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NON-EXPONENTIAL DISCOUNTING PORTFOLIO MANAGEMENT WITH HABIT FORMATION

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Abstract. This paper studies the portfolio management problem for an individual with a non-exponential discount function and habit formation in finite time. The investor receives a deterministic income, invests in risky assets, buys insurance and consumes continuously. The objective is to maximize the utility of excessive consumption, heritage and terminal wealth. The non-exponential discounting makes the optimal strategy adopted by a naive person time-inconsistent. The equilibrium for a sophisticated person is Nash subgame perfect equilibrium, and the sophisticated person is time-consistent. We calculate the analytical solution for both the naive strategy and equilibrium strategy in the CRRA case and compare the results of the two strategies. By numerical simulation, we find that the sophisticated individual will spend less on consumption and insurance and save more than the naive person. The difference in the strategies of the naive and sophisticated person decreases over time. Furthermore, if an individual of either type is more patient in the future or has a greater tendency toward habit formation, he/she will consume less and buy less insurance, and the degree of inconsistency will also be increased. The sophisticated person's consumption and habit level are initially lower than those of a naive person but are higher in later periods.

1. Introduction.

Merton[21] studied the optimal consumption and investment strategy for a individual in continuous time. He assumed the individual's income generated by returns on two assets: a risky asset and a risk-free asset. In addition, this is the first article to introduce the portfolio optimization problem. Later, Richard[23] introduced the uncertain lifetime into the model and allowed the individual to purchase life insurance to mitigate the risk of death. The individual's income comes from not only returns on assets but also his/her wage, which is certain

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during his/her whole life. In addition, he/she extended one risky asset to n risky assets to prove the Tobin-Markowitz separation theorem. This is the benchmark work in the study of the optimal life insurance rule for uncertain-lived individuals.

Habit formation is an important individual characteristic. Many researchers, such as Lally et al.[17], have found evidence of habit formation in field experiments. The existence of habit indicates that the current consumption of the individual is affected by his/her past consumption. For example, an individual who is accustomed to eating one apple a day will feel disappointed if he/she does not eat an apple today but will feel happier if he/she has two apples. People commonly become accustomed to daily consumption patterns. Habit formation was first used to explain the equity premium puzzle by Constantinides[8]. Detemple and Zapatero[9] studied the optimal consumption-portfolio policies for individuals with habit. Liu et al. [19] revisited Detemple and Zapatero[9]'s work by including life insurance. Diaz et al.[10] studied the role of habit formation in savings decisions, and they found that habit formation increases precautionary savings. The model becomes more reasonable when we consider habit formation for the individual.

Numerous works have studied the portfolio management problem based on maximizing the individual's intertemporal utility, and they solve for the optimal strategy by using the Hamilton-Jacobi-Bellman (HJB) equation. The key assumption in using the HJB equation is that the individual discount function is exponential. However, the results from some experimental studies have shown that this assumption is not suited to human behaviour. Loewenstein and Prelec[20] suggested that the discount rate decreases with time. Ainslie[1] noted that the discount function is almost hyperbolic for humans. For more works on hyperbolic discounting, see [15], [16], [3]. When we discard exponential discounting, using the HJB equation to solve the problem will lead to time inconsistency.

The time-inconsistency problem is widely studied in many fields. This problem occurs when
(i) the objective function depends on the initial point (t, x), for example, when the general form of the discount function (see [11],[27], [31]) or wealth-dependent risk aversion (see [18], [4], [29]) is used in the model or
(ii) the terminal evaluation is allowed to be a nonlinear function of $E_{t,x}[X_T^{\mathbf{u}}]$. The classic example is the mean-variance portfolio (see [28], [14], [25]).

This means that the objective function is of the form

$$J(t,x;\mathbf{u}) = E\left[\int_t^T H(t,x,s,X_s^{\mathbf{u}},\mathbf{u}_s(X_s^{\mathbf{u}}))ds + F(t,x,X_T^{\mathbf{u}})\right] + G(t,x,E_{t,x}[X_T^{\mathbf{u}}]),$$

where $G(t, x, y)$ is a non-linear function of y.

Such an objective function breaks the recursion of the value function, and the Bellman equation loses its effect in this case. If we still use the HJB equation to obtain the optimal strategy, we will find that the individual will not follow it in later periods. We call this a time-inconsistency problem. Strotz[24] introduced three types of individuals in the time-inconsistent problem. The first one will 'precommit' his/her future action at the very first so that he/she can maximize the utility at the beginning. But such decision may not be optimal in the future. The second one,i.e. the so called naive person, will attempt to optimize his/her intertemporal utility at every time. Both the two type individual cannot recognize the time inconsistency in his/her decision, so their decisions are time-inconsistent. The third type of individual is the sophisticated person, who realizes the time inconsistency and considers the problem as a game between him and himself in the future. The sophisticated person will find a strategy that he/she will follow in the future once he/she makes his/her decision. Such strategy is a time-consistent strategy.

The discussion about the difference between the naive and sophisticated person is popular recently. Zhao et al[30] compared the difference between the naive insurer and sophisticated insurer in consumption-investment-reinsurance problem. Chen et al[6] discussed the dividend strategy for the naive and sophisticated managers. Chen and Li[7] studied the consumption, investment and life insurance strategy for naive and sophisticated agents. In our paper, we shall also focus on the difference between the naive and sophisticated strategies.

Since the sophisticated strategy is a solution of a game, it is also called equilibrium strategy in many literatures. There are two kinds of equilibrium strategies. One is the open-loop control, which is mainly used in linear-quadratic control problem (see [12], [13], [26]) and the main way to get an open-loop control is stochastic maximum principle. Alia[2] found the open-loop control of Merton's problem with non-constant discounting. The other is the close-loop control, Björk et al[5] derived the extended HJB equation to solve the close-loop control in a general time-inconsistent problem by using Nash subgame perfect theory.

In this paper, we consider the close-loop equilibrium strategy for the consumption, investment and insurance purchasing problem with an uncertain lifetime within $[0, T]$. At time t, the individual receives a deterministic rate of income and decides how much to invest in risky assets and how much to spend to consume and buy life insurance. To make the model more reasonable, we make two changes to the lifecycle model. We substitute the exponential discount function in the tradition lifecycle model with a general form. This change produces a time-inconsistent problem in our model. The portfolio management problem with non-exponential discounting function was studied in [11]. We will extend the problem to individual with habit formation. Following [9], the consumption will only bring utility to decision maker if the consumption exceeds the habit formation when we take the habit formation into consideration. In other words, habit formation is regarded as the living standards of the individual, which is the weighted average of past consumption. Thus, the utility of the individual consists of excess consumption, legacy and wealth at the final time T. Furthermore, we derive both time-inconsistent strategy for the naive individual and time-consistent strategy for the sophisticated individual, while most articles in literature such as [11] only consider equilibrium strategy. Then, we obtain the naive strategy and the equilibrium strategy in the case of the CRRA utility to observe the difference between the two strategies. Numerical simulation results are also provided to demonstrate the impacts of habit as well as the discount function, and the difference between the strategies of the naive person and the sophisticated person.

The remainder of this paper is organized as follows. In section 2, we describe some main assumptions of the model. In section 3, we derive the HJB equation for the time-inconsistent strategy and the extended HJB equation for the time-consistent strategy. Section 4 addresses the CRRA utility and shows the difference between the naive strategy and the equilibrium strategy. Section 5 presents some numerical results. In section 6, we draw our conclusions.

2. **The Model.** We consider the consumption-investment-insurance choice in a financial market for an individual during period $[0,T]$. $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbf{P})$ is a filtered complete probability space, where $\mathscr{F}_t$ is the information about the market available up to time t. $[0,T]$ is a fixed time horizon. All the processes introduced below are defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbf{P})$ and assumed to be adapted to $\{\mathscr{F}_t\}_{t\in[0,T]}$.

2.1. **Market.** Assume that the financial market consists of a savings account and one stock (the risky asset). The individual can invest part of his/her wealth into the risky asset and place the remaining wealth into the savings account. Here, we only consider the case of one risky asset because the case of additional risky assets in the market can be easily solved by the Tobin-Maekowitz separation theorem if we solve the problem with one risky asset. The savings account accrues with an interest rate $r > 0$. The stock price per share follows exponential Brownian motion:

$$dS(t) = S(t)(\alpha dt + \sigma dW(t)),$$

where $W(t)$ is a standard Brownain motion, α and σ are constants.

Assume that the individual is alive at time 0 and has a lifetime denoted by τ, which is a random variable independent of the Brownian motion $W(t)$. Given that the hazard function is $\lambda(t)$, the probability that $\tau > s$ given $\tau > t$ can be written as:

$$P(\tau > s | \tau > t) = e^{-\int_t^s \lambda(\mu) d\mu}.$$

The insurance company sells the life insurance with infinitesimal horizon to the individual with a premium rate of $\eta(t)$, which means that the insured should pay $\eta(t)$ per unit of insurance. The individual spends $p(t)$ per unit time covered by life insurance. In other words, an individual at time t will leave $X(t) + \frac{p(t)}{\eta(t)}$ to his/her beneficiaries if he/she dies immediately after purchasing life insurance.

2.2. **The characteristics of the individual.** To make the model more realistic, we consider three characteristics of the individual. Those characteristics describe the risk preference, time preference and habit formation for the individual.

First, the individual has a strictly increasing and strictly concave differentiable real-valued utility function $U(x)$ defined on $[0, +\infty)$. The utility function means that the individual prefers greater consumption, and he/she is risk averse; this is a classic assumption made in many studies.

Second, the individual has an inner habit $H(t)$, which is defined as $H(t) = e^{-at}H_0 + b\int_0^t e^{a(s-t)}c(s)\,ds$. Thus, $H(t)$ satisfies

$$dH(t) = [bc(t) - aH(t)]dt. \tag{1}$$

We can regard $H(t)$ as a living standard for the individual. In addition, an individual with habit $H(t)$ will only obtain utility from the exceeded consumption $c(t) - H(t)$. From the definition of $H(t)$, we can see that $H(t)$ is actually a weighted average of past consumption.

