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Stochastic LQ Problem with Delayed Control

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Abstract: A discrete-time stochastic LQ problem with multiplicative noises and state transmission delay is studied in this paper, which does not require any definiteness constraint on the cost weighting matrices. Necessary and sufficient conditions are derived for the case with a fixed time-state initial pair. A set of coupled discrete-time Riccati-like equations can be derived to characterize the existence and the form of the delayed optimal control. Furthermore, the convexity of the cost functional is fully characterized via certain properties of the solution of the Riccati-like equations.

Key Words: Indefinite stochastic linear-quadratic optimal control, input delay, forward-backward stochastic difference equation, convexity

1 Introduction

Linear-quadratic (LQ, for short) optimal control was pioneered by Kalman [12] in 1960, which is now a classical yet fundamental problem in control theory. Extension to stochastic LQ problems was first carried out by Wonham [27] in 1968, and has received considerable interests and efforts since then. A common assumption of most literature on stochastic LQ problems is that the state weighting matrices are nonnegative definite and the control weighting matrices are positive definite. Contrary to this, Chen, Li and Zhou [8] revealed in 1998 that a stochastic LQ problem with multiplicative noises might still be solvable even if the cost weighting matrices are indefinite. More about this kind of LQ problems can be found in [1] [2] [11] [20] and references therein. Recently, some researchers are interested in the so-called mean-field LQ problems [15] [16] [21] [25] [29] [30]. An important feature of mean-field control problems is that the expected values of the state and control enter nonlinearly into the cost functional, which will bring new phenomena and new theoretical difficulties.

Note that all the aforementioned papers are free of time delay. If time delay happens to appear in the system state, the control input and/or the information-transmission channel, it is much more complicated and challenging to design the optimal control of the corresponding LQ problems. Such kind of LQ problems have been extensively studied since 1970's; see, for example, [3] [9] [13] [23] [32] or other related literature [4] [14] [17] [22]. Concerned with a deterministic LQ problem with input delay, it is shown [23] that the delayed optimal control is obtained by invoking the Smith predictor theory, and that the optimal gains are same to those of the LQ problem without input delay.

Unfortunately, the results about deterministic LQ problems (with input delay) cannot be directly generalized to the stochastic setting. In [32], the authors considered a discrete-time stochastic LQ problem with input delay and multiplicative noises, and showed that the optimal control (if exists) is a linear feedback of d-step-lagged conditional expectation of current states and that the optimal gains are computed via a set of coupled discrete-time Riccati-like equations. Here, the set of discrete-time Riccati-like equations differs significantly from what we have in hand the standard discrete-time Riccati equation. The reason for this, to the authors' knowledge, is related to a poor property of this class of stochastic systems, namely that the known separation principle does not hold for stochastic systems with multiplicative noises. Therefore, for a stochastic LQ problem with multiplicative noises, we can have a control by replacing the current states in the optimal control with their d-step-lagged conditional expectations; however, the obtained control is not optimal for the corresponding stochastic LQ problem with d-step input delay.

It is worth pointing out that the stochastic systems with multiplicative noises have been extensively studied in the past half century. From the viewpoint of mathematics, almost all the theories about stochastic differential equations (SDEs, for short) are for the case with multiplicative noises, and there are lots of practical motivations to study such kind of SDEs. The study of controlled systems with multiplicative noises is also popular in the control community; a recent small collection in the literature related to our paper includes [1] [2] [6] [7] [8] [11] [15] [18] [20] [26] [29].

In this paper, a general discrete-time stochastic LQ problem with multiplicative noises and state transmission delay is thoroughly investigated, whose cost weighting matrices for the state and control are allowed to be indefinite. Apart from intending to generalize the existing results to the joint case with indefiniteness and time delay, the topic of this paper is also partially motivated by recent progresses in network control system [5] [10] [24] and other related areas. The contributions of this paper are listed as follows:

For the case with a fixed time-state initial pair, the solvability of Problem (LQ) at that initial pair is shown to be equivalent to that a stationary condition and a convexity condition are satisfied, with the backward state of a forward-backward stochastic difference equation (FBSΔE, for short) being involved in the stationary condition. Further, a set of

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coupled discrete-time Riccati-like equations is introduced, by which we can express the backward state of the FBSΔE via its forward state. Moreover, equivalent characterizations of the stationary condition and the convexity condition are derived via certain properties of the solution of the Riccati-like equations.

