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Dynamic programming for optimal ship refueling decision

Abstract**:** This study investigates an optimal control policy for a liner ship to decide at which ports and how much fuel the liner ship should be refueled under stochastic fuel consumption in each leg and stochastic fuel price at each port. Based on some properties proved in this study, a dynamic programming algorithm is then designed to obtain some important threshold values, which are used in the optimal control policy for ship refueling decision. Extensive experiments show that the proposed method can obtain the optimal decision within a reasonable time (about 170 seconds) for various scales of problem instances (up to 30 ports) as well as various settings of probability distributions. In addition, some comparative experiments also show that the proposed optimal decision policy can save at least 8% fuel consumption cost by comparing with some relatively simple rules and save about 1% cost on average by comparing with some brilliantly-designed rules.

Keywords**:** maritime transportation; dynamic programming; optimal control; liner shipping.

1. Introduction

The fluctuating world oil price and world maritime transportation market bring a lot of uncertainties to operations management of shipping liners. This study investigates an optimal control policy for a liner ship to decide at which ports and how much fuel the liner ship should be refueled. This paper studies this refueling decision problem for a liner ship. Suppose a ship visits a given sequence of ports. Due to the limitation of its tank volume, the fuel in the tank usually cannot support the ship to fulfill the whole sailing process without refueling at some ports in the voyage. Then where the ship should be refueled during its voyage as well as how much fuel the ship should be refueled at a port becomes an important decision for shipping companies as the bunker fuel expenditure is a main part of the total operation cost of a liner ship (or cruise ship).

If all the fuel consumption during each leg in the voyage can be exactly estimated and the fuel price at each port in the voyage is fixed, this decision problem is trivial. In reality, the fuel price is usually different at each port and also fluctuates frequently. But the voyage time is usually a relatively long time, which implies the fuel price may have a significant dynamic nature during a ship's voyage.

The fluctuation of fuel price is common in reality. The fuel price may sharply decline (or increase) at the same port in one month. Moreover, the fuel price at different ports may be quite different even on the same day. For example, the fuel price gap between Singapore and Vancouver was $54/ton on 15 Dec, 2015. In addition, a liner ship's voyage usually covers a relatively long period. There are usually significant changes of fuel price during liner ships' voyages in reality. Therefore, the ship refueling decision policy should also consider the stochastic feature of the fuel price at each port.

If the fuel price at each port can be exactly predicted, it is also not difficult to solve the optimal refuel decision for the ship in order to minimize the total fuel cost. However, both the fuel consumption during each leg and the fuel price at each port during the voyage are stochastic. Although the route length of each leg from one port to another port is deterministic, the fuel consumption in the leg is uncertain because it is influenced by many factors such as sailing speed, draft, trim, weather/sea conditions (e.g., wind, waves, sea currents and sea water temperature) and the consumption of power for all types of facilities on the ship. Due to the unpredictable number of containers the ship needs to unload or load at each port, the ship's dwell time at ports becomes uncertain. Then the ship must adjust its sailing speed so as to meet its scheduled arrival time for its next port; otherwise the ship may be punished by waiting a long time at the anchorage of ports. The sailing speed significantly affects the fuel consumption of a leg. In addition, the uncertain weight of cargos or passengers (mainly for cruise ships) during each leg will influence the ship's draft, which also further incurs the uncertain fuel consumption for each leg. Moreover, the weather/sea conditions are difficult to capture. All of these factors make the fuel consumption during each leg become uncertain.

Based on the above analysis, if both the fuel consumption during each leg and the fuel price at each port during the voyage are stochastic, how to obtain the optimal policy for refueling a liner ship is an interesting problem for the shipping liner. This paper makes an explorative study on this problem. Several properties are proved in this study. On the basis of these properties, a dynamic programming algorithm is then proposed to obtain some important threshold values, which are used in the optimal control policy for ship refueling decision. Numerical experiments are conducted on the basis of a real world example of a cruise ship. The numerical results validate the effectiveness of the proposed method.

2. Literature review

The refueling decision is similar to the inventory replenishment problem to some extent if the fuel in a ship's tanker is regarded as the parts in a warehouse. The optimal refueling decision can also be investigated by borrowing the idea of the optimal inventory control (Lee et al. 2006), for which the methodology of dynamic programming is widely used (Meng and Wang 2011; Chen et al. 2004). Different from the optimal inventory control, the study on the optimal refueling decision policy for a liner ship needs to consider special features originating from the maritime shipping backgrounds. This study concerns two factors: one is the fuel consumption of ships, the other is the fuel prices. Related works are discussed mainly through these two aspects.

The fuel consumption is mainly influenced by the speed, draft, trim, and weather/sea conditions. Qi and Song (2012) and Wang and Meng (2012) studied ship schedule design problems with considering the speed of a ship; uncertain port time is taken into account in these two studies. Lindstad et al. (2013) considered the factor of ship draft to design a new bulk ship for decreasing fuel cost. Yang et al. (2014) identified the optimal trim configuration to improve ship energy efficiency. Zhen et al. (2016, 2017) examined port operations considering vessels' fuel consumption. Plenty of recent studies considered weather conditions and optimized ship routes to minimize the bunker fuel consumption (Lin et al. 2013; Zhang et al. 2013; Fang and Lin 2015). Besikci et al. (2016) developed an artificial neural network model to predict fuel consumption for various operational conditions, which can be used on a real time basis for energy efficient ship operations. By using the shipping log data available in practice, Meng et al. (2016) put forward a practical method to combine the fuel consumption rate with a lot of determinants such as sailing speed, draft, sea, and weather conditions.

