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An Extension of the Christofides Heuristic for the Generalized Multiple Depot Multiple Traveling Salesmen Problem

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Abstract

We study a generalization of the classical traveling salesman problem, where multiple salesmen are positioned at different depots, of which only a limited number (k) can be selected to service customers. For this problem, only two 2-approximation algorithms are available in the literature. Here, we improve on these algorithms by showing that a non-trivial extension of the well-known Christofides heuristic has a tight approximation ratio of $2 - 1/(2k)$. In doing so, we develop a body of analysis which can be used to build new approximation algorithms for other vehicle routing problems.

Keywords: Approximation algorithm; multiple depots; traveling salesman problem

1. Introduction

We study a generalization of the classical traveling salesman problem (TSP), commonly referred to as the *Generalized Multiple Depot Multiple Traveling Salesmen Problem*, or GMDMTSP for short (Malik et al., 2007). Given a set of multiple salesmen positioned at different depots, the objective of the GMDMTSP is to select at most $k \geq 1$ salesmen to service customers situated at different cities through closed walks (or cycles) so as to minimize the total travel distance. This problem has a wide range of applications, and can be found, for example, in routing unmanned aerial vehicles (Chandler and Pachter, 1998; Chandler et al., 2002), as well as in location and routing optimization for air ambulance services (Carnes and Shmoys, 2011; Prodhon and Prins, 2014).

The GMDMTSP can be defined on a complete undirected graph $G = (V, E)$ with a vertex set V and an edge set E. Let n indicate the number of vertices. Let $D \subseteq V$ denote a set of depots where each depot represents the base location of a distinct salesman. Let $I := V \setminus D$ denote a set of customers where each customer in I is located in a city. Each edge $(u, v) \in E$ has a non-negative length $\ell(u, v)$, indicating the distance of the locations of u and v. We assume that the edge lengths are symmetric and satisfy the triangle inequality. Let k indicate the maximum number of salesmen that can be selected to visit customers, where $1 \leq k \leq |D|$. A feasible solution is thus a collection of at most k cycles that include each customer exactly once, and where each cycle begins and ends at a distinct depot. The objective in the GMDMTSP is to find a feasible solution that minimizes the total cycle length.

As with the TSP, the GMDMTSP is strongly NP-hard. Hence, it is of practical interest to develop constant ratio approximation algorithms. Here, we recall that, for a minimization problem, an algorithm is a ρ-approximation algorithm with an approximation ratio ρ if it has a polynomial running time and always provides a solution with a value no more than ρ times the minimum objective value. The ratio ρ is tight if there exists an instance for which the solution obtained has a value equal to ρ times the minimum objective value.

For the GMDMTSP, only two 2-approximation algorithms are available in the literature (Malik et al., 2007; Carnes and Shmoys, 2011). Both have polynomial time complexities, with one being nearly $O(n^4)$, and the other being $O(n^2 \log n)$. Here, we improve on these algorithms by providing a new non-trivial extension of the well-known Christofides heuristic of the TSP (Christofides, 1976) which has a tight approximation ratio of $2 - 1/(2k)$ and time complexity nearly $O(n^4)$.

The rest of the paper is organized as follows: Following a literature review in Section 2 and preliminaries in Section 3, we develop an extension of the Christofides heuristic for the GMDMTSP in Section 4, and go on to prove that it has a tight approximation ratio of $2 - 1/(2k)$ in Section 5. The proof requires an inequality, which is shown in Section 6. We conclude the paper in Section 7.

2. Literature Review

For the GMDMTSP, only two 2-approximation algorithms are known, and they both extend a tree algorithm of the TSP. This algorithm for the TSP has three steps (Papadimitriou and Steiglitz, 1998; Rosenkrantz et al., 1977): (1) Find a minimum spanning tree (MST) of the given graph; (2) construct an Eulerian multigraph by duplicating all edges of the MST; and (3) find an Eulerian closed walk of the multigraph, remove repeated vertices of the closed walk, and return the resulting cycle. In the TSP, since the MST can be no longer than the optimal cycle, the tree algorithm has an approximation ratio of 2. Since the MST and the Eulerian closed walk can be obtained in $O(n^2)$ time and $O(n)$ time, respectively, the tree algorithm for the TSP runs in $O(n^2)$ time.

Malik et al. (2007) extended the tree algorithm for the GMDMTSP by introducing a *degree constrained spanning forest (DCSF) w.r.t.* (G, D, k), which is defined as a spanning forest of G that covers all the vertices in V, with each tree of the forest containing a distinct depot in D as its root, and with the total degree of all the roots not exceeding k. It is known that a shortest DCSF F^* w.r.t. (G, D, k) is a lower bound on the optimal solution of the GMDMTSP, and that F^* can be computed in $O(n^4 \alpha^2 \log^2 \alpha)$ time by a Lagrangian relaxation method (Malik et al., 2007). Here $\alpha := \alpha(n^2, n)$ is the functional inverse of Ackermann's function, which grows very slowly and can be considered as a constant (Chazelle, 2000). Thus, by replacing the MST with F^*, the tree algorithm can be extended to return a feasible solution of at most twice the length of the optimal solution to the GMDMTSP, in $O(n^4\alpha^2 \log^2 \alpha)$ time, which is nearly $O(n^4)$ time.

Carnes and Shmoys (2011) developed another 2-approximation algorithm for the GMDMTSP, which they studied as a variant of the classical location-routing problem (Laporte et al., 1988; Goemans and Williamson, 1995; Mina et al., 1998). Their algorithm also extended the tree algorithm, but applied a primal and dual schema to obtain a DCSF w.r.t. (G, D, k), which may not be a shortest DCSF but is a lower bound on the optimal solution of the GMDMTSP. This algorithm is equivalent to a truncated version of the well known Kruskal's minimum spanning tree algorithm, for which the best implementation requires $O(n^2 \log n)$ time (Cormen et al., 2001).

To improve on the existing best 2-approximation for the GMDMTSP, Malik et al. (2007) suggested extending the well-known Christofides heuristic of the TSP. The Christofides heuristic of the TSP improves on the tree algorithm by revising only Step 2, where it adds to the MST only edges of a minimum-weight perfect matching for vertices of odd degree in the MST. Since the number of vertices of odd degree in the MST is even, by short-cutting the optimal TSP tour, one can obtain the union of two disjoint perfect matchings on these vertices. It follows, by the triangle inequality, that the length of the minimum-weight perfect matching obtained in Step 2 is not longer than half of the optimal TSP tour. This guarantees that the Christofides heuristic achieves a superior ratio of 3/2 for the TSP. Since there are at most n vertices of odd degree in the MST, the number of edges that connect these vertices cannot exceed $n(n-1)/2$. Thus, the minimum-weight perfect matching can be obtained in $O(n^3)$ time (Gabow, 1990; Cook and Rohe, 1999), implying that the Christofides heuristic of the TSP runs in $O(n^3)$ time. For the GMDMTSP, it is natural to extend the Christofides heuristic of the TSP by replacing the MST with the shortest DCSF F^* w.r.t. (G, D, k). However, as pointed out by Malik et al. (2007), the worst-case analysis of this extended heuristic is challenging, since it needs to bound the length of a minimum-weight perfect matching for vertices of odd degree in F^*, for which no effective approach is available in the literature. In this paper, we now develop several new approaches to bound the edges of this matching, which allows us to show that the extended Christofides heuristic achieves a tight approximation ratio of $2 - 1/(2k)$.

Extensions of the Christofides heuristic have been proved to guarantee approximation ratios that are less than 2 for some special cases of the GMDMTSP and their variants. For the multiple depot multiple TSP (MDMTSP), a special case of the GMDMTSP with $k = |D|$, Xu et al. (2011) showed that when $k \geq 2$, the Christofides heuristic can be extended to achieve a tight approximation ratio of $2 - 1/k$ in $O(n^3)$ time, by replacing the MST with a shortest *constrained spanning forest (CSF) w.r.t.* (G, D), where a CSF w.r.t. (G, D) is defined as a spanning forest of G that covers all the vertices in V, with each tree containing a distinct depot in D. Noting that, unlike the DCSF, the total degree of the roots of a CSF may exceed k, the worst-case analysis for the extended Christofides heuristic of the MDMTSP cannot be directly applied to that of the GMDMTSP. Besides this, Rathinam and Sengupta (2010) have extended the Christofides heuristic to obtain a 3/2-approximation algorithm that runs in $O(n^3)$ time for a two-depot Hamiltonian path problem, which determines paths instead of cycles for salesmen. The analysis of the approximation ratio in their work is manageable, since it needs only to consider a two-depot case with $k = |D| = 2$. In addition, Xu and Rodrigues (2015) have developed a 3/2-approximation algorithm for the MDMTSP, but whose running time is $O(n^{3k})$, which is exponential in k. In this paper, we develop a $[2 - 1/(2k)]$-approximation algorithm for the GMDMTSP with a polynomial running time of only about $O(n^4)$.

3. Preliminaries

Recall the definitions of *walk*, *tree*, *rooted tree*, *forest*, *matching*, and *perfect matching* in Diestel (2010). A walk, $(v_1 v_2 ... v_t v_{t+1})$ where $t \geq 0$, is a *closed walk* if $v_1 = v_{t+1}$. A walk with no repeated vertices is a *path*. A closed walk with no repeated vertices except its start and end vertices is a *cycle*. A multigraph is an undirected graph that may contain multiple edges between a pair of vertices. A connected multigraph is *Eulerian* if the degree of each vertex is even. Every Eulerian multigraph has an *Eulerian closed walk*, that is, a closed walk containing every edge (Diestel, 2010).

Consider any two vertices u and v of a rooted tree T with the root of T denoted by r. If u is on the unique path that connects r and v in T, then u is an *ancestor* of v, and v is a *descendant* of u. If u is an ancestor of v and (u, v) is in T, then u is the *parent* of v, and v is a *child* of u. Moreover, throughout the paper, for any subgraph H of G, we use $V(H)$, $E(H)$, and $\ell(H)$ to denote the vertex set, the edge set, and the total length of edges of H. For any edge subset W with $V(W) \subseteq V(H)$, we use $H - W$ to denote a graph on $V(H)$ with an edge set equal to $E(H) \setminus W$, and we use $H + W$ to denote a graph on $V(H)$ with an edge set equal to $E(H) \cup W$. As in Section 2, $\alpha := \alpha(n^2, n)$ indicates the functional inverse of Ackermann's function, which, as pointed out, grows very slowly and can be considered as a constant (Chazelle, 2000).

In this paper, we use F^* to indicate a shortest DCSF w.r.t. (G, D, k) as defined in Section 2, and use a cycle collection $\mathcal{C}^{\text{opt}}$ to indicate an optimal solution to the GMDMTSP.

4. An Extension of the Christofides Heuristic

We elaborate on our extension of the Christofides heuristic for the GMDMTSP in Algorithm 1. It first computes a shortest DCSF F^* w.r.t. (G, D, k), and obtains a vertex set $\text{Odd}(F^*)$ that contains all the vertices of odd degree in F^*. It then computes a minimum-weight perfect matching $M^*(F^*)$ in the subgraph of G induced by $\text{Odd}(F^*)$, and adds to F^* every edge of $M^*(F^*)$ (or a copy of the edge if the edge is in F^*). As a result, it obtains a new multigraph on V, in which each vertex is guaranteed to have an even degree.

Thus, each connected component of the multigraph is Eulerian, and must have an Eulerian closed walk. By removing repeated vertices and redundant depots in these closed walks, Algorithm 1 obtains and returns a collection of cycles, denoted by $\mathcal{C}(F^*)$.

Algorithm 1 (An Extended Christofides Heuristic for the GMDMTSP)**.**

1. Compute a shortest DCSF F^* w.r.t. (G, D, k), and let $\text{Odd}(F^*)$ denote a vertex set that contains all the vertices of odd degree in F^*;
2. Compute a minimum-weight perfect matching $M^*(F^*)$ in the subgraph induced by $\text{Odd}(F^*)$, and add to F^* every edge e of $M^*(F^*)$ (or a copy of e if $e \in E(F^*)$), so as to obtain a new multigraph on V;
3. Let $\mathcal{C}(F^*)$ denote the collection of cycles to be output, which is empty initially. For each connected component of the multigraph constructed above, if it contains at least one customer, then (i) find its Eulerian closed walk, (ii) use shortcuts to obtain a cycle by removing repeated vertices and keeping exactly one depot in the closed walk, and then (iii) add the resulting cycle to $\mathcal{C}(F^*)$. Return $\mathcal{C}(F^*)$.

We now establish Theorem 1 below to show that Algorithm 1 always returns a feasible solution $\mathcal{C}(F^*)$ to the GMDMTSP with $\ell(\mathcal{C}(F^*)) \leq \ell(F^*)+\ell(M^*(F^*))$ in about $O(n^4)$ time, given that α, as we explained in Section 3, can be considered as a constant.

Theorem 1. *Algorithm 1 runs in $O(n^4\alpha^2 \log^2 \alpha)$ time, and the cycle collection $\mathcal{C}(F^*)$ that it returns is a feasible solution to the GMDMTSP with $\ell(\mathcal{C}(F^*)) \leq \ell(F^*) + \ell(M^*(F^*))$.*

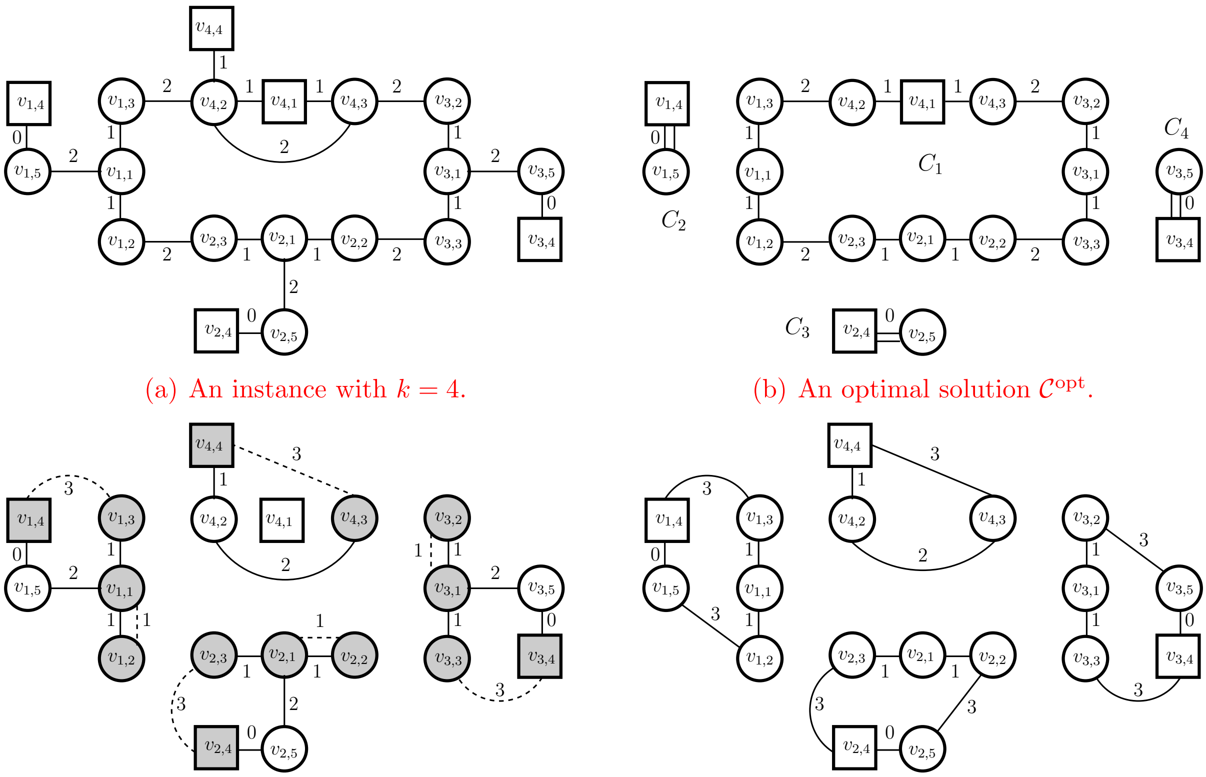
Proof. As mentioned in Section 2, the shortest DCSF F^* w.r.t. (G, D, k) can be obtained in $O(n^4\alpha^2 \log^2 \alpha)$ time by a Lagrangian relaxation method (Malik et al., 2007). Since the perfect matching $M^*(F^*)$ and Eulerian closed walks can be computed in $O(n^3)$ time and $O(n)$ time, respectively, we can say that Algorithm 1 runs in $O(n^4\alpha^2 \log^2 \alpha)$ time.