2 Problem formulation

Consider the following controlled stochastic difference equation (SΔE, for short)

$$\begin{cases} X_{k+1} = \big(A_k X_k + B_k u_k\big) \\ \qquad\quad + \big(C_k X_k + D_k u_k\big) w_k, \\ X_t = x, \ k \in \mathbb{T}_t \triangleq \{t, t+1, ..., N-1\}, \end{cases} \tag{2.1}$$

where $t \in \mathbb{T} \triangleq \{0, 1, ..., N-1\}$, $A_k, C_k \in \mathbb{R}^{n\times n}$, $B_k, D_k \in \mathbb{R}^{n\times m}$ are deterministic matrices. The noise $\{w_k, k \in \mathbb{T}\}$ is assumed to be a martingale difference sequence defined on a probability space $(\Omega, \mathcal{F}, P)$ with

$$\mathbb{E}_{k+1}[w_{k+1}] = 0, \ \ \mathbb{E}_{k+1}[(w_{k+1})^2] = 1, \ k \in \mathbb{T}. \tag{2.2}$$

Here, $\mathbb{E}_{k+1}$ is the conditional mathematical expectation $\mathbb{E}[\,\cdot\,|\mathcal{F}_{k+1}]$ with respect to $\mathcal{F}_{k+1} = \sigma\{w_l, l = 0, 1, \cdots, k\}$, and $\mathcal{F}_0$ is understood as $\{\emptyset, \Omega\}$. Introduce the following cost functional associated with (2.1)

$$\begin{aligned} J(t, x; u) = & \sum_{k=t}^{N-1} \mathbb{E}\big[X_k^T Q_k X_k + u_k^T R_k u_k\big] \\ & + \mathbb{E}\big[X_N^T G X_N\big], \end{aligned} \tag{2.3}$$

where $Q_k, R_k, k \in \mathbb{T}_t$, G are deterministic symmetric matrices of appropriate dimensions. Note, here, that we do not pose any definiteness constraints on the cost weighting matrices.

As mentioned in Introduction, time delays arise naturally in many engineering fields such as chemical processes and communication systems, *etc*. Throughout this paper, we assume that there is a transmission delay of d steps (with $d \geq 2$); such kind of delays arise frequently in network control systems [5] [10] [24]. Hence, at stage $k \in \{t, ...t+d\}$ the controller's decision information set remains $\mathcal{F}_t$ as no new information is available; and for $k \in \mathbb{T}_{t+d} = \{t + d, ..., N-1\}$ the information set should be $\mathcal{F}_{k-d}$. To define the set of admissible controls, let us review the following spaces. For $t = 0, ..., N$, let

$$L^2_{\mathcal{F}}(t; \mathbb{R}^m) = \left\{ \zeta \in \mathbb{R}^m \;\middle|\; \begin{array}{l} \zeta \text{ is } \mathcal{F}_t\text{-measurable,} \\ \text{and } \mathbb{E}|\zeta|^2 < \infty \end{array} \right\}, \tag{2.4}$$

and

$$\begin{aligned} & L^2_{\mathcal{F}}(\mathbb{T}_t^{-d}; \mathbb{R}^m) \\ & = \left\{ \nu \;\middle|\; \begin{array}{l} \nu = \{\nu_k, k \in \mathbb{T}_t^{-d}\}, \\ \nu_k \text{ is } \mathcal{F}_k\text{-measurable}, k \in \mathbb{T}_t^{-d}, \\ \text{and } \sum_{k=t}^{N-1-d} \mathbb{E}|\nu_k|^2 < \infty \end{array} \right\} \end{aligned} \tag{2.5}$$

with

$$\mathbb{T}_t^{-d} = \{t, ..., N-1-d\}.$$

Introduce the set of admissible controls

$$\mathcal{U}_{ad}^t = \left\{ u \;\middle|\; \begin{array}{l} u = \{u_t, u_{t+1}, ..., u_{N-1}\}, \\ u \in \big(L^2_{\mathcal{F}}(t; \mathbb{R}^m)\big)^d \times L^2_{\mathcal{F}}(\mathbb{T}_t^{-d}; \mathbb{R}^m) \end{array} \right\}. \tag{2.6}$$

The following optimal control problem will be studied in this paper.

