A(t) = \mathbf{1}_n' \left(\sigma(t)\sigma(t)'\right)^{-1} \mathbf{1}_n > 0$, $B(t) = b(t)' \left(\sigma(t)\sigma(t)'\right)^{-1} b(t) \geq 0$. Furthermore, since $\left(\sigma(t)\sigma(t)'\right)^{-1}$ is positive definite, there exists a non-singular matrix Z such that $\left(\sigma(t)\sigma(t)'\right)^{-1} = Z'Z$. Then, we have

$$A(t) = (Z\mathbf{1}_n)'Z\mathbf{1}_n, \ B(t) = (Zb(t))'Zb(t), \ C(t) = (Z\mathbf{1}_n)'Zb(t).$$

By the Cauchy-Schwarz inequality, it follows that

$$C^2(t) = \left((Z\mathbf{1}_n)'Zb(t)\right)^2 \leq \left((Z\mathbf{1}_n)'Z\mathbf{1}_n\right)(Zb(t))'Zb(t) = A(t)B(t),$$

which means that $D(t) \geq 0$.

On the other hand, let $t \in \mathcal{N}$. By the definition of $\mathcal{N}$ in (5), we have $b(t) \neq \mathbf{0}_n$, where $\mathbf{0}_n$ is the zero vector of order n. Hence $B(t) = b(t)' \left(\sigma(t)\sigma(t)'\right)^{-1} b(t) > 0$, which further yields $C(t)b(t) - B(t)\mathbf{1}_n \neq \mathbf{0}_n$. So we have

$$\begin{aligned} 0 <& (C(t)b(t) - B(t)\mathbf{1}_n)' \left(\sigma(t)\sigma(t)'\right)^{-1} (C(t)b(t) - B(t)\mathbf{1}_n) \\ =& B(t)\left(B(t)A(t) - C^2(t)\right) = B(t)D(t). \end{aligned} \tag{25}$$

This implies that $D(t) > 0$. □

Proposition 3.2. $0 < \frac{\psi^2(0)}{\varphi(0)} < e^{-\int_0^T \frac{D(t)}{A(t)}dt} < \vartheta(0) < 1.$

Proof. By Proposition 3.1, $A(t) > 0, D(t) \geq 0$ for all $t \in [0, T]$, and $D(t) > 0$ for all $t \in \mathcal{N}$. Noting that $e^{-\int_t^T \frac{D(s)+1}{A(s)}ds} > 0$, it follows from (24) and Assumption 2.2 that

$$\vartheta(0) = 1 - \int_0^T e^{-\int_t^T \frac{D(s)+1}{A(s)}ds} \frac{D(t)}{A(t)} dt \leq 1 - \int_{\mathcal{N}} e^{-\int_t^T \frac{D(s)+1}{A(s)}ds} \frac{D(t)}{A(t)} dt < 1.$$

Since $e^{-\int_z^T \frac{D(s)+1}{A(s)}ds} < e^{-\int_z^T \frac{D(s)}{A(s)}ds}$, we have

$$\int_{\mathcal{N}} e^{-\int_t^T \frac{D(s)+1}{A(s)}ds} \frac{D(t)}{A(t)} dt < \int_{\mathcal{N}} e^{-\int_t^T \frac{D(s)}{A(s)}ds} \frac{D(t)}{A(t)} dt. \tag{26}$$

Let $\bar{\mathcal{N}} = [0, T] - \mathcal{N} = \{t \,|t \in [0, T] \text{ and } t \notin \mathcal{N}\}$. Then, it is obvious that

$$\int_{\bar{\mathcal{N}}} e^{-\int_t^T \frac{D(s)+1}{A(s)}ds} \frac{D(t)}{A(t)} dt \leq \int_{\bar{\mathcal{N}}} e^{-\int_t^T \frac{D(s)}{A(s)}ds} \frac{D(t)}{A(t)} dt. \tag{27}$$

Adding inequality (26) to inequality (27) yields

$$\int_0^T e^{-\int_t^T \frac{D(s)+1}{A(s)}ds} \frac{D(t)}{A(t)} dt < \int_0^T e^{-\int_t^T \frac{D(s)}{A(s)}ds} \frac{D(t)}{A(t)} dt.$$

Therefore, by (24), we have

$$\begin{aligned} \vartheta(0) =& 1 - \int_0^T e^{-\int_t^T \frac{D(s)+1}{A(s)}ds} \frac{D(t)}{A(t)} dt > 1 - \int_0^T e^{-\int_t^T \frac{D(s)}{A(s)}ds} \frac{D(t)}{A(t)} dt \\ =& 1 - e^{-\int_t^T \frac{D(s)}{A(s)}ds} \Big|_0^T = 1 - e^0 + e^{-\int_0^T \frac{D(s)}{A(s)}ds} = e^{-\int_0^T \frac{D(s)}{A(s)}ds}. \end{aligned}$$

Namely, $\vartheta(0) > e^{-\int_0^T \frac{D(t)}{A(t)}dt}$. On the other hand, it is known from (23) that

$$0 < \frac{\psi^2(0)}{\varphi(0)} = e^{-\int_0^T \frac{D(t)+1}{A(t)}dt} < e^{-\int_0^T \frac{D(t)}{A(t)}dt}.$$