Few of the fuel consumption related works have mentioned the refueling decisions. Kim et al. (2012) considered the optimum ship speed, refueling ports and amounts of fuel for a given ship's route, and developed an epsilon-optimal algorithm to solve the problem. Its main contribution lies in considering a lot of realistic factors such as bunker prices, carbon taxes, and greenhouse gas emissions. Yao et al. (2012) designed a bunker fuel management strategy for a single shipping liner service, which includes refueling ports selection, refueling amounts determination and ship speeds adjustment; the complexity of this problem mainly lies in the fact that these three decisions are highly interrelated. Although the above studies considered a lot of complex factors, the fuel price at each port is given as deterministic parameters in their problems, which is different from the setting of uncertain fuel price in this study. In reality, contracts are often used for bunker purchasing, ensuring supply and often giving a discounted price (Pedrielli et al. 2015). Plum et al. (2014) proposed a mixed-integer programming model for bunker purchasing with contract and designed a column generation algorithm to solve the model. Ghosh et al. (2015) studied a refueling decision problem, which considered some special issues. For example, the ship has a contract with fuel supplier; ship can choose to refuel at the contract price or the spot price at a port; there is a penalty if the ship does not use up a certain amount of fuel stated in the contract. For the above two contract related studies, the second one considered more complex (realistic) factors in contract design, but is mainly oriented for one ship route; while the first one is oriented for a shipping network. Considering the real sailing speed may deviate from the planned one, Wang and Meng (2015) examined the ship speed and refueling decision in a liner shipping network, and developed a mixed-integer nonlinear optimization model under the worst-case bunker consumption scenario. One of its main contributions lies in proposing a close-form expression for the worst-case bunker consumption. Sheng et al. (2015) proposed a dynamic (s,S) policy for a liner shipping refueling and speed determination problem. Two variations of the progressive hedging algorithm were designed to tackle large-scale problem instances. Meng et al. (2015) considered different bunker prices at different ports for a tramp ship routing decision problem. The main contribution is the design of a branch-and-price based exact solution method for maximizing the total profit by routing ships to carry the given cargoes as well as determining the amount of bunker refueled at each port.

By comparing with the related literature, the contribution of this study mainly includes two aspects:

(1) The problem setting in this paper is much more general than those in the refueling decision related literature, which usually assumes that the fuel price at each port is known as deterministic parameters, and the fuel consumption at each leg is usually defined as a function of the sailing speed. However, this paper investigates a more general ship refueling decision problem, in which both the fuel price at each port and the fuel consumption at each leg are stochastic parameters.

(2) This paper proposes an optimal refueling decision policy for liner ships when facing the stochastic parameters of the fuel price at each port and the fuel consumption at each leg. In operations research (OR) related literature, the usual practice of modeling a problem with uncertain parameters is to use stochastic programming, based on which it may be difficult to design exact solution methods and then solve the original problem in large-scale instances. This study focuses on a rather general problem background and uses a dynamic programming methodology to obtain an optimal refueling policy. For practitioners, the optimal refueling policy is easier to use in realistic decision environment than the mathematical programming model based method as we demonstrate that the optimal refueling policy is a threshold-based one.

This study does not consider factors such as the special contracts of refueling or carbon emission. However, this study offers an optimal solution method for a simple but still practical problem.

3. Mathematical model

This section addresses a dynamic programming model for the ship refueling decision problem under uncertain fuel consumption and price. First some notations are defined as follow:

Index and sets

i index of a port or a leg. In a ship route, leg i is from port i to port $i+1$.

I set of the ports or set of the legs.

Decision variables

x_i amount of fuel added at port i.

Parameters

n number of ports, $n := |I|$.

V capacity of the tank for storing fuel.

p_i *actual* fuel price when the ship arrives at port i.

$\tilde{p}_i$ *stochastic* fuel price when the ship arrives at port i.

$\overline{p}_i$ expected value of $\tilde{p}_i$, that is $\overline{p}_i = \mathbb{E}\{\tilde{p}_i\}$.

p_i^{low} lower bound (LB) of $\tilde{p}_i$.

p_i^{high} upper bound (UB) of $\tilde{p}_i$.

q_i^c *actual* amount of fuel consumed in leg i.

$\tilde{q}_i^c$ *stochastic* amount of fuel consumed in leg i.

q_i^{cMin} LB of $\tilde{q}_i^c$.

q_i^{cMax} UB of $\tilde{q}_i^c$.

q_i^r *actual* amount of remaining fuel when the ship arrives at port i.

$\tilde{q}_i^r$ *stochastic* amount of remaining fuel when the ship arrives at port i.

Constraints and objective

We have the following relations due to the conservation of fuel:

$$\tilde{q}_{i+1}^r = \tilde{q}_i^r + x_i - \tilde{q}_i^c \tag{1}$$

Eq. (1) states the remaining fuel when the ship arrives at the port $i+1$ equals to the remaining fuel at the port i plus the fuel added at the port i and minus the fuel consumption in leg i (from the port i to the port $i+1$).

For a realization, we have:

$$q_{i+1}^r = q_i^r + x_i - q_i^c \tag{2}$$

Eq. (2) states the case in a realization; its meaning is similar as the above explanation for Eq. (1).

We assume that if $q_i^r < q_i^{cMax}$, the fuel in the tank is added to at least q_i^{cMax} no matter how much the fuel price at port i is. Hence

$$x_i \geq \max\left\{0\ ;\ q_i^{cMax} - q_i^r\right\} \tag{3}$$

$$x_i \leq V - q_i^r \tag{4}$$

Eq. (3) states that the amount of the added fuel at the port i should guarantee the ship can arrive at the next port; Eq. (4) ensures the capacity of the tank for storing fuel should not be exceeded after refueling at the port i.

Two functions are defined as follows:

$c_i\left(p_i, q_i^r, x_i\right)$ the minimum total expected cost from port i to the last port if the fuel price at port i is p_i, the amount of the remaining fuel at port i is q_i^r, and the amount of fuel added at port i is x_i.

$u_i\left(p_i, q_i^r\right)$ the minimum total expected cost from port i to the last port if the fuel price is p_i, the amount of the remaining fuel is q_i^r, and we add the optimal amount of the fuel in the tank. That is:

$$u_i\left(p_i, q_i^r\right) = \min_{x_i \in \left[\max\left\{0;\ q_i^{cMax} - q_i^r\right\},\ V - q_i^r\right]} \left\{c_i\left(p_i, q_i^r, x_i\right)\right\} \tag{5}$$

The Bellman equations for the problem are:

$$\begin{aligned} c_i\left(p_i, q_i^r, x_i\right) &= x_i \cdot p_i + \mathbb{E}\left\{u_{i+1}\left(\tilde{p}_{i+1}, \tilde{q}_{i+1}^r\right)\right\} \\ &= x_i \cdot p_i + \mathbb{E}\left\{u_{i+1}\left(\tilde{p}_{i+1}, q_i^r + x_i - \tilde{q}_i^c\right)\right\} \end{aligned} \tag{6}$$

The Bellman equation writes the minimum expected cost at a certain port in terms of the payoff from some initial refueling decisions and the value of the remaining decision problem that results from those initial refueling decisions. This breaks a dynamic optimization problem into simpler subproblems. The Bellman equation is widely used in the dynamic programming related studies (Hsu et al. 2011; Huang and Liang 2011; Meng and Wang 2011; Zhen 2012).