Problem (LQ). *For a time-state initial pair (t, x), find a $\bar{u} \in \mathcal{U}_{ad}^t$ such that*

$$J(t, x; \bar{u}) = \inf_{u \in \mathcal{U}_{ad}^t} J(t, x; u). \tag{2.7}$$

Noting that the initial pair (t, x) is specialized, hereafter the above problem will be called as Problem (LQ) for the initial pair (t, x). Furthermore, any $\bar{u}$ satisfying (2.7) is called an optimal control of Problem (LQ) for the initial pair (t, x).

Definition 2.1. *Problem (LQ) is said to be (uniquely) solvable at (t, x) if there exists a (unique) $\bar{u} \in \mathcal{U}_{ad}^t$ such that (2.7) holds.*

3 Main results

In this section, we will study Problem (LQ) for the fixed initial pair (t, x).

Theorem 3.1. *The following statements are equivalent.*
(i) Problem (LQ) is solvable at (t, x).
(ii) The following assertions hold.

a) There exists a $u^{t,x,*} \in \mathcal{U}_{ad}^t$ such that the stationary condition

$$\begin{aligned} & R_k u_k^{t,x,*} + B_k^T \mathbb{E}_{k-d} Z_{k+1}^{t,x,*} \\ & + D_k^T \mathbb{E}_{k-d}(Z_{k+1}^{t,x,*} w_k) = 0, \quad a.s., \quad k \in \mathbb{T}_t \end{aligned} \tag{3.1}$$

is satisfied, where $Z^{t,x,*}$ is the backward state of the following FBSΔE

$$\begin{cases} X_{k+1}^{t,x,*} = \big(A_k X_k^{t,x,*} + B_k u_k^{t,x,*}\big) \\ \qquad\qquad + \big(C_k X_k^{t,x,*} + D_k u_k^{t,x,*}\big) w_k, \\ Z_k^{t,x,*} = Q_k X_k^{t,x,*} + A_k^T \mathbb{E}_k Z_{k+1}^{t,x,*} \\ \qquad\qquad + C_k^T \mathbb{E}_k (Z_{k+1}^{t,x,*} w_k), \\ X_t^{t,x,*} = x, \quad Z_N^{t,x,*} = G X_N^{t,x,*}, \quad k \in \mathbb{T}_t. \end{cases} \tag{3.2}$$

b) The convexity condition

$$\inf_{u \in \mathcal{U}_{ad}^t} J(t, 0; u) \geq 0 \tag{3.3}$$

holds.

Under any of above conditions, $u^{t,x,*}$ in (ii) is an optimal control of Problem (LQ) for the initial pair (t, x).

Remark 3.2. *Throughout this paper, $\mathbb{E}_{k-d}$ is understood as $\mathbb{E}_t$ if $k \in \{t, ..., t+d-1\}$ (i.e., $k - d < t$). Hence, for $k = t, ..., t+d-1$, (3.1) reads as*

$$R_k u_k^{t,x,*} + B_k^T \mathbb{E}_t Z_{k+1}^{t,x,*} + D_k^T \mathbb{E}_t (Z_{k+1}^{t,x,*} w_k) = 0, \ \ a.s.$$

Different from the general maximum principle of [33], Theorem 3.1 provides necessary and sufficient conditions on the existence of the optimal control of Problem (LQ) for the initial pair (t, x).

Recall the pseudo-inverse of a matrix. By [19], for a given matrix $M \in \mathbb{R}^{n\times m}$, there exists a unique matrix in $\mathbb{R}^{m\times n}$ denoted by $M^\dagger$ such that

$$\begin{cases} M M^\dagger M = M, \ \ M^\dagger M M^\dagger = M^\dagger, \\ (M M^\dagger)^T = M M^\dagger, \ \ (M^\dagger M)^T = M^\dagger M. \end{cases} \tag{3.4}$$

This $M^\dagger$ is called the Moore-Penrose inverse of M. The following lemma is from [1].