Finally, the individual at time t discounts the intertemporal utility at time s with the discount function $\phi(s-t)$. Many studies set $\phi(s-t) = e^{-\delta(s-t)}$. Exponential discounting means that the individual shows no difference in the time delay when comparing goods at two time points. Muellbauer[22] has suggested that people may exhibit a common difference effect when discounting the utility in the future. For example, if an individual with an exponential discount function feels no difference between an apple now and two apples tomorrow, he/she also will feel no difference between an apple 50 days later and two apples 51 days later. However, in reality, the individual may prefer two apples 51 days later. Such property cannot be described by an exponential discount function; thus, we do not limit the discount function to an exponential form and use a general form instead of the exponential form in the following analysis.

2.3. **Wealth dynamics.** The strategy for the decision maker is the set of control strategy $\{c(t,x,h),\theta(t,x,h),p(t,x,h)\}$. Given $X(t)=x$, at time $s\in[t,T]$ the individual is continuously investing $\theta(s,X(s),H(s))$ in the stock and bond, consuming $c(s,X(s),H(s))$ and spending $p(s,X(s),H(s))$ on life insurance and receiving income at the continuous deterministic rate $i(s)$ during period $[t,T]$. Thus, the individual's wealth at time s satisfies

$$\begin{aligned}dX(s)=&r[X(s)-\theta(s,X(s),H(s))]ds+\theta(s,X(s),H(s))[\alpha ds+\sigma dW(s)]+i(s)ds\\&-c(s,X(s),H(s))ds-p(s,X(s),H(s))ds\\=&[rX(s)+\theta(s,X(s),H(s))\mu+i(s)-c(s,X(s),H(s))-p(s,X(s),H(s))]ds\\&+\theta(s,X(s),H(s))\sigma dW(s),\quad X(t)=x,\end{aligned}$$

where $\mu=\alpha-r$ is the excess return on the stock.

Let $\mathbf{u}(t,x,h)=\{c(t,x,h),\theta(t,x,h),p(t,x,h)\}$.

Definition 2.1. For $t\in[0,T]$, a map $\mathbf{u}:[t,T]\times\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}^+\times\mathbb{R}^2$ is called an admissible strategy, if it satisfies£º

(i) For each initial point $(t,x,h)\in[0,T]\times\mathbb{R}\times\mathbb{R}^+$, the pair of SDEs for $X(s)$ and $H(s)$

$$\begin{aligned}dX(s)=&[rX(s)+\theta(s,X(s),H(s))\mu+i(s)-c(s,X(s),H(s))-p(s,X(s),H(s))]ds\\&+\sigma\theta(s,X(s),H(s))dW(s),\quad X(t)=x,\\dH(s)=&[bc(s,X(s),H(s))-aH(s)]ds,\quad H(t)=h\end{aligned}$$

has a pair of unique strong solution.

(ii) $c(s,y,k)\geq k$, $y+\frac{p(s,y,k)}{\eta(s)}\geq 0$ for all $(s,y,k)\in[t,T]\times\mathbb{R}\times\mathbb{R}^+$ a.s.;

(iii) $E[\int_t^T|\theta(s,X(s),H(s))\sigma(s)|^2ds]<+\infty$; $E[\int_t^T|c(s,X(s),H(s))|ds]<+\infty$ and such that $X(T)\geq 0$ a.s.

Set $\mathscr{A}$ as the collection of all admissible strategies.

As mentioned above, the discount function here is not limited to an exponential form; thus, for a naive individual who always wants to maximize his/her intertemporal utility, the strategy that he/she applies is time-inconsistent. The sophisticated person will realize the time-inconsistent problem faced by the naive individual; therefore, he/she will attempt to apply a time-consistent strategy instead of simply maximizing the utility. Then, we look for the equilibrium strategy for the sophisticated person based on Nash subgame perfect theory to compare with the naive strategy.

2.4. **Intertemporal utility.** The individual evaluates the performance of a strategy with investment, consumption and insurance by the expected utility criterion. Given the wealth and habit formation $X(t)=x$, $H(t)=h$ at time t, respectively, for an admissible strategy $\mathbf{u}$, we denote the intertemporal utility by

$$J(t,x,h;\mathbf{u})\triangleq E\bigg[\int_t^{T\wedge\tau}\phi(s-t)U(c(s,X(s),H(s))-H(s))ds$$

$$+\phi(\tau-t)U\Big(X(\tau)+\frac{p(\tau,X(\tau),H(\tau))}{\eta(\tau)}\Big)\mathrm{I}_{\{\tau\le T\}}+$$
$$\phi(T-t)U(X(T))\mathrm{I}_{\{\tau>T\}}|\tau>t,X(t)=x,H(t)=h\Big].$$

Because the death time is independent of the Brownian motion that drives the stock price, we can rewrite a simplified expression for $J(t,x,h;\mathbf{u})$ as

$$J(t,x,h;\mathbf{u}) = E_{t,x,h}\Big[\int_t^T Q(s,t)U(c(s,X(s),H(s))-H(s))ds+ \int_t^T \lambda(s)Q(s,t)U\Big(X(s)+\frac{p(s,X(s),H(s))}{\eta(s)}\Big)ds+Q(T,t)U(X(T))\Big], \tag{2}$$

where $Q(s,t)=\phi(s-t)e^{-\int_t^s\lambda(u)du}$, $Q(t,t)=1$ and $E_{t,x,h}$ is the expectation conditioned on $X(t)=x$ and $H(t)=h$.

3. Time-inconsistent strategy vs time-consistent strategy.

In this section, we will derive two types of strategy: the naive strategy for a naive person and the equilibrium strategy for a sophisticated person.

3.1. Time-inconsistent strategy and HJB equation.

To obtain the naive strategy, we first need to derive the t-optimal strategy, which is a strategy $\mathbf{u}^{t,x,h}$ made by the naive individual at time t to maximize his/her intertemporal utility $J(t,x,h;\mathbf{u})$. Then, the naive strategy is constructed based on the t-optimal strategy.

For any given pair (t,x,h), and $s>t$ we define

$$J^{t,x,h}(s,y,k;\mathbf{u}) = E_{s,y,k}\Big[\int_s^T Q(v,t)U(c(v,X(v),H(v))-H(v))dv+ \int_s^T Q(v,t)\lambda(v)U\Big(X(v)+\frac{p(v,X(v),H(v))}{\eta(v)}\Big)dv+Q(T,t)U(X(T))\Big].$$

Problem 1. *For any given initial pair $(s,y,k)\in[t,T]\times\mathbb{R}^2$, find a $\hat{\mathbf{u}}^{t,x,h}\in\mathscr{A}$ such that*

$$J^{t,x,h}(s,y,k;\hat{\mathbf{u}}^{t,x,h}) = \sup_{\mathbf{u}\in\mathscr{A}} J^{t,x,h}(s,y,k;\mathbf{u}).$$

As t,x,h are constants here, Problem 1 is actually a time-consistent problem, so we can derive the HJB equation to obtain $\mathbf{u}^{t,x,h}$. Define the optimal function as $V^{t,x,h}(s,y,k)=J^{t,x,h}(s,y,k;\hat{\mathbf{u}}^{t,x,h})$. From dynamic programming principle, $V^{t,x,h}(s,y,k)$ satisfies the following HJB equation:

$$\begin{cases}\sup_{\mathbf{u}\in\mathscr{A}}\{Q(s,t)U(c-k)+Q(s,t)\lambda(s)U\big(y+\frac{p}{\eta(s)}\big)+V_s^{t,x,h}+V_k^{t,x,h}[bc-ak]+\\ V_y^{t,x,h}[ry+\theta\mu+i(s)-c-p]+\frac{1}{2}V_{yy}^{t,x,h}\theta^2\sigma^2\}=0,\\ V(T,y,k)=Q(T,t)U(y).\end{cases} \tag{3}$$

Thus,

$$\hat{\mathbf{u}}^{t,x,h} = \arg\sup\{Q(s,t)U(c-k)+Q(s,t)\lambda(s)U\big(y+\frac{p}{\eta(s)}\big)+V_s^{t,x,h}+ V_k^{t,x,h}[bc-ak]+V_y^{t,x,h}[ry+\theta\mu+i(s)-c-p]+\frac{1}{2}V_{yy}^{t,x,h}\theta^2\sigma^2\}.$$

Then, we define the t-optimal value function

$$\hat{V}^{t,x,h}(s,y,k;\mathbf{u}) \triangleq \frac{V^{t,x,h}(s,y,k;\mathbf{u})}{Q(s,t)}.$$

We can rewrite equation (3) as

$$\begin{cases} \sup_{\mathbf{u}\in\mathscr{A}}\{U(c-k)+\lambda(s)U\big(y+\frac{p}{\eta(s)}\big)+\hat{V}^{t,x,h}_s+\hat{V}^{t,x,h}\frac{Q_1(s,t)}{Q(s,t)}+\hat{V}^{t,x,h}_k[bc-ak]+ \\ \hat{V}^{t,x,h}_y[ry+\theta\mu+i(s)-c-p]+\frac{1}{2}\hat{V}^{t,x,h}_{yy}\theta^2\sigma^2\}=0, \\ \hat{V}^{t,x,h}(T,y,k)=U(y). \end{cases} \tag{4}$$

When we return to time t, we can see $J^{t,x,h}(t,x,h;\mathbf{u})=J(t,x,h;\mathbf{u})$. Therefore,

$$\begin{aligned} J^{t,x,h}(t,x,h;\hat{\mathbf{u}}^{t,x,h}) &= \sup_{\mathbf{u}\in\mathscr{A}} J^{t,x,h}(t,x,h;\mathbf{u}) \\ &= \sup_{\mathbf{u}\in\mathscr{A}} J(t,x,h;\mathbf{u}) \end{aligned}$$

Proposition 1. *If there exists $\hat{V}^{t,x,h}\in C^{1,2,1}([t,T]\times\mathbf{R}\times\mathbf{R}^+)$ satisfy equation (4) and*

$$\begin{cases} \hat{c}^{t,x,h}(s,y,k)=I\big[\hat{V}^{t,x,h}_y-b\hat{V}^{t,x,h}_k\big]+k, \\ \hat{p}^{t,x,h}(s,y,k)=\eta(s)\big[I(\hat{V}^{t,x,h}_y\frac{\eta(s)}{\lambda(s)})-y\big], \\ \hat{\theta}^{t,x,h}(s,y,k)=-\frac{\mu\hat{V}^{t,x,h}_y}{\sigma^2\hat{V}^{t,x,h}_{yy}}, \end{cases} \tag{5}$$

where $I(U'(x))=x$, we have $J(t,x,h;\hat{\mathbf{u}}^{t,x,h})=\sup_{\mathbf{u}(\cdot)\in\mathscr{A}}J(t,x,h;\mathbf{u})$.