The boundary condition is:

$$u_n(p_n, q_n^r) = 0 \tag{7}$$

Then according to the above definition, the objective of the problem is to find the optimal refueling policy to minimize $u_1(p_1, q_1^r)$.

4. Optimal refueling policy

This section addresses some properties of the optimal refueling policy. We define $x_i^*(p_i, q_i^r)$ as the optimal amount of fuel to refill at port i if the fuel price is p_i and the amount of the remaining fuel is q_i^r.

Lemma 1: At port $n-1$ the optimal refueling policy is:

$$x_{n-1}^*(p_{n-1}, q_{n-1}^r) = \begin{cases} 0, & if\ q_{n-1}^r \geq q_{n-1}^{cMax} \\ q_{n-1}^{cMax} - q_{n-1}^r, & if\ q_{n-1}^r < q_{n-1}^{cMax} \end{cases} \tag{8}$$

Proof: As leg $n-1$ (from port $n-1$ to port n) is the last leg, the ship needs not to add fuel if its remaining fuel is enough for the last leg (i.e., $q_{n-1}^r \geq q_{n-1}^{cMax}$); otherwise, the ship needs to add fuel to the level of 'q_{n-1}^{cMax}' at port $n-1$ so as to guarantee the fuel is enough for the last leg, which means the amount of fuel added at port $n-1$ is '$q_{n-1}^{cMax} - q_{n-1}^r$'. ■

Lemma 2: If the fuel price at a port is higher than the expected price at the next port when the ship arrives at the port, we will only add fuel to the maximum fuel consumption of the next leg. The optimal refueling policy at port i ($i = 1,2,\cdots,n-2$) is:

$$x_i^*(p_i, q_i^r) = \max\{0\ ;\ q_i^{cMax} - q_i^r\} \qquad if\ p_i > \overline{p}_{i+1} \tag{9}$$

Proof: We prove it by contradiction. Suppose an optimal policy, denoted by $\hat{x}_i(p_i, q_i^r)$, is "$p_i > \overline{p}_{i+1}$ and $\hat{x}_i(p_i, q_i^r) > q_i^{cMax} - q_i^r$". We slightly revise the policy as follows: $x_i(p_i, q_i^r) = q_i^{cMax} - q_i^r$,

$x_{i+1}\left(p_{i+1},q_{i+1}^{r}\right)=\hat{x}_{i+1}\left(p_{i+1},q_{i+1}^{r}\right)+[\hat{x}_i\left(p_i,q_i^r\right)-\left(q_i^{cMax}-q_i^r\right)]$, which means we refill $\hat{x}_i\left(p_i,q_i^r\right)-\left(q_i^{cMax}-q_i^r\right)$ less fuel at port i and refill $\hat{x}_i\left(p_i,q_i^r\right)-\left(q_i^{cMax}-q_i^r\right)$ more fuel at port $i+1$ than the policy $\hat{x}_i\left(p_i,q_i^r\right)$. The policies for ports $1,2\cdots i-1$ and ports $i+2,i+3\cdots n-1$ are not changed.

The fuel cost of the new policy is lower than the policy $\hat{x}_i\left(p_i,q_i^r\right)$ by $p_i[\hat{x}_i\left(p_i,q_i^r\right)-\left(q_i^{cMax}-q_i^r\right)]$ at port i and higher than the latter by a random value $\tilde{p}_{i+1}[\hat{x}_i\left(p_i,q_i^r\right)-\left(q_i^{cMax}-q_i^r\right)]$ due to the randomness of the fuel price at port $i+1$. Therefore, the total expected fuel cost saved by the new policy is:

$$
\begin{aligned}
&p_i\left[\hat{x}_i\left(p_i,q_i^r\right)-\left(q_i^{cMax}-q_i^r\right)\right]-\mathbb{E}\left\{\tilde{p}_{i+1}\left[\hat{x}_i\left(p_i,q_i^r\right)-\left(q_i^{cMax}-q_i^r\right)\right]\right\}\\
&=\left(p_i-\mathbb{E}\{\tilde{p}_{i+1}\}\right)\left[\hat{x}_i\left(p_i,q_i^r\right)-\left(q_i^{cMax}-q_i^r\right)\right]\\
&=\left(p_i-\overline{p}_{i+1}\right)\left[\hat{x}_i\left(p_i,q_i^r\right)-\left(q_i^{cMax}-q_i^r\right)\right]
\end{aligned}
$$

Recall the assumption of the policy $\hat{x}_i\left(p_i,q_i^r\right)$ at the beginning of the proof, "$p_i>\overline{p}_{i+1}$ and $\hat{x}_i\left(p_i,q_i^r\right)>q_i^{cMax}-q_i^r$". Hence, the above formula is positive, meaning that the new policy is better. This contradicts the assumption that $\hat{x}_i\left(p_i,q_i^r\right)$ is an optimal policy. ■

Lemma 3: Given p_i and q_i^r, the $c_i\left(p_i,q_i^r,x_i\right)$ in Eq. (6) is a convex function of x_i.

Proof: Suppose that when $x_i=x_i^{(1)}$, the optimal policy is $x_j^{(1)*}(p_j,q_j^r)$, $j=i+1\cdots n-1$; when $x_i=x_i^{(2)}$, the optimal policy is $x_j^{(2)*}(p_j,q_j^r)$, $j=i+1\cdots n-1$.