Lemma 3.3. *Let matrices L, M and N be given with appropriate size. Then, $LXM = N$ has a solution X if and only if $LL^\dagger NMM^\dagger = N$. Moreover, the solution of $LXM = N$ can be expressed as $X = L^\dagger NM^\dagger + Y - L^\dagger LYMM^\dagger$, where Y is a matrix with appropriate size.*

If $M = I$ in Lemma 3.3, then $LL^\dagger N = N$ is equivalent to $\mathrm{Ran}(N) \subset \mathrm{Ran}(L)$. Here, $\mathrm{Ran}(N)$ is the range of N. We now introduce the following discrete-time Riccati-like iterations:

$$\begin{cases} P_k^{(0)} = Q_k + A_k^T\big(P_{k+1}^{(0)} + P_{k+1}^{(1)}\big)A_k \\ \qquad\quad + C_k^T P_{k+1}^{(0)} C_k, \\ P_k^{(i)} = A_k^T P_{k+1}^{(i+1)} A_k, \quad i = 1, ..., d-1, \\ P_k^{(d)} = -H_k^T W_k^\dagger H_k, \\ P_N^{(0)} = G, \quad P_N^{(j)} = 0, \quad j = 1, ..., d, \\ k \in \mathbb{T}_{t+d} = \{t+d, ..., N-1\}, \end{cases} \tag{3.5}$$

$$\begin{cases} P_k^{(0)} = Q_k + A_k^T\big(P_{k+1}^{(0)} + P_{k+1}^{(1)}\big)A_k \\ \qquad\quad + C_k^T P_{k+1}^{(0)} C_k, \\ P_k^{(i)} = A_k^T P_{k+1}^{(i+1)} A_k, \quad i = 1, ..., k-t-1, \\ P_k^{(k-t)} = A_k^T P_{k+1}^{(k+1-t)} A_k - H_k^T W_k^\dagger H_k, \\ k \in \{t+2, ..., t+d-1\}, \end{cases} \tag{3.6}$$

and

$$\begin{cases} \begin{cases} P_{t+1}^{(0)} = Q_{t+1} + A_{t+1}^T\big(P_{t+2}^{(0)} + P_{t+2}^{(1)}\big)A_{t+1} \\ \qquad\quad + C_{t+1}^T P_{t+2}^{(0)} C_{t+1}, \\ P_{t+1}^{(1)} = A_{t+1}^T P_{t+2}^{(2)} A_{t+1} - H_{t+1}^T W_{t+1}^\dagger H_{t+1}, \end{cases} \\ P_t^{(0)} = Q_t + A_t^T\big(P_{t+1}^{(0)} + P_{t+1}^{(1)}\big)A_t + C_t^T P_{t+1}^{(0)} C_t \\ \qquad\quad - H_t^T W_t^\dagger H_t, \end{cases} \tag{3.7}$$

where

$$W_k = \begin{cases} R_k + \sum_{i=0}^{d} B_k^T P_{k+1}^{(i)} B_k \\ \quad + D_k^T P_{k+1}^{(0)} D_k, \quad k \in \mathbb{T}_{t+d}, \\ R_k + \sum_{i=0}^{k+1-t} B_k^T P_{k+1}^{(i)} B_k \\ \quad + D_k^T P_{k+1}^{(0)} D_k, \quad k \in \{t, ..., t+d-1\}, \end{cases} \tag{3.8}$$

and

$$H_k = \begin{cases} \sum_{i=0}^{d} B_k^T P_{k+1}^{(i)} A_k \\ \quad + D_k^T P_{k+1}^{(0)} C_k, \quad k \in \mathbb{T}_{t+d}, \\ \sum_{i=0}^{k+1-t} B_k^T P_{k+1}^{(i)} A_k \\ \quad + D_k^T P_{k+1}^{(0)} C_k, \quad k \in \{t, ..., t+d-1\}. \end{cases} \tag{3.9}$$

Based on the solution of (3.5)-(3.7), an equivalent characterization of the stationary condition is derived.

Theorem 3.4. *The following statements are equivalent.*

(i) The stationary condition of (3.1) is satisfied for some $u^{t,x,*} \in \mathcal{U}_{ad}^t$.