Thus, $\hat{\mathbf{u}}^{t,x,h}$ is the t-optimal strategy. From the t-optimal strategy, we can see that, if the naive individual at time t wants to maximize his/her intertemporal utility $J(t,x,h;\mathbf{u})$, he/she needs to insist on $\hat{\mathbf{u}}^{t,x,h}$. Given that $X(s)=y$ and $H(s)=k$, it means that the strategy he/she should take at time $s>t$ is $\hat{\mathbf{u}}^{t,x,h}(s,y,k)$. However, at time s, the naive individual will also want to maximize his/her intertemporal utility $J(s,y,k;\mathbf{u})$; then, he/she will take $\hat{\mathbf{u}}^{s,y,k}(s,y,k)$ at that time, which may not be the same as $\hat{\mathbf{u}}^{t,x,h}(s,y,k)$.

Following equation (4), we can obtain the HJB equation for the s-optimal value function $\hat{V}^{s,y,k}(v,z,l)$, $v\geq s$ as

$$\begin{aligned} &\sup_{\mathbf{u}\in\mathscr{A}}\{U(c-l)+\lambda(v)U\big(z+\frac{p}{\eta(v)}\big)+\hat{V}^{s,y,k}_v+\hat{V}^{s,y,k}\frac{Q_1(v,s)}{Q(v,s)}+ \\ &\hat{V}^{s,y,k}_l[bc-al]+\hat{V}^{s,y,k}_z[rz+\theta\mu+i(v)-c-p]+\frac{1}{2}\hat{V}^{s,y,k}_{zz}\theta^2\sigma^2\}=0 \end{aligned} \tag{6}$$

From equation (6), we can derive the s-optimal strategy $\mathbf{u}^{s,y,k}(v,z,l)$. Let $v=s, z=y, l=k$, and we obtain $\hat{\mathbf{u}}^{s,y,k}(s,y,k)$. Comparing (4) and (6), if we want $\hat{\mathbf{u}}^{t,x,h}(s,y,k)$ and $\hat{\mathbf{u}}^{s,y,k}(s,y,k)$ to be equal, we need $\frac{Q_1(s,s)}{Q(s,s)}=\frac{Q_1(s,t)}{Q(s,t)}$. By the definition of $Q(s,t)$, we will have $\phi'(s-t)=\phi'(0)\phi(s-t)$, which means that $\phi(t)$ is an exponential function. Hence, $\hat{\mathbf{u}}^{t,x,h}(s,y,k)$ and $\hat{\mathbf{u}}^{s,y,k}(s,y,k)$ are equal only if the discount function is exponential. As we do not limit the discount to being exponential, the individual may change his/her strategy in the future, and we can see time inconsistency appears in the t-optimal strategy $\hat{\mathbf{u}}^{t,x,h}$.

As the naive person wants to maxmize his intertemporal utility every time, he will take only $\mathbf{u}^{t,x,h}(t,x,h)$ at time t. Defining $\hat{\mathbf{u}}(t,x,h)=\hat{\mathbf{u}}^{t,x,h}(t,x,h)$, the naive strategy is $\hat{\mathbf{u}}(t,x,h)$ for $t\in[0,T]$.

3.2. **Time-consistent strategy and extended HJB equation.** From the above analysis, we explain why the t-optimal strategy is time-inconsistent and derive the naive strategy based on the t-optimal strategy. Because we do not force the individual to commit to a decision made previously, we need to find a strategy whereby a sophisticated individual will not have an incentive to change his/her strategy in the future. This is the so-called time-consistent equilibrium strategy. The definition of the equilibrium strategy follows [5], which is given below:

Definition 3.1. For any fixed $(t,x,h)\in[0,T]\times\mathbb{R}^2$, consider an admissible strategy $\bar{\mathbf{u}}$. For any fixed admissible control strategy u and $\varepsilon>0$, one can define the control strategy $\mathbf{u}_\varepsilon$ by

$$\mathbf{u}_\varepsilon(s,y,k)=\begin{cases}\bar{\mathbf{u}}(s,y,k) & for \quad t+\varepsilon\le s\le T,\\ u(s,y,k) & for \quad t\le s\le t+\varepsilon,\end{cases}$$

where u is any strategy such that $\mathbf{u}_\varepsilon\in\mathscr{A}$.
If

$$\lim_{\varepsilon\to0}\inf\frac{J(t,x,h;\bar{\mathbf{u}})-J(t,x,h;\mathbf{u}_\varepsilon)}{\varepsilon}\ge0,$$

then $\bar{\mathbf{u}}$ is called an equilibrium control strategy, and the equilibrium value function is given by $V(t,x,h)=J(t,x,h;\bar{\mathbf{u}})$

A sophisticated person who recognizes the time inconsistency in the t-optimal strategy regards portfolio management as a non-cooperative game among his/her selves at different times in $[0,T]$. For example, at time t, he/she is player t. All players from 0 to T need to negotiate with one another to find a strategy whereby no one has an incentive to defect. The definition of the equilibrium strategy follows the Nash subgame perfect equilibrium strategy. For any time t, if players other than player t apply the equilibrium strategy, player t's best choice is also the equilibrium strategy. Thus, once the individual decides to apply the equilibrium strategy, he/she will follow it in the future and have no incentive to change strategies. This is why the equilibrium strategy is time-consistent.

Definition 3.2. Given the objective functional $J(t,x,h;\mathbf{u})$ as (2), the extended HJB equation for V is given by

$$\begin{cases}\sup\limits_{\mathbf{u}\in\mathscr{A}}\big\{U(c-h)+\lambda(t)U\big(x+\dfrac{p}{\eta(t)}\big)+A^uV(t,x,h)\big\}=\\ \qquad E_{t,x,h}\Big[\displaystyle\int_t^T\frac{\partial Q(s,t)}{\partial t}U(\bar{c}(s,\bar{X}(s),\bar{H}(s))-\bar{H}(s))+\\ \qquad \dfrac{\partial Q(s,t)}{\partial t}\lambda(s)U\left(\bar{X}(s)+\dfrac{\bar{p}(s,\bar{X}(s),\bar{H}(s))}{\eta(s)}\right)ds+\dfrac{\partial Q(T,t)}{\partial t}U(\bar{X}(T))\Big],\\ V(T,x,h)=U(x).\end{cases}\tag{7}$$

where
(i)

$$A^uV(t,x,h)\triangleq V_t+\big[rx+\theta\mu+i(t)-c-p\big]V_x+(bc-ah)V_h+\frac{1}{2}\theta^2\sigma^2V_{xx};$$

(ii)$\bar{X}(s)$ and $\bar{H}(s)$ are the wealth and habit formation under $\bar{\mathbf{u}}$, which satisfy:

$$\begin{aligned} d\bar{X}(s) =& [r\bar{X}(s) + \bar{\theta}(s,\bar{X}(s),\bar{H}(s))\mu + i(s) - \bar{c}(s,\bar{X}(s),\bar{H}(s)) - \bar{p}(s,\bar{X}(s),\bar{H}(s))]dt \\ & + \bar{\theta}(s,\bar{X}(s),\bar{H}(s))\sigma dW(s) \quad \bar{X}(t) = x; \\ d\bar{H}(s) =& [b\bar{c}(s,\bar{X}(s),\bar{H}(s)) - a\bar{H}(s)]dt \quad \bar{H}(t) = h. \end{aligned}$$

(iii)$\bar{\mathbf{u}}(s,y,k) = \{\bar{c}(s,y,k), \bar{\theta}(s,k,y), \bar{p}(s,k,y)\}$ attains the supremum in (7);

Theorem 3.3. *(Verification Theorem) Assume that $V \in C^{1,2,1}([0,T]\times\mathbb{R}\times\mathbb{R}^+)$ is a classical solution to the extended HJB equation given in Definition 3.2, and the supremum is attained for each $(t,x,h) \in [0,T]\times\mathbb{R}\times\mathbb{R}^+$ given a control strategy $\bar{\mathbf{u}} \in \mathscr{A}$. Then $\bar{\mathbf{u}}$ is the equilibrium control strategy, and V is the corresponding equilibrium value function.*

The proof of Theorem 3.3 is given in the appendix.

4. **CRRA case.**

In this section, we are going to focus on the CRRA case, and we will find the analytical solution of both the time-inconsistent and time-consistent strategy to compare the difference between them. For the CRRA case, we assume that $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with $\gamma > 0$ and $\gamma \neq 1$.