Then for any $0<\lambda<1$, a policy "$x_i=\lambda x_i^{(1)}+(1-\lambda)\,x_i^{(2)}$, and $x_j=\lambda x_j^{(1)*}(p_j,q_j^r)+(1-\lambda)\,x_j^{(2)*}(p_j,q_j^r)$, $j=i+1,\cdots,n-1$" is one feasible policy due to the linearity of the constraints. The linearity of the cost function means the cost of the policy $\lambda x_j^{(1)*}(p_j,q_j^r)+(1-\lambda)x_j^{(2)*}(p_j,q_j^r)$ for $x_i=\lambda x_i^{(1)}+(1-\lambda)\,x_i^{(2)}$ is equal to $\lambda c_i(p_i,q_i^r,x_i^{(1)})+(1-\lambda)c_i(p_i,q_i^r,x_i^{(2)})$. Hence, $c_i(p_i,q_i^r,\lambda x_i^{(1)}+(1-\lambda)\,x_i^{(2)})$ is smaller than or equal to: $\lambda c_i(p_i,q_i^r,x_i^{(1)})+(1-\lambda)c_i(p_i,q_i^r,x_i^{(2)})$. ■

Proposition 1: The optimal refueling policy is a threshold-based policy and depends on the price p_i. That is, there exist functions $w_i(p_i) := x_i^*(p_i, 0)$ which act as the thresholds such that the optimal policy satisfies:

$$x_i^*(p_i, q_i^r) = \begin{cases} 0, & if\ q_i^r \ge w_i(p_i) \\ w_i(p_i) - q_i^r, & if\ q_i^r < w_i(p_i) \end{cases} \tag{10}$$

where $i = 1, 2, \cdots, n-1$.

In words, when the fuel price at port i is p_i, the optimal policy aims to refuel the ship so that the amount of fuel in the tank reaches $w_i(p_i)$ when the ship leaves port i; if the amount of fuel is not less than $w_i(p_i)$ when the ship arrives at port i, then the ship is not refueled.

Proof: Lemma 1 shows the proposition holds at port $n-1$ with $w_{n-1}(p_{n-1}) = q_{n-1}^{cMax}$; Lemma 2 shows the proposition holds at port $i = 1, 2, \cdots, n-2$ when $p_i > \overline{p}_{i+1}$ with $w_i(p_i) = q_i^{cMax}$; we next need to prove the proposition holds for port $i = 1, 2, \cdots, n-2$ when $p_i \le \overline{p}_{i+1}$ by showing that $w_i(p_i) := x_i^*(p_i, 0)$ is the threshold.

Consider a particular port i with $p_i \le \overline{p}_{i+1}$.

(i) If $q_i^r = 0$, then we should refill $x_i^*(p_i, 0)$ amount of fuel, $x_i^*(p_i, 0) \ge q_i^{cMax}$.

(ii) If $0 < q_i^r := \hat{q}_i^r \le x_i^*(p_i, 0)$, then we can decompose the refueling process when $q_i^r > 0$ into two stages. In the first stage, we refill $\hat{q}_i^r$ amount of fuel. After the first stage, the system is the same as the one with $q_i^r = \hat{q}_i^r$. Hence, when $q_i^r = \hat{q}_i^r$, the optimal amount of fuel to add is $x_i^*(p_i, 0) - \hat{q}_i^r$.

(iii) If $q_i^r := \hat{q}_i^r > x_i^*(p_i, 0)$, the proposition states the optimal solution $x_i^*(p_i, \hat{q}_i^r) = 0$. We prove it by contradiction. Suppose $\hat{x}_i(p_i, \hat{q}_i^r)$ is an optimal policy with $\hat{x}_i(p_i, \hat{q}_i^r) > 0$, which implies:

$$c_i(p_i, \hat{q}_i^r, 0) > c_i(p_i, \hat{q}_i^r, \hat{x}_i(p_i, \hat{q}_i^r)). \tag{11}$$

According to Eq. (6), we have: $c_i\left(p_i,0,\hat{q}_i^r\right) = \hat{q}_i^r \cdot p_i + \mathbb{E}\left\{u_{i+1}\left(\tilde{p}_{i+1},0 + \hat{q}_i^r - \tilde{q}_i^c\right)\right\}$ and $c_i\left(p_i,\hat{q}_i^r,0\right) = 0 \cdot p_i + \mathbb{E}\left\{u_{i+1}\left(\tilde{p}_{i+1},\hat{q}_i^r + 0 - \tilde{q}_i^c\right)\right\}$, which means:

$$c_i\left(p_i,\hat{q}_i^r,0\right) = c_i\left(p_i,0,\hat{q}_i^r\right) - p_i \cdot \hat{q}_i^r \tag{12}$$

Similarly, we can also derive:

$$c_i(p_i,\hat{q}_i^r,\hat{x}_i\left(p_i,\hat{q}_i^r\right)) = c_i(p_i,0,\hat{q}_i^r + \hat{x}_i\left(p_i,\hat{q}_i^r\right)) - p_i \cdot \hat{q}_i^r \tag{13}$$

Then based on the Eq. (12 – 13), the Eq. (11) turns to:

$$c_i\left(p_i,0,\hat{q}_i^r\right) > c_i(p_i,0,\hat{q}_i^r + \hat{x}_i\left(p_i,\hat{q}_i^r\right)) \tag{14}$$

Moreover, the definition of the optimal solution $x_i^*\left(p_i,0\right)$ implies:

$$c_i\left(p_i,0,\hat{q}_i^r\right) > c_i(p_i,0,x_i^*\left(p_i,0\right)). \tag{15}$$

In addition, $x_i^*\left(p_i,0\right) < \hat{q}_i^r < \hat{q}_i^r + \hat{x}_i\left(p_i,\hat{q}_i^r\right)$. The above inequalities (14 – 15) contradict the convexity of $c_i\left(p_i,q_i^r,x_i\right)$ proved in Lemma 3. ■

Proposition 2: If the fuel price at port i is lower than the expected prices at several latter ports $(i+1,i+2,\cdots,j)$ and higher than the expected price at port $j+1$ when the ship arrives at port i, the fuel in the tank should be refilled to no more than the sum of the maximal fuel consumption on leg $i,i+1,\cdots,j$ when the ship departs from port i. That is,

$$x_i^*\left(\hat{p}_i,0\right) \le \left(q_i^{cMax} + q_{i+1}^{cMax} + \cdots + q_j^{cMax}\right), \text{ if } \hat{p}_i > \overline{p}_{j+1} \tag{16}$$

where $\hat{p}_i$, similar to p_i, also represents a realization of the random fuel price at when the ship arrives at port i. The difference is that the domain of p_i is larger than that of $\hat{p}_i$, as $\hat{p}_i$ only represents those realizations satisfying some further requirements (e.g., higher than the expected fuel price at the next port).