(ii) The following condition

$$H_k \mathbb{E}_{k-d} X_k^{t,x,*} \in Ran(W_k), \quad a.s., \quad k \in \mathbb{T}_t \tag{3.10}$$

is satisfied, where $W_k, H_k, k \in \mathbb{T}_t$, are given in (3.8) and (3.9), and $X^{t,x,*}$ is given by the forward SΔE of

$$\begin{cases} X_{k+1}^{t,x,*} = \big(A_k X_k^{t,x,*} + B_k u_k^{t,x,*}\big) \\ \qquad\qquad + \big(C_k X_k^{t,x,*} + D_k u_k^{t,x,*}\big) w_k, \\ Z_k^{t,x,*} = Q_k X_k^{t,x,*} + A_k^T \mathbb{E}_k Z_{k+1}^{t,x,*} \\ \qquad\qquad + C_k^T \mathbb{E}_k(Z_{k+1}^{t,x,*} w_k), \\ X_t^{t,x,*} = x, \quad Z_N^{t,x,*} = G X_N^{t,x,*}, \quad k \in \mathbb{T}_t \end{cases} \tag{3.11}$$

with

$$u_k^{t,x,*} = -W_k^\dagger H_k \mathbb{E}_{k-d} X_k^{t,x,*}, \qquad k \in \mathbb{T}_t. \tag{3.12}$$

Furthermore, the backward state $Z^{t,x,*}$ of (3.11) has the following expression

$$Z_k^{t,x,*} = \begin{cases} P_k^{(0)} X_k^{t,x,*} + P_k^{(1)} \mathbb{E}_{k-1} X_k^{t,x,*} + \cdots \\ \quad + P_k^{(k-t)} \mathbb{E}_t X_k^{t,x,*}, \quad k \in \{t, ..., t+d-1\}, \\ P_k^{(0)} X_k^{t,x,*} + P_k^{(1)} \mathbb{E}_{k-1} X_k^{t,x,*} + \cdots \\ \quad + P_k^{(d)} \mathbb{E}_{k-d} X_k^{t,x,*}, \quad k \in \mathbb{T}_{t+d}, \end{cases}$$

where $P^{(i)}, i = 0, ..., d$, are given in (3.5)-(3.7).

We now study the convexity condition. In what follows, the functional $u \mapsto J(t,x;u)$ is called convex if (3.3) holds. Let X^0 be the solution of (2.1) with $x = 0$, i.e.,

$$\begin{cases} X_{k+1}^0 = \big(A_k X_k^0 + B_k u_k\big) \\ \qquad\quad + \big(C_k X_k^0 + D_k u_k\big) w_k, \\ X_t^0 = 0, \quad k \in \mathbb{T}_t. \end{cases} \tag{3.13}$$

Lemma 3.5. *For any $u \in \mathcal{U}_{ad}^t$, it holds that*

$$\begin{aligned} J(t,0;u) = \sum_{k=t}^{N-1} \mathbb{E}\Big\{ & (\mathbb{E}_{k-d} X_k^0)^T H_k^T W_k^\dagger H_k \mathbb{E}_{k-d} X_k^0 \\ & + 2(H_k \mathbb{E}_{k-d} X_k^0)^T u_k + u_k^T W_k u_k \Big\} \end{aligned} \tag{3.14}$$

with X^0 given in (3.13).

Proof. By adding to and subtracting

$$\begin{aligned} & \sum_{k=t}^{N-1} \mathbb{E}\Big\{ \sum_{i=0}^{d} (\mathbb{E}_{k+1-i} X_{k+1}^0)^T P_{k+1}^{(i)} \mathbb{E}_{k+1-i} X_{k+1}^0 \\ & - \sum_{i=0}^{d} (\mathbb{E}_{k-i} X_k^0)^T P_k^{(i)} \mathbb{E}_{k-i} X_k^0 \Big\} \\ & + \sum_{k=t}^{t+d-1} \mathbb{E}\Big\{ \sum_{i=0}^{k+1-t} (\mathbb{E}_{k+1-i} X_{k+1}^0)^T P_{k+1}^{(i)} \mathbb{E}_{k+1-i} X_{k+1}^0 \\ & - \sum_{i=0}^{k-t} (\mathbb{E}_{k-i} X_k^0)^T P_k^{(i)} \mathbb{E}_{k-i} X_k^0 \Big\} \end{aligned}$$