To show that $\mathcal{C}(F^*)$ is a feasible solution of the GMDMTSP, consider the shortest DCSF F^*. By definition, F^* covers all customers, and each connected component of F^* contains a distinct depot. Moreover, among all connected components of F^*, at most k contain at least one customer. Since these are also true for the multigraph obtained in Step 2 of Algorithm 1, $\mathcal{C}(F^*)$ must contain at most k cycles and covers every customer, where each cycle includes a distinct depot. Hence, $\mathcal{C}(F^*)$ is a feasible solution to the GMDMTSP.

Moreover, by the triangle inequality, we know that $\ell(\mathcal{C}(F^*))$ cannot be longer than the total length of the edges of the multigraph that is obtained in Step 2 of Algorithm 1. This implies that $\ell(\mathcal{C}(F^*)) \leq \ell(F^*) + \ell(M^*(F^*))$. Thus, Theorem 1 is proved. □

We next apply Algorithm 1 to an instance shown in Figure 1(a) with $k = 4$, where rectangles are depots, circles are customers, and lines (with numbers) are edges (with their lengths). For edges that are not shown in Figure 1(a), their lengths are equal to the lengths of the shortest paths between their endpoints, so as to satisfy the triangle inequality. For

Figure 1: Illustration of Algorithm 1.



(a) An instance with $k = 4$.

(b) An optimal solution $\mathcal{C}^{\text{opt}}$.

(c) A multigraph obtained by adding edges in $M^*(F^*)$ (shown in dashed lines) to F^* (shown in solid lines).

(d) $\mathcal{C}(F^*)$ returned by Algorithm 1.

example, we have $\ell(v_{1,4}, v_{1,3}) = \ell(v_{1,4}v_{1,5}v_{1,1}v_{1,3}) = 3$. As shown in Figure 1(b), it can be seen that the length of the optimal solution equals 16, i.e., $\ell(\mathcal{C}^{\text{opt}}) = 16$.

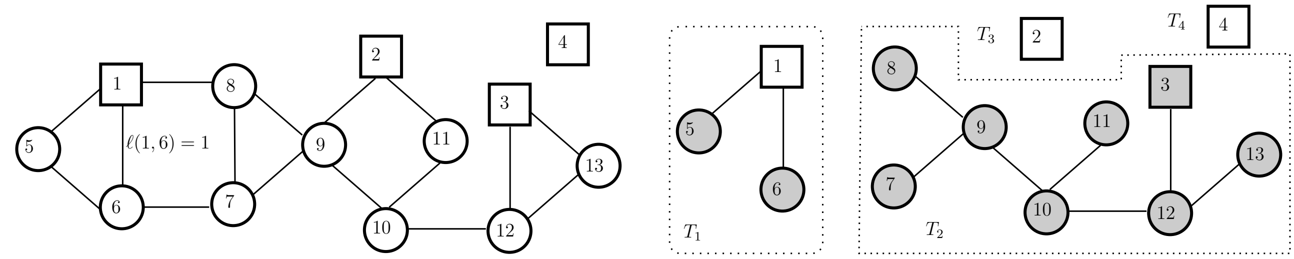
By Step 1 of Algorithm 1, we obtain a shortest DCSF F^* w.r.t. (G, D, k) with $\ell(F^*) = 15$, which is shown in solid lines in Figure 1(c). It can be seen that except for $v_{1,5}$, $v_{2,5}$, $v_{3,5}$, $v_{4,2}$ and $v_{4,1}$, all other vertices are of odd degree in F^*, which form the set $\text{Odd}(F^*)$, and are shown in gray in Figure 2(c). Thus, by Step 2 of Algorithm 1, we obtain a minimum-weight perfect matching $M^*(F^*)$ for the vertices in $\text{Odd}(F^*)$ with $\ell(M^*(F^*)) = 15$, which is shown in dashed lines in Figure 1(c). By adding edges in $M^*(F^*)$ to F^*, we obtain a multigraph with four connected components with each having an Eulerian closed walk. Based on these closed walks, from Step 3 of Algorithm 1 we find a feasible solution $\mathcal{C}(F^*)$ with $\ell(\mathcal{C}^*(F^*)) = 30$, as shown in Figure 1(d). Since $\ell(\mathcal{C}^{\text{opt}}) = 16$, we know that the approximation ratio of Algorithm 1 is at least $30/16 = 15/8 = 2 - 1/(2 \times 4)$.

We note that it is possible to extend the example above for each $k \geq 1$, to show that the approximation ratio of Algorithm 1 is at least $2 - 1/(2k)$. See Online Appendix A.

5. Approximation Ratio

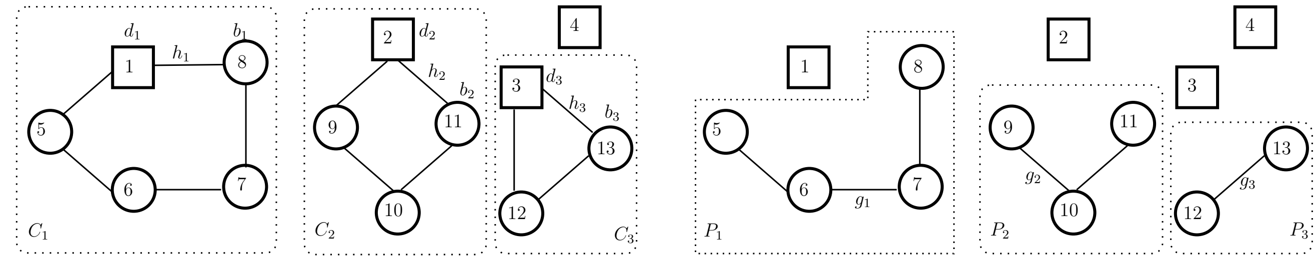
In this section, we will show that Algorithm 1 has a tight approximation ratio of $2 - 1/(2k)$. For this, we need to show that $\ell(\mathcal{C}(F^*)) \leq [2 - 1/(2k)]\ell(\mathcal{C}^{\text{opt}})$. Since $\ell(\mathcal{C}(F^*)) \leq \ell(F^*) + \ell(M^*(F^*))$ (by Theorem 1), we will need to bound both $\ell(F^*)$ and $\ell(M^*(F^*))$.

Figure 2: An example which illustrates the analysis in Section 5 and Section 6.



(a) An instance with $k = 3$ and $D = \{1, 2, 3, 4\}$, where edges that are shown in solid lines are of length equal to 1, and other edges are of length equal to 2.

(b) $F^* = \{T_1, T_2, T_3, T_4\}$ shown in solid lines, and $\mathrm{Odd}(F^*)$ shown in gray.



(c) $\mathcal{C}^{\text{opt}} = \{C_1, C_2, C_3\}$ shown in solid lines with $m = 3$.

(d) $\mathcal{P} = \{P_1, P_2, P_3\}$ shown in solid lines.

We begin with some notation. It can be assumed, without loss of generality, that each cycle in the optimal solution $\mathcal{C}^{\text{opt}}$ contains at least one customer, since if $\mathcal{C}^{\text{opt}}$ has a cycle C that contains no customer, we can exclude C from $\mathcal{C}^{\text{opt}}$. Label the cycles in $\mathcal{C}^{\text{opt}}$ as $C_1, C_2, ..., C_m$ where $m = |\mathcal{C}^{\text{opt}}| \le k$. For each C_j with $1 \le j \le m$, let d_j indicate the depot of C_j. Among the two edges of C_j that d_j is incident with, let $h_j := (d_j, b_j)$ indicate the longer edge (breaking ties arbitrarily), where b_j denotes the endpoint of h_j other than the depot d_j. Since C_j contains only one depot and at least one customer, b_j must be a customer in I. From each C_j, we can obtain a path P_j by removing d_j and the two edges that d_j is incident with. As a result, P_j consists of only customers of C_j. Let $\mathcal{P} := \{P_j : 1 \le j \le m\}$. For each path $P_j \in \mathcal{P}$, let g_j indicate the longest edge of P_j (breaking ties arbitrarily), and we take $\ell(g_j) = 0$ when P_j contains only a single customer.

Example 1. *To illustrate the notation introduced above, let us consider an instance shown in Figure 2(a). Its shortest DCSF F^* w.r.t. (G, D, k) is shown in Figure 2(b). Its optimal solution $\mathcal{C}^{\text{opt}} = \{C_1, C_2, C_3\}$ is shown in Figure 2(c). By definition, it can be seen that $h_1 = (1, 8)$, $h_2 = (2, 11)$, and $h_3 = (3, 13)$. As a result, we have $d_1 = 1$, $b_1 = 8$, $d_2 = 2$, $b_2 = 11$, $d_3 = 3$, and $b_3 = 13$. From $\mathcal{C}^{\text{opt}}$ we know that $\mathcal{P} = \{P_1, P_2, P_3\}$ with $P_1 = (8, 7, 6, 5)$, $P_2 = (11, 10, 9)$, and $P_3 = (12, 13)$, as shown in Figure 2(d). By definition, $g_1 = (6, 7)$, $g_2 = (9, 10)$, and $g_3 = (12, 13)$.*

We first derive two upper bounds on $\ell(F^*)$, which are given below in (1) and (2). Notice that each cycle C_j in $\mathcal{C}^{\text{opt}}$ for $1 \le j \le m$ can be transformed to a tree by deleting edge h_j of C_j. For example, see cycle C_1 in Figure 2(c), which can be transformed to a tree by deleting $h_1 = (1, 8)$. Thus, from $\mathcal{C}^{\text{opt}}$ we can obtain a DCSF F' w.r.t. (G, D, k) by first deleting edges h_j for all $1 \le j \le m$, and then including all the depots that are not in $\mathcal{C}^{\rm opt}$. Thus, $\ell(F') \le \sum_{j=1}^m \ell(C_j) - \sum_{j=1}^m \ell(h_j) = \ell(\mathcal{C}^{\rm opt}) - \sum_{j=1}^m \ell(h_j)$. By $\ell(F^*) \le \ell(F')$, we have

$$\ell(F^*) \le \ell(\mathcal{C}^{\rm opt}) - \sum_{j=1}^{m} \ell(h_j). \tag{1}$$

It is also possible to obtain another DCSF F'' w.r.t. (G, D, k) from $\mathcal{P}$ as follows. Notice that each path $P_j \in \mathcal{P}$ for $1 \le j \le m$ can be transformed to a tree by first including the depot d_j, joining the start and end points of P_j, and then removing edge g_j and adding edge h_j. For example, see path $P_1 = (8, 7, 6, 5)$ in Figure 2(d), which can be transformed to a tree by adding depot $d_1 = 1$, adding edge $(5, 8)$, deleting edge $g_1 = (6, 7)$, and adding edge $h_1 = (1, 8)$. Thus, we can obtain a DCSF F'' from $\mathcal{P}$ by first including all depots in D, joining the start and end points of each $P_j \in \mathcal{P}$, and then removing each edge g_j and adding each edge h_j for $1 \le j \le m$. For each P_j with $1 \le j \le m$, let C'_j indicate the cycle that is obtained by joining the start and end points of P_j. By the triangle inequality we have that $\ell(C'_j) \le \ell(C_j)$ for $1 \le j \le m$. Thus, we obtain that $\ell(F'') = \sum_{j=1}^m \ell(C'_j) + \sum_{j=1}^m [\ell(h_j) - \ell(g_j)] \le \sum_{j=1}^m \ell(C_j) + \sum_{j=1}^m [\ell(h_j) - \ell(g_j)] = \ell(\mathcal{C}^{\rm opt}) + \sum_{j=1}^m [\ell(h_j) - \ell(g_j)]$, which, together with $\ell(F^*) \le \ell(F'')$, implies that

$$\ell(F^*) \le \ell(\mathcal{C}^{\rm opt}) + \sum_{j=1}^{m} [\ell(h_j) - \ell(g_j)]. \tag{2}$$

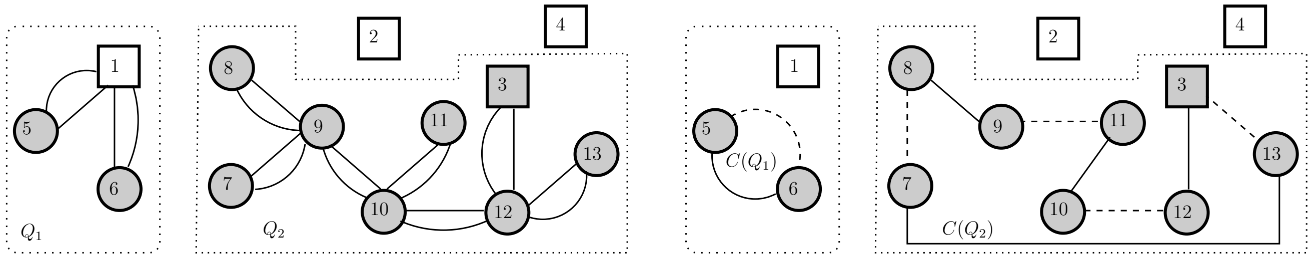
We next provide two upper bounds on $\ell(M^*(F^*))$, which are given below in (3) and (4):

$$\begin{aligned} \ell(M^*(F^*)) &\le \ell(\mathcal{C}^{\rm opt}) + \sum_{j=1}^{m} [\ell(h_j) - \ell(g_j)], & (3) \\ \ell(M^*(F^*)) &\le \ell(\mathcal{C}^{\rm opt})/2 + \sum_{j=1}^{m} [\ell(h_j) + (k-1)\ell(g_j)]. & (4) \end{aligned}$$

It is easy to prove (3) as follows. (See Figure 3 for illustration.) By duplicating all the edges of F^* we can obtain a multigraph, in which each vertex has an even degree. Thus, each connected component Q of the multigraph has an Eulerian closed walk. From this closed walk, using shortcuts to remove repeated vertices, we can find a cycle $C(Q)$ that visits only vertices in $V(Q) \cap {\rm Odd}(F^*)$. Since each tree in F^* contains an even number of vertices of odd degree in F^*, we know that $|V(Q) \cap {\rm Odd}(F^*)|$ is even. Thus, $C(Q)$ consists of two disjoint perfect matching for vertices in $V(Q) \cap {\rm Odd}(F^*)$, the shorter of which is denoted by $L(Q)$. By combining $L(Q)$ for all the connected components Q of the multigraph, we obtain a perfect matching for vertices in ${\rm Odd}(F^*)$. Thus, due to the triangle inequality and (2), we have $\ell(M^*(F^*)) \le 2\ell(F^*)/2 = \ell(F^*) \le \ell(\mathcal{C}^{\rm opt}) + \sum_{j=1}^m [\ell(h_j) - \ell(g_j)]$. (3) is proved.

Compared with the above proof of (3), the proof of (4) is much more complicated. We

Figure 3: Illustration of the proof of (3), using the instance in Figure 2, with vertices of $\mathrm{Odd}(F^*)$ in gray.



(a) Duplicating all the edges of F^* in Figure 2(b).

(b) Shortcutting the Eulerian closed walk of each connected component in Figure 3(a) by removing repeated vertices to obtain two disjoint matchings for vertices in $\mathrm{Odd}(F^*)$, as shown in dashed and solid lines.

therefore present it later in Section 6.