4.1. **Naive strategy for CRRA case.**

To obtain the naive strategy, we need to derive the t-optimal strategy first. We attempt an optimal value function of the form $\hat{V}^{t,x,h}(s,y,k) = \hat{A}^t(s)U[y + \hat{B}(s)k + \hat{D}(s)]$. According to (5), the t-optimal strategy would be

$$\begin{cases} \hat{c}^{t,x,h}(s,y,k) = [\hat{A}^t(s) - b\hat{A}^t(s)\hat{B}(s)]^{-\frac{1}{\gamma}}[y + \hat{B}(s)k + \hat{D}(s)] + k, \\ \hat{p}^{t,x,h}(s,y,k) = (\dfrac{\hat{A}^t(s)}{\lambda(s)})^{-\frac{1}{\gamma}}\eta^{\frac{\gamma-1}{\gamma}}(s)[y + \hat{B}(s)k + \hat{D}(s)] - y\eta(s), \\ \hat{\theta}^{t,x,h}(s,y,k) = \dfrac{\mu}{\sigma^2\gamma}[y + \hat{B}(s)k + \hat{D}(s)]. \end{cases}$$

Substituting the t-optimal strategy into the HJB equation (4), we obtain

$$\begin{cases} \hat{D}'(s) - [r + \eta(s)]\hat{D}(s) + i(s) = 0, \\ \hat{D}(T) = 0, \end{cases}$$

$$\begin{cases} \hat{B}'(s) - [r + \eta(s) - b + a]\hat{B}(s) - 1 = 0, \\ \hat{B}(T) = 0, \end{cases}$$

and

$$\begin{cases} \hat{A}^t_s(s) + \hat{A}^t(s)^{\frac{\gamma-1}{\gamma}}\gamma\Big\{[1 - b\hat{B}(s)]^{\frac{\gamma-1}{\gamma}} + \lambda^{\frac{1}{\gamma}}(s)\eta^{\frac{\gamma-1}{\gamma}}(s)\Big\} \\ \quad + (1-\gamma)\Big[\dfrac{1}{2}\dfrac{\mu^2}{\sigma^2\gamma} + \dfrac{Q_1(s,t)}{(1-\gamma)Q(s,t)} + \eta(s) + r\Big]\hat{A}^t(s) = 0, \\ \hat{A}^t(T) = 1. \end{cases}$$

Solving the ODEs above, we have

$$\begin{cases}\hat{D}(s)=\displaystyle\int_s^T i(u)e^{-\int_s^u[r+\eta(\xi)]d\xi}du,\\ \hat{B}(s)=-\displaystyle\int_s^T e^{-\int_s^u[r+\eta(\xi)-b+a]d\xi}du,\\ \hat{A}^t(s)=\Bigg[e^{\int_s^T\frac{1-\gamma}{\gamma}\left[\frac{1}{2}\frac{\mu^2}{\sigma^2\gamma}+\frac{\phi'(u-t)}{(1-\gamma)\phi(u-t)}-\frac{\lambda(u)}{1-\gamma}+\eta(u)+r\right]du}+\\ \displaystyle\int_s^T\left\{[1-b\hat{B}(u)]^{\frac{\gamma-1}{\gamma}}+\lambda^{\frac{1}{\gamma}}(u)\eta^{\frac{\gamma-1}{\gamma}}(u)\right\}e^{\int_s^u\frac{1-\gamma}{\gamma}\left[\frac{1}{2}\frac{\mu^2}{\sigma^2\gamma}+\frac{\phi'(\xi-t)}{(1-\gamma)\phi(\xi-t)}-\frac{\lambda(\xi)}{1-\gamma}+\eta(\xi)+r\right]d\xi}du\Bigg]^{\gamma}.\end{cases}$$

Thus, the naive strategy would be

$$\begin{cases}\hat{c}(t,x,h)=[\hat{A}^t(t)-b\hat{A}^t(t)\hat{B}(t)]^{-\frac{1}{\gamma}}[x+\hat{B}(t)h+\hat{D}(t)]+h,\\ \hat{p}(t,x,h)=(\dfrac{\hat{A}^t(t)}{\lambda(t)})^{-\frac{1}{\gamma}}\eta^{\frac{\gamma-1}{\gamma}}(t)[x+\hat{B}(t)h+\hat{D}(t)]-x\eta(t),\\ \hat{\theta}(t,x,h)=\dfrac{\mu}{\sigma^2\gamma}[x+\hat{B}(t)h+\hat{D}(t)].\end{cases}\tag{8}$$

4.2. **Equilibrium strategy for CRRA case.** According to Theorem 3.3, we assume an equilibrium value function of the form $V(t,x,h)=\bar{A}(t)U[x+\bar{B}(t)h+\bar{D}(t)]$. Then, we will have

$$\begin{cases}\bar{c}(t,x,h)=I[V_x-bV_h]+h=[\bar{A}(t)-b\bar{A}(t)\bar{B}(t)]^{-\frac{1}{\gamma}}[x+\bar{B}(t)h+\bar{D}(t)]+h,\\ \bar{p}(t,x,h)=\eta(t)\left[I(V_x\dfrac{\eta(t)}{\lambda(t)})-x\right]=\left(\dfrac{\bar{A}(t)}{\lambda(t)}\right)^{-\frac{1}{\gamma}}\eta^{\frac{\gamma-1}{\gamma}}(t)[x+\bar{B}(t)h+\bar{D}(t)]-x\eta(t),\\ \bar{\theta}(t,x,h)=-\dfrac{\mu V_x}{\sigma^2V_{xx}}=\dfrac{\mu}{\sigma^2\gamma}[x+\bar{B}(t)h+\bar{D}(t)].\end{cases}\tag{9}$$

Under the equilibrium strategy, the equilibrium wealth $\bar{X}(t)$ satisfies

$$\begin{aligned}d\bar{X}(t)=&\Bigg[[r+\eta(t)]\bar{X}(t)+\left(\frac{\mu^2}{\sigma^2\gamma}-[\bar{A}(t)-b\bar{A}(t)\bar{B}(t)]^{-\frac{1}{\gamma}}-(\frac{\bar{A}(t)}{\lambda(t)})^{-\frac{1}{\gamma}}\eta^{\frac{\gamma-1}{\gamma}}(t)\right)[\bar{X}(t)\\&+\bar{B}(t)\bar{H}(t)+\bar{D}(t)]+i(s)-\bar{H}(t)\Bigg]dt+\frac{\mu}{\sigma\gamma}[\bar{X}(t)+\bar{B}(t)\bar{H}(t)+\bar{D}(t)]dW(t).\end{aligned}$$

In addition, the equilibrium habit under the equilibrium strategy would follow

$$d\bar{H}(t)=b[\bar{A}(t)-b\bar{A}(t)\bar{B}(t)]^{-\frac{1}{\gamma}}[\bar{X}(t)+\bar{B}(t)\bar{H}(t)+\bar{D}(t)]dt+(b-a)\bar{H}(s)dt.$$

Let $Y(t)=\bar{X}(t)+\bar{B}(t)\bar{H}(t)+\bar{D}(t)$. Then, $Y(\cdot)$ satisfies

$$\begin{aligned}dY(t)=&\,d\bar{X}(t)+\bar{B}(t)d\bar{H}(t)+\bar{B}'(t)\bar{H}(t)ds+\bar{D}'(t)ds\\=&\,Y(t)\left\{\left[r+\eta(s)+\frac{\mu^2}{\sigma^2\gamma}-\bar{A}^{-\frac{1}{\gamma}}(s)\left\{[1-b\bar{B}(s)]^{\frac{\gamma-1}{\gamma}}+\lambda^{\frac{1}{\gamma}}(s)\eta^{\frac{\gamma-1}{\gamma}}(s)\right\}\right]ds+\frac{\mu}{\sigma\gamma}dW(t)\right\}\\&+\left[-(r+\eta(t))\bar{D}(t)+i(t)+\bar{D}'(t)\right]ds+\bar{H}(t)[\bar{B}'(t)-(r+\eta(t)-b+a)\bar{B}(t)-1]ds.\end{aligned}\tag{10}$$

Here, we choose $\bar{D}(\cdot)$,$\bar{B}(\cdot)$ satisfying

$$\begin{cases}\bar{D}'(t)-[r+\eta(t)]\bar{D}(t)+i(t)=0,\\ \bar{D}(T)=0,\end{cases}$$

and

$$\begin{cases}\bar{B}'(t)-[r+\eta(t)-b+a]\bar{B}(t)-1=0,\\ \bar{B}(T)=0.\end{cases}$$

From the equations above, we know that $\bar{D}(t)$ and $\bar{B}(t)$ are exactly the same as $\hat{D}(t)$ and $\hat{B}(t)$ in the naive strategy.

Define $\beta(s)=r+\eta(s)+\frac{\mu^2}{\sigma^2\gamma}-\bar{A}^{-\frac{1}{\gamma}}(s)\Big\{[1-b\bar{B}(s)]^{\frac{\gamma-1}{\gamma}}+\lambda^{\frac{1}{\gamma}}(s)\eta^{\frac{\gamma-1}{\gamma}}(s)\Big\}$. Given that $\bar{X}(t)=x$ and $\bar{H}(t)=h$, let $Y(t)=x+\bar{B}(t)h+\bar{D}(t)=y$, from (10), we obtain

$$Y(s)=ye^{\int_t^s[\beta(u)-\frac{\mu^2}{2\sigma^2\gamma^2}]du+\int_t^s\frac{\mu}{\sigma\gamma}dW(u)},\quad s>t. \tag{11}$$

As a result, $\bar{X}(T)=Y(T)=ye^{\int_t^T[\beta(u)-\frac{\mu^2}{2\sigma^2\gamma^2}]du+\int_t^T\frac{\mu}{\sigma\gamma}dW(u)}$.