Proof: The conclusion holds trivially if $q_i^{cMax} + q_{i+1}^{cMax} + \cdots + q_j^{cMax} \ge V$. If $q_i^{cMax} + q_{i+1}^{cMax} + \cdots + q_j^{cMax} < V$, we prove the case by contradiction.

Suppose that π is an optimal policy and the threshold with policy π for port $k = 1,2...n-1$ is $w_k^{\pi}(p_k)$ if the price is p_k. Suppose that there exists a particular port i and port $j+1$, and a particular $\hat{p}_i > \overline{p}_{j+1}$ such that:

$$w_i^{\pi}(\hat{p}_i) > q_i^{cMax} + q_{i+1}^{cMax} + \cdots + q_j^{cMax}.$$

Then we define $\Delta := w_i^{\pi}(\hat{p}_i) - \left(q_i^{cMax} + q_{i+1}^{cMax} + \cdots + q_j^{cMax}\right) > 0$

and put forward a new policy π' (that may not be threshold-based) as follows:

$$x_i^{\pi'}(\hat{p}_i,0) = q_i^{cMax} + q_{i+1}^{cMax} + \cdots + q_j^{cMax} = x_i^{\pi}(\hat{p}_i,0) - \Delta,$$

$$x_k^{\pi'}(p_k,q_k) = x_k^{\pi}(p_k,q_k + \Delta),\ k = i+1...j,\ p_k^{low} \le p_k \le p_k^{high},\ 0 \le q_k \le V,$$

$$x_{j+1}^{\pi'}(p_{j+1},q_{j+1}) = x_{j+1}^{\pi}(p_{j+1},q_{j+1} + \Delta) + \Delta,\ p_{j+1}^{low} \le p_{j+1} \le p_{j+1}^{high},\ 0 \le q_{j+1} \le V.$$

For any sample path (i.e., realization) of $(\tilde{p}_k, k = i+1...j, j+1; \tilde{q}_k^c, k = i, i+1...j)$, by the definition of the two policies, the remaining fuel at port $k = i+1...j$ with policy π' is smaller than that with policy π by Δ; hence, the policy π' purchases less fuel than π at port i by Δ, the same amount of fuel as π at port $k = i+1...j$, and more fuel than π at port $j+1$ by Δ. This means, for any sample path, policy π' pays less fuel cost than π at port i by $\hat{p}_i\Delta$, but more fuel cost than π at port $j+1$ by a random number $\tilde{p}_{j+1}\Delta$. Taking the expectation over all sample paths, the expected cost of policy π' is smaller than that of π by $(\hat{p}_i - \overline{p}_{j+1})\Delta$. ■

Hypothesis: It is tempting to conclude that if the fuel price at port i is lower than the expected prices at several latter ports $(i+1, i+2, \cdots, j)$ and higher than the expected price at port $j+1$ when the ship arrives at the port, the fuel in the tank should be refilled to at least the sum of the minimal fuel consumption on leg $i, i+1, \cdots, j$ when the ship departs from port i, that is,

$$x_i^*(p_i,0) \ge (q_i^{cMin} + q_{i+1}^{cMin} + \cdots + q_j^{cMin}),$$

if $p_i < \min\{\overline{p}_{i+1}, \overline{p}_{i+2}, \cdots, \overline{p}_j\}$ and $p_i > \overline{p}_{j+1}$.

Unfortunately, such an intuition is incorrect, as demonstrated by the example below.

A counter example against the hypothesis: Suppose that the distances on leg $i, i+1, \cdots, j-1$ are all very short, which can be taken as 0. So we can think there is no fuel consumption on these legs. There is fuel consumption on leg j; the prices at port $i, i+1, \cdots, j$ all follow uniform distribution $[0, 1]$; the price at port $j+1$ is $\frac{1}{2}$ (in fact, we can consider the price at port $j+1$ as $\frac{1}{2} - \varepsilon$ so that it is the lowest, ε being a very small positive number and $\lim_{\varepsilon \to 0} \left(\frac{1}{2} - \varepsilon\right) = \frac{1}{2}$); the price at port i is $\frac{1}{2}$ when the ship arrives at the port (i.e., the price is actually $\frac{1}{2} - \frac{1}{2}\varepsilon$, so that it is lower than the following ports, with $\lim_{\varepsilon \to 0} \left(\frac{1}{2} - \frac{1}{2}\varepsilon\right) = \frac{1}{2}$). We analyze the problem backward. (i) We can consider that if the ship arrives at port j and has not been refilled any fuel at previous ports, we have to refill fuel at port j, and the expected cost at port j (denoted by $\overline{p}_j'$) is $\frac{1}{2}$ since the price follows uniform distribution $[0, 1]$. (ii) If the ship arrives at port $j-1$ and has not been refilled any fuel at previous ports, we should refill fuel at port $j-1$ when the price is lower than the expected cost (i.e., $\frac{1}{2}$) at port j, and we should not refill any fuel at port $j-1$ when the price is higher than the expected cost (i.e., $\frac{1}{2}$) at port j. The probability of the price between 0 and $\frac{1}{2}$ is $\frac{1}{2}$, and the expected price in this area is $\frac{1}{4}$ (i.e., $\frac{0+\frac{1}{2}}{2} = \frac{1}{4}$); the probability of the price between $\frac{1}{2}$ and 1 is also $\frac{1}{2}$, and the expected price in this area is $\frac{1}{2}$ equal to the expected cost at port j. Hence, we can calculate that the expected cost at port $j-1$ is $\frac{3}{8}$ (i.e., $\frac{1}{2} \cdot \frac{1}{4} + \left(1 - \frac{1}{2}\right) \cdot \frac{1}{2} = \frac{3}{8}$). (iii) In the same way, we only refill fuel at port $j-2$ if the price is lower than the expected cost at port $j-1$ (i.e., $\frac{3}{8}$), the expected price at port $j-2$ is $\frac{39}{128}$ (i.e., $\frac{3}{8} \cdot \frac{3}{16} + \left(1 - \frac{3}{8}\right) \cdot \frac{3}{8} = \frac{39}{128}$). (4) We can infer that the expected cost at port $j-s$ ($j-s \geq i$) is $\overline{p}_{j-s+1}' \cdot \frac{\overline{p}_{j-s+1}'}{2} + \left(1 - \overline{p}_{j-s+1}'\right) \cdot \overline{p}_{j-s+1}'$. If there are infinitely many ports between port i and j, we only refill fuel at port i when the price is 0 or infinitely close to 0. As $\frac{1}{2} \gg 0$, we will not refill any fuel at port i in the optimal policy. ■

Proposition 3: The threshold functions $w_i(p_i)$ in Proposition 1 are monotonically decreasing. That is: $w_i(p_i + \Delta p_i) \le w_i(p_i)$, here $\Delta p_i > 0$.