Based on the above bounds on $\ell(F^*)$ and $\ell(M^*(F^*))$, we can derive the tight approximation ratio of Algorithm 1 as follows:

Theorem 2. *Algorithm 1 has a tight approximation ratio of* $2 - 1/(2k)$.

Proof. By Theorem 1, $\ell(\mathcal{C}(F^*)) \leq \ell(F^*) + \ell(M^*(F^*))$. Thus, by (1) and (3), we have that

$$\ell(\mathcal{C}(F^*)) \leq 2\ell(\mathcal{C}^{\mathrm{opt}}) - \sum_{j=1}^{m} \ell(g_j), \tag{5}$$

and by (1) and (4), we have that

$$\ell(\mathcal{C}(F^*)) \leq 3\ell(\mathcal{C})^{\mathrm{opt}}/2 + \sum_{j=1}^{m}(k-1)\ell(g_j). \tag{6}$$

Thus, multiplying (5) by $(k-1)$, and adding it to (6), we have that

$$\ell(\mathcal{C}(F^*)) \leq \{[(3/2) + 2(k-1)]/k\}\ell(\mathcal{C}^{\mathrm{opt}}) = [2 - 1/(2k)]\ell(\mathcal{C}^{\mathrm{opt}}).$$

Hence, Algorithm 1 achieves an approximation ratio of $2 - 1/(2k)$, which is also tight, due to the example in Online Appendix A. □

6. Proof of Inequality (4)

In this section, we are going to prove the inequality (4), for which we only need to show that $\ell(\mathcal{C}^{\mathrm{opt}})/2 + \sum_{j=1}^{m}[\ell(h_j) + (k-1)\ell(g_j)]$ is an upper bound on $\ell(M^*(F^*))$. For this, we need the following notion of an *auxiliary edge subset*, which extends the notation introduced in Xu et al. (2011).

Definition 1. *An auxiliary edge subset* $\hat{A}$ *w.r.t.* $(F^*, \mathcal{C}^{\mathrm{opt}})$ *is an edge subset of* E *such that each connected component of the graph* $(V, E(\mathcal{C}^{\mathrm{opt}}) \cup \hat{A})$ *contains an even number of vertices of odd degree in* F^*.

Example 2. *To illustrate the definition of an auxiliary edge subset, consider the instance in Figure 2(a), for which* $F^* = \{T_1, T_2, T_3, T_4\}$ *and* $\mathrm{Odd}(F^*) = \{3, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ *are shown in Figure 2(b). Consider the optimal solution* $\mathcal{C}^{\mathrm{opt}} = \{C_1, C_2, C_3\}$ *shown in Figure 2(c). It can be seen that* C_1 *contains four vertices in* $\mathrm{Odd}(F^*)$*,* C_2 *contains three vertices in* $\mathrm{Odd}(F^*)$*, and* C_3 *contains three vertices in* $\mathrm{Odd}(F^*)$*. Thus, since* $E(\mathcal{C}^{\mathrm{opt}}) \cup \emptyset = E(\mathcal{C}^{\mathrm{opt}})$*, according to Definition 1 we know that* $\emptyset$ *is not an auxiliary edge subset w.r.t.* $(F^*, \mathcal{C}^{\mathrm{opt}})$*. Next, consider an edge subset* $\hat{A} = \{(12, 10)\}$*. It can be seen that the edge* $(12, 10)$ *connects* C_2 *and* C_3*. Thus,* $(V, E(\mathcal{C}^{\mathrm{opt}}) \cup \{(12, 10)\})$ *contains two connected components with one component including four and the other including six vertices in* $\mathrm{Odd}(F^*)$*. Hence, from Definition 1 we know that* $\hat{A} = \{(12, 10)\}$ *is an auxiliary edge subset w.r.t.* $(\mathcal{F}^*, \mathcal{C}^{\mathrm{opt}})$*.*

As defined in Algorithm 1, $\mathrm{Odd}(F^*)$ represents the set of vertices of odd degree in F^*. Definition 1 indicates that $\hat{A}$ is an auxiliary edge subset w.r.t. $(F^*, \mathcal{C}^{\mathrm{opt}})$ if and only if for each connected component Q of $(V, E(\mathcal{C}^{\mathrm{opt}}) \cup \hat{A})$, the number of its vertices that have odd degree in F^* is even, i.e., $|V(Q) \cap \mathrm{Odd}(F^*)|$ is even. Based on this, we can follow an argument similar to that given in Xu et al. (2011) to establish Lemma 1 below, showing that $\ell(\mathcal{C}^{\mathrm{opt}})/2 + \ell(\hat{A})$ is an upper bound on $\ell(M^*(F^*))$.

Lemma 1. *If* $\hat{A}$ *is an auxiliary edge subset w.r.t.* $(F^*, \mathcal{C}^{\mathrm{opt}})$*, then*

$$\ell(M^*(F^*)) \leq \ell(\mathcal{C}^{\mathrm{opt}})/2 + \ell(\hat{A}). \tag{7}$$

Proof Sketch. (See Online Appendix B for details.) By duplicating edges in $\hat{A}$ and adding these edges to $\mathcal{C}^{\mathrm{opt}}$, we obtain a multigraph H on V, which has the same connected components as $(V, E(\mathcal{C}^{\mathrm{opt}}) \cup \hat{A})$, and satisfies $\ell(H) = \ell(\mathcal{C}^{\mathrm{opt}}) + 2\ell(\hat{A})$. It can be seen that each vertex of H has an even degree. Thus, for each connected component of H, there exists an Eulerian closed walk. We can show that these closed walks can be transformed to two disjoint perfect matchings of vertices in $\mathrm{Odd}(F^*)$, the shorter of which, denoted by M, satisfies that $\ell(M) \leq \ell(H)/2 = \ell(\mathcal{C}^{\mathrm{opt}})/2 + \ell(\hat{A})$. This, together with $\ell(M^*(F^*)) \leq \ell(M)$, completes the proof of (7) of Lemma 1. □

Lemma 1 implies that to prove (4), it is sufficient to establish the following theorem.

Theorem 3. *There exists an auxiliary edge subset* $\hat{A}$ *w.r.t.* $(F^*, \mathcal{C}^{\mathrm{opt}})$ *such that* $\ell(\hat{A}) \leq \sum_{j=1}^{m} [\ell(h_j) + (k-1)\ell(g_j)]$*.*

The proof of Theorem 3 is outlined as follows. In Section 6.1, we will first construct an auxiliary edge subset A w.r.t. $(F^*, \mathcal{C}^{\mathrm{opt}})$ that contains only edges in S, where

$$S := E(F^*) \setminus E(\mathcal{P}), \tag{8}$$

and $\mathcal{P}$, as defined in Section 5, is a collection of paths $P_1, P_2, ..., P_m$ that are obtained by removing depots and their incident edges from cycles $C_1, C_2, ..., C_m$ of $\mathcal{C}^{\text{opt}}$. We will show that our construction of A guarantees certain properties, which are useful for the proof of Theorem 3. In Section 6.2, we will derive several upper bounds on the lengths of the edges in S. Since $A \subseteq S$, these bounds are also applicable to the edges in A. In Section 6.3, based on the properties and bounds obtained, we will derive an upper bound on $\ell(A)$. Using this upper bound, we can complete the proof of Theorem 3 in Section 6.4 by showing that either A satisfies $\ell(A) \leq \sum_{j=1}^{m}[\ell(h_j) + (k-1)\ell(g_j)]$, or there exist edges $(s,t) \in A$ and $(s',t') \in S$, such that $A' := A \setminus \{(s,t)\} \cup \{(s',t')\}$ is an auxiliary edge subset w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$ that satisfies $\ell(A') \leq \sum_{j=1}^{m}[\ell(h_j) + (k-1)\ell(g_j)]$.

6.1. Construction of an auxiliary edge subset A from S

For the construction, we first derive a sufficient condition for an edge subset to be an auxiliary edge subset w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$. The condition is based on the following observations: (i) Each tree in F^* must contain an even number of vertices of odd degree in F^*; and (ii) for any edge subset $\hat{A}$, every two vertices u and v that belong to the same connected component of $(V, E(\mathcal{P}) \cup \hat{A})$ must belong to the same connected component of $(V, E(\mathcal{C}^{\text{opt}}) \cup \hat{A})$.

Lemma 2. *An edge subset $\hat{A} \subseteq E$ is an auxiliary edge subset w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$, if every two vertices u and v that belong to the same tree in F^* also belong to the same connected component of $(V, E(\mathcal{P}) \cup \hat{A})$.*

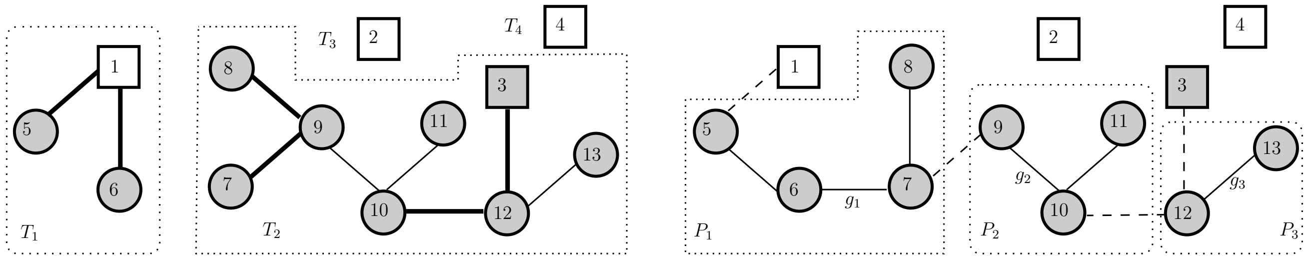
Proof. If the condition is satisfied, then due to observation (ii) above, every two vertices u and v that belong to the same tree in F^* must also belong to the same connected component of $(V, E(\mathcal{C}^{\text{opt}}) \cup \hat{A})$. Thus, in view of observation (i) above, we find that every connected component of $(V, E(\mathcal{C}^{\text{opt}}) \cup \hat{A})$ must contain an even number of vertices of odd degree in F^*. Hence, by Definition 1, $\hat{A}$ is an auxiliary edge subset w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$. □

Remark 1. *The sufficient condition specified in Lemma 2 may not be necessary for an edge subset to be an auxiliary edge subset. For example, the auxiliary edge subset $\hat{A} = \{(12, 10)\}$ in Example 2 does not satisfy the condition, since customers 9 and 7 of T_2 of F^* belong to two different connected components of $(V, E(\mathcal{P}) \cup \{(12, 10)\})$. However, $\{(1, 5), (3, 12), (9, 7), (12, 10)\}$, which is also an auxiliary edge subset, satisfies the condition. Moreover, the sufficient condition is still valid even if it is relaxed by replacing $\mathcal{P}$ with $\mathcal{C}^{\text{opt}}$. In Lemma 2, we use $\mathcal{P}$ in the condition so as to simplify our further proof of Theorem 3.*

Lemma 2 implies that an auxiliary edge subset A w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$ can be constructed by iteratively adding to A an edge of F^* that joins customers in two different connected components of $(V, E(\mathcal{P}) \cup A)$. Such edges added to A must be in $S = E(F^*) \setminus E(\mathcal{P})$.

To guarantee certain properties, which are useful for the further proof of Theorem 3, our construction of A examines edges of S in a particular predetermined sequence. To determine

Figure 4: Illustration of Algorithm 2 for the construction of an auxiliary edge subset A w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$, using the instance in Figure 2, with vertices of $\text{Odd}(F^*)$ in gray.



(a) $S = E(F^*) \setminus E(\mathcal{P}) = \{(1,5),(1,6),(3,12),(12,10),(9,7),(9,8)\}$ shown in bold lines.

(b) $A = \{(1,5),(3,12),(9,7),(12,10)\}$ shown in dashed lines, which connect P_1, P_2, and P_3.

the sequence, we need the following notation. Denote the trees in F^* by $T_1, T_2, ..., T_{|D|}$. For each vertex v, define the *depth of* v as the number of edges on the unique path that connects v to the root of the tree in F^* that contains v. For each edge (u, v) of F^*, where u is the parent of v, the *depth of* (u, v) is defined to be the same as the depth of u.

As shown in Algorithm 2 below, in our construction of A, we examine edges in $E(T_i) \setminus E(\mathcal{P})$ sequentially in a non-decreasing order of their depths, for each tree T_i, with $i = 1, 2, ..., |D|$, respectively. For each edge (u, v) examined, we add (u, v) to A only when u and v belong to different connected components of $(V, E(\mathcal{P}) \cup A)$. Thus, by Lemma 2, the edge subset A returned by Algorithm 2 must be an auxiliary edge subset w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$. From Step 2 of Algorithm 2 and (8), we know that $A \subseteq E(F^*) \setminus E(\mathcal{P}) = S$.

Algorithm 2 (Construction of an auxiliary edge subset A w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$)**.**

1. Start with $A = \emptyset$.
2. For each tree T_i, where $i = 1, 2, ..., |D|$, examine edges $(u, v) \in E(T_i) \setminus E(\mathcal{P})$ in a non-decreasing order of their depths, and if u and v belong to different connected components of $(V, E(\mathcal{P}) \cup A)$, then add (u, v) to A.
3. Return A.

Example 3. *To illustrate Algorithm 2, let us apply it to the instance in Figure 2(a) to construct an auxiliary edge subset A w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$, where $F^* = \{T_1, T_2, T_3, T_4\}$, $\mathcal{C}^{\text{opt}}$, and $\mathcal{P}$, are shown in Figures 2(b)(c)(d), and $S = E(F^*) \setminus (E(\mathcal{P}) = \{(1,5),(1,6),(3,12),(12,10),(9,7),(9,8)\}$ is shown in bold lines in Figure 4(a). Algorithm 2 starts with $A = \emptyset$, and examines the edges of T_1, T_2, T_3, and T_4, respectively. For T_1, the first edge to be examined is $(1,5)$ with a depth of 0, and it is added to A, since vertices 1 and 5 belong to different connected components of $(V, E(\mathcal{P}) \cup A)$. The second edge to be examined is $(1,6)$, but this edge is not added to A, since vertices 1 and 6 belong to the same connected component of $(V, E(\mathcal{P}) \cup A)$. Similarly, edges $(3,12)$, $(12,10)$, $(9,7)$, and $(9,8)$ of T_2 are examined sequentially, and only $(3,12)$, $(12,10)$, and $(9,7)$ are added to A. Thus, since $E(T_3) = E(T_4) = \emptyset$, Algorithm 2 returns $A = \{(1,5),(3,12),(9,7),(12,10)\}$, which, as shown in*

Figure 4(b), connects all the paths of $\mathcal{P}$, and satisfies the condition specified in Lemma 2. Hence, it is an auxiliary edge subset w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$. $\square$

Algorithm 2 guarantees certain properties for the auxiliary edge subset A that it returns. To illustrate these properties, we need to introduce the following notation. As shown earlier, $A \subseteq S$. Consider each edge $(u, v) \in S$, where u is the parent of v in F^*. Since v is a customer in I, there exists a unique path in $\mathcal{P}$ that contains v. We refer to this path as the *associated path of* (u, v), denoted by $P_{j(u,v)} \in \mathcal{P}$ with $1 \leq j(u, v) \leq m$. For example, from Figure 2(b) and Figure 2(d) we know that P_1 is the associated path of $(1, 5)$, $(1, 6)$, $(9, 7)$ and $(9, 8)$, P_2 is the associated path of $(12, 10)$, and P_3 is the associated path of $(3, 12)$.

Let R denote the set of root edges of F^*, i.e.,

$$R := \{(d, x) \in E(F^*) : d \in D\}. \tag{9}$$

For example, from Figure 2(b) it can be seen that $R = \{(1, 5), (1, 6), (3, 12)\}$. Since the total degree of roots in F^* does not exceed k, we have $|R| \leq k$. For each customer $v \in I$, there exists a unique root edge on the path from a depot to v in F^*, which we refer to as the *root edge of* v. Thus, for each edge $(u, v) \in E(F^*)$, the root edge of (u, v) is defined as being the same as the root edge of v. For example, from Figure 2(b) it can be seen that (3,12) is the root edge of (3,12), (12,10), (10,11), (10,9), (9,7), (9,8), and (12,13).