from $J(t,0;u)$, we have (noting $X_t^0=0$)

$$
\begin{aligned}
J(t,0;u) = &\sum_{k=t+d}^{N-1} \mathbb{E}\Big\{(X_k^0)^T\big[Q_k + A_k^T\big(P_{k+1}^{(0)} + P_{k+1}^{(1)}\big)A_k^T \\
&+ C_k^T P_{k+1}^{(0)} C_k - P_k^{(0)}\big]X_k^0 \\
&+ \sum_{i=1}^{d-1} (\mathbb{E}_{k-i}X_k^0)^T\big[A_k^T P_{k+1}^{(i+1)} A_k - P_k^{(i)}\big]\mathbb{E}_{k-i}X_k^0 \\
&- (\mathbb{E}_{k-d}X_k^0)^T P_k^{(d)} \mathbb{E}_{k-d}X_k^0 \\
&+ 2(H_k\mathbb{E}_{k-d}X_k^0)^T u_k + u_k^T W_k u_k\Big\} \\
&+ \sum_{k=t+2}^{t+d-1} \mathbb{E}\Big\{(X_k^0)^T\big[Q_k + A_k^T\big(P_{k+1}^{(0)} + P_{k+1}^{(1)}\big)A_k^T \\
&+ C_k^T P_{k+1}^{(0)} C_k - P_k^{(0)}\big]X_k^0 \\
&+ \sum_{i=1}^{k-t-1} (\mathbb{E}_{k-i}X_k^0)^T\big[A_k^T P_{k+1}^{(i+1)} A_k - P_k^{(i)}\big]\mathbb{E}_{k-i}X_k^0 \\
&+ (\mathbb{E}_t X_k^0)^T\big(A_k^T P_{k+1}^{(k+1-t)} A_k - P_k^{(k-t)}\big)\mathbb{E}_t X_k^0 \\
&+ 2(H_k\mathbb{E}_t X_k^0)^T u_k + u_k^T W_k u_k\Big\} \\
&+ \mathbb{E}\Big\{(X_{t+1}^0)^T\big[Q_{t+1} + A_{t+1}^T\big(P_{t+2}^{(0)} + P_{t+2}^{(1)}\big)A_{t+1}^T \\
&+ C_{t+1}^T P_{t+2}^{(0)} C_{t+1} - P_{t+1}^{(0)}\big]X_{t+1}^0 \\
&+ (\mathbb{E}_t X_{t+1}^0)^T\big(A_{t+1}^T P_{t+2}^{(2)} A_{t+1} - P_{t+1}^{(1)}\big)\mathbb{E}_t X_{t+1}^0 \\
&+ 2(H_{t+1}\mathbb{E}_t X_{t+1}^0)^T u_{t+1} \\
&+ u_{t+1}^T W_{t+1} u_{t+1}\Big\} + \mathbb{E}\Big\{(X_t^0)^T\big[Q_t + A_t^T\big(P_{t+1}^{(0)} \\
&+ P_{t+1}^{(1)}\big)A_t^T + C_t^T P_{t+1}^{(0)} C_t - P_t^{(0)}\big]X_t^0 \\
&+ 2(H_t X_t^0)^T u_t + u_t^T W_t u_t\Big\} \\
= &\sum_{k=t}^{N-1} \mathbb{E}\Big\{(\mathbb{E}_{k-d}X_k^0)^T H_k^T W_k^\dagger H_k \mathbb{E}_{k-d}X_k^0 \\
&+ 2(H_k\mathbb{E}_{k-d}X_k^0)^T u_k + u_k^T W_k u_k\Big\}.
\end{aligned}
$$

This completes the proof. □

Based on above preparations, we have the following theorem.