By substituting (9) and (11) into the extended HJB equation (7), we can obtain the ODE of $\bar{A}(\cdot)$ as

$$\begin{cases}\bar{A}'(t)+\bar{A}^{\frac{\gamma-1}{\gamma}}(t)\big\{[1-b\bar{B}(t)]^{\frac{\gamma-1}{\gamma}}+\lambda^{\frac{1}{\gamma}}(t)\eta^{\frac{\gamma-1}{\gamma}}(t)\big\}+(1-\gamma)[\beta(t)-\dfrac{1}{2}\dfrac{\mu^2}{\sigma^2\gamma}]\bar{A}(t)=\\ \quad\displaystyle\int_t^T\frac{\partial Q(s,t)}{\partial t}\bar{A}^{\frac{\gamma-1}{\gamma}}(s)e^{\int_t^s(1-\gamma)[\beta(\xi)-\frac{1}{2}\frac{\mu^2}{\sigma^2\gamma}]d\xi}\big\{[1-b\bar{B}(s)]^{\frac{\gamma-1}{\gamma}}+\lambda^{\frac{1}{\gamma}}(s)\eta^{\frac{\gamma-1}{\gamma}}(s)\big\}ds+\\ \quad\dfrac{\partial Q(T,t)}{\partial t}e^{\int_t^T(1-\gamma)[\beta(\xi)-\frac{1}{2}\frac{\mu^2}{\sigma^2\gamma}]d\xi},\\ \bar{A}(T)=1,\end{cases} \tag{12}$$

which can be rewritten as

$$\begin{aligned}\bar{A}(t)=&\int_t^T Q(s,t)e^{\int_t^s(1-\gamma)[\beta(\xi)-\frac{1}{2}\frac{\mu^2}{\sigma^2\gamma}]d\xi}\bar{A}^{\frac{\gamma-1}{\gamma}}(s)\big\{[1-b\bar{B}(s)]^{\frac{\gamma-1}{\gamma}}+\lambda^{\frac{1}{\gamma}}(s)\eta^{\frac{\gamma-1}{\gamma}}(s)\big\}ds\\ &+Q(T,t)e^{\int_t^T(1-\gamma)[\beta(\xi)-\frac{1}{2}\frac{\mu^2}{\sigma^2\gamma}]d\xi}.\end{aligned} \tag{13}$$

By the advantage of the following theorem, we can show that the solution $\bar{A}(\cdot)$ of (13) is unique.

Theorem 4.1. *Assume that $Z_t(s)\in C[t,T]$ and $f\in C([0,T])$ are both positive, then there exists a unique positive solution of $A(t)\in C([0,T])$ such that*

$$A(t)=\int_t^T Z_t(s)e^{\int_t^s(1-\gamma)[\beta(\xi)-\frac{1}{2}\frac{\mu^2}{\sigma^2\gamma}]d\xi}A^{\frac{\gamma-1}{\gamma}}(s)ds+f(t)e^{\int_t^T(1-\gamma)[\beta(\xi)-\frac{1}{2}\frac{\mu^2}{\sigma^2\gamma}]d\xi}.$$

Proof. Let

$$\tilde{A}(t)=A(t)e^{-\int_t^T(1-\gamma)[\beta(\xi)-\frac{1}{2}\frac{\mu^2}{\sigma^2\gamma}]d\xi}=\int_t^T Z_t(s)A^{-\frac{1}{\gamma}}(s)\tilde{A}(s)ds+f(t)$$

and

$$M=\max_{0\le t\le s\le T}Z_t(s)A^{-\frac{1}{\gamma}}(s).$$

Define $\mathcal{T}\tilde{A}(t) = \int_t^T Z_t(s)A^{-\frac{1}{\gamma}}(s)\tilde{A}(s)ds + f(t)$; then,

$$\begin{aligned}|\mathcal{T}\tilde{A}_1(t) - \mathcal{T}\tilde{A}_2(t)| &= |\int_t^T Z_t(s)A^{-\frac{1}{\gamma}}(s)[\tilde{A}_1(s) - \tilde{A}_2(s)]ds| \\ &\leq \int_t^T |Z_t(s)A^{-\frac{1}{\gamma}}(s)||\tilde{A}_1(s) - \tilde{A}_2(s)|ds \\ &\leq M\left\|\tilde{A}_1 - \tilde{A}_2\right\|_\infty (T-t).\end{aligned}$$

Furthermore,

$$\begin{aligned}|\mathcal{T}^2\tilde{A}_1(t) - \mathcal{T}^2\tilde{A}_2(t)| &= |\int_t^T Z_t(s)A^{-\frac{1}{\gamma}}(s)[T\tilde{A}_1(s) - T\tilde{A}_2(s)]ds| \\ &\leq \int_t^T |Z_t(s)A^{-\frac{1}{\gamma}}(s)||T\tilde{A}_1(s) - T\tilde{A}_2(s)|ds \\ &\leq M^2\left\|\tilde{A}_1 - \tilde{A}_2\right\|_\infty \int_t^T (T-s)ds \\ &= M^2\left\|\tilde{A}_1 - \tilde{A}_2\right\|_\infty \frac{(T-t)^2}{2}.\end{aligned}$$

By mathematical induction, we can conclude that

$$|\mathcal{T}^n\tilde{A}_1(t) - \mathcal{T}^n\tilde{A}_2(t)| = \frac{[M(T-t)]^n}{n!}\left\|\tilde{A}_1 - \tilde{A}_2\right\|_\infty.$$

Thus, there exists an N such that $\frac{[M(T-t)]^N}{N!} < 1$. According to the contracting mapping principle, we know that $\mathcal{T}^N$ is a contracting mapping. Therefore, there exists a unique $\tilde{A}$ that satisfies $\mathcal{T}\tilde{A} = \tilde{A}$, which implies that there exists a unique solution for

$$A(t) = \int_t^T Z_t(s)e^{\int_t^s (1-\gamma)[\beta(\xi)-\frac{1}{2}\frac{\mu^2}{\sigma^2\gamma}]d\xi}A^{\frac{\gamma-1}{\gamma}}(s)ds + f(t)e^{\int_t^T (1-\gamma)[\beta(\xi)-\frac{1}{2}\frac{\mu^2}{\sigma^2\gamma}]d\xi}$$

□

Now, let $Z_t(s) = Q(s,t)\{[1-b\bar{B}(s)]^{\frac{\gamma-1}{\gamma}} + \lambda^{\frac{1}{\gamma}}(s)\eta^{\frac{\gamma-1}{\gamma}}(s)\}$ and $f(t) = Q(T,t)$ in the theorem above, we can obtain the uniqueness of $\bar{A}(t)$ in (13).

Here, we obtain the equilibrium strategy of the individual at time t, which is

$$\begin{cases}\bar{c}(t,x,h) = [\bar{A}(t) - b\bar{A}(t)\bar{B}(t)]^{-\frac{1}{\gamma}}[x + \bar{B}(t)h + \bar{D}(t)] + h, \\ \bar{p}(t,x,h) = (\dfrac{\bar{A}(t)}{\lambda(t)})^{-\frac{1}{\gamma}}\eta^{\frac{\gamma-1}{\gamma}}(t)[x + \bar{B}(t)h + \bar{D}(t)] - x\eta(t), \\ \bar{\theta}(t,x,h) = \dfrac{\mu}{\sigma^2\gamma}[x + \bar{B}(t)h + \bar{D}(t)].\end{cases} \tag{14}$$

4.3. **Difference between the naive strategy and equilibrium strategy.** Given $X(t) = x$ and $H(t) = h$, we call $Y(t) = x + \hat{B}(t)h + \hat{D}(t) = x + \bar{B}(t)h + \bar{D}(t)$ the available asset because both the naive strategy and the equilibrium strategy are based on $Y(t)$. $Y(t)$ consists of three parts:
(1) the wealth of the individual at time t, x;
(2) the accumulated value of the discounted income from time t to T, $\hat{D}(t)$ or $\bar{D}(t)$, which is also called individual human capital in many studies; and

(3) the debt ensuring that the individual can keep his/her habit after time t, $\hat{B}(t)h$ or $\bar{B}(t)h$.

Compared with the result in Richard[23], when the individual makes his/her decision, he/she will not only consider his/her wealth and the future income but also the reserve fund for maintaining future habit formation. From (11), we know that $Y(t) > 0$ for both the naive strategy and the equilibrium strategy. In other words, $x + \bar{D}(t) > |\bar{B}(t)h|$ and $x + \hat{D}(t) > |\hat{B}(t)h|$, which means that the individual will set his/her habit within a reasonable range based on his/her income and wealth. From (8) and (14), it is obvious that $\hat{c}(t, \hat{X}(t), \hat{H}(t)) > \hat{H}(t)$ and $\bar{c}(t, \bar{X}(t), \bar{H}(t)) > \bar{H}(t)$. Therefore, habit formation serves as the standard of living for the individual. Moreover, the standard of living will increase over time if $a > b$ because $dH(t) > 0$ here. In addition, the parameter a, b in habit formation also appears in the expression $\hat{A}^t(t)$ and $\bar{A}(t)$. When we set $a = b = 0$, $\hat{A}^t(t)$ and $\bar{A}(t)$ will return to the expression without considering habit formation.

The naive strategy and the equilibrium strategy are very similar: they are both based on the available asset $Y(t)$ at time t. The consumption consists of the habit and the extra consumption, which is part of the available asset. The role of insurance here is to translate future assets into a legacy for inheritance, and the legacy at time t is $Z(t) = \hat{A}^t(t)^{-\frac{1}{\gamma}}(\frac{\lambda(t)}{\eta(t)})^{-\frac{1}{\gamma}}Y(t)$ or $Z(t) = \bar{A}^{-\frac{1}{\gamma}}(t)(\frac{\lambda(t)}{\eta(t)})^{-\frac{1}{\gamma}}Y(t)$ based on the available asset. Thus, the premium is also a function of the available asset. Because we choose a constant asset return rate here, the investment in the risky asset is a constant proportion of the available asset at all times.

However, clearly, the naive strategy is not exactly the same as the equilibrium strategy. The only difference between them is in the multiplier in the value function, which is $\hat{A}^t(t)$ in the optimal value function and $\bar{A}(t)$ in the equilibrium value function.