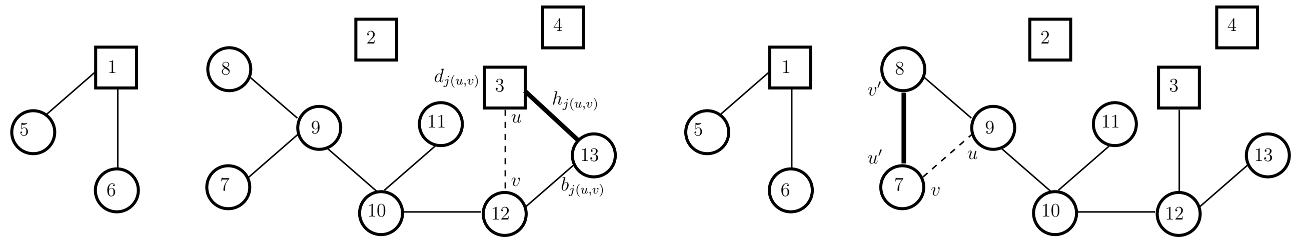
Lemma 3 below provides three properties of A and $A \cup R$.

Lemma 3. *The following hold: (i) $(A \cup R) \subseteq S$; (ii) $|A \cup R| \leq 2k - 1$; and (iii) edges in $A \cup R$ that have the same associated path must have different root edges in F^*.*

Proof Sketch. (See Online Appendix C for details.) By the definitions of R and S, and noting that $A \subseteq S$, it is easy to verify property (i). Noting that in Step 2 of Algorithm 2, only edges that join different connected components of $(V, E(\mathcal{P}) \cup A)$ are added to A, we can show that $|A \setminus R| \leq |\mathcal{P}| - 1 \leq k - 1$. This, together with $|R| \leq k$, implies that $|A \cup R| = |A \setminus R| + |R| \leq 2k - 1$, and proves property (ii). Property (iii) can be proved by contradiction, noting that Algorithm 2 constructs A by examining edges of each tree of F^* in a non-decreasing order of their depths. $\square$

Remark 2. *The three properties in Lemma 3 are later used in the proof of Theorem 3. In fact, we know that Theorem 3 can be proved if it can be shown that the auxiliary edge subset A returned by Algorithm 2 satisfies $\ell(A) \leq \sum_{j=1}^{m}[\ell(h_j) + (k-1)\ell(g_j)]$. For this, property (i) in Lemma 3 indicates that we can use bounds in Section 6.2 below to bound the lengths of edges in A, from which we can then derive an upper bound on $\ell(A)$ in Section 6.3 using $\ell(g_j)$ for $1 \leq j \leq m$ and using $\sum_{j=1}^{m} \ell(h_j)$. In Section 6.4, we will use properties (ii) and (iii) in Lemma 3 to bound the number of times that each $\ell(g_j)$ appears in the upper bound on $\ell(A)$, so as to obtain a condition for A, as well as to gain some insight on how to modify A, if necessary, to satisfy $\ell(A) \leq \sum_{j=1}^{m}[\ell(h_j) + (k-1)\ell(g_j)]$.*

Figure 5: Illustration of proofs of Lemma 4 and Lemma 6, using the instance in Figure 2.



(a) $F = (F^* - \{(u,v)\}) + \{h_{j(u,v)}\}$, as defined in the proof of Lemma 4, where $(u,v) = (3,12)$ and $h_{j(u,v)} = (3,13)$.

(b) $F = (F^* - \{(u,v)\}) + \{(u',v')\}$, as defined in the proof of Lemma 6, where $(u,v) = (9,7)$, $(u',v') = (7,8)$ is on $P_{j(u,v)} = P_1$, and $u' = 7$ is a descendant of $v = 7$ in F^* but $v' = 8$ is not.

6.2. Upper bounds on the lengths of the edges in S

Consider the auxiliary edge subset A returned by Algorithm 2, the edge subset S defined in (8), and the edge subset R defined in (9). In view of property (i) in Lemma 3, we have that $\ell(A) \leq \ell(A \cup R)$ and $(A \cup R) \subseteq S$. Thus, to derive an upper bound on $\ell(A)$ later in Section 6.3 for the proof of Theorem 3, we will use $\ell(g_j)$ and $\ell(h_j)$ with $1 \leq j \leq m$ to derive upper bounds on the lengths of different edges in S, as follows.

First, consider any edge $(u,v) \in S$, where u is the parent of v in F^*, such that $b_{j(u,v)}$ of the associated path $P_{j(u,v)}$ of (u,v), with $1 \leq j(u,v) \leq m$, is a descendant of v in F^*. We refer to such an edge $(u,v) \in S$ as a *b-type edge*, and use $B \subseteq S$ to denote the set of all b-type edges. For example, consider the DCSF F^* in Figure 2(b), and the edge set S shown in Example 3. From Figures 2(b)(c)(d) we know that edge $(12,10) \in S$ is a b-type edge, since its associated path is P_2, and $b_2 = 11$ is a descendant of 10 in F^*. Similarly, $(9,8)$ and $(3,12)$ are also b-type edges, and thus $B = \{(12,10),(9,8),(3,12)\}$. For such a b-type edge (u,v), Lemma 4 shows that $\ell(h_{j(u,v)})$ is an upper bound on $\ell(u,v)$ if u is a depot or $|R| < k$.

Lemma 4. *For each b-type edge $(u,v) \in B$, where u is the parent of v in F^*, if $u \in D$ or $|R| < k$, then $\ell(u,v) \leq \ell(h_{j(u,v)})$.*

Proof. Let $F := (F^* - \{(u,v)\}) + \{h_{j(u,v)}\} = (F^* - \{(u,v)\}) + \{(d_{j(u,v)}, b_{j(u,v)})\}$. (See Figure 5(a) for illustration.) Since (u,v) is a b-type edge, $b_{j(u,v)}$ is a descendant of v in F^*. Hence, if $u \in D$ or $|R| < k$, then since $d_{j(u,v)} \in D$, it can be seen that F is also a DCSF w.r.t. (G,D,k). Thus, $\ell(F^*) \leq \ell(F)$, which implies that $\ell(u,v) \leq \ell(h_{j(u,v)})$. □

Next, for any b-type edge $(u,v) \in B \setminus R$ and any edge $(u',v') \in R \setminus B$, Lemma 5 below shows that $\ell(g_{j(u',v')}) + \ell(h_{j(u,v)})$ is an upper bound on $\ell(u',v') + \ell(u,v)$.

Lemma 5. *Consider any b-type edge $(u,v) \in B \setminus R$, where u is the parent of v in F^*. Then, for each $(u',v') \in R \setminus B$, where u' is the parent of v' in F^*, it satisfies that $\ell(u',v') + \ell(u,v) \leq \ell(g_{j(u',v')}) + \ell(h_{j(u,v)})$.*

Proof Sketch. (See Online Appendix D for details.) Consider $F := (F^* - \{(u, v)\}) + \{h_{j(u,v)}\}$. It can be shown that F is a CSF w.r.t. (G, D), with the total degree of the roots no more than $k + 1$. Moreover, it can be shown that for any edge $(u', v') \in R \setminus B$, there exists an edge (x, y) on $P_{j(u',v')}$ with $x \in I$ and $y \in I$, such that (u', v') is the root edge of x but is not the root edge of y in F. From this it can be seen that $F' := (F - \{(u', v')\}) + \{(x, y)\}$ is a DCSF w.r.t. (G, D, k). This implies that $\ell(F^*) \leq \ell(F')$, and so $\ell(u, v) + \ell(u', v') \leq \ell(h_{j(u,v)}) + \ell(x, y)$. Hence, since edge (x, y) is on $P_{j(u',v')}$, which implies that $\ell(x, y) \leq \ell(g_{j(u',v')})$, we obtain that $\ell(u', v') + \ell(u, v) \leq \ell(g_{j(u',v')}) + \ell(h_{j(u,v)})$, which completes the proof of Lemma 5. □

Moreover, consider any edge $(u, v) \in S$, where u is the parent of v in F^*, such that the associated path $P_{j(u,v)}$ of (u, v) contains a customer that is not a descendant of v in F^*. We refer to such an edge $(u, v) \in S$ as a *y-type edge*, and use $Y \subseteq S$ to denote the set of all y-type edges. For example, consider the DCSF F^* shown in Figure 2(b), and the edge set S shown in Example 3. From Figures 2)(b)(c)(d) we know that $(9, 7) \in S$ is a y-type edge, since its associated path P_1 contains 8, which is not a descendant of 7 in F^*. Similarly, $(1, 5)$, $(1, 6)$ and $(9, 8)$ are all y-type edges, and thus $Y = \{(1, 5), (1, 6), (9, 7), (9, 8)\}$. For such a y-type edge (u, v), Lemma 6 below shows that $\ell(g_{j(u,v)})$ is an upper bound on $\ell(u, v)$.

Lemma 6. *For each y-type edge $(u, v) \in Y$, where u is the parent of v in F^*, it satisfies that $\ell(u, v) \leq \ell(g_{j(u,v)})$.*

Proof. Since (u, v) is a y-type edge, by definition we know that path $P_{j(u,v)}$ must contain a customer that is not a descendant of v in F^*. Thus, since v is a customer on $P_{j(u,v)}$, it can be seen that $P_{j(u,v)}$ must contain an edge (u', v'), such that u' is a descendant of v in F^* but v' is not. (See Figure 5(b) for illustration.) Thus, u' and v' must both be in I. Therefore, $F := (F^* - \{(u, v)\}) + \{(u', v')\}$ is also a DCSF w.r.t. (G, D, k). Hence, $\ell(F^*) \leq \ell(F)$, and so $\ell(u, v) \leq \ell(u', v') \leq \ell(g_{j(u,v)})$. □

In view of Lemma 7 below, the bound presented in Lemma 6 above is also valid for edges in $R \setminus B$ and $S \setminus B$.

Lemma 7. $(R \setminus B) \subseteq (S \setminus B) \subseteq Y$.

Proof. By (i) of Lemma 3, $R \subseteq S$, which implies that $(R \setminus B) \subseteq (S \setminus B)$. For each $(u, v) \in S \setminus B$, where u is the parent of v in F^*, we know that the customer $b_{j(u,v)}$ of $P_{j(u,v)}$ is not a descendant of v in F^*. Thus, by definition we know that (u, v) is a y-type edge in Y. This implies that $(S \setminus B) \subseteq Y$, and completes the proof of Lemma 7. □

6.3. An upper bound on $\ell(A)$

We now establish Theorem 4 below to derive an upper bound on $\ell(A)$ for the auxiliary edge subset A returned by Algorithm 2, by using $\ell(g_j)$ for $1 \leq j \leq m$ and $\sum_{j=1}^{m} \ell(h_j)$.

Theorem 4. *Consider the edge subset S defined in (8), the edge subset R defined in (9), and the set B of all b-type edges of S. Then, the auxiliary edge subset A w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$ returned by Algorithm 2 satisfies:*

$$\ell(A) \leq \sum_{e \in (A \cup R) \setminus (R \cap B)} \ell(g_{j(e)}) + \sum_{j=1}^{m} \ell(h_j). \tag{10}$$

To prove Theorem 4, we only need to prove (10). For this, we notice that $\ell(A) \leq \ell(A \cup R) = \ell(R) + \ell(A \setminus R) = \ell(R) + \ell(A \setminus R \setminus Y) + \ell((A \setminus R) \cap Y)$. Define

$$H := A \setminus R \setminus Y, \tag{11}$$

and note that

$$[(A \setminus R) \cap Y] = (A \setminus R) \setminus [(A \setminus R) \setminus Y] = (A \setminus R \setminus H). \tag{12}$$

We have that

$$\ell(A) \leq \ell(R) + \ell(H) + \ell(A \setminus R \setminus H). \tag{13}$$

Moreover, (12) implies that every $e \in A \setminus R \setminus H$ is a y-type edge, which, together with Lemma 6, implies that $\ell(A \setminus R \setminus H) \leq \sum_{e \in A \setminus R \setminus H} \ell(g_{j(e)})$. Thus, from (13) we obtain that

$$\ell(A) \leq \ell(R) + \ell(H) + \sum_{e \in A \setminus R \setminus H} \ell(g_{j(e)}). \tag{14}$$

Therefore, to prove (10) we need to derive an upper bound on $\ell(R) + \ell(H)$.

To derive an upper bound on $\ell(R) + \ell(H)$, we first examine some properties of H. For each edge $(u, v) \in H$, where u is the parent of v in F^*, we know from (11) that $(u, v) \in A \setminus R$ and $(u, v) \in A \setminus Y$. Thus, $u \in I$, and customers on $P_{j(u,v)}$ are all descendants of v in F^*.

Example 4. *To illustrate other properties of H, consider the instance given in Figure 2, for which we know from Example 3 that $A = \{(1,5), (3,12), (9,7), (12,10)\}$ and $S = \{(1,5), (1,6), (3,12), (12,10), (9,7), (9,8)\}$. As we have shown, $R = \{(1,5), (1,6), (3,12)\}$, $B = \{(12,10), (9,8), (3,12)\}$, and $Y = \{(1,5), (1,6), (9,7), (9,8)\}$. Thus, by (11), $H = \{(12,10)\}$. Moreover, we have $R \cap B = \{(3,12)\}$, $(S \setminus R) \cap B = \{(12,10), (9,8)\}$, and $H \cup (R \cap B) = \{(12,10), (3,12)\}$. Thus, we can see that (i) H and $R \cap B$ are disjoint; (ii) $H \subseteq (S \setminus R) \cap B$; and (iii) different edges in $H \cup (R \cap B)$ have different associated paths.*

By Lemma 8 below, the three properties of H observed in Example 4 are in fact valid in general.

Lemma 8. *The following hold: (i) H and $R \cap B$ are disjoint; (ii) $H \subseteq (S \setminus R) \cap B$; and (iii) different edges in $H \cup (R \cap B)$ have different associated paths.*

Proof Sketch. (See Online Appendix E for details.) We can verify properties (i) and (ii) by the definition of H in (11). To prove (iii), notice that (ii) implies that $H \cup (R \cap B) \subseteq B$, and $H = A \setminus R \setminus Y$ implies that $H \cup (R \cap B) \subseteq (A \cup R)$. Based on these and the fact shown in (iii) of Lemma 3 that edges in $A \cup R$ that have the same associated path must have different root edges in F^*, we can prove property (iii) by contradiction. □

By properties (i) and (iii) of Lemma 8, and by $|\mathcal{P}| \leq k$, we find that

$$|H| + |R \cap B| \leq |\mathcal{P}| \leq k. \tag{15}$$

Moreover, for each $(u, v) \in H$, where u is the parent of v in F^*, since $H \subseteq B$ (due to (ii) of Lemma 8), we have that (u, v) is a b-type edge, implying that $b_{j(u,v)}$ is a descendant of v in F^*. This, together with $H \subseteq (B \setminus R)$ (due to (ii) of Lemma 8) and with (15), allows us to utilize Lemmas 4–6 to derive an upper bound on $\ell(R) + \ell(H)$ as shown below in Lemma 9.

Lemma 9. $\ell(R) + \ell(H) \leq \sum_{e \in R \setminus B} \ell(g_{j(e)}) + \sum_{e \in R \cap B} \ell(h_{j(e)}) + \sum_{e \in H} \ell(h_{j(e)})$.