To sum up, the proposition is proved. □

According to Proposition 3.2, $\frac{\vartheta(0)}{1-\vartheta(0)} > 0$. Therefore, setting $u = u_{\sigma_{\min}} := \frac{\psi(0)}{\vartheta(0)} x_0$, we obtain the global minimum variance of the terminal wealth

$$\mathrm{Var}[X^{\min}(T)] = \left(\varphi(0) - \frac{\psi^2(0)}{\vartheta(0)}\right) x_0^2, \tag{28}$$

where $X^{\min}$ is the wealth process corresponding to the optimal investment strategy π^* given by (21) with $\mathrm{E}[X^*(T)] = u_{\sigma_{\min}}$ Obviously, rational investors would not select the expected terminal wealth less than $u_{\sigma_{\min}}$. Proposition 3.2 shows that $\vartheta(0) > \frac{\psi^2(0)}{\varphi(0)} > 0$, that is, $\varphi(0) - \frac{\psi^2(0)}{\vartheta(0)} > 0$. Since $x_0 > 0$, (28) implies that $\mathrm{Var}[X^{\min}(T)] > 0$.

Let $\tilde{\sigma}[X^*(T)]$ be the standard deviation of terminal wealth, i.e., $\tilde{\sigma}[X^*(T)] = \sqrt{\mathrm{Var}[X^*(T)]}$. Noting that $\mathrm{E}[X^*(T)] = u$. then, the efficient frontier (22) can be rewritten as

$$\tilde{\sigma}[X^*(T)] = \sqrt{\frac{\vartheta(0)}{1-\vartheta(0)}\left(\mathrm{E}[X^*(T)] - \frac{\psi(0)x_0}{\vartheta(0)}\right)^2 + \left(\varphi(0) - \frac{\psi^2(0)}{\vartheta(0)}\right)x_0^2}, \tag{29}$$

where $\mathrm{E}[X^*(T)] \geq \frac{\psi(0)}{\vartheta(0)}x_0$. Because $\left(\varphi(0) - \frac{\psi^2(0)}{\vartheta(0)}\right)x_0^2 > 0$, the efficient frontier is the upper branch of a hyperbola in the standard derivation-mean plane, which is the same as that of the static case (see Merton [17]).

In the static case, the Sharpe ratio is defined as the excess return (or risk premium) per unit of deviation in a portfolio (see Sharpe [21] and [22]). Similarly, in a continuous-time setting, the Sharpe ratio can be defined as

$$\frac{\mathrm{E}[X(T)] - x_0 R_f(T)}{\tilde{\sigma}[X(T)]},$$

where $R_f(T) = e^{\int_0^T r(s)ds}$ is the risk-free return rate over the entire horizon $[0, T]$, and $r(t)$ is the (instantaneous) risk-free interest rate at time t.

Now, we study the optimal (highest or maximum) Sharpe ratio generated by only risky assets in our continuous-time setting.

Case 1: $\frac{\psi(0)}{\vartheta(0)} \leq e^{\int_0^T r(t)dt}$. In this case, point $\left(0,\, x_0 e^{\int_0^T r(t)dt}\right)$ lies on or above the asymptotic line $\mathrm{E}[X^*(T)] = \frac{\psi(0)x_0}{\vartheta(0)} + \sqrt{\frac{1-\vartheta(0)}{\vartheta(0)}}\tilde{\sigma}[X^*(T)]$ of the efficient frontier (22) in the standard derivation-mean plane. Then there exists no tangent line for the efficient frontier (22) passing through the point $\left(0,\, x_0 e^{\int_0^T r(t)dt}\right)$. In this case, the Sharpe ratio cannot reach its maximum value, but its supremum exists and is given by the slope of this asymptotic line, $\sqrt{\frac{1-\vartheta(0)}{\vartheta(0)}}$. We can refer this supremum as the optimal Sharpe ratio when there are only risky assets, that is

$$Shp_{opt} = \sqrt{\frac{1-\vartheta(0)}{\vartheta(0)}}. \tag{30}$$

Case 2: $\frac{\psi(0)}{\vartheta(0)} > e^{\int_0^T r(t)dt}$. In this case, we can derive a tangent line of the efficient frontier (22) passing through the point $\left(0,\, x_0 e^{\int_0^T r(t)dt}\right)$. Then, the slope of this tangent line is the optimal Sharpe ratio generated by only risky assets, and is given by

$$Shp_{opt} = \sqrt{\frac{1-\vartheta(0)}{\vartheta(0)} + \frac{\left(\frac{\psi(0)}{\vartheta(0)} - e^{\int_0^T r(s)ds}\right)^2}{\varphi(0) - \frac{\psi^2(0)}{\vartheta(0)}}}. \tag{31}$$