Proof: According to the definition '$w_i(p_i) := x_i^*(p_i, 0)$' in Proposition 1, the proof of '$w_i(p_i + \Delta p_i) \le w_i(p_i)$' is equivalent to the proof of '$x_i^*(p_i + \Delta p_i, 0) \le x_i^*(p_i, 0)$', which can be proved by contradiction.

Suppose that $x_i^*(p_i, 0)$ is an optimal policy at port i for a particular p_i; and '$x_i^*(p_i, 0) + \Delta x_i$, here $\Delta x_i > 0$' is an optimal policy when p_i increases by Δp_i, i.e., we have: $x_i^*(p_i + \Delta p_i, 0) = x_i^*(p_i, 0) + \Delta x_i$. We will prove that such an optimal policy does not exist.

As $x_i^*(p_i, 0)$ is the optimal policy for p_i, we obtain that:

$$c_i\left(p_i, 0, x_i^*(p_i, 0) + \Delta x_i\right) > c_i\left(p_i, 0, x_i^*(p_i, 0)\right) \tag{17}$$

The above inequality can be decomposed into the following specific form:

$$\left(x_i^*(p_i, 0) + \Delta x_i\right) \cdot p_i + \mathbb{E}\left\{u_{i+1}\left(\tilde{p}_{i+1}, \left(x_i^*(p_i, 0) + \Delta x_i\right) - \tilde{q}_i^c\right)\right\} > x_i^*(p_i, 0) \cdot p_i + \mathbb{E}\left\{u_{i+1}\left(\tilde{p}_{i+1}, x_i^*(p_i, 0) - \tilde{q}_i^c\right)\right\}$$

Then, we propose a new policy that refills $x_i^*(p_i, 0)$ amount of fuel for the ship at port i when the fuel price increases by Δp_i, and the minimal expected cost will be $x_i^*(p_i, 0) \cdot (p_i + \Delta p_i) + \mathbb{E}\left\{u_{i+1}\left(\tilde{p}_{i+1}, x_i^*(p_i, 0) - \tilde{q}_i^c\right)\right\}$. Moreover, according to Eq. (17), we can obtain:

$$\left(x_i^*(p_i, 0) + \Delta x_i\right) \cdot (p_i + \Delta p_i) + \mathbb{E}\left\{u_{i+1}\left(\tilde{p}_{i+1}, \left(x_i^*(p_i, 0) + \Delta x_i\right) - \tilde{q}_i^c\right)\right\}$$
$$> x_i^*(p_i, 0) \cdot (p_i + \Delta p_i) + \mathbb{E}\left\{u_{i+1}\left(\tilde{p}_{i+1}, x_i^*(p_i, 0) - \tilde{q}_i^c\right)\right\} \tag{18}$$

Eq. (18) implies that $x_i^*(p_i, 0) + \Delta x_i$ is not an optimal policy when p_i increases by Δp_i because refilling $x_i^*(p_i, 0)$ is a better policy. Hence, when the fuel price increases, the threshold will not increase. ■

5. Dynamic programming algorithm

The fuel price at each port and the fuel consumption on each leg are stochastic parameters with known probability distribution functions in this problem. For ease of calculation, we can discretize the fuel price and the amount of fuel consumption. Then a dynamic programming based algorithm is designed to obtain the optimal refueling decision policy, the core of which is to find the threshold functions $w_i(p_i)$. It should be noted that Proposition 1 and Proposition 3 are used in the algorithm to reduce the search space.

Algorithm 1: *Dynamic programming algorithm to find the threshold functions* $w_i(p_i)$.

Step 0: Define the step size of fuel price α (e.g., α dollars). We assume that p_i^{low}/α and p_i^{high}/α are both integers.
Define a set $P_i := \{p_i^{low}, p_i^{low} + \alpha, p_i^{low} + 2\alpha, \cdots, p_i^{high}\}$ and the fuel prices at each port are divided into discrete values in the set.
Define the step size of amount of fuel β. We assume that V/β is an integer. Define a set $Q := \{0, \beta, 2\beta, \cdots, V\}$ and the amounts of fuel are divided into discrete values in the set.
$x_n^*(p_n, q_n^r) = 0$, $q_n^r \in Q$, which means we will not refill any fuel at the last port.

Step 1: According to Lemma 1, the threshold for the optimal policy at port $n-1$ is:
$w_{n-1}(p_{n-1}) = q_{n-1}^{cMax}$, $p_{n-1} \in P_{n-1}$,
and the cost function is:
$u_{n-1}(p_{n-1}, q_{n-1}^r) = p_{n-1} \max\{0; w_{n-1}(p_{n-1}) - q_{n-1}^r\}$, $p_{n-1} \in P_{n-1}$.
Set port $i \leftarrow n-1$.

Step 2: Examine the optimal policy for port i.
Step 2.1: if $p_i \geq \overline{p}_{i+1}$
According to Lemma 2, we can know the threshold is
$w_i(p_i) = q_i^{cMax}$ for $p_i \in P_i$, $p_i \geq \overline{p}_{i+1}$
and the cost function:
$u_i(p_i, q_i^r) = p_i \max\{0; w_i(p_i) - q_i^r\} +$
$\sum_{y=p_{i+1}^{low}}^{p_{i+1}^{high}-1} \sum_{z=q_i^{cMin}}^{q_i^{cMax}-1} \Pr(y \leq \delta < y + \alpha) \cdot \Pr(z \leq \varphi < z + \beta) \cdot u_{i+1}(y, q_i^r + x_i - z)$
(19)
In Eq. (19), $q_i^r \in Q$, $p_i \in P_i$, $p_i \geq \overline{p}_{i+1}$.
Step 2.2: if $p_i < \overline{p}_{i+1}$
According to Proposition 1, $w_i(p_i) = x_i^*(p_i, 0)$
(i) When $p_i = 0$, we have

$$w_i(p_i) = x_i^*(p_i,0) \in \operatorname{argmin}_{x_i \in Q, x_i \geq q_i^{cMax}}$$
$$\left[p_i x_i + \sum_{y=p_{i+1}^{low}}^{p_{i+1}^{high}-1} \sum_{z=q_i^{cMin}}^{q_i^{cMax}-1} \Pr(y \leq \delta < y+\alpha) \cdot \Pr(z \leq \varphi < z+\beta) \cdot u_{i+1}(y, x_i - z)\right] \quad (20)$$

In Eq. (20), $p_i \in P_i$, $p_i < \overline{p}_{i+1}$.