Proof Sketch. (See Online Appendix F for details.) Since $|R| \leq k$, we can prove Lemma 9 by taking into account the following two cases. For Case 1, where $|R| < k$, noting that $\ell(R) + \ell(H) = \ell(R \setminus B) + \ell(R \cap B) + \ell(H)$ and $H \subseteq B$, we can prove Lemma 9 by applying Lemma 4 to bound $\ell(R \cap B) + \ell(H)$, and applying Lemma 6 and Lemma 7 to bound $\ell(R \setminus B)$. For Case 2, where $|R| = k$, it can be seen from (15) that $|H| \leq k - |R \cap B| = |R| - |R \cap B| = |R \setminus B|$. This allows us to arbitrarily select $|H|$ different edges of $R \setminus B$, so as to form a subset $\hat{R}$ of $R \setminus B$ with $|\hat{R}| = |H|$. Thus, by Lemma 5, $\ell(\hat{R}) + \ell(H) \leq \sum_{e \in \hat{R}} \ell(g_{j(e)}) + \sum_{e \in H} \ell(h_{j(e)})$. With this and Lemmas 4–6, noting that $\ell(R) + \ell(H) = \ell(R \setminus B \setminus \hat{R}) + \ell(R \cap B) + [\ell(\hat{R}) + \ell(H)]$ we can prove Lemma 9 for Case 2. □

We are now ready to prove Theorem 4.

Proof of Theorem 4. As shown in (14), $\ell(A) \leq \ell(R) + \ell(H) + \sum_{e \in A \setminus R \setminus H} \ell(g_{j(e)})$. Thus, by the upper bound on $\ell(R) + \ell(H)$ shown in Lemma 9, we obtain that

$$\ell(A) \leq \sum_{e \in R \setminus B} \ell(g_{j(e)}) + \sum_{e \in R \cap B} \ell(h_{j(e)}) + \sum_{e \in H} \ell(h_{j(e)}) + \sum_{e \in A \setminus R \setminus H} \ell(g_{j(e)}). \tag{16}$$

Due to (i) and (iii) of Lemma 8, we have that

$$\sum_{e \in R \cap B} \ell(h_{j(e)}) + \sum_{e \in H} \ell(h_{j(e)}) = \sum_{e \in (R \cap B) \cup H} \ell(h_{j(e)}) \leq \sum_{j=1}^{m} \ell(h_j). \tag{17}$$

Moreover, since $R \setminus B$ and $A \setminus R \setminus H$ are disjoint, and since $[(R \setminus B) \cup (A \setminus R \setminus H)] \subseteq [(A \cup R) \setminus (R \cap B)]$, we have that

$$\sum_{e \in R \setminus B} \ell(g_{j(e)}) + \sum_{e \in A \setminus R \setminus H} \ell(g_{j(e)}) \leq \sum_{e \in (A \cup R) \setminus (R \cap B)} \ell(g_{j(e)}). \tag{18}$$

Therefore, from (16), (17) and (18), we can obtain (10) as follows:

$$\ell(A) \leq \sum_{e \in R \setminus B} \ell(g_{j(e)}) + \sum_{j=1}^{m} \ell(h_j) + \sum_{e \in A \setminus R \setminus H} \ell(g_{j(e)}) \leq \sum_{e \in (A \cup R) \setminus (R \cap B)} \ell(g_{j(e)}) + \sum_{j=1}^{m} \ell(h_j),$$

and so Theorem 4 is proved. □

6.4. Proof of Theorem 3

In the following, we will prove Theorem 3, to show that there exists an auxiliary edge subset w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$ of length not greater than $\sum_{j=1}^{m}[(k-1)\ell(g_j) + \ell(h_j)]$. As explained earlier, in view of Lemma 1, this is sufficient for the proof of the inequality (4).

The proof of Theorem 3 for $k = 1$ is trivial, since when $k = 1$, it can be easily verified that the set R of root edges in F^* is an auxiliary edge subset w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$ that satisfies $\ell(R) \leq \sum_{j=1}^{m}[(k-1)\ell(g_j) + \ell(h_j)]$. See Online Appendix G.

Thus, in the following, we only need to consider the situation where $k \geq 2$. To prove Theorem 3 for this situation, we first derive a sufficient condition for the auxiliary edge subset A constructed by Algorithm 2 to satisfy $\ell(A) \leq \sum_{j=1}^{m}[(k-1)\ell(g_j) + \ell(h_j)]$, and then show that when this condition is not true, A can be transformed to another auxiliary edge subset A' that satisfies $\ell(A') \leq \sum_{j=1}^{m}[(k-1)\ell(g_j) + \ell(h_j)]$.

For the auxiliary edge subset A constructed by Algorithm 2, we know from Theorem 4 that $\ell(A) \leq \sum_{e \in (A \cup R) \setminus (R \cap B)} \ell(g_{j(e)}) + \sum_{j=1}^{m} \ell(h_j)$. Notice that (i) of Lemma 3 implies that $[(A \cup R) \setminus (R \cap B)] \subseteq S$. Thus, each edge $e \in [(A \cup R) \setminus (R \cap B)]$ has an associated path $P_{j(e)} \in \mathcal{P}$. We now partition the set $[(A \cup R) \setminus (R \cap B)]$ into m disjoint subsets, denoted by $A_1, A_2, ..., A_m$, where each A_j for $1 \leq j \leq m$ consists of all those edges in $[(A \cup R) \setminus (R \cap B)]$ that have P_j as their associated path. Accordingly, by Theorem 4 we have that

$$\ell(A) \leq \sum_{e \in (A \cup R) \setminus (R \cap B)} \ell(g_{j(e)}) + \sum_{j=1}^{m} \ell(h_j) = \sum_{j=1}^{m} |A_j| \ell(g_j) + \sum_{j=1}^{m} \ell(h_j). \tag{19}$$

Moreover, due to (iii) of Lemma 3 and $|R| \leq k$ we know that $|A_j| \leq |R| \leq k$ for $1 \leq j \leq m$. Without loss of generality, let P_1 denote the path that maximizes $|A_j|$ over all $P_j \in \mathcal{P}$ for $1 \leq j \leq m$. Thus, we have $|A_j| \leq |A_1| \leq k$ for $2 \leq j \leq m$.

Example 5. *To illustrate the partition $A_1, A_2, ..., A_m$ of $(A \cup R) \setminus (R \cap B)$, let us consider the instance in Figure 2, where $m = 3$, and the values of A, R, and B are referred to in Ex-*

ample 4. By definition, it can be seen that $(A\cup R)\setminus(R\cap B)=\{(1,5),(12,10),(9,7),(1,6)\}$, *and from Figure 2(d) we have* $A_1=\{(1,5),(1,6),(9,7)\}$, $A_2=\{(12,10)\}$, *and* $A_3=\emptyset$.

It can now be seen that $|A_1|\le k-1$ is a sufficient condition for A to satisfy $\ell(A)\le\sum_{j=1}^{m}[(k-1)\ell(g_j)+\ell(h_j)]$. This is because if $|A_1|\le k-1$, which implies that $|A_j|\le|A_1|\le k-1$ for $2\le j\le m$, then from (19) we can obtain that $\ell(A)\le\sum_{j=1}^{m}[(k-1)\ell(g_j)+\ell(h_j)]$, and thus Theorem 3 is proved.

If the condition $|A_1|\le k-1$ is not true, then $|A_1|\ge k$, which, together with $|A_1|\le k$ and $k\ge 2$ as shown above, implies that $|A_1|=k\ge 2$. For this case, where $|A_1|=k\ge 2$, since (ii) of Lemma 3 indicates that $|A|\le|A\cup R|\le 2k-1$, we find that for $2\le j\le m$,

$$|A_j|\le|(A\cup R)\setminus(R\cap B)|-|A_1|\le(2k-1)-k=k-1. \tag{20}$$

Thus, by (19), we can obtain the following upper bound on $\ell(A)$:

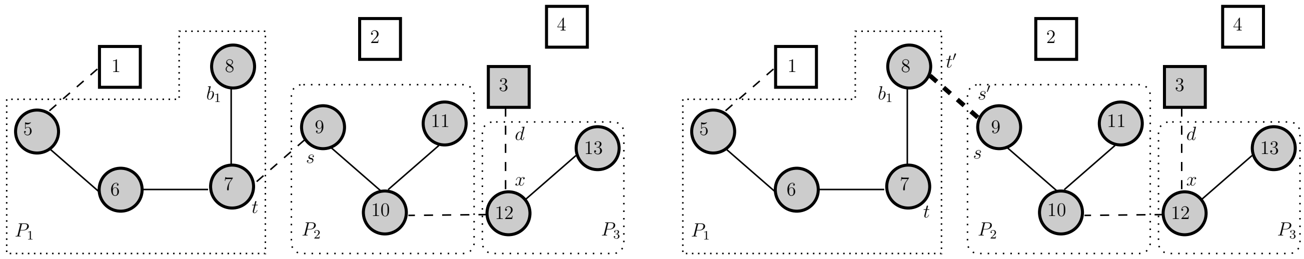
$$\ell(A)\le|A_1|\ell(g_1)+\sum_{j=2}^{m}|A_j|\ell(g_j)+\sum_{j=1}^{m}\ell(h_j)\le k\ell(g_1)+\sum_{j=2}^{m}(k-1)\ell(g_j)+\sum_{j=1}^{m}\ell(h_j). \tag{21}$$

Notice that the upper bound on $\ell(A)$ shown in (21) is greater by $\ell(g_1)$ than the upper bound that we need for the proof of Theorem 3. In the following, we are going to show that for this case, A can be transformed to A' by replacing an edge $(s,t)\in A_1\setminus R$ with a new edge $(s',t')\in S\setminus R$, such that (i) $A'=A\setminus\{(s,t)\}\cup\{(s',t')\}$ is also an auxiliary edge subset (as shown in Lemma 10 below), and that (ii) $\ell(A')\le(|A_1|-1)\ell(g_1)+\sum_{j=2}^{m}|A_j|\ell(g_j)+\sum_{j=1}^{m}\ell(h_j)$ (as shown in Lemma 11 below), which, together with (20), are sufficient to complete the proof of Theorem 3.

First, let us determine which edge $(s,t)\in A_1\setminus R$ is to be replaced. By (iii) of Lemma 3, edges in A_1 have different root edges in F^*. Notice that $|R|\le k$ and $|A_1|=k$, implying that for each edge in $|R|$, it must be a root edge of an edge in A_1. Thus, there must exist an edge $(s,t)\in A_1$, where s is the parent of t in F^*, such that both the edge (s,t) of A_1 and the customer b_1 of path P_1 have the same root edge in F^*, which is denoted by (d,x) with $d\in D$ and $x\in I$. See Figure 6(a). It can also be shown that $(s,t)\in A_1\setminus R$. By contradiction, suppose that (s,t) is not in $A_1\setminus R$. Thus, since $(s,t)\in A_1$, we know that $(s,t)\in R$, which implies that $(s,t)=(d,x)$. Therefore, since (s,t) has P_1 as its associated path, by definition we know that (s,t) is a b-type edge. Thus, $(s,t)\in R\cap B$, which contradicts that $(s,t)\in A_1\subseteq[(A\cup R)\setminus(R\cap B)]$. Hence, we have $(s,t)\in A_1\setminus R$.

Next, we can establish Lemma 10 below to determine which edge (s',t') to replace (s,t) for the construction of a new auxiliary edge subset A', with a guarantee that the customer b_1 of P_1 is a descendant of t' in F^*. This ensures that such an edge (s',t') has a similar property to that of a b-type edge, and accordingly, as we will explain later, bounds shown in Lemma 4 and Lemma 5 for the b-type edges can be extended for (s',t'), which will be

Figure 6: Illustration of the construction of A' for the proof of Theorem 4, using the instance in Figure 2.



(a) Edges in A obtained by Algorithm 2 shown in dashed lines, where $(s,t) = (9,7) \in A_1 \setminus R$, and $(d,x) = (3,12)$ is the root edge of both (s,t) and $b_1 = 8$.

(b) Edges in $A' = A \setminus \{(s,t)\} \cup \{(s',t')\}\}$ shown in dashed lines, where $(s,t) = (9,7)$, and $(s',t') = (9,8) \in S \setminus R$ is on the path from $x = 12$ to $b_1 = 8$ in F^*.

useful in the later proof of Lemma 11 for a bound on $\ell(A')$.

Lemma 10. *There exists an edge $(s',t') \in S \setminus R$, where $s' \in I$ is the parent of t' in F^*, such that b_1 is a descendant of t' in F^*, and that $A' := A \setminus \{(s,t)\} \cup \{(s',t')\}$ is an auxiliary edge subset w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$.*

Proof Sketch. (See Online Appendix H for details.) First, by $(s,t) \in A_1 \setminus R$ and $A_1 \subseteq [(A \cup R) \setminus (R \cap B)]$ we have $(s,t) \in A$. Thus, it can be seen from Step 2 of Algorithm 2 that both s and t belong to the same connected component of $(V, E(\mathcal{P}) \cup A)$, denoted by G_{st}, and that G_{st} must be split into two connected components in $(V, E(\mathcal{P}) \cup (A \setminus \{(s,t)\}))$, one containing s, denoted by G_s, and the other containing t, denoted by G_t.

Next, as we have shown earlier, (d,x) is the root edge of both (s,t) and b_1 in F^*, and vertices s, t, and b_1 are all customers in I. Thus, s, t, and b_1 are all descendants of x in F^*. Let L denote the path from x to b_1 in F^*. We can show that there must exist an edge (s',t') on L, where s' is the parent of t' in F^*, such that $s' \in V(G_s)$ and $t' \in V(G_t)$. (See Figure 6(b) for illustration.) This implies that $(s',t') \in E(L) \setminus E(\mathcal{P})$, which, together with $E(L) \subseteq E(F^*)$ and $S = E(F^*) \setminus E(\mathcal{P})$, implies that $(s',t') \in S$. In addition, since x and b_1 are the endpoints of the path L in F^*, we obtain that $(s',t') \in S \setminus R$, $s' \in I$, and b_1 is a descendant of t' in F^*. Based on these, we can verify that $A' = A \setminus \{(s,t)\} \cup \{(s',t')\}$ is an auxiliary edge subset w.r.t $(F^*, \mathcal{C}^{\text{opt}})$, which completes the proof of Lemma 10. $\square$

Now, consider the auxiliary edge subset $A' = A \setminus \{(s,t)\} \cup \{(s',t')\}$ as defined in Lemma 10. We can establish Lemma 11 below to derive an upper bound on $\ell(A')$ shown in (22), which is smaller by $\ell(g_j)$ than the upper bound on $\ell(A)$ shown in (19).

Lemma 11. *The auxiliary edge subset A' w.r.t. $(F^*, \mathcal{C}^{\text{opt}})$ satisfies that*

$$\ell(A') \leq (|A_1| - 1)\ell(g_1) + \sum_{j=2}^{m} |A_j| \ell(g_j) + \sum_{j=1}^{m} \ell(h_j). \tag{22}$$

Proof Sketch. (See Online Appendix I for details.) Since $A_1, A_2, ..., A_m$ are m disjoint subsets that form a partition of $(A \cup R) \setminus (R \cap B)$, and since $(s,t) \in A_1$, we find that

$\sum_{e\in(A\cup R)\setminus(R\cap B)\setminus\{(s,t)\}} \ell(g_{j(e)}) = (|A_1|-1)\ell(g_1) + \sum_{j=2}^{m} |A_j|\ell(g_j)$. Thus, to prove (22) for Lemma 11, it is sufficient to show that $\ell(A') \le \sum_{e\in(A\cup R)\setminus(R\cap B)\setminus\{(s,t)\}} \ell(g_{j(e)}) + \sum_{j=1}^{m} \ell(h_j)$. We can show this by first extending the bounds for the b-type edges (in Lemma 4 and Lemma 5) to the edge (s', t'), and then following an argument similar to the proof of the upper bound on $\ell(A)$ given in (10) of Theorem 4. □

Hence, for the case with $|A_1| = k \ge 2$, we have shown in Lemma 11 that $A' = A \setminus \{(s,t)\} \cup \{(s',t')\}$, as defined in Lemma 10, is an auxiliary edge subset that satisfies $\ell(A') \le (|A_1|-1)\ell(g_1) + \sum_{j=2}^{m} |A_j|\ell(g_j) + \sum_{j=1}^{m} \ell(h_j)$. Thus, from (20) we obtain that $\ell(A) \le \sum_{j=1}^{m} [(k-1)\ell(g_j) + \ell(h_j)]$, and so Theorem 3 is proved. This completes the proof of (4).