The corresponding expected terminal wealth is

$$\mathrm{E}[X^{shp_{opt}}(T)] := \left(\frac{(1-\vartheta(0))\left(\varphi(0)\vartheta(0) - \psi^2(0)\right)}{\vartheta(0)\left(\psi(0) - \vartheta(0)e^{\int_0^T r(s)ds}\right)} + \frac{\psi(0)}{\vartheta(0)}\right)x_0. \tag{32}$$

Substituting $u = \mathrm{E}[X^{shp_{opt}}(T)]$ into (21) and simplifying the equation, we obtain the corresponding optimal investment strategy

$$\pi^{shp_{opt}}(t) = \frac{\left(\sigma(t)\sigma(t)'\right)^{-1}}{A(t)} \times \left(\left(X^*(t) + \frac{\left(\psi(0)e^{\int_0^T r(s)ds} - \varphi(0)\right)x_0\psi(t)}{\varphi(t)}\right)(C(t)\mathbf{1}_n - A(t)b(t)) + X^*(t)\mathbf{1}_n\right). \tag{33}$$

To sum up, we have the following theorem.

Theorem 3.2. *In our continuous-time setting, the optimal Sharpe ratio with only risky assets is given by*

$$Shp_{opt} = \begin{cases} \sqrt{\frac{1-\vartheta(0)}{\vartheta(0)}}, & \text{if } \frac{\psi(0)}{\vartheta(0)} \le e^{\int_0^T r(t)dt}, \\ \sqrt{\frac{1-\vartheta(0)}{\vartheta(0)} + \frac{\left(\frac{\psi(0)}{\vartheta(0)} - e^{\int_0^T r(s)ds}\right)^2}{\varphi(0) - \frac{\psi^2(0)}{\vartheta(0)}}}, & \text{if } \frac{\psi(0)}{\vartheta(0)} > e^{\int_0^T r(t)dt}. \end{cases} \tag{34}$$

4. **Enhancement of the optimal Sharpe ratio with inclusion of a risk-free asset.** From (8), the optimal Sharpe ratio generated by both risky and risk-free assets can be derived as

$$Shpf_{opt} = \sqrt{\frac{1 - e^{-\int_0^T \rho(s)ds}}{e^{-\int_0^T \rho(s)ds}}} = \sqrt{e^{\int_0^T \rho(s)ds} - 1}. \tag{35}$$

Proposition 4.1. *For $t \in [0, T]$, we have $\rho(t) = \frac{D(t)+(A(t)r(t)-C(t))^2}{A(t)} \ge \frac{D(t)}{A(t)}$.*

Proof. By (20) and (9), we have

$$\begin{aligned} &\rho(t) = (b(t) - \mathbf{1}_n r(t))'(\sigma(t)\sigma'(t))^{-1}(b(t) - \mathbf{1}_n r(t)) \\ &= b'(t)(\sigma(t)\sigma'(t))^{-1}b(t) + r^2(t)\mathbf{1'}_n(\sigma(t)\sigma'(t))^{-1}\mathbf{1}_n - 2r(t)\mathbf{1'}_n(\sigma(t)\sigma'(t))^{-1}b(t) \\ &= B(t) + A(t)r^2(t) - 2r(t)C(t) = \frac{D(t)+(A(t)r(t)-C(t))^2}{A(t)} \ge \frac{D(t)}{A(t)}, \end{aligned}$$

which completes the proof. □

Proposition 4.2. $\frac{\vartheta(0)}{1-\vartheta(0)} > \frac{e^{-\int_0^T \rho(s)ds}}{1-e^{-\int_0^T \rho(s)ds}}$.

Proof. According to Propositions 3.2 and 4.1, we have

$$1 > \vartheta(0) > e^{-\int_0^T \frac{D(s)}{A(s)}ds} \ge e^{-\int_0^T \rho(s)ds} > 0,$$

which gives $\frac{\vartheta(0)}{1-\vartheta(0)} > \frac{e^{-\int_0^T \rho(s)ds}}{1-e^{-\int_0^T \rho(s)ds}}$. □

Lemma 4.1. $(e^x - 1)(e^y - 1) \ge \left(e^{\sqrt{xy}} - 1\right)^2$ *for any real numbers $x \ge 0$ and $y \ge 0$.*

Lemma 4.2. *Suppose that $f(t)$ and $g(t)$ are measurable real-valued functions on $[0, T]$. Then*

$$\sqrt{\int_0^T f^2(t)dt \int_0^T g^2(t)dt} \ge \int_0^T |f(t)g(t)|\, dt.$$

Lemma 4.1 follows immediately from Lemma A.1 in Chiu and Zhou [5], and Lemma 4.2 can be easily obtained by the well-known Hölder inequality.

Lemma 4.3. *Suppose that* m_0, m_1, z_0, z_1 *and* n_1 *are all given constants satisfying* $m_1 \neq m_0$ *and* $m_1 m_0 > 0$, *then two parabolas* $y = m_0(z - z_0)^2$ *and* $y = m_1(z - z_1)^2 + n_1$ *have no common point if and only if*

$$(z_1 - z_0)^2 < \left(\frac{1}{m_0} - \frac{1}{m_1}\right) n_1.$$

Lemma 4.3 can be easily proved by the well-known Vieta theorem, which addresses the relations between the coefficients of a polynomial and its roots.