If we set $U(x) = \ln(x)$, we can easily obtain the expression for $\hat{A}^t(t)$ and $\bar{A}(t)$ by setting $\gamma = 1$. In fact, we have in this case

$$\hat{A}^t(t) = \int_t^T [1 + \lambda(u)] e^{\int_t^u \frac{\phi'(\xi - t)}{\phi(\xi - t)} - \lambda(\xi) d\xi} du + e^{\int_t^T \frac{\phi'(u-t)}{\phi(u-t)} - \lambda(u) du},$$

and

$$\bar{A}(t) = \int_t^T [1 + \lambda(u)] \phi(u - t) e^{-\int_t^u \lambda(\xi) d\xi} du + \phi(T - t) e^{-\int_t^T \lambda(u) du}.$$

Hence,

$$\begin{aligned}
\bar{A}(t) - \hat{A}^t(t) = & \int_t^T [1 + \lambda(u)] e^{-\int_t^u \lambda(\xi) d\xi} [\phi(u - t) - e^{\int_t^u \frac{\phi'(\xi - t)}{\phi(\xi - t)} d\xi}] du + \\
& [\phi(T - t) - e^{\int_t^T \frac{\phi'(u-t)}{\phi(u-t)}}] e^{-\int_t^T \lambda(u) du} \\
= & \int_t^T [1 + \lambda(u)] e^{-\int_t^u \lambda(\xi) d\xi} [\phi(u - t) - e^{\ln[\phi(u-t)]}] du + \\
& [\phi(T - t) - e^{\ln[\phi(T-t)]}] e^{-\int_t^T \lambda(u) du} \\
= & 0.
\end{aligned}$$

Therefore, when $U(x) = \ln(x)$, we can see that $\hat{A}^t(t) = \bar{A}(t)$, which means that the naive strategy and equilibrium strategy are the same. Furthermore, from the HJB equation (4) and extended HJB equation (7), we know that the naive strategy is exactly the same as the equilibrium strategy if the discount function is exponential.

However, expect for the case in which $U(x) = \ln(x)$ and we have an exponential discount function, the difference between $\bar{A}(t)$ and $\hat{A}^t(t)$ is quite complex. We cannot obtain an analytical solution for $\bar{A}(t)$ in the time-consistent strategy case. Hence, we need to analyse the impact of different habit formation settings and different discount functions on the difference between $\bar{A}(t)$ and $\hat{A}^t(t)$ by numerical simulation, which we do in the following section.

5. **Numerical results.** As discussed above, when we consider an individual with non-exponential discounting, the naive strategy and the equilibrium strategy are different. In addition, the difference mainly appears in the multipliers $\bar{A}(t)$ and $\hat{A}^t(t)$. We call the difference between $\hat{A}^t(t)$ and $\bar{A}(t)$ the degree of inconsistency. The closer $\hat{A}^t(t)$ and $\bar{A}(t)$ are, the more similar the two strategies are. Thus, the sophisticated individual who applies the time-consistent strategy would very likely maximize their utility. In other words, it would be less expensive for the naive individual at time t to force an individual (himself as well) to apply the strategy that he/she made at time t. We are going to analyse the impact of the discount function and habit on the degree of inconsistency using a numerical method in this section.

The basic parameters are as follows: $T = 40$, $r = 0.03$, $\alpha = 0.3$, $\sigma = 2.5$, $X(0) = 200000$, $H(0) = 0.11 * X(0)$, $\gamma = 2.2$, $a = 0.1$, $b = 0.093$, $i(t) = 50000e^{0.04t}$, $\lambda(t) = 0.001 + e^{-9.5+0.1t}$, $\eta(t) = \lambda(t)$, and $\phi(t) = (1 + 0.4t)^{log(0.9)/log(1.4)}$.

When we compare different cases, we will change the corresponding parameters.

5.1. **The impact of the discount function.** The discount function that we use here is the hyperbolic discount function. The form of the hyperbolic discount function is $\phi(t) = (1 + \rho t)^{-\frac{\beta}{\rho}} \quad \rho > 0, \beta > 0$. As previously mentioned, the hyperbolic discount function exhibits a common difference effect. It can also be described by the following equation:

$$U(x) = U(y)\phi(s) \quad \Leftrightarrow \quad U(x)\phi(t) < U(y)\phi(t + s).$$

In other words, we can rewrite it as:

$$U(x) = U(y)\phi(s) \quad \Leftrightarrow \quad U(x)\phi(t) = U(y)\phi(kt + s) \quad k > 1.$$

This means that after waiting a time delay t, to make x and y equal, the time interval should be longer than s. When we take the hyperbolic discount function and the time interval $s = 1$, we have $k = 1 + \rho > 1$. A larger k means that the individual is more sensitive to the time delay occurring early, which means that the individual is more patient in the future. When $\rho \to 0$, $\phi(t) \to e^{-\beta t}$. Thus, for an exponential discount function, $k = 1$.

Then, we take three cases, $k = 1$, $k = 1.1$, and $k = 1.2$, to calculate $\bar{A}(t)$ and $\hat{A}^t(t)$. The results are shown in figure 1.

As we can see, $\bar{A}(t)$ coincides with $\hat{A}^t(t)$ in the case of $k = 1$, as we have shown before. From figure 1, we can see that $\bar{A}(t)$ is greater than $\hat{A}^t(t)$ when $k > 1$. Both $\bar{A}(t)$ and $\hat{A}^t(t)$ increase with increasing k. From the naive strategy and equilibrium strategy, i.e., (8) and (14), consumption and insurance are decreased with increasing multiplier. Because the multiplier of the equilibrium strategy is higher than the naive strategy, the sophisticated individual will consume less and purchase less life insurance. Applying the equilibrium strategy means that the individual needs to make an agreement with their future self; thus, he/she needs to give up some of his/her utility to the future. Therefore, it is reasonable that the sophisticated person

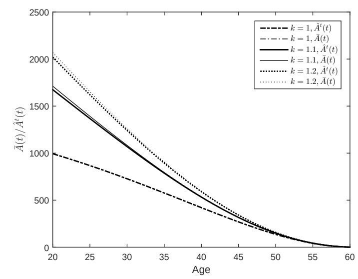


FIGURE 1. The impact of k on $\bar{A}$ and $\hat{A}$

needs to spend less on consumption and insurance than does the naive person. However, if the sophisticated individual and the naive individual have the same initial wealth and habits, the sophisticated individual may be richer when he/she is older because he/she saved more in the past; then, he/she will consume more later.

We can also find that the degree of inconsistency decreases over time. As the individual becomes older, the difference between the naive strategy and the equilibrium strategy tends to decrease. This result agrees with intuition. Because we utilize the hyperbolic discount function here, the discount rate $\frac{\phi'(t)}{\phi(t)}$ decreases with time. The naive individual does not consider the change in the discount rate. However, he/she would actually be more patient than he/she thought in the future. That is why the inconsistency appears. When he/she is young, the change in the discount rate in the future is huge, whereas when he/she is older, the change is smaller. Thus, the degree of inconsistency decreases over time.

Figure 2 shows the degree of inconsistency for three types of individuals. The image clearly shows that as k increases, the degree of inconsistency increases because a larger k indicates the more patient that he/she will be in the future, and the change in the discount rate between now and the future would also be larger. Thus, an increase in the degree of inconsistency is rational.

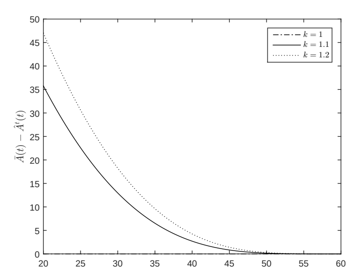


FIGURE 2. The impact of k in the degree of inconsistency

5.2. **The impact of habit formation.** There are two parameters in the habit formation function: a and b. From (1), we know that a larger b or a smaller a results in a higher habit level. We say that an individual with a higher b and a lower a has a greater tendency to form a consumption habit. In this section, we want to see how the two parameters affect $\bar{A}(t)$ and $\hat{A}^t(t)$. We take three types of habit formation functions to compare the difference between them. The first type is a benchmark, with $a = 0.1$ and $b = 0.1$. For the second type, we increase a to 0.11 but do not change b. For the third type, we increase b to 0.11 and keep a unchanged. Then, we calculate $\bar{A}(t)$ and $\hat{A}^t(t)$ for these three types of individuals. The results are given in figure 3.

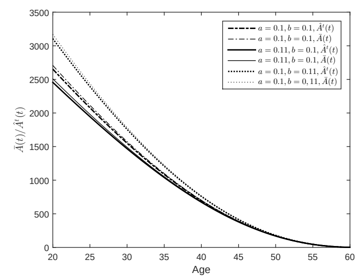


FIGURE 3. The impact of habit on $\bar{A}$ and $\hat{A}$

We can see that $\bar{A}(t)$ is higher than $\hat{A}^t(t)$ for all three types of individuals all the time, and the difference decreases with time. This coincides with what we found previously. Moreover, both $\bar{A}(t)$ and $\hat{A}^t(t)$ increase as a decreases and b increases. This is highly similar to the situation where k is increased. The degree of inconsistency for the three types of individuals is shown in figure 4. We can see that an individual with a greater tendency to form a habit will see a larger difference between the naive strategy and the equilibrium strategy. This is also the same as when k increases. An intuitive explanation is that an individual with a higher k and higher habit level is more concerned about the future.