7. Conclusions

In this work, we studied the GMDMTSP to find a solution where at most k salesmen as used. For this, we developed a non-trivial extension of the well-known Christofides heuristic which achieves a tight approximation ratio of $2 - 1/(2k)$. This is an improvement on the current best 2-approximation algorithms available in the literature. Moreover, the analysis developed here can be applied to other generalized multi-depot vehicle routing problems. See Online Appendix J for two such examples. Arising out of this work, a natural question for future research would be to find improved approximation algorithms for this problem. Since the best approximation algorithm known for the classical TSP achieves an approximation ratio of 3/2, It will be of significant theoretical and practical interest to determine whether or not there exists a 3/2-approximation algorithm for the GMDMTSP that runs in polynomial time.

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Online Appendix A. A Problem Instance Showing the Lower Bound on the Approximation Ratio of Algorithm 1

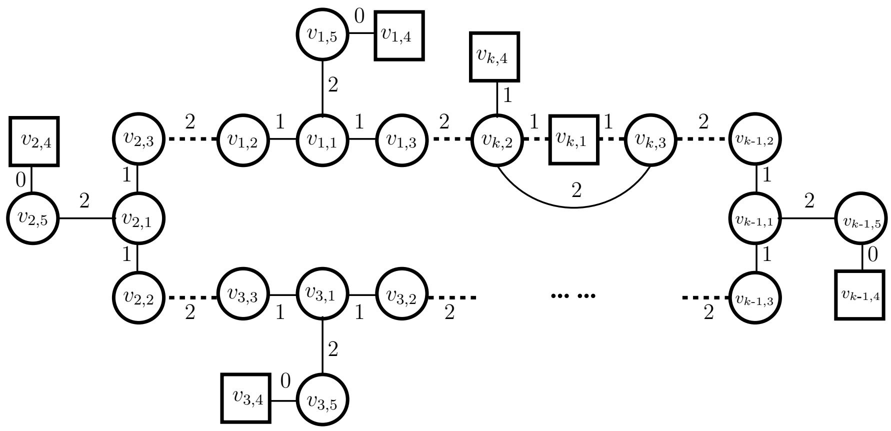
The instance below shows that the approximation ratio of Algorithm 1 is at least $2 - 1/(2k)$, for each $k \geq 1$. Consider the following complete graph $G = (V, E)$. Set the vertex set $V = \bigcup_{i=1}^{k} V_i$ where $V_i = \{v_{i,j} : 1 \leq j \leq 5\}$ for $1 \leq i \leq k-1$, and $V_k = \{v_{k,j} : 1 \leq j \leq 4\}$. Take the depot set $D = \{v_{i,4} : 1 \leq i \leq k\} \cup \{v_{k,1}\}$, so that $I = V \setminus D$ contains $|I| = 4k - 2$ customers. Next, we set lengths for the following edges, as shown in Figure A.7: $\ell(v_{i,1}, v_{i,2}) = \ell(v_{i,1}, v_{i,3}) = 1$, for $1 \leq i \leq k$; $\ell(v_{i,4}, v_{i,5}) = 0$ and $\ell(v_{i,1}, v_{i,5}) = \ell(v_{i,2}, v_{i+1,3}) = 2$, for $1 \leq i \leq k-1$; $\ell(v_{k,2}, v_{1,3}) = 2$; and $\ell(v_{k,2}, v_{k,4}) = 1$. Take lengths of the other edges of G to be the lengths of the shortest paths between their endpoints, so as to satisfy the triangle inequality. For example, we have $\ell(v_{k,2}, v_{k,3}) = \ell(v_{k,2}v_{k,1}v_{k,3}) = \ell(v_{k,2}, v_{k,1}) + \ell(v_{k,1}, v_{k,3}) = 2$.

For the instance above, we first show as follows that $\mathcal{C}^{\text{opt}} = \{C_1, ..., C_k\}$ is an optimal solution, where $C_i = (v_{i,4}v_{i,5}v_{i,4})$ for $1 \leq i \leq k-1$ and $C_k = (v_{k,1}v_{k,2}v_{1,3}v_{1,1}v_{1,2}v_{2,3}v_{2,1}v_{2,2}\cdots v_{k-1,3}v_{k-1,1}v_{k-1,2}v_{k,3}v_{k,1})$. It can be seen that $\mathcal{C}^{\text{opt}}$ is a feasible solution with $\ell(\mathcal{C}^{\text{opt}}) = 4k$. Consider any feasible solution $\mathcal{C}$. To show that $\mathcal{C}^{\text{opt}}$ is an optimal solution, it is sufficient to show that $\ell(\mathcal{C}) \geq 4k$. Let $E_0 = \{e \in E : \ell(e) = 0\}$ and $E_1 = \{e \in E : \ell(e) = 1\}$. We have $|E_0| = k - 1$ and $|E_1| = 2k + 1$. Since $(v_{k,2}, v_{k,4}) \in E_1$ and $(v_{k,2}, v_{k,1}) \in E_1$ join $v_{k,2}$ with different depots, at most one of them can appear in $\mathcal{C}$. Thus, since $\mathcal{C}$ must contain at least $|I| + 1$ edges, we have $\ell(\mathcal{C}) \geq 0 \times |E_0| + 1 \times (|E_1| - 1) + 2 \times [|I| + 1 - |E_0| - (|E_1| - 1)] = (2k + 1 - 1) + 2[4k - 2 + 1 - (k - 1) - 2k] = 4k = \ell(\mathcal{C}^{\text{opt}})$. Hence, $\mathcal{C}^{\text{opt}}$ is optimal.

Apply Algorithm 1 on the instance above. Let us consider a spanning forest $F^* = \{T_1, T_2, ..., T_k, T_{k+1}\}$, as shown in solid lines in Figure A.7, where each tree $T_i = (V_i, \{(v_{i,4}, v_{i,5}), (v_{i,5}, v_{i,1}), (v_{i,1}, v_{i,2}), (v_{i,1}, v_{i,3})\})$ for $1 \leq i \leq k-1$, tree $T_k = (\{v_{k,2}, v_{k,3}, v_{k,4}\}, \{(v_{k,4}, v_{k,2}), (v_{k,2}, v_{k,3})\})$, and tree $T_{k+1} = (\{v_{k,1}\}, \emptyset)$. We now show as follows that F^* is a shortest DCSF w.r.t. (G, D, k). To show this, since we know that F^* is a DCSF with $\ell(F^*) = 4k - 1$, we only need to show as follows that for any DCSF F, it satisfies that $\ell(F) \geq 4k - 1$. Let x denote the number of edges in E_0 that appear in F. Since $(v_{k,2}, v_{k,4}) \in E_1$ and $(v_{k,2}, v_{k,1}) \in E_1$ join $v_{k,2}$ with different depots, at most one of them can appear in F. Thus, since F must contain exactly $|I|$ edges, we find that $\ell(F) \geq 0 \times x + 1 \times (|E_1| - 1) + 2 \times [|I| - x - (|E_1| - 1)] = (2k + 1 - 1) + 2[4k - 2 - x - (2k + 1 - 1)] = 6k - 4 - 2x$. Thus, if $x \leq k - 2$, then $\ell(F) \geq 4k > 4k - 1 = \ell(F^*)$, and hence F^* is a shortest DCSF. Otherwise, $x \geq k - 1$, and thus by $x \leq |E_0| = k - 1$, we have $x = k - 1$. Hence, F must contain all $(k - 1)$ edges of E_0. Since these $(k-1)$ edges are all incident on depots, at most one edge among $(v_{k,4}, v_{k,2})$, $(v_{k,1}, v_{k,2})$, and $(v_{k,1}, v_{k,3})$ of E_1 can be included in F. Thus, $\ell(F) \geq 0 \times (k-1) + 1 \times (|E_1| - 2) + 2 \times [|I| - (k-1) - (|E_1| - 2)] = (2k+1-2) + 2[4k-2-(k-1)-(2k+1-2)] = 4k-1 = \ell(F^*)$, and so F^* is a shortest DCSF w.r.t. (G, D, k).

From F^* we find that $\text{Odd}(F^*) = \{v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4} : 1 \leq i \leq k - 1\} \cup \{v_{k,3}, v_{k,4}\}$, and so $|\text{Odd}(F^*)| = 4k - 2$. We next show that $M^*(F^*) := (\bigcup_{i=1}^{k-1}\{(v_{i,1}, v_{i,2}), (v_{i,4}, v_{i,3})\}) \cup$

Figure A.7: Illustration of the instance to show the lower bound on the approximation ratio of Algorithm 1.



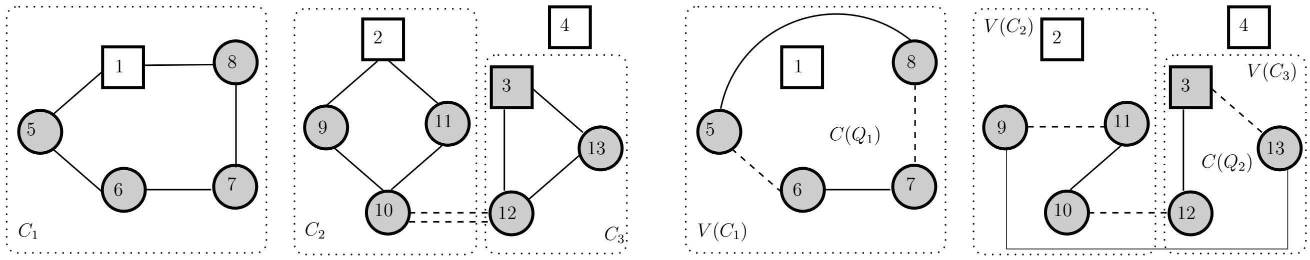
$\{(v_{k,4}, v_{k,3})\}$ is a minimum-weight perfect matching for $\mathrm{Odd}(F^*)$. It can be seen that $M^*(F^*)$ is a perfect matching for $\mathrm{Odd}(F^*)$ with $\ell(M^*(F^*)) = 4k - 1$. Consider any perfect matching M for $\mathrm{Odd}(F^*)$. We have $2|M| = |\mathrm{Odd}(F^*)|$. To show that $M^*(F^*)$ is minimum, it only needs to be shown that $\ell(M) \geq 4k - 1$. Let W denote the set of edges in M of length equal to 1. It can be seen that every edge $e \in W$ must cover a vertex $v_{i,1}$ for some i with $1 \leq i \leq k-1$, and that no edge $e \in W$ covers any $v_{i,4}$ with $1 \leq i \leq k$. By the triangle inequality, we have $\ell(e) \geq 2$ for every edge $e \in M \setminus W$. Let Z denote the set of edges in M that cover $v_{i,4}$ for some i with $1 \leq i \leq k$, such that either $i = k$, or $v_{i,1} \in V(W)$. Thus, $|Z| = |W| + 1$, and Z and W are disjoint. Moreover, for every edge $e \in Z$, using the triangle inequality, it can be seen that $\ell(e) \geq 3$, and that if $e = (v_{i,4}, v_{i',4})$ with $1 \leq i < i' \leq k$, then $\ell(e) \geq 6$. This implies that $\ell(Z) \geq 3|Z|$. Thus, we find that $\ell(M) = \ell(W) + \ell(Z) + \ell(M \setminus W \setminus Z) \geq 1 \times |W| + 3 \times |Z| + 2 \times |M \setminus W \setminus Z| = |W| + 3(|W|+1) + 2[|M| - |W| - (|W|+1)] = 2|M| + 1 = |\mathrm{Odd}(F^*)| + 1 = 4k - 1 = \ell(M^*(F^*))$, and so $M^*(F^*)$ is a minimum-weight perfect matching for $\mathrm{Odd}(F^*)$.

By adding $M^*(F^*)$ to F^*, we obtain a multigraph where every vertex is of even degree. Among the $k+1$ connected components of the multigraph, k components include at least one customer. Thus, from this multigraph, we can obtain k closed walks, including $(v_{i,4}v_{1,5}v_{i,2}v_{i,1}v_{i,2}v_{1,3}v_{i,4})$ for $1 \leq i \leq k-1$, and $(v_{k,4}v_{k,2}v_{k,3}v_{k,4})$, which include every customer. Thus, by removing repeated vertices, Algorithm 1 returns a feasible solution $\mathcal{C}(F^*) = \{C'_1, ..., C'_k\}$, where $C'_i = (v_{i,4}v_{i,5}v_{i,2}v_{i,1}v_{i,3}v_{i,4})$ for $1 \leq i \leq k-1$, and $C'_k = (v_{k,4}v_{k,2}v_{k,3}v_{k,4})$. Hence, $\ell(\mathcal{C}(F^*)) = \sum_{i=1}^{k-1} \ell(C'_i) + \ell(C'_k) = 8(k-1) + 6 = [2 - 1/(2k)]\ell(\mathcal{C}^{\mathrm{opt}})$, and so the approximation ratio of Algorithm 1 is at least $2 - 1/(2k)$. □

Online Appendix B. Proof of Lemma 1

Consider any auxiliary edge subset $\hat{A}$ w.r.t. $(F^*, \mathcal{C}^{\mathrm{opt}})$. As illustrated in Figure B.8(a), by duplicating edges in $\hat{A}$ and adding these edges to $\mathcal{C}^{\mathrm{opt}}$, we obtain a multigraph H on V, which has the same connected components as $(V, E(\mathcal{C}^{\mathrm{opt}}) \cup \hat{A})$, and satisfies $\ell(H) = \ell(\mathcal{C}^{\mathrm{opt}}) + 2\ell(\hat{A})$. It can also be seen that each vertex of H has an even degree. Thus, for each connected component Q of H, there exists an Eulerian closed walk, which covers all

Figure B.8: Illustration of the proof of Lemma 1, using the instance in Figure 2, where vertices in $\mathrm{Odd}(F^*)$ are in gray, and $\hat{A} = \{(12, 10)\}$, as shown in Example 2, is an auxiliary edge subset w.r.t $(F^*, \mathcal{C}^{\mathrm{opt}})$.



(a) Duplicating the edge in $\hat{A} = \{(12, 10)\}$, and adding them to $\mathcal{C}^{\mathrm{opt}}$.

(b) Shortcutting the Eulerian closed walk of each connected component in Figure B.8(a) by removing repeated vertices to obtain two disjoint matchings for vertices in $\mathrm{Odd}(F^*)$, as shown in dashed and solid lines.

vertices of Q. As illustrated in Figure B.8(b), from this closed walk, using shortcuts to remove repeated vertices, we can find a cycle $C(Q)$ that visits only vertices in $V(Q) \cap \mathrm{Odd}(F^*)$. Since $\hat{A}$ is an auxiliary edge subset w.r.t. $(F^*, \mathcal{C}^{\mathrm{opt}})$, by Definition 1 we know that each connected component Q of H contains an even number of vertices in $\mathrm{Odd}(F^*)$. Thus, $|V(Q) \cap \mathrm{Odd}(F^*)|$ must be even. This implies that $C(Q)$ consists of two disjoint perfect matchings for vertices in $V(Q) \cap \mathrm{Odd}(F^*)$, the shorter of which is denoted by $L(Q)$. Therefore, we obtain $\ell(L(Q)) \le \ell(C(Q))/2 \le \ell(Q)/2$.