Proposition 4.3. *If* $\int_0^T \frac{C(s)}{A(s)} ds > \int_0^T r(s) ds$, *then the following inequality holds*

$$\left(e^{\int_0^T \frac{C(s)}{A(s)} ds} - e^{\int_0^T r(s) ds}\right)^2 < \left(e^{\int_0^T \frac{(A(s)r(s) - C(s))^2}{A(s)} ds} - 1\right)\left(e^{\int_0^T \frac{1}{A(s)} ds} - 1\right) e^{\int_0^T \frac{2C(s)}{A(s)} ds}. \tag{36}$$

Proof. According to Lemma 4.1, we have

$$\left(e^{\sqrt{\left(\int_0^T \frac{(C(s) - A(s)r(s))^2}{A(s)} ds\right) \int_0^T \frac{1}{A(s)} ds}} - 1\right)^2 \leq \left(e^{\int_0^T \frac{(C(s) - A(s)r(s))^2}{A(s)} ds} - 1\right)\left(e^{\int_0^T \frac{1}{A(s)} ds} - 1\right). \tag{37}$$

By Lemma 4.2 and the assumption of this proposition, it follows that

$$\begin{aligned} &\sqrt{\int_0^T \frac{(C(s) - A(s)r(s))^2}{A(s)} ds \int_0^T \frac{1}{A(s)} ds} \\ \geq &\int_0^T \left| \frac{C(s) - A(s)r(s)}{\sqrt{A(s)}} \frac{1}{\sqrt{A(s)}} \right| ds = \int_0^T \left| \frac{C(s) - A(s)r(s)}{A(s)} \right| ds \\ \geq &\int_0^T \frac{C(s) - A(s)r(s)}{A(s)} ds = \int_0^T \frac{C(s)}{A(s)} ds - \int_0^T r(s) ds > 0. \end{aligned}$$

This implies

$$\left(e^{\int_0^T \frac{C(s) - A(s)r(s)}{A(s)} ds} - 1\right)^2 \leq \left(e^{\sqrt{\left(\int_0^T \frac{(C(s) - A(s)r(s))^2}{A(s)} ds\right) \int_0^T \frac{1}{A(s)} ds}} - 1\right)^2. \tag{38}$$

Inequality (38) along with inequality (37) gives

$$\left(e^{\int_0^T \frac{C(s) - A(s)r(s)}{A(s)} ds} - 1\right)^2 \leq \left(e^{\int_0^T \frac{(C(s) - A(s)r(s))^2}{A(s)} ds} - 1\right)\left(e^{\int_0^T \frac{1}{A(s)} ds} - 1\right). \tag{39}$$

On the other hand, the assumption $\int_0^T \frac{C(s)}{A(s)} ds > \int_0^T r(s) ds$ implies

$$e^{\int_0^T \frac{C(s) - A(s)r(s)}{A(s)} ds} > e^{\int_0^T \frac{A(s)r(s) - C(s)}{A(s)} ds} > 0.$$

Hence,

$$e^{\int_0^T \frac{C(s) - A(s)r(s)}{A(s)} ds} + e^{\int_0^T \frac{A(s)r(s) - C(s)}{A(s)} ds} > 2\sqrt{e^{\int_0^T \frac{C(s) - A(s)r(s)}{A(s)} ds} e^{\int_0^T \frac{A(s)r(s) - C(s)}{A(s)} ds}} = 2.$$

Namely,

$$e^{\int_0^T \frac{C(s) - A(s)r(s)}{A(s)} ds} - 1 > 1 - e^{\int_0^T \frac{A(s)r(s) - C(s)}{A(s)} ds} = 1 - \frac{e^{\int_0^T r(s) ds}}{e^{\int_0^T \frac{C(s)}{A(s)} ds}} > 0.$$

Therefore, we have

$$\left(1 - e^{\int_0^T \frac{A(s)r(s) - C(s)}{A(s)} ds}\right)^2 < \left(e^{\int_0^T \frac{C(s) - A(s)r(s)}{A(s)} ds} - 1\right)^2. \tag{40}$$

By inequalities (39) and (40), we have

$$\left(1-e^{\int_0^T \frac{A(s)r(s)-C(s)}{A(s)}ds}\right)^2 < \left(e^{\int_0^T \frac{(C(s)-A(s)r(s))^2}{A(s)}ds}-1\right)\left(e^{\int_0^T \frac{1}{A(s)}ds}-1\right). \tag{41}$$

The above inequality multiplied by $e^{\int_0^T \frac{2C(s)}{A(s)}ds}(>0)$ yields inequality (36). □

Theorem 4.4. *The efficient frontier (22) with only risky assets is strictly separated from (namely, not tangent to) the efficient frontier (8) with both risky and risk-free assets.*

Proof. We prove this theorem in two cases.