Proposition 3 implies $w_i(p_i)$ is monotonically decreasing when p_i increases. Hence, when $p_i \geq \alpha$, we only need to consider those x_i that are not greater than $x_i^*(p_i - \alpha, 0)$. Then the Eq. (20) can be replace by

$$w_i(p_i) = x_i^*(p_i,0) \in \operatorname{argmin}_{x_i \in Q, x_i \geq q_i^{cMax}, x_i \leq x_i^*(p_i - \alpha, 0)}$$
$$\left[p_i x_i + \sum_{y=p_{i+1}^{low}}^{p_{i+1}^{high}-1} \sum_{z=q_i^{cMin}}^{q_i^{cMax}-1} \Pr(y \leq \delta < y+\alpha) \cdot \Pr(z \leq \varphi < z+\beta) \cdot u_{i+1}(y, x_i - z)\right] \quad (21)$$

In Eq. (21), $p_i \in P_i, p_i < \overline{p}_{i+1}$.

In both cases, the cost function is Eq. (19).

Step 3: If $i = 1$, output the optimal policy and stop. Otherwise, set $i \leftarrow i-1$ and go to Step 2. ■

6. Numerical experiments

This study applies the above method to a cruise ship's itinerary in Mediterranean Sea, and conducts some numerical experiments. The parameters used in this study are either real data or estimated from real data.

Fig. 1 shows the ship's voyage. Table 1 shows this service, which contains seven ports of call, arrival time, departure time and travel time, etc. The cruise itinerary lasts for seven days. The ship in this study is called MSC Preziosa, which is 1092 feet long and has a gross tonnage of 137,936 tons. The maximum passenger capacity is 3959. In addition, the fuel capacity of the ship is 3500 tons. The speed is between 18 and 24 knots. Considering the influence of several determinants, the mean fuel consumption per hour follows the Truncated Normal Distribution (Ghosh et al. 2015), the mean value is 2800 gallons and the standard deviation is 300 gallons; here the mean value is based on real data and the standard deviation is estimated according to some experts in the shipping industry. Then the fuel consumption on each leg equals the mean consumption rate multiplied by sailing time.

As aforementioned in Section 1, the fuel consumption in the leg is uncertain because it is influenced by many factors such as sailing speed, draft, trim, weather/sea conditions (e.g., wind, waves, sea currents and sea water temperature). Table 1 implies the average speed of the ship during a leg is constant. It should be noted that due to the uncertain draft, trim, weather/sea conditions, the instantaneous speed of the ship may vary in a stochastic manner during the leg, although the average speed in the leg is constant. In reality, the fuel consumption rate is significantly influenced by a ship's instantaneous speed. In the experiments, we use a random number, which follows the Truncated Normal Distribution with the mean 2800 and the standard deviation 300, to reflect the stochastic fuel consumption per hour, which actually involves the multiple uncertain factors (i.e., the uncertain draft, trim, weather/sea conditions) affecting the ship's instantaneous speed. Then we use the random number to multiply the constant leg duration time (shown in Table 1) to obtain the fuel consumption in the leg, which is also a stochastic value.

The bunker fuel prices at the six ports are different, and the prices at the same port are also different on different days. According to the fuel prices at the six ports in the last three months of 2015, we assume the prices follow Uniform Distribution, and the prices are set in Table 1.

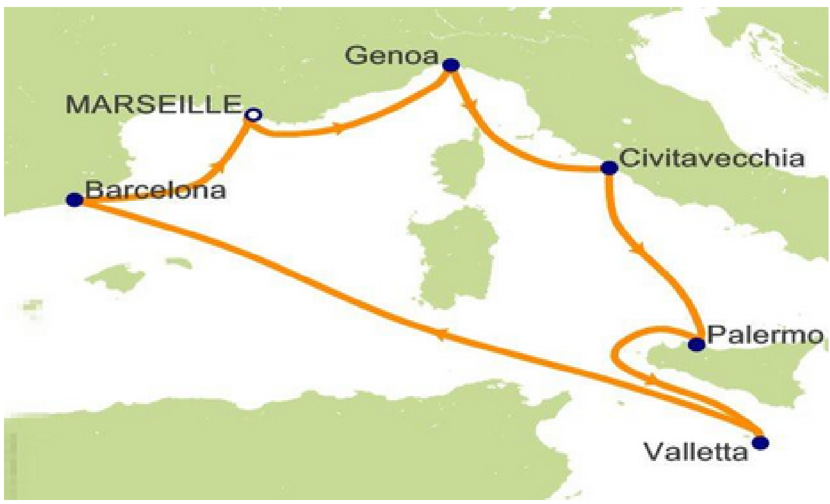

Fig. 1: An example of the cruise itinerary of Mediterranean Sea

Table 1: A Cruise Itinerary of Mediterranean Sea

Day	Port	Arrive	Depart	Travel Time (h)	Distribution fuel prices (USD)
1	Marseille		4:00 PM		U(150,260)
2	Genoa	8:00 AM	6:00 PM	16	U(150,270)
3	Civitavecchia	8:00 AM	6:00 PM	14	U(150,270)
4	Palermo	10:00 AM	5:00 PM	16	U(150,270)
5	Valletta	10:00 AM	6:00 PM	17	U(140,260)
7	Barcelona	9:00 AM	6:00 PM	39	U(170,250)
8	Marseille	9:00 AM		15	