Furthermore, by combining $L(Q)$ for all connected components Q of H, we can obtain a perfect matching M for vertices in $\mathrm{Odd}(F^*)$. Since $\ell(L(Q)) \le \ell(Q)/2$ for each connected component Q of H, we have $\ell(M) \le \ell(H)/2 = [\ell(\mathcal{C}^{\mathrm{opt}}) + 2\ell(\hat{A})]/2 = \ell(\mathcal{C}^{\mathrm{opt}})/2 + \ell(\hat{A})$. This, together with $\ell(M^*(F^*)) \le \ell(M)$, implies that $\ell(M^*(F^*)) \le \ell(\mathcal{C}^{\mathrm{opt}})/2 + \ell(\hat{A})$, which completes the proof of (7) of Lemma 1. $\square$

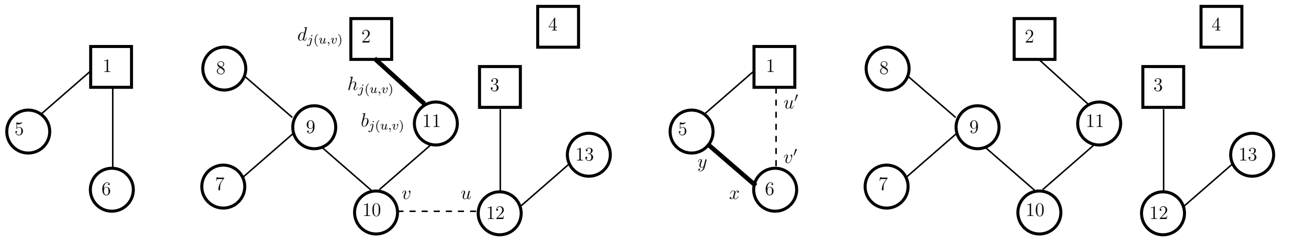
Online Appendix C. Proof of Lemma 3

Consider the auxiliary edge subset A w.r.t. $(F^*, \mathcal{C}^{\mathrm{opt}})$ constructed by Algorithm 2. To prove (i) of Lemma 3, we note that by definition, $\mathcal{P}$ contains no depot, which, together with (9), implies that $R \subseteq E(F^*) \setminus E(\mathcal{P}) = S$. Thus, since $A \subseteq S$, we know that $(A \cup R) \subseteq S$, and so (i) of Lemma 3 is proved.

To prove (ii) of Lemma 3, we note from Step 2 of Algorithm 2 that for each edge $(u, v) \in A \setminus R$ with $u \in I$ and $v \in I$, u and v must belong to two different paths of $\mathcal{P}$ that are not connected by any edges that are added to A before (u, v). Thus, at most $|\mathcal{P}| - 1$ such edges can be in A, and so $|A \setminus R| \le |\mathcal{P}| - 1 \le k - 1$. Hence, since $|R| \le k$, we obtain that $|A \cup R| = |A \setminus R| + |R| \le 2k - 1$, and so (ii) of Lemma 3 is proved.

To prove (iii) of Lemma 3, consider any two edges (u, v) and (u', v') in $A \cup R$ that have the same associated path $P \in \mathcal{P}$, where u is the parent of v in F^*, and u' is the parent of v' in F^*. By contradiction, suppose that (u, v) and (u', v') have the same root edge (d, x) for $d \in D$ and $x \in I$. Without loss of generality, suppose that (u, v) is examined before (u', v') in Step 2 of Algorithm 2. Moreover, since in Step 2 of Algorithm 2, edges in each tree of

Figure D.9: Illustration of the proof of Lemma 5, using the instance in Figure 2.



(a) $F := (F^* - \{(u,v)\}) + \{h_{j(u,v)}\}$, where $(u,v) = (12,10)$ and $h_{j(u,v)} = (2,11)$.

(b) $F' := (F - \{(u',v')\}) + \{(x,y)\}$, where $(u',v') = (1,6)$, $(x,y) = (6,5)$ is on $P_{j(u',v')} = P_1$, and (u',v') is the root edge of x but not of y.

F^* are examined in a non-decreasing order of their depths, we know that (d,x) must be examined before (u',v'). Thus, we obtain that $u' \in I$.

Consider the time when (u',v') is examined, and let A'' denote the set of edges added to A before (u',v'). From Step 2 of Algorithm 2 we know that, by this time, all edges on the path from x to v and the path from x to u' in F^* must have been examined. Thus, u' must belong to the same connected component of $(V, E(\mathcal{P}) \cup A'')$ as are all the vertices on the associated path P of (u,v) and (u',v'). Therefore, since P contains v', we find that (u',v') cannot be added to A, which, together with $(u',v') \in A \cup R$, implies that $(u',v') \in R \setminus A$. Thus, $u' \in D$, which contradicts that $u' \in I$. Hence, (u,v) and (u',v') must have different root edges, and so (iii) of Lemma 3 is proved. $\square$

Online Appendix D. Proof of Lemma 5

Consider any b-type edge $(u,v) \in B \setminus R$, where u is the parent of v in F^*, and consider any $(u',v') \in R \setminus B$, where u' is the parent of v' in F^*. To prove Lemma 5, we only need to show as follows that $\ell(u',v') + \ell(u,v) \leq \ell(g_{j(u',v')}) + \ell(h_{j(u,v)})$.

First, consider $F := (F^* - \{(u,v)\}) + \{h_{j(u,v)}\}$. Due to $(u,v) \in B$, $b_{j(u,v)}$ is a descendant of v in F^*. Thus, since $h_{j(u,v)}$ joins $b_{j(u,v)}$ and the depot $d_{j(u,v)}$ of path $P_{j(u,v)}$, it can be seen that F is a CSF w.r.t. (G,D), with the total degree of the roots no more than $k+1$. (See Figure D.9(a) for illustration.)

Next, from F we can construct a DCSF F' w.r.t. (G,D,k) as follows. Since $(u,v) \in B \setminus R$, we have $u \in I$. Since $(u',v') \in R \setminus B$, we have $u' \in D$. Thus, $(u,v) \neq (u',v')$, which implies that (u',v') is in both F^* and F. Moreover, by $(u',v') \in R \setminus B$, we know that (u',v') is not a b-type edge, and thus, by the definition of b-type edge, (u',v') cannot be the root edge of $b_{j(u',v')}$ in F^*. Let e^* indicate the root edge of $b_{j(u',v')}$ in F^*. We have $(u',v') \neq e^*$. Let e denote the root edge of $b_{j(u',v')}$ in F. From $F = (F^* - \{(u,v)\}) + \{h_{j(u,v)}\}$ it can be seen that e must either equal e^* or equal $h_{j(u,v)}$. Thus, we know $e \neq (u',v')$, because, if $e = (u',v')$, then since $(u',v') \neq e^*$, we have $e = h_{j(u,v)}$, which, together with $e = (u',v') \in R \subseteq E(F^*)$ and $F = (F^* - \{(u,v)\}) + \{h_{j(u,v)}\}$, implies that $E(F) \subseteq E(F^*)$, and so $e = (u',v')$ must also be the root edge of $b_{j(u',v')}$ in F^*, leading to a contradiction. Thus, $e \neq (u',v')$, which implies that (u',v') cannot be the root edge of $b_{j(u',v')}$ in F. However, (u',v') is the root edge of v' in F. Thus, since both $b_{j(u',v')}$ and v' are on the path $P_{j(u',v')}$, there must exist an edge (x, y) on $P_{j(u',v')}$ with $x \in I$ and $y \in I$, such that (u', v') is the root edge of x but is not the root edge of y in F. Hence, $F' := (F - \{(u', v')\}) + \{(x, y)\}$ is a CSF w.r.t. (G, D). Noting that $u' \in D$, $x \in I$, and $y \in I$, we find that the total degree of the roots in F' cannot exceed k. Therefore, F' is a DCSF w.r.t. (G, D, k). (See Figure D.9(b).)

Furthermore, since F^* is a shortest DCSF w.r.t. (G, D, k), we obtain that $\ell(F^*) \leq \ell(F')$, which implies that $\ell(u, v) + \ell(u', v') \leq \ell(h_{j(u,v)}) + \ell(x, y)$. Since edge (x, y) is on $P_{j(u',v')}$, which implies that $\ell(x, y) \leq \ell(g_{j(u',v')})$, we obtain that $\ell(u', v') + \ell(u, v) \leq \ell(g_{j(u',v')}) + \ell(h_{j(u,v)})$. This completes the proof of Lemma 5. □

Online Appendix E. Proof of Lemma 8

Consider the set $H = A \setminus R \setminus Y$ defined in (11). We prove properties (i), (ii), and (iii) of H as follows to establish Lemma 8. First, properties (i) and (ii) can be verified by the definition of H. To see this, consider each $(u, v) \in H$, where u is the parent of v in F^*. By $H = A \setminus R \setminus Y$, we have that H and $R \cap B$ are disjoint. Thus, (i) of Lemma 8 is proved. By $H = A \setminus R \setminus Y$, we also have that (u, v) is not a y-type edge. Thus, $b_{j(u,v)}$ must be a descendant of v in F^*, which implies that $(u, v) \in B$, and so $H \subseteq B$. Therefore, by (11) and $A \subseteq S$, we have $H \subseteq (A \setminus R \setminus Y) \cap B \subseteq (S \setminus R) \cap B$. Thus, (ii) of Lemma 8 is proved.

Next, to prove (iii) of Lemma 8, notice that $H \cup (R \cap B) \subseteq B$ (since $H \subseteq B$), and that $H \cup (R \cap B) \subseteq (A \cup R)$ (since $H \subseteq A$). By contradiction, suppose that $H \cup (R \cap B)$ contains two edges (u, v) and (u', v') both having P_j as the associated path, where u is the parent of v, and u' is the parent of v' in F^*. From $H \cup (R \cap B) \subseteq B$ we know that both (u, v) and (u', v') are b-type edges. Thus, b_j is a descendant of both v and v' in F^*. It follows that both (u, v) and (u', v') must have the same root edge in F^*. This, together with the fact that $H \cup (R \cap B) \subseteq (A \cup R)$ and (iii) of Lemma 3, implies that (u, v) and (u', v') must have different associated paths. This leads to a contradiction, and proves (iii) of Lemma 8. □

Online Appendix F. Proof of Lemma 9

Since $|R| \leq k$, we can prove Lemma 9 by taking into account the following two cases. For Case 1, where $|R| < k$, since (ii) of Lemma 8 implies that $H \subseteq B$, by Lemma 4 we have that $\ell(H) \leq \sum_{e \in H} \ell(h_{j(e)})$. Moreover, by Lemma 6 and Lemma 7, $\ell(R \backslash B) \leq \sum_{e \in R \backslash B} \ell(g_{j(e)})$. By Lemma 4, $\ell(R \cap B) \leq \sum_{e \in R \cap B} \ell(h_{j(e)})$. Thus, since $\ell(R) + \ell(H) = \ell(R \backslash B) + \ell(R \cap B) + \ell(H)$, we obtain that $\ell(R) + \ell(H) \leq \sum_{e \in R \backslash B} \ell(g_{j(e)}) + \sum_{e \in R \cap B} \ell(h_{j(e)}) + \sum_{e \in H} \ell(h_{j(e)})$. Lemma 9 is proved for Case 1.

For Case 2, where $|R| = k$, since (15) shows that $|H| + |R \cap B| \leq k = |R|$, we have $|H| \leq |R| - |R \cap B| = |R \setminus B|$. This allows us to arbitrarily select $|H|$ different edges of $R \setminus B$, so as to form a subset $\hat{R}$ of $R \setminus B$ such that $|\hat{R}| = |H|$. By Lemma 5, $\ell(\hat{R}) + \ell(H) \leq \sum_{e \in \hat{R}} \ell(g_{j(e)}) + \sum_{e \in H} \ell(h_{j(e)})$. Moreover, by Lemma 7 and Lemma 6, $\ell(R \setminus B \setminus \hat{R}) \leq \sum_{e \in R \setminus B \setminus \hat{R}} \ell(g_{j(e)})$. By Lemma 4, $\ell(R \cap B) \leq \sum_{e \in R \cap B} \ell(h_{j(e)})$. Thus, since $\hat{R} \subseteq (R \setminus B)$

implies that $\ell(R)+\ell(H)=\ell(R\setminus B)+\ell(R\cap B)+\ell(H)=[\ell(R\setminus B\setminus\hat{R})+\ell(\hat{R})]+\ell(R\cap B)+\ell(H)=\ell(R\setminus B\setminus\hat{R})+\ell(R\cap B)+[\ell(\hat{R})+\ell(H)]$, we obtain that $\ell(R)+\ell(H)\leq\sum_{e\in R\setminus B}\ell(g_{j(e)})+\sum_{e\in R\cap B}\ell(h_{j(e)})+\sum_{e\in H}\ell(h_{j(e)})$. Lemma 9 is proved for Case 2. $\square$

Online Appendix G. Proof of Theorem 3 for $k = 1$

To prove Theorem 3 for the case with $k=1$, we only need to show that when $k=1$, the set R of root edges in F^* is an auxiliary edge subset w.r.t. $(F^*, \mathcal{C}^{\mathrm{opt}})$ and satisfies that $\ell(R)\leq\sum_{j=1}^{m}[(k-1)\ell(g_j)+\ell(h_j)]$. Since $k=1$, we know that $\mathcal{P}$ contains only one path P_1, which must include all the customers. Thus, $m=|\mathcal{P}|=1$. From $k=1$, it can also be seen that R must contain only one root edge e' of F^*, and we use d' to denote the depot that e' is incident on. Notice that both F^* and $E(\mathcal{P})\cup R$ have only one connected component that connects r' and all the customers in I. Thus, by Lemma 2, R is an auxiliary edge subset w.r.t. $(F^*, \mathcal{C}^{\mathrm{opt}})$. Moreover, since e' is the only root edge of F^*, it must be a b-type edge. Thus, by Lemma 4 and $k=m=1$, we have $\ell(R)=\ell(e')\leq\ell(h_1)=\sum_{j=1}^{m}[(k-1)\ell(g_j)+\ell(h_j)]$, and so Theorem 3 is proved for the case with $k=1$. $\square$

Online Appendix H. Proof of Lemma 10

First, as we have shown earlier, $(s,t)\in A_1\setminus R$. Thus, since $A_1\subseteq[(A\cup R)\setminus(R\cap B)]$, we have $(s,t)\in A\setminus R$. Since $(s,t)\in A$, it can be seen from Step 2 of Algorithm 2 that both s and t belong to the same connected component of $(V, E(\mathcal{P})\cup A)$, denoted by G_{st}, and that G_{st} must be split into two connected components in $(V, E(\mathcal{P})\cup(A\setminus\{(s,t)\}))$, one containing s, denoted by G_s, and the other containing t, denoted by G_t.

Next, we can determine edge (s',t') as follows. As shown earlier, (d,x) is the root edge of both (s,t) and b_1 in F^*, and vertices s, t, and b_1 are all customers in I. Thus, s, t, and b_1 are all descendants of x in F^*. Let L denote the path from x to b_1 in F^*. We have that vertices s, t, and vertices in $V(L)$ (including b_1 and x) are in the same connected component of F^*. Thus, since A satisfies the condition in Lemma 2, we obtain that $V(L)\subseteq V(G_{st})$. Moreover, from Step 2 of Algorithm 2 it can be seen that for every edge e on the path from x to s in F^*, since its depth is smaller than the depth of (s,t), edge e must be examined before (s,t) is added to A. This implies that x and s must belong to the same connected component of $(V, E(\mathcal{P})\cup(A\setminus\{(s,t)\}))$. Thus, $x\in V(G_s)$. It can also be seen that since both b_1 and t are on P_1, they must belong to the same connected component of $(V, E(\mathcal{P})\cup(A\setminus\{(s,t)\}))$. Thus, $b_1\in V(G_t)$. Therefore, since $x\in V(G_s)$ and $b_1\in V(G_t)$ are the endpoints of the path L, there must exist an edge (s',t') on L, where s' is the parent of t' in F^*, such that $s'\in V(G_s)$ and $t'\in V(G_t)$. (See Figure 6(b) for illustration.)