Theorem 3.6. *The following statements are equivalent.*
(i) Problem (LQ) is solvable at (t,x).
(ii) The following assertions hold

a) The solution of Riccati-like equations (3.5)-(3.7) has the property $W_k \geq 0, k \in \mathbb{T}_t$.

b) For any $u \in \mathcal{U}_{ad}^t$, the condition

$$H_k\mathbb{E}_{k-d}X_k^{x,u} \in Ran(W_k), \quad a.s., \quad k \in \mathbb{T}_t \tag{3.15}$$

is satisfied, where $X^{x,u}$ is the solution of the following S∆E

$$
\begin{cases}
X_{k+1}^{x,u} = \big(A_k X_k^{x,u} - B_k W_k^\dagger H_k \mathbb{E}_{k-d} X_k^{x,u} \\
\qquad\quad + B_k u_k\big) + \big(C_k X_k^{x,u} + D_k u_k \\
\qquad\quad - D_k W_k^\dagger H_k \mathbb{E}_{k-d} X_k^{x,u}\big) w_k, \\
X_t^{x,u} = x, \quad k \in \mathbb{T}_t.
\end{cases} \tag{3.16}
$$

Under any of above conditions, the following control

$$u_k^{t,x,*} = -W_k^\dagger H_k \mathbb{E}_{k-d} X_k^{t,x,*}, \qquad k \in \mathbb{T}_t$$

is an optimal control of Problem (LQ) for the initial pair (t,x), where $X^{t,x,*}$ is given by

$$
\begin{cases}
X_{k+1}^{t,x,*} = \big(A_k X_k^{t,x,*} - B_k W_k^\dagger H_k \mathbb{E}_{k-d} X_k^{t,x,*}\big) \\
\qquad\quad + \big(C_k X_k^{t,x,*} - D_k W_k^\dagger H_k \mathbb{E}_{k-d} X_k^{t,x,*}\big) w_k, \\
X_t^{t,x,*} = x, \quad k \in \mathbb{T}_t
\end{cases}
$$

4 Conclusion

In this paper, an indefinite stochastic LQ problem with state transmission delay and multiplicative noises is studied. For the case with a fixed initial pair, a set of discrete-time Riccati-like equations is introduced, which is used to characterize the existence of the delayed optimal control of Problem (LQ). For future research, the infinite-horizon stochastic LQ problem with input delay should be investigated.

References

[1] M. Ait Rami, X. Chen, and X. Y. Zhou. Discrete-time indefinite LQ control with state and control dependent noise. *Journal of Global Optimization*, 2002, vol.23, 245-265.

[2] M. Ait Rami, X. Chen, J.B. Moore, and X.Y. Zhou. Solvability and asymptotic behavior of generalized Riccati equations arising in indefinite stochastic LQ controls. *IEEE Transactions on Automatic Control*, 2001, vol.46, 428-440.

[3] Y. Alekal, P. Brunovsky, D. H. Chyung, and E. B. Lee. The quadratic problem for systems with time delays. *IEEE Transactions on Automatic Control*, 1971, vol.16, 673-687.

[4] E. Altman, T. Basar, and R. Srikant. Congestion control as a stochastic control problem with action delays. *Automatica*, 1999, vol.35, 1937-1950.

[5] J. Baillieul and P. Antsaklis. Control and communication challenges in networked real-time systems. *Proceedings of the IEEE*, 2007, vol.95, no.1, 9-28.

[6] J. M. Bismut. Linear quadratic optimal stochastic control with random coefficients. *SIAM Journal on Control and Optimization*, 1976, vol.14, 419-444.

[7] L. Chen and Z. Wu. Maximum principle for stochastic optimal control problem with delay and application. *Automatica*, 2010, vol.46, 1074-1080.

[8] S. Chen, X. Li, and X. Y. Zhou. Stochastic linear quadratic regulators with indefinite control weight costs. *SIAM Journal on Control and Optimization*, 1998, vol.36, 1685-1702.

[9] M. C. Delfour. The linear-quadratic optimal control problem with delays in state and control variables: a state space approach. *SIAM Journal on Control and Optimization*, 1986, vol.24, 835-883.

[10] J. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. *Proceedings of the IEEE*, 2007, vol.95, no.1, 138-162, 2007.

[11] Y. Hu and X. Y. Zhou. Indefinite stochastic Riccati equations. *SIAM Journal on Control and Optimization*, 2003, vol.42, 123-137.

[12] R.E. Kalman. Contribution to the Theory of Optimal Control. *Boletin de la Sociedad Matematica Mexicana*, 1960, vol.5, no.2, 102-119.