Case 1: $\frac{\psi(0)}{\vartheta(0)} \le e^{\int_0^T r(t)dt}$**.** For convenience of expression, let $u = \mathrm{E}[X^*(T)]$. In this case, the common domain for the two efficient frontiers (22) and (8) should be $\Phi = \left\{u\in\mathbb{R}\,\middle|\, u\ge x_0 e^{\int_0^T r(t)dt}\right\}$. Notice that $x_0>0$, so for any $u\in\Phi$, we have $u-\frac{\psi(0)}{\vartheta(0)}x_0 \ge u - e^{\int_0^T r(t)dt}x_0 \ge 0$. By Propositions 3.2 and 4.2, it follows that

$$\varphi(0)-\frac{\psi^2(0)}{\vartheta(0)}>0 \text{ and } \frac{\vartheta(0)}{1-\vartheta(0)} > \frac{e^{-\int_0^T \rho(s)ds}}{1-e^{-\int_0^T \rho(s)ds}}.$$

Therefore,

$$\frac{\vartheta(0)}{1-\vartheta(0)}\left(u-\frac{\psi(0)x_0}{\vartheta(0)}\right)^2+\left(\varphi(0)-\frac{\psi^2(0)}{\vartheta(0)}\right)x_0^2 > \frac{e^{-\int_0^T \rho(s)ds}}{1-e^{-\int_0^T \rho(s)ds}}\left(u-x_0e^{\int_0^T r(t)dt}\right)^2.$$

This means that efficient frontiers (22) and (8) have no common point.

Case 2: $\frac{\psi(0)}{\vartheta(0)} > e^{\int_0^T r(t)dt}$. According to Proposition 3.2 and (23), we have

$$0 < e^{\int_0^T r(s)ds} < \frac{\psi(0)}{\vartheta(0)} < \frac{e^{\int_0^T \frac{C(s)-D(s)}{A(s)}ds}}{e^{-\int_0^T \frac{D(s)}{A(s)}ds}} = e^{\int_0^T \frac{C(s)}{A(s)}ds}. \tag{42}$$

This implies

$$\left(\frac{\psi(0)}{\vartheta(0)}-e^{\int_0^T r(s)ds}\right)^2 < \left(e^{\int_0^T \frac{C(s)}{A(s)}ds}-e^{\int_0^T r(s)ds}\right)^2 \tag{43}$$

and $\int_0^T \frac{C(s)}{A(s)}ds > \int_0^T r(s)ds$. Then, by Proposition 4.3, we get

$$\left(e^{\int_0^T \frac{C(s)}{A(s)}ds}-e^{\int_0^T r(s)ds}\right)^2 < \left(e^{\int_0^T \frac{(A(s)r(s)-C(s))^2}{A(s)}ds}-1\right)\left(e^{\int_0^T \frac{1}{A(s)}ds}-1\right)e^{\int_0^T \frac{2C(s)}{A(s)}ds}. \tag{44}$$

Again by Proposition 3.2 and (23), it follows that

$$\frac{\psi^2(0)}{\vartheta(0)} < \frac{\psi^2(0)}{e^{-\int_0^T \frac{D(s)}{A(s)}ds}} = \frac{e^{\int_0^T \frac{2C(s)-2D(s)}{A(s)}ds}}{e^{-\int_0^T \frac{D(s)}{A(s)}ds}} = e^{\int_0^T \frac{2C(s)-D(s)}{A(s)}ds}, \tag{45}$$

which along with (23) gives

$$\varphi(0)-\frac{\psi^2(0)}{\vartheta(0)} > e^{\int_0^T \frac{2C(s)-D(s)+1}{A(s)}ds}-e^{\int_0^T \frac{2C(s)-D(s)}{A(s)}ds} > 0. \tag{46}$$

According to Propositions 3.2 and 4.1, we obtain

$$\frac{1-e^{-\int_0^T \rho(s)ds}}{e^{-\int_0^T \rho(s)ds}}-\frac{1-\vartheta(0)}{\vartheta(0)} = e^{\int_0^T \rho(s)ds}-\frac{1}{\vartheta(0)} > e^{\int_0^T \frac{D(s)+(A(s)r(s)-C(s))^2}{A(s)}ds}-e^{\int_0^T \frac{D(s)}{A(s)}ds} \ge 0. \tag{47}$$

Inequalities (46) and (47) yield

$$\begin{aligned}
&\left(\frac{1-e^{-\int_0^T \rho(s)ds}}{e^{-\int_0^T \rho(s)ds}} - \frac{1-\vartheta(0)}{\vartheta(0)}\right)\left(\varphi(0) - \frac{\psi^2(0)}{\vartheta(0)}\right) \\
> &\left(e^{\int_0^T \frac{D(s)+(A(s)r(s)-C(s))^2}{A(s)}ds} - e^{\int_0^T \frac{D(s)}{A(s)}ds}\right)\left(e^{\int_0^T \frac{2C(s)-D(s)+1}{A(s)}ds} - e^{\int_0^T \frac{2C(s)-D(s)}{A(s)}ds}\right) \\
= &\left(e^{\int_0^T \frac{(A(s)r(s)-C(s))^2}{A(s)}ds} - 1\right)\left(e^{\int_0^T \frac{1}{A(s)}ds} - 1\right) e^{\int_0^T \frac{2C(s)}{A(s)}ds}.
\end{aligned} \tag{48}$$