The results in Fig. 2 show that the bunker threshold equals the maximum consumption of the next leg when the fuel price at a port is higher than the average price at the next port. The maximum consumption of the next leg is the minimum bunker quantity needed to be held in the tank when the ship departs from each port. However, when the fuel price at a port is equal to or lower than the average price at the next port, the bunker threshold is equal to or more than the maximum fuel consumption of the next leg. According to Proposition 3, for a port, the lower the price is, the higher the bunker threshold will be. As shown in Fig. 2, the bunker threshold is a strictly decreasing function before the fuel price at a port reaches the average price at the next port, and then becomes a constant after that. Because different ports have different average fuel prices, the turning points of bunker thresholds (the leftmost points from which the thresholds are constant) at different ports are different. The sailing time between Valletta and Barcelona is the longest among all legs, which means that the leg between Valletta and Barcelona will consume the most fuel and the minimum bunker threshold of Valletta is the highest for all minimum bunker thresholds at different ports. The sailing times on the other legs are similar, which means the minimum bunker thresholds of the other ports are similar. The bunker thresholds are the same in different prices at Barcelona because it is the penultimate port and we assume no fuel will be used after the last leg. In addition, when the fuel price at a port decreases from the price of the turning point, the bunker threshold at the port first increase very fast and then more and more slowly. In other words, the bunker threshold is very sensitive when the fuel price at the port is close to and smaller than the turning point.

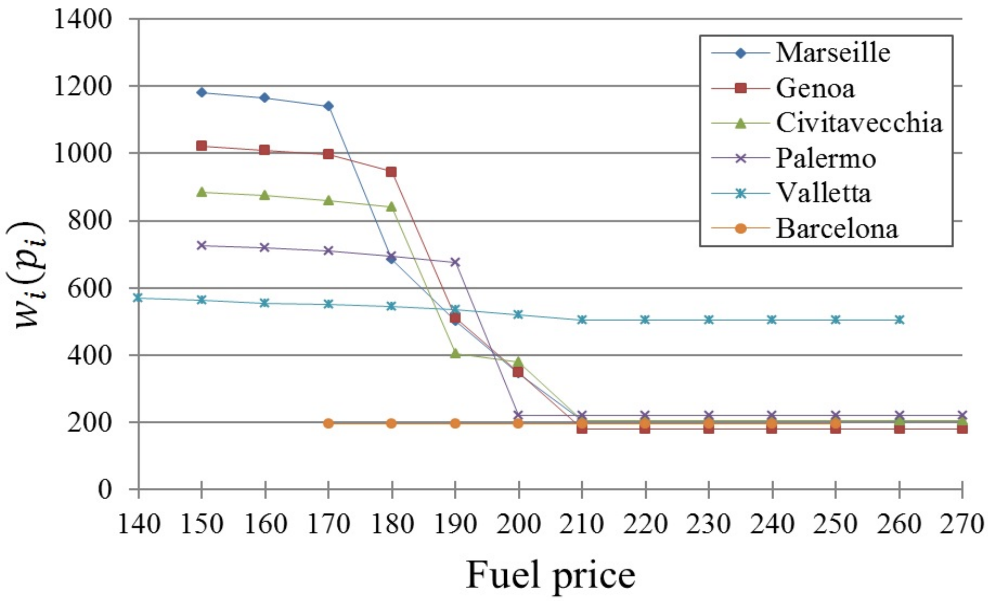

Fig. 2: The bunker thresholds (tons) at six ports

In order to further investigate the performance of the proposed method, some numerical experiments on container liners are also conducted. This study collects a 62 days' liner itinerary operated by Maersk Line (Fig. 3) and some parameters (Table 2) to conduct several numerical experiments. The parameters in Table 2 contain ten ports of call, arrival time, departure time, travel time and the distribution of fuel prices. We assume the prices follow Uniform Distribution according to the fuel prices at the eight ports in the last three months of 2015. We consider a ship called Adrian Maersk deployed in the itinerary, which is 352 m in length and 43 m in breadth. The gross tonnage is 93496 tons and the deadweight is 109000 tons. Moreover, the fuel capacity of the ship is 4500 tons. The average speed is 21 knots and the maximum speed is 25 knots. Considering the influence of speed and other determinants, we assume the mean fuel consumption per day follows the Truncated Normal Distribution (Ghosh et al. 2015), in which the mean value is 90 tons and the standard deviation is 10 tons. Then the fuel consumption on each leg equals the mean consumption rate multiplied by sailing time.

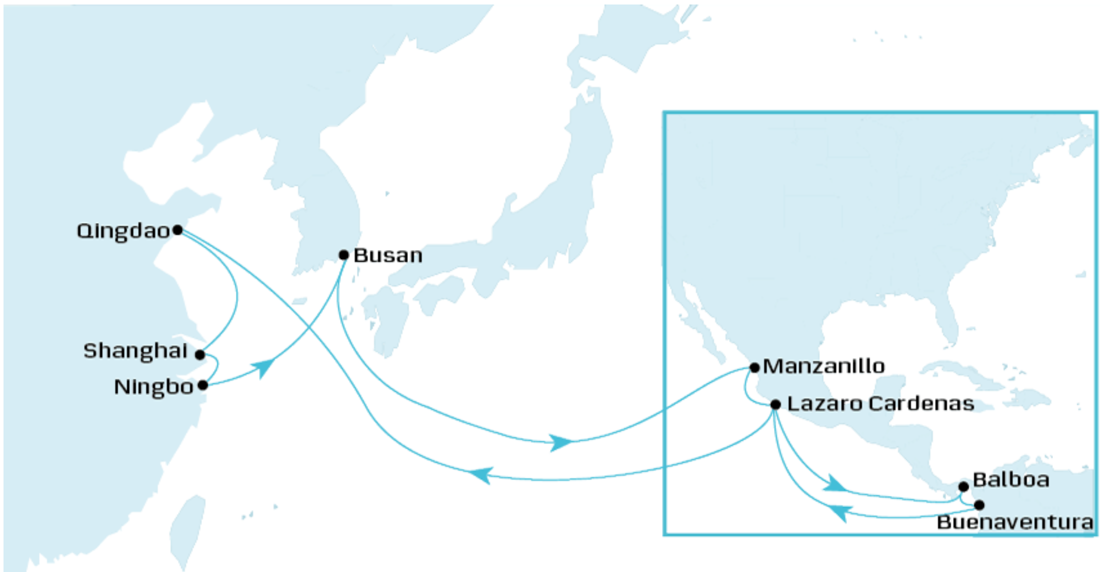

Fig. 3: An example of the liner itinerary operated by Maersk Line

Table 2: A liner itinerary of Maersk