[13] H. N. Koivo and E. B. Lee. Controller synthesis for linear systems with retarded state and control variables and quadratic cost. *Automatica*, 1972, vol.8, 203-208.

[14] A. Kojima and S. Ishijima. Formulas on preview and delayed H^∞ control. *IEEE Transactions on Automatic Control*, 2006, vol.51, 1920-1937.

[15] Y. H. Ni, R. J. Elliott, and X. Li. Discrete-time mean-field stochastic linear-quadratic optimal controls, II: Infinite time horizon. *Automatica*, 2015, vol.57, 65-77.

[16] Y. H. Ni, J. F. Zhang, and X. Li. Indefinite mean-field stochastic linear-quadratic optimal control. *IEEE Transactions on Automatic Control*, 2015, vol.60, no.7,1786-1800.

[17] W. H. Mao, F. Q. Deng, and A. H. Wan. Robust H_2/H_∞ global linearization filter design for nonlinear stochastic time-varying delay systems. *SCIENCE CHINA Information Sciences*, 2016, vol.59, no.3: 032204, Doi: 10.1007/s11432-015-5386-7.

[18] S. Peng. A general stochastic maximum principle for optimal control problems. *SIAM Journal on Control and Optimization*, 1990, vol.28, no.4, 966-979.

[19] R. Penrose. A generalized inverse of matrices. *Mathematical Proceedings of the Cambridge Philosophical Society*, 1955, vol.52, 17-19.

[20] J. R. Sun, X. Li, and J. M. Yong. Open-Loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems. *SIAM Journal on Control and Optimization*, 2016, vol. 54, no. 5, 2274-2308.

[21] J. R. Sun and J. M. Yong. Mean-field stochastic linear quadratic optimal control problems: open-loop solvabilities. To appear in *ESAIM: Control Optimisation and Calculus of Variations*, 2016, DOI: 10.1051/cocv/2016023.

[22] G. Tadmor and L. Mirkin. H^∞ control and estimation with preview-part I: matrix ARE solutions in continuous time. *IEEE Transactions on Automatic Control*, 2005, vol.50, 19-28.

[23] K. Watanabe and M. Ito. A process-model control for linear systems with delay. *IEEE Transactions on Automatic Control*, 1981, vol.26, 1261-1269.

[24] L. Wei, H. Zhang, and M. Fu. Quantized stabilization for stochastic discrete-time systems with multiplicative noises. *International Journal of Robust and Nonlinear Control*, 2013, vol.23, no.6, 591-601.

[25] G. C. Wang, C. H. Zhang, and W. H. Zhang. Stochastic maximum principle for mean-field type optimal control under partial information. *IEEE Transactions on Automatic Control*, 2014, vol.59, 522-528.

[26] Y. Q. Wang, D. H. Yang, J, M. Yong, and Z. Y. Yu. Exact controllability of linear stochastic differential equations and related problems. *arXiv*: 1603.07789.

[27] W. M. Wonham. On a matrix Riccati equation of stochastic control. *SIAM Journal on Control and Optimization*, 1068, vol.6, 312-326.

[28] D. Yao, S. Zhang, and X. Y Zhou. Stochastic LQ control via primal—dual semidefinite programming. *SIAM Review*, 2004, vol.46, 87-111.

[29] J. M. Yong. A linear-quadratic optimal control problem for mean-field stochastic differential equations. *SIAM Journal on Control Optimization*, 2013, vol.51, 2809-2838.

[30] J. M. Yong. Linear-quadratic optimal control problems for mean-field stochastic differential equations—time-consistent solutions. Electronically published by *Transactions of the American Mathematical Society*, DOI: 10.1090/tran/6502, 2015.

[31] J. M. Yong and X. Y. Zhou. *Stochastic controls: Hamiltonian systems and HJB equations*. Springer, New York, 1999.

[32] H. Zhang, L. Li, J. J. Xu, and M. Y. Fu. Linear quadratic regulation and stabilization of discrete-time systems with delay and multiplicative noise. *IEEE Transactions on Automatic Control*, 2015, vol.60, 2599-2613.

[33] H. Zhang, H. Wang, and L. Li. Adapted and casual maximum principle and analytical solution to optimal control for stochastic multiplicative noise systems with multiple input-delays. *IEEE CDC*, 2011, 2122-2127.