By inequalities (43), (44) and (48), and noting that $x_0 > 0$, we have

$$\left(\frac{\psi(0)x_0}{\vartheta(0)} - x_0 e^{\int_0^T r(s)ds}\right)^2 < \left(\frac{1-e^{-\int_0^T \rho(s)ds}}{e^{-\int_0^T \rho(s)ds}} - \frac{1-\vartheta(0)}{\vartheta(0)}\right)\left(\varphi(0) - \frac{\psi^2(0)}{\vartheta(0)}\right) x_0^2. \tag{49}$$

According to Lemma 4.3, efficient frontiers (22) and (8) have no common point. □

Now we can obtain the following result about the relationship between the two optimal Sharpe ratios with and without a risk-free asset.

Theorem 4.5. *In our continuous-time setting, the optimal Sharpe ratio $Shpf_{opt}$ generated by both risky and risk-free assets is strictly greater than the optimal Sharpe ratio Shp_{opt} generated by only risky assets.*

Proof. We consider two cases:

Case 1: $\frac{\psi(0)}{\vartheta(0)} \leq e^{\int_0^T r(t)dt}$. By Proposition 4.2, $\frac{1-e^{-\int_0^T \rho(s)ds}}{e^{-\int_0^T \rho(s)ds}} > \frac{1-\vartheta(0)}{\vartheta(0)}$. Then, according to (35) and (34), we have $Shpf_{opt} > Shp_{opt}$.

Case 2: $\frac{\psi(0)}{\vartheta(0)} > e^{\int_0^T r(t)dt}$. From the proof of Theorem 4.4, inequality (49) holds in this case. This further shows that

$$\sqrt{\frac{1-e^{-\int_0^T \rho(s)ds}}{e^{-\int_0^T \rho(s)ds}}} > \sqrt{\frac{1-\vartheta(0)}{\vartheta(0)} + \frac{\left(\frac{\psi(0)}{\vartheta(0)} - e^{\int_0^T r(s)ds}\right)^2}{\varphi(0) - \frac{\psi^2(0)}{\vartheta(0)}}}.$$

Again by (35) and (34), $Shpf_{opt} > Shp_{opt}$ holds in this case. □

Theorems 4.4 and 4.5 imply that the continuous-time M-V efficient frontier with both risky and risk-free assets is strictly above the one with only risky assets. Theorem 4.5 also suggests a positive enhancement of the optimal Sharpe ratio with inclusion of a risk-free asset. Chiu and Zhou [5] call this enhancement of the Sharpe ratio the *premium of dynamic trading* of the risk-free asset. They show that the *premium of dynamic trading* is positive, but do not provide a computational expression for it. By (34) and (35), we obtain an expression for the *premium of dynamic trading* $PDT := Shpf_{opt} - Shp_{opt}$ as follows

$$PDT = \begin{cases} \sqrt{e^{\int_0^T \rho(s)ds} - 1} - \sqrt{\frac{1-\vartheta(0)}{\vartheta(0)}}, & \text{if } \frac{\psi(0)}{\vartheta(0)} \leq e^{\int_0^T r(t)dt} \\ \sqrt{e^{\int_0^T \rho(s)ds} - 1} - \sqrt{\frac{1-\vartheta(0)}{\vartheta(0)} + \frac{\left(\frac{\psi(0)}{\vartheta(0)} - e^{\int_0^T r(s)ds}\right)^2}{\varphi(0) - \frac{\psi^2(0)}{\vartheta(0)}}}, & \text{if } \frac{\psi(0)}{\vartheta(0)} > e^{\int_0^T r(t)dt}. \end{cases} \tag{50}$$

Theorem 4.5 shows that PDT is strictly positive.

5. **Empirical analysis.** In this section, using real data from the China equity market, we perform an out-of-sample empirical analysis to test our theory results. Some interesting numerical comparisons with the equally weighted strategy (also called naive 1/N strategy, see DeMiguel et al. [11] for details) are also considered in the final part of this section.

We take Shanghai Stock Exchange (SSE) 50 Index Components as the risky assets to carry out our empirical analysis. They are representative stocks in the China stock markets. Our data set is obtained from historical daily return information from January 5 2010 to July 28 2015. The sample size is 1350. Though there are 50 stocks in the SSE 50 Index Components, but the SSE 50 Index Components change from time to time since its inception. To keep consistent, we choose 44 stocks that have been staying in the component list during our sample period. We obtained daily data from the center for research in China Stock Market & Accounting Research (CSMAR) Database. In addition, since the M-V models in this paper are continuous-time models, so we take logarithm transform for these original data. Suppose that the 1350 daily returns are observed at $t_1,\ t_2,\ \cdots,\ t_{1350}$, and the corresponding daily return vectors of these 44 stocks are denoted by $R_1,\ R_2,\ \cdots,\ R_{1350}$. In this empirical example we take one year as a unit time, namely, the length of one day is $\frac{1}{365}$. It is well known that, in practice, investors cannot re-balance their positions continuously. In this example, we discretize the trading time, and suppose that investors can re-balance their positions one time at the beginning of every trading day. Suppose that initial wealth $x_0 = 1$ and take the one-year deposit rate of the bank during March 2011 in China as the risk-free (annual) interest rate, that is, $r(t) = 3\% = 0.03$. We adopt the optimal investment strategy obtained in this paper which maximizes the Sharpe ratio in continuous-time setting. More specifically, when there are only risky assets in the investment opportunity set, for Case 1 ($\frac{\psi^2(0)}{\vartheta(0)} \le e^{\int_0^T r(t)dt}$), we use the investment strategy (21) with $u = 100$ (a big enough number); for Case 2 ($\frac{\psi^2(0)}{\vartheta(0)} > e^{\int_0^T r(t)dt}$), we use the investment strategy (33). When there are both risk-free and risky assets in the investment opportunity set, we choose the efficient strategy (7) with the same u as that in case with only risky assets, because in this case the optimal Sharpe ratio is independent of the selection of u. Our out-of-sample analysis scheme is stated as follows.