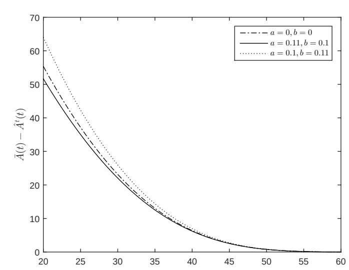
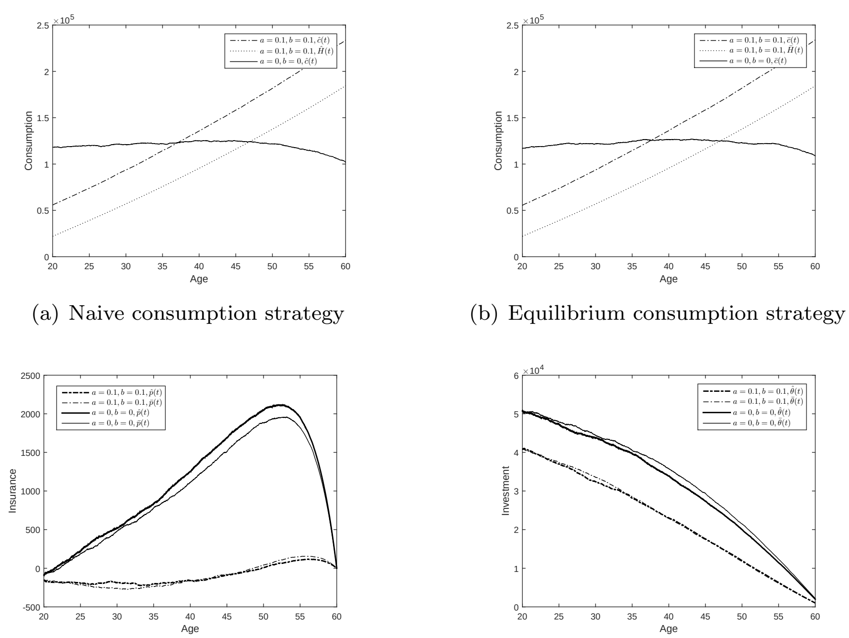


FIGURE 4. The impact of habit on the degree of inconsistency

From (8) and (14), we can know that individual with the same available asset who have a greater tendency to form habits will spend less of their available asset for consumption and life insurance in both naive and equilibrium strategy because both $\bar{A}(t)$ and $\hat{A}^t(t)$ is higher. The income of the individual is deterministic in our model; thus, the resources of the individual are limited. Without habits, an individual has no limit on consumption during their entire life. After considering the habit, the individual should take the habit as the lowest standard line for consumption. An individual with a higher tendency to form a habit will have a higher standard of living in the future, which means that he/she needs to commit more resources to the future; thus, he/she will spend less now and save more for the future.

In figure 5, we show the strategy in different ages for an individual with the first type habit formation($a = 0.1, b = 0.1$) and the individual without habit formation($a = 0, b = 0, H(0) = 0$). It illustrates our analysis above. In figure 5(a) and figure 5(b), for both naive strategy and equilibrium strategy, the consumption ($\hat{c}(t)$ or $\bar{c}(t)$) need to be higher than the corresponding strategies with habit formation ($\hat{H}(t)$ or $\bar{H}(t)$). The consumption for individual with habit formation is lower than the case without habit formation in the early period, and higher in later period. In figure 5(c), the individual with habit will buy less life insurance compare with the individual without habit in both naive strategy and stochastic strategy. Figure 5(d) shows that the individual with habit will be more prudent in risk investment and have more precautionary saving for future, which is also shown by Díaz et al[10].



(a) Naive consumption strategy

(b) Equilibrium consumption strategy

(c) Insurance strategy

(d) Investment strategy

Figure 5. Strategy for individual with or without habit

5.3. **Habit formation for naive or sophisticated individual.** Figure 6 presents the consumption and habit formation for a naive person and for a sophisticated person with the basic parameters.

As we found previously, the naive person will consume more if he/she has the same wealth and habit as the sophisticated person. In this example, the naive person and the sophisticated person have the same initial wealth and habit; thus, the naive person will consume more initially. In addition, because the habit consists of past consumption, the habit of the naive person will be larger than that of the

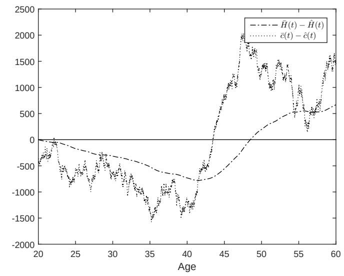


FIGURE 6. Example of $\bar{H}(t) - \hat{H}(t)$ and $\bar{c}(t) - \hat{c}(t)$

sophisticated person. Figure 6 clearly shows that $\bar{c}(t) - \hat{c}(t) < 0$ before age 45 and $\bar{H}(t) - \hat{H}(t) < 0$ before age 50. Because the sophisticated person saves more at a younger age, he/she will be richer at later times and consume more. Then, the habit of the sophisticated person will be larger than that of the naive person at later times. Finally, the time at which $\bar{H}(t)$ surpasses $\hat{H}(t)$ is later than the time at which $\bar{c}(t)$ surpasses $\hat{c}(t)$.

6. **Conclusions.** In this article, we consider the effect of both habit formation and the general discount function in the consumption-investment-insurance problem. If the discount function is exponential, the optimal strategy is time-consistent, which means that the individual follows a strategy that continuously maximizes intertemporal utility. However, when we replace the exponential discount function with a non-exponential discount function, the optimal strategy becomes time-inconsistent. We consider two types of individuals: a naive individual and a sophisticated individual. Both the time-inconsistent strategy and the time-consistent strategy for the individual are derived. Instead of using the HJB equation to obtain the naive strategy, we derive the extended HJB equation to obtain the equilibrium strategy.

We also obtain the analytical solution of both the naive strategy for the naive person and the equilibrium strategy for the sophisticated person in the CRRA case. The two strategies are very similar and are both based on the available asset $Y(t)$. There is only one difference, which appears in the multiplier: $\hat{A}^t(t)$ for the naive strategy and $\bar{A}(t)$ for the equilibrium strategy. Because they are too complex to analyse in mathematical form, we use a numerical simulation for analysis.

We use the hyperbolic discount function $\phi(t) = (1+\rho t)^{-\frac{\beta}{\rho}}$ in the numerical simulation to describe the common difference effect. In addition, we use $k = 1+\rho > 1$ to show the attitude for the future of the individual. A higher k means that the individual is more patient in the future. From the numerical results, we found that the individual will spend less for consumption and insurance if he/she applies the time-consistent strategy rather than the time-inconsistent strategy because the individual needs to give up some utility to make an agreement with his/her future self. In addition, over time, the degree of inconsistency decreases. Moreover, we found that the increase in k has the same effect as an increase in the habit level. Specifically, if the individual is more patient in the future or has a higher tendency to form a habit, he/she will spend less for consumption and insurance at a younger age compared to in later years. We can say that this type of individual is more concerned about the future. We also determine the consumption and habit formation for the naive person and for the sophisticated person with the same parameter. In addition, our example agrees with the analysis that the sophisticated person would spend less and save more in early times and be richer and consume more in later times. Thus, the habit formation of the sophisticated person is smaller in early times than the naive person and larger when they are older.

7. **Appendix.** **Proof.** The proof of this theorem can be done in two steps.
Step 1: we show that V is the value function corresponding to $\bar{\mathbf{u}}$.
By Dynkin's formulation, we have

$$\begin{aligned} V(t,x,h) =& E_{t,x,h}[V(T,\bar{X}(T),\bar{H}(T))] - E_{t,x,h}\left[\int_t^T A^{\bar{\mathbf{u}}}V(t,\bar{X}(s),\bar{H}(s))ds\right] \\ =& E_{t,x,h}[U(\bar{X}(T))] - E_{t,x,h}\left[\int_t^T A^{\bar{\mathbf{u}}}V(t,\bar{X}(s),\bar{H}(s))ds\right]. \end{aligned} \tag{15}$$

It follows from (7) that

$$\begin{aligned} A^{\bar{\mathbf{u}}}V(t,x,h) =& E_{t,x,h}\Bigg[\int_t^T \frac{\partial Q(s,t)}{\partial t}U(\bar{c}(s,\bar{X}(s),\bar{H}(s)) - \bar{H}(s)) + \\ & \frac{\partial Q(s,t)}{\partial t}\lambda(s)U\bigg(\bar{X}(s) + \frac{\bar{p}(s,\bar{X}(s),\bar{H}(s))}{\eta(s)}\bigg)ds + \frac{\partial Q(T,t)}{\partial t}U(\bar{X}(T))\Bigg] - \\ & U(\bar{c}(t,x,h) - h) - \lambda(t)U\bigg(x + \frac{\bar{p}(t,x,h)}{\eta(t)}\bigg). \end{aligned}$$

Thus,

$$\begin{aligned} & E_{t,x,h}\left[\int_t^T A^{\bar{\mathbf{u}}}V(s,\bar{X}(s),\bar{H}(s))ds\right] \\ &= E_{t,x,h}\Bigg[\int_t^T \int_s^T \frac{\partial Q(z,s)}{\partial s}U(\bar{c}(z,\bar{X}(z),\bar{H}(z)) - \bar{H}(z)) + \\ & \frac{\partial Q(z,s)}{\partial s}\lambda(z)U\bigg(\bar{X}(z) + \frac{\bar{p}(z,\bar{X}(z),\bar{H}(z))}{\eta(z)}\bigg)dzds + \\ & \int_t^T \frac{\partial Q(T,s)}{\partial s}U(\bar{X}(T)) - U(\bar{c}(s,\bar{X}(s),\bar{H}(s)) - \bar{H}(s)) - \\ & \lambda(s)U\bigg(\bar{X}(s) + \frac{\bar{p}(s,\bar{X}(s),\bar{H}(s))}{\eta(s)}\bigg)ds\Bigg] \\ &= E_{t,x,h}\Bigg[\int_t^T \int_z^t \frac{\partial Q(z,s)}{\partial s}U(\bar{c}(z,\bar{X}(z),\bar{H}(z)) - \bar{H}(z)) + \\ & \frac{\partial Q(z,s)}{\partial s}\lambda(z)U\bigg(\bar{X}(z) + \frac{\bar{p}(z,\bar{X}(z),\bar{H}(z))}{\eta(z)}\bigg)dsdz + \int_t^T \frac{\partial Q(T,s)}{\partial s}U(\bar{X}(T))ds - \\ & \int_t^T U(\bar{c}(s,\bar{X}(s),\bar{H}(s)) - \bar{H}(s)) + \lambda(s)U\bigg(\bar{X}(s) + \frac{\bar{p}(s,\bar{X}(s),\bar{H}(s))}{\eta(s)}\bigg)ds\Bigg] \\ &= E_{t,x,h}\Bigg[\int_t^T [1 - Q(z,t)]U(\bar{c}(z,\bar{X}(z),\bar{H}(z)) - \bar{H}(z)) + \\ & [1 - Q(z,t)]\lambda(z)U\bigg(\bar{X}(z) + \frac{\bar{p}(z,\bar{X}(z),\bar{H}(z))}{\eta(z)}\bigg)dz + [1 - Q(T,t)]U(\bar{X}(T)) - \end{aligned}$$

$$\int_t^T U(\bar{c}(s,\bar{X}(s),\bar{H}(s))-\bar{H}(s))+\lambda(s)U\bigg(\bar{X}(s)+\frac{\bar{p}(s,\bar{X}(s),\bar{H}(s))}{\eta(s)}\bigg)ds\bigg]$$
$$=E_{t,x,h}\bigg[\int_t^T[1-Q(z,t)]U(\bar{c}(z,\bar{X}(z),\bar{H}(z))-\bar{H}(z))+$$
$$[1-Q(z,t)]\lambda(z)U\bigg(\bar{X}(z)+\frac{\bar{p}(z,\bar{X}(z),\bar{H}(z))}{\eta(z)}\bigg)dz+[1-Q(T,t)]U(\bar{X}(T))-$$
$$\int_t^T U(\bar{c}(s,\bar{X}(s),\bar{H}(s))-\bar{H}(s))+\lambda(s)U\bigg(\bar{X}(s)+\frac{\bar{p}(s,\bar{X}(s),\bar{H}(s))}{\eta(s)}\bigg)ds\bigg]$$
$$=-J(t,x,h;\bar{\mathbf{u}})+U(\bar{X}(T)).$$