Moreover, by the definition of (s',t') above, it can be seen that s' and t' must belong to different connected components of $(V, E(\mathcal{P})\cup(A\setminus\{(s,t)\}))$. Thus, $(s',t')\in E(L)\setminus E(\mathcal{P})$, which, together with $E(L)\subseteq E(F^*)$ and $S=E(F^*)\setminus E(\mathcal{P})$, implies that $(s',t')\in S$. In addition, since x and b_1 are the endpoints of the path L in F^*, we obtain that $(s',t') \in S \setminus R$, $s' \in I$, and b_1 is a descendant of t' in F^*.

Now, let us consider $A' := A \setminus \{(s,t)\} \cup \{(s',t')\}$. Since $s' \in V(G_s)$ and $t' \in V(G_t)$, $(V, E(\mathcal{P}) \cup A')$ must have the same connected components as $(V, E(\mathcal{P}) \cup A)$. Thus, the same as with A, the edge subset A' must also satisfy the condition in Lemma 2, implying that A' is an auxiliary edge subset w.r.t $(F^*, \mathcal{C}^{\text{opt}})$. This completes the proof of Lemma 10. $\square$

Online Appendix I. Proof of Lemma 11

Consider the auxiliary edge subset $A' := A \setminus \{(s,t)\} \cup \{(s',t')\}$ as defined in Lemma 10. To prove Lemma 11, we only need to show as follows that A' satisfies (22), i.e., $\ell(A') \leq (|A_1| - 1)\ell(g_1) + \sum_{j=2}^{m} |A_j|\ell(g_j) + \sum_{j=1}^{m} \ell(h_j)$.

Since $A_1, A_2, ..., A_m$ are m disjoint subsets that form a partition of $(A \cup R) \setminus (R \cap B)$, and since $(s,t) \in A_1$, we find that $\sum_{e \in (A \cup R) \setminus (R \cap B) \setminus \{(s,t)\}} \ell(g_{j(e)}) = (|A_1| - 1)\ell(g_1) + \sum_{j=2}^{m} |A_j|\ell(g_j)$. Thus, to prove (22) for Lemma 11, it is sufficient to show that

$$\ell(A') \leq \sum_{e \in (A \cup R) \setminus (R \cap B) \setminus \{(s,t)\}} \ell(g_{j(e)}) + \sum_{j=1}^{m} \ell(h_j). \tag{I.1}$$

Our proof of (I.1) for $\ell(A')$ is similar to the proof of (10) in Theorem 4 for $\ell(A)$. Recall that $H = A \setminus R \setminus Y$ as defined in (11) for the proof of (10). Thus, we define that

$$H' := H \cup \{(s',t')\}. \tag{I.2}$$

Note that $(A' \setminus R \setminus H') = [A \setminus \{(s,t)\} \cup \{(s',t')\} \setminus R \setminus H \setminus \{(s',t')\}] \subseteq (A \setminus R \setminus H \setminus \{(s,t)\})$. Thus, since $\ell(A') \leq \ell(R) + \ell(H') + \ell(A' \setminus R \setminus H')$, we obtain that

$$\ell(A') \leq \ell(R) + \ell(H') + \ell(A \setminus R \setminus H \setminus \{(s,t)\}). \tag{I.3}$$

Moreover, from (12) we know that each $e \in (A \setminus R \setminus H \setminus \{(s,t)\})$ is a y-type edge, which, together with Lemma 6 and (I.3), implies the following upper bound on $\ell(A')$,

$$\ell(A') \leq \ell(R) + \ell(H') + \sum_{e \in A \setminus R \setminus H \setminus \{(s,t)\}} \ell(g_{j(e)}), \tag{I.4}$$

which is similar to the upper bound on $\ell(A)$ shown in (14).

We are now going to derive an upper bound on $\ell(R) + \ell(H')$, which is similar to the upper bound on $\ell(R) + \ell(H)$ in Lemma 9. First, noting that $H' = H \cup \{(s',t')\}$, we can prove as follows that

$$|H'| + |R \cap B| \leq k, \tag{I.5}$$

which strengthens the inequality $|H| + |R \cap B| \leq k$ in (15). To prove (I.5), we know from definition of A_1 and (iii) of Lemma 3 that there exist at least $|A_1| = k \geq 2$ vertices on P_1 that have different root edges. Since $A_1 \subseteq (A \cup R) \setminus (R \cap B)$, it can be seen from the definition of A_1 and (iii) of Lemma 3 that edges in A_1 all have P_1 as their associated path, but have different root edges in F^*. Thus, since $|A_1| = k \geq |R|$ and $(R \cap B) \cap A_1 = \emptyset$, by (iii) of Lemma 8 we obtain that no edge in $R \cap B$ can have P_1 as its associated path. Moreover, since there are at least two vertices on P_1 that have different root edges, every edge that has P_1 as its associated path must be a y-type edge. Thus, since $H = A \setminus R \setminus Y$, no edge in H can have P_1 as its associated path. Hence, we have obtained that no edge in H or $R \cap B$ can have P_1 as its associated path. Therefore, by (i) and (iii) of Lemma 8 we have $|H| + |R \cap B| \leq |\mathcal{P}| - 1 \leq k - 1$. Thus, we obtain (I.5) by $|H'| + |R \cap B| \leq |H| + 1 + |R \cap B| \leq k - 1 + 1 = k$.

Next, let us examine edge (s', t'), which, by definition, is in $S \setminus R$. Due to Lemma 10, we know that b_1 of path P_1 is a descendant of t' in F^*, and so (s', t') is similar to a b-type edge. Thus, although (s', t') may not have P_1 as its associated path, it can be verified that the inequalities in Lemma 4 and Lemma 5 for edges (u, v) with $j(u, v)$ replaced by 1 are also valid for (s', t'). Note that by (ii) of Lemma 8, $H \subseteq B \setminus R$, which, together with $(s', t') \in S \setminus R$, implies that $H' \subseteq (B \cup \{(s', t')\}) \setminus R$. Thus, define $i(u, v) := j(u, v)$ for $(u, v) \in H$, and $i(s', t') := 1$. We have that $b_{i(u,v)}$ is a descendant of v in F^* for each $(u, v) \in H'$, and that Lemma 4 and Lemma 5 are also valid for each $(u, v) \in H'$ if they are revised by replacing $j(u, v)$ with $i(u, v)$.

Accordingly, by replacing H with H', replacing $j(e)$ with $i(e)$ for $e \in H'$, and applying the revised Lemma 4 and Lemma 5 for $e \in H'$, we can follow an argument similar to the proof of Lemma 9 to obtain the following upper bound on $\ell(R) + \ell(H')$:

$$\ell(R) + \ell(H') \leq \sum_{e \in R \setminus B} \ell(g_{j(e)}) + \sum_{e \in R \cap B} \ell(h_{j(e)}) + \sum_{e \in H'} \ell(h_{i(e)}), \tag{I.6}$$

which is similar to the upper bound on $\ell(R) + \ell(H)$ in Lemma 9, with H replaced by H', and $j(e)$ replaced by $i(e)$ for $e \in H'$.

From (I.4) and (I.6) we have that

$$\ell(A') \leq \sum_{e \in R \setminus B} \ell(g_{j(e)}) + \sum_{e \in R \cap B} \ell(h_{j(e)}) + \sum_{e \in H'} \ell(h_{i(e)}) + \sum_{e \in A \setminus R \setminus H \setminus \{(s,t)\}} \ell(g_{j(e)}). \tag{I.7}$$

Since, as shown above, no edge in $R \cap B$ or H can have P_1 as its associated path, by the definition of $i(e)$ for $e \in H'$, and by (iii) of Lemma 8, we find that

$$\sum_{e \in R \cap B} \ell(h_{j(e)}) + \sum_{e \in H'} \ell(h_{i(e)}) = \sum_{e \in R \cap B} \ell(h_{j(e)}) + \sum_{e \in H} \ell(h_{j(e)}) + \ell(h_{i(s',t')}) \leq \sum_{j=1}^{m} \ell(h_j). \tag{I.8}$$

Moreover, $(s,t) \in A_1 \setminus R$ implies that $(R \setminus B) \cup (A \setminus R \setminus H \setminus \{(s,t)\})$ is a subset of $(A \cup R) \setminus (R \cap B) \setminus \{(s,t)\}$. Thus, since $(R \setminus B)$ and $(A \setminus R \setminus H \setminus \{(s,t)\})$ are disjoint,

$$\sum_{e \in R \setminus B} \ell(g_{j(e)}) + \sum_{e \in A \setminus R \setminus H \setminus \{(s,t)\}} \ell(g_{j(e)}) \leq \sum_{e \in (A \cup R) \setminus (R \cap B) \setminus \{(s,t)\}} \ell(g_{j(e)}). \tag{I.9}$$

Therefore, by (I.7), (I.8), and (I.9), we can obtain (I.1) as follows:

$$\ell(A') \leq \sum_{e \in R \setminus B} \ell(g_{j(e)}) + \sum_{j=1}^{m} \ell(h_j) + \sum_{e \in A \setminus R \setminus H \setminus \{(s,t)\}} \ell(g_{j(e)}) \leq \sum_{e \in (A \cup R) \setminus (R \cap B) \setminus \{(s,t)\}} \ell(g_{j(e)}) + \sum_{j=1}^{m} \ell(h_j),$$

which completes the proof of Lemma 11. $\square$

Online Appendix J. Applications to Other Vehicle Routing Problems

The analysis developed earlier for the GMDMTSP can be applied to other generalized multi-depot vehicle routing problems. We do this here by describing two such applications.

Online Appendix J.1. The generalized multi-depot multi-stacker-crane problem

The stacker crane problem (SCP) is defined on a mixed graph $G = (V, Z \cup E)$, where Z is a set of directed arcs, and E a set of undirected edges. The problem seeks the shortest cycle which includes each arc of Z exactly once. Thus, given a depot set $D \subset V$ and $k \geq 1$, the generalized multi-depot multi-stacker-crane problem (GMDMSCP) seeks the shortest collection of at most k cycles that start and end at distinct depots in D, with the objective to visit each arc of Z exactly once. Frederickson et al. (1978) provided two heuristics for the SCP, known as LARGEARCS and SMALLARCS. The better solution returned by these heuristics achieves an approximation ratio of 9/5. Here, we show it possible to extend these heuristics to solve the GMDMSCP based on the results we obtained for the GMDMTSP.

Without loss of generality, assume that arcs of Z are not incident to any depot of D (otherwise, we can construct an equivalent instance by making a copy of each depot and move all edges incident to the depot to its copies, such that the resulting instance satisfies the assumption). Use $\mathcal{C}^{\text{opt}}$ to denote the optimum solution to the GMDMSCP.

The LARGEARCS in Frederickson et al. (1978) inserts into G a minimum bipartite matching M between heads and tails of arcs of Z, contracts connected components of the resulting graph to obtain a contracted graph $\hat{G}$, and then constructs a minimum spanning tree of the contracted graph. To solve the GMDMSCP, we construct a shortest DCSF F^* w.r.t. $(\hat{G}, D, k)$. It is easy to verify that $\ell(F^*) \leq \ell(\mathcal{C}^{\text{opt}}) - \ell(Z)$. Following the same steps as in LARGEARCS in Frederickson et al. (1978), we can construct from F^*, M, and Z a solution $\mathcal{C}_L$ to the GMDMSCP such that $\ell(\mathcal{C}_L) \leq 3\ell(\mathcal{C}^{\text{opt}}) - 2\ell(Z)$.

The SMALLARCS heuristic in Frederickson et al. (1978) contracts arcs of Z to obtain a contracted graph G', and then applies the Christofides heuristic to obtain a solution to the TSP on G'. To solve the GMDMSCP, we take G' to include depots of D, and then apply Algorithm 1 to obtain a solution $\mathcal{C}'$ to the GMDMTSP on G' and D. By Theorem 1, $\ell(\mathcal{C}') \leq [2 - 1/(2k)][\ell(\mathcal{C}^{\text{opt}}) - \ell(Z)]$. Following the same steps as in SMALLARCS, we can then construct from $\mathcal{C}'$ and Z a solution $\mathcal{C}_S$ to the GMDMSCP with $\ell(\mathcal{C}_S) \leq [2 - 1/(2k)]\ell(\mathcal{C}^{\text{opt}}) + [1/(2k)]\ell(Z)$.

Hence, the shorter of the solutions L and S returned by LARGEARCS and SMALLARCS achieves an approximation ratio of $[2 - 1/(4k+1)]$ for the GMDMSCP.

Online Appendix J.2. The generalized clustered MDMTSP

Given a collection of clusters that forms a partition of the vertex set, the clustered TSP seeks to determine the shortest cycle that visits every vertex exactly once and vertices of each cluster consecutively. Thus, given a depot set $D \subset V$ and $k \geq 1$, the generalized clustered MDMTSP (GCMDMTSP) seeks the shortest collection of at most k cycles that start and end at distinct depots in D, so as to visit each customer in $V \setminus D$ exactly once, with vertices of each cluster being visited consecutively. Anily et al. (1999); Guttmann-Beck et al. (2000) proposed several approximation algorithms for the clustered TSP. These algorithms can be extended for the GCMDMTSP.

For example, consider the case with given start and end vertices for each cluster. In Guttmann-Beck et al. (2000), a (21/11)-approximation scheme is used to solve the corresponding variant of the clustered TSP. The first step of their method is to determine a Hamiltonian path for each cluster with the start and end vertices of the cluster as its endpoints. We use $\mathcal{P}$ to denote the collection of these paths. The second step solves the stacker-crane problem on a graph with directed arcs from start to end vertices of the clusters, and as a result, it obtains a tour that can be transformed to a solution of the clustered TSP by replacing directed arcs with the corresponding Hamiltonian paths of the first step. To solve the GCMDMTSP, we solve in the second step the GMDMSCP on the same graph, depot set D, and k, and we denote the approximation solution obtained for the GMDMSCP by $\mathcal{C}_{\text{SCP}}$. Let $\mathcal{C}$ represent the approximation solution obtained for the GCMDMTSP, $\mathcal{C}^{\text{opt}}$ the optimal solution to the GCMDMTSP, Y the set of edges of $\mathcal{C}^{\text{opt}}$ that connect different clusters, X the set of edges in $\mathcal{C}^{\text{opt}}$ but not in Y, and Z the set of directed arcs from the start to the end vertices of the clusters. By following a similar analysis as in Guttmann-Beck et al. (2000) and using the analysis developed in Section Online Appendix J.1 for the GMDMSCP, we can obtain the following: (i) $\ell(\mathcal{C}) = \ell(\mathcal{P}) - \ell(Z) + \ell(\mathcal{C}_{\text{SCP}})$; (ii) $\ell(\mathcal{P}) \leq \min\{2\ell(X) - \ell(Z), (3/2)\ell(X) + (1/2)\ell(Z)\} \leq \alpha[2\ell(X) - \ell(Z)] + (1-\alpha)[(3/2)\ell(X) + (1/2)\ell(Z)]$ for every $0 \leq \alpha \leq 1$; (iii) $\ell(\mathcal{C}_{\text{SCP}}) \leq \min\{3\ell(Y) + \ell(Z), [2 - (1/2k)]\ell(Y) + 2\ell(Z)\} \leq \beta[3\ell(Y) + \ell(Z)] + (1-\beta)\{[2 - (1/2k)]\ell(Y) + 2\ell(Z)\}$ for every $0 \leq \beta \leq 1$. By choosing $\alpha = (8k+1)/(8k+3)$ and $\beta = 3/(8k+3)$, we have $\ell(\mathcal{C}) = \ell(\mathcal{P}) - \ell(Z) + \ell(\mathcal{C}_{\text{SCP}}) \leq [2 - 1/(8k+3)][\ell(X) + \ell(Y)] = [2 - 1/(8k+3)]\ell(\mathcal{C}^{\text{opt}})$. This achieves an approximation ratio of $[2 - 1/(8k+3)]$ for the GCMDMTSP with given start and end vertices of each cluster.