We first use observations from t_1 to t_{180} to estimate relevant market parameters $b(t)$ and $\sigma(t)$ in our model. Since there are about 245 trading days for one year in China. Multiplying computational results by 245, we obtain estimations of the annual appreciation rate $b(t)$ and volatility rate matrix $\sigma(t)$ for the 44 stocks. Using the realized daily returns $R_1,\ R_2,\ R_{180}$, we perform the above investment strategies over 5-day horizon. There are $180/5 = 36$ pairs of terminal wealths corresponding to the strategies with and without the risk-free asset, respectively. Using these 36 pairs of terminal wealths, we compute the sample means and variances of terminal wealth for both case with and without a risk-free asset. Based on the computational results for sample means and variances above, we further calculate the first pair of empirical Sharp ratios, $\widehat{Shpf}_{opt}$ and $\widehat{Shp}_{opt}$, for both case with and without a risk-free asset as follows

$$\widehat{Shpf}_{opt} = 1.2623, \widehat{Shp}_{opt} = 1.2397.$$

The above approach only produces one pair of observed Sharpe ratios for comparison. Next, use observations from t_{31} to t_{210} to estimate parameters and perform strategies over 5-day horizon to produce another 36 pairs of terminal wealths. Compute the empirical Sharpe ratios $\widehat{Shpf}_{opt}$ and $\widehat{Shp}_{opt}$ again. By repeating this procedure until the last time window of t_{1171} to t_{1350}, we obtain 40 pairs of empirical Sharpe ratios, $\widehat{Shpf}_{opt}$ and $\widehat{Shp}_{opt}$. They are showed in Table 1.

For each pair of empirical Sharpe ratios, $\widehat{Shpf}_{opt}$ and $\widehat{Shp}_{opt}$, we calculate the empirical *premium of dynamic trading,* $\widehat{PDT} = \widehat{Shpf}_{opt} - \widehat{Shp}_{opt}$. What is more, when $\widehat{Shpf}_{opt} > \widehat{Shp}_{opt}$, we label $I_{Shpf} = 1$, otherwise, we label $I_{Shpf} = 0$. The computational results for both $\widehat{PDT}$ and I_{Shpf} are also listed in Table 1.

From Table 1, we find that, there are 32 pairs of empirical Sharpe ratios satisfy that the empirical Sharpe ratio with inclusion of a risk-free asset is greater than the empirical Sharpe ratio with only risky assets, there are 8 pairs of empirical Sharpe ratios satisfy that the empirical Sharpe ratio with inclusion of a risk-free asset is less than that with only risky assets. Namely, empirical probability of $\widehat{Shpf}_{opt} > \widehat{Shp}_{opt}$ is

$$\hat{P}rob(\widehat{Shpf}_{opt} > \widehat{Shp}_{opt}) = \frac{32}{40} = 80\%,$$

while, the empirical probability of $\widehat{Shpf}_{opt} < \widehat{Shp}_{opt}$ is

$$\hat{P}rob(\widehat{Shpf}_{opt} < \widehat{Shp}_{opt}) = \frac{8}{40} = 20\%.$$

The empirical results shed light on that the probability of $\widehat{Shpf}_{opt} > \widehat{Shp}_{opt}$ is greater than the probability of $\widehat{Shpf}_{opt} < \widehat{Shp}_{opt}$. Namely, in most cases, the empirical Sharpe ratio with both risk-free and risky assets is greater than that with only risky assets, and the *premium of dynamic trading* is greater than zero. This partially supports our results. We admit that there still are cases, in which the empirical Sharpe ratio with inclusion of a risk-free asset is less than the empirical Sharpe ratio with only risky asset, i.e., $\widehat{Shpf}_{opt} < \widehat{Shp}_{opt}$. We think that the main reasons lie on that, i) we use discretized investment strategies and daily realized returns of stocks to approximate continuous-time investment strategies and stocks' returns; ii) the optimal investment strategies obtained in our model are based on the assumption that the prices of stocks follow geometric Brownian motions, but in the real world, the prices of stocks may not follow geometric Brownian motions.