That is,

$$E_{t,x,h}\bigg[\int_t^T A^{\bar{\mathbf{u}}}V(s,\bar{X}(s),\bar{H}(s))ds\bigg]=-J(t,x,h;\bar{\mathbf{u}})+U(\bar{X}(T)).$$

Substituting $E_{t,x,h}\bigg[\int_t^T A^{\bar{\mathbf{u}}}V(s,\bar{X}(s),\bar{H}(s))ds\bigg]$ into (15), we obtain

$$V(t,x,h)=J(t,x,h;\bar{\mathbf{u}}).$$

Step 2: We show that $\bar{\mathbf{u}}$ is the equilibrium control strategy which is defined in Definition 3.1.

Let $\mathbf{u}_\varepsilon(s,y,k)=\begin{cases}\bar{\mathbf{u}}(s,y,k), & t+\varepsilon<s\le T\\ u(s,y,k), & t\le s\le t+\varepsilon.\end{cases}$, where u is any strategy that makes $\mathbf{u}_\varepsilon$ an admissible strategy.

To simplify the expression, We set $X^{\mathbf{u}_\varepsilon}_{t,x,h}(s)$ and $H^{\mathbf{u}_\varepsilon}_{t,x,h}(s)$ as the corresponding processes under the control strategy $\mathbf{u}_\varepsilon$ given $X(t)=x$ and $H(t)=h$.

Let $\{c(s),\theta(s),p(s)\}=u(s,X^{\mathbf{u}_\varepsilon}_{t,x,h}(s),H^{\mathbf{u}_\varepsilon}_{t,x,h}(s))$ for $t\le s<t+\varepsilon$.While in $t+\varepsilon<s\le T$, we have $\{\bar{c}(s),\bar{\theta}(s),\bar{p}(s)\}=\bar{\mathbf{u}}(s,X^{\mathbf{u}_\varepsilon}_{t,x,h}(s),H^{\mathbf{u}_\varepsilon}_{t,x,h}(s))$. Set $X_\varepsilon=X^{\mathbf{u}_\varepsilon}_{t,x,h}(t+\varepsilon)$ and $H_\varepsilon=X^{\mathbf{u}_\varepsilon}_{t,x,h}(t+\varepsilon)$.

Moreover, $V(t,x,h)=J(t,x,h;\bar{\mathbf{u}})$.

$$\begin{aligned}
&J(t,x,h;\mathbf{u}_\varepsilon)\\
=&E_{t,x,h}\bigg[\int_t^{t+\varepsilon}Q(s,t)U(c(s)-H^{\mathbf{u}_\varepsilon}_{t,x,h}(s))ds\\
&+\int_t^{t+\varepsilon}\lambda(s)Q(s,t)U\bigg(X^{\mathbf{u}_\varepsilon}_{t,x,h}(s)+\frac{p(s)}{\eta(s)}\bigg)ds\\
&+\int_{t+\varepsilon}^T Q(s,t)U(\bar{c}(s)-H^{\mathbf{u}_\varepsilon}_{t,x,h}(s))ds+\int_{t+\varepsilon}^T\lambda(s)Q(s,t)U\bigg(X^{\mathbf{u}_\varepsilon}_{t,x,h}(s)+\frac{\bar{p}(s)}{\eta(s)}\bigg)ds\\
&+Q(T,t)U(X^{\mathbf{u}_\varepsilon}_{t,x,h}(T))\bigg]\\
=&[U(c(t)-h)+\lambda(t)U\big(x+\frac{p(t)}{\eta(t)}\big)]\varepsilon+E_{t,x,h}\bigg[J(t+\varepsilon,X_\varepsilon,H_\varepsilon;\bar{\mathbf{u}})\\
&+E_{t+\varepsilon,X_\varepsilon,H_\varepsilon}\big[\int_{t+\varepsilon}^T(Q(s,t)-Q(s,t+\varepsilon))U(\bar{c}(s)-H^{\bar{\mathbf{u}}}_{t+\varepsilon,X_\varepsilon,H_\varepsilon}(s))ds\\
&+\int_{t+\varepsilon}^T\lambda(s)(Q(s,t)-Q(s,t+\varepsilon))U\bigg(X^{\bar{\mathbf{u}}}_{t+\varepsilon,X_\varepsilon,H_\varepsilon}(s)+\frac{\bar{p}(s)}{\eta(s)}\bigg)ds+
\end{aligned}$$

$$
\begin{aligned}
&(Q(T,t)-Q(T,t+\varepsilon))U(X^{\bar{\mathbf{u}}}_{t+\varepsilon,X_\varepsilon,H_\varepsilon}(T))\Big]\Bigg]\\
=&[U(c(t)-h)+\lambda(t)U\big(x+\frac{p(t)}{\eta(t)}\big)]\varepsilon+E_{t,x,h}\Bigg[V(t+\varepsilon,X_\varepsilon,H_\varepsilon)\\
&+E_{t+\varepsilon,X_\varepsilon,H_\varepsilon}\Big[\int_{t+\varepsilon}^{T}(Q(s,t)-Q(s,t+\varepsilon))U(\bar{c}(s)-H^{\bar{\mathbf{u}}}_{t+\varepsilon,X_\varepsilon,H_\varepsilon}(s))ds+\\
&\int_{t+\varepsilon}^{T}\lambda(s)(Q(s,t)-Q(s,t+\varepsilon))U\Bigg(X^{\bar{\mathbf{u}}}_{t+\varepsilon,X_\varepsilon,H_\varepsilon}(s)+\frac{\bar{p}(s)}{\eta(s)}\Bigg)ds+\\
&(Q(T,t)-Q(T,t+\varepsilon))U(X^{\bar{\mathbf{u}}}_{t+\varepsilon,X_\varepsilon,H_\varepsilon}(T))\Big]\Bigg].
\end{aligned}
$$

Hence, we will have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{J(t,x,h;\mathbf{u}_\varepsilon)-V(t,x,h)}{\varepsilon}\\
=&U(c(t)-h)+\lambda(t)U(x+\frac{p(t)}{\eta(t)})+\lim_{\varepsilon\to 0}\frac{E_{t,x,h}[V(t+\varepsilon,X_\varepsilon,H_\varepsilon)]-V(t,x,h)}{\varepsilon}\\
&+\lim_{\varepsilon\to 0}\frac{E_{t,x,h}E_{t+\varepsilon,X_\varepsilon,H_\varepsilon}[\int_{t+\varepsilon}^{T}(Q(s,t)-Q(s,t+\varepsilon))U(\bar{c}(s)-H^{\bar{\mathbf{u}}}_{t+\varepsilon,X_\varepsilon,H_\varepsilon}(s))ds]}{\varepsilon}\\
&+\lim_{\varepsilon\to 0}\frac{E_{t,x,h}E_{t+\varepsilon,X_\varepsilon,H_\varepsilon}[\int_{t+\varepsilon}^{T}\lambda(s)(Q(s,t)-Q(s,t+\varepsilon))U(X^{\bar{\mathbf{u}}}_{t+\varepsilon,X_\varepsilon,H_\varepsilon}(s)+\frac{\bar{p}(s)}{\eta(s)})ds]}{\varepsilon}\\
&+\lim_{\varepsilon\to 0}\frac{E_{t,x,h}E_{t+\varepsilon,X_\varepsilon,H_\varepsilon}[(Q(T,t)-Q(T,t+\varepsilon))U(X_{t+\varepsilon,X^{\mathbf{u}_\varepsilon}_{t,x,h},H^{\mathbf{u}_\varepsilon}_{t,x,h}}(T))]}{\varepsilon}\\
=&U(c(t)-h)+\lambda(t)U(x+\frac{p(t)}{\eta(t)})+A^{u}_{t,x,h}V(t,x,h)-\\
&E_{t,x,h}[\int_{t}^{T}\frac{\partial Q(s,t)}{\partial t}U(\bar{c}(s)-\bar{H}(s))ds+\int_{t}^{T}\lambda(s)\frac{\partial Q(s,t)}{\partial t}U(\bar{X}(s)+\frac{\bar{p}(s)}{\eta(s)})ds\\
&+\frac{\partial Q(T,t)}{\partial t}U(\bar{X}(T))].
\end{aligned}
$$

From equation (7), we know that $\lim_{\varepsilon\to 0}\frac{J(t,x,h;\mathbf{u}_\varepsilon)-J(t,x,h;\bar{\mathbf{u}})}{\varepsilon}\leq 0$. Therefore, $\bar{\mathbf{u}}$ satisfies the Definition3.1, which is an equilibrium strategy, and $V(t,x,h)=J(t,x,h;\bar{\mathbf{u}})$ is the equilibrium value function.

□

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