On the other hand, DeMiguel et al. [11] shows that in a static setting (single period) the strategy investing equal proportion of money into each risky asset (the so-called naive 1/N strategy) outperforms all the strategies with and without the risk-free asset, in the sense of Sharpe ratio. In the following, based on the data above, we will report the empirical analysis for the naive $1/N$ strategy in dynamic setting[3]. In exactly the same empirical analysis scheme above, we also obtained 40 empirical Sharpe ratios $\widehat{Shp}_{1/N}$ with naive $1/N$ strategy. The corresponding computational results are also presented in Table 1. Comparing the empirical Sharpe ratios based on continuous-time model for both case with and without a risk-free asset to the empirical Sharpe ratio of the naive $1/N$ strategy suggested by DeMiguel et al. [11], we find that the empirical Sharpe ratio of the $1/N$ strategy is the smallest one (see Table 1 for more details). Our finding implies that DeMiguel et al.

[3]We point out that using the $1/N$ strategy in dynamic setting still means one trades on each trading day. Because the stock prices evolve randomly one has to trade at every instant to ensure that the fraction of wealth invested in each security equals to $1/N$.

Table 1. Computational results for the out-of-sample empirical analysis

i	$\widehat{Shpf}_{opt}$	$\widehat{Shp}_{opt}$	$\widehat{Shp}_{1/N}$	I_{Shpf}	$\widehat{PDT}$
1	1.2623	1.2397	-0.3364	1	0.0226
2	1.1306	1.1307	-0.1763	0	-0.0001
3	1.3555	1.3595	-0.2105	0	-0.0040
4	1.1508	1.1485	-0.0030	1	0.0023
5	1.4455	1.4416	0.0944	1	0.0040
6	0.8880	0.8882	0.0659	0	-0.0001
7	0.9247	0.9243	0.1214	1	0.0004
8	0.9042	0.9036	-0.0065	1	0.0006
9	0.9748	1.0066	-0.0623	0	-0.0318
10	1.0651	1.0712	-0.1153	0	-0.0060
11	1.1288	1.1171	-0.2061	1	0.0118
12	1.1166	1.1051	-0.2055	1	0.0115
13	1.0595	1.0398	-0.1820	1	0.0196
14	0.8222	0.8001	-0.0792	1	0.0222
15	0.8377	0.8350	-0.0858	1	0.0027
16	0.8781	0.8348	-0.1900	1	0.0432
17	0.8107	0.7647	-0.0821	1	0.0460
18	0.9910	0.9260	-0.0264	1	0.0650
19	0.8887	0.8830	-0.0565	1	0.0057
20	0.7912	0.7805	-0.0090	1	0.0107
21	0.8506	0.8502	0.0811	1	0.0004
22	1.0396	1.0423	0.0582	0	-0.0026
23	1.2097	1.2066	-0.0026	1	0.0032
24	1.1486	1.1012	-0.0619	1	0.0474
25	0.8786	0.8499	0.0616	1	0.0287
26	0.7611	0.7211	-0.0889	1	0.0400
27	0.7547	0.7319	-0.1268	1	0.0228
28	0.8216	0.7850	-0.0742	1	0.0366
29	0.8815	0.8742	-0.0023	1	0.0073
30	0.9891	0.9116	-0.0144	1	0.0775
31	0.9342	0.8812	-0.1807	1	0.0530
32	0.9017	0.9005	0.0095	1	0.0011
33	0.9755	0.9778	0.0552	0	-0.0023
34	1.1543	1.1563	0.1036	0	-0.002
35	0.9607	0.9481	0.3367	1	0.0126
36	1.0940	1.0934	0.4308	1	0.0006
37	1.3044	1.2959	0.4813	1	0.0085
38	1.2116	1.2009	0.5951	1	0.0107
39	0.8597	0.8548	0.4308	1	0.0049
40	0.8278	0.8141	0.1292	1	0.0137

[11] missed the important point of considering dynamic strategies, which is a very important result.

6. **Conclusion.** This paper establishs a continuous-time M-V model with only risky assets in a new way. We obtain the closed-form expression for the efficient frontier with only risky assets, and then find the explicit relations between the efficient frontiers and hence between the optimal Sharpe ratios with and without a risk-free asset. Our findings include: (i) in the case with only risky assets, the global minimum variance is strictly larger than zero and the efficient frontier is a branch of a hyperbola in the standard derivation-mean plane; (ii) the efficient frontier with only risky assets is no longer tangent to the one with both risky and risk-free assets; (iii) inclusion of a risk-free asset can strictly enhance the optimal Sharpe ratio. These results are tested by an out-of-sample empirical analysis based on real data of Chinese stock market. Comparisons between the empirically optimal Sharpe ratios in continuous-time markets and the empirical Sharpe ratio of the $1/N$ strategy is also provided. Empirical results indicate that both the empirically optimal Sharpe ratios in continuous-time markets with and without a risk-free assets are greater than the one of the naive $1/N$ strategy. Our work can be extended in several ways. For example, it can be extended to the case with random market parameters, such as stochastic interest rate and stochastic volatility, or Markov regime switching market environment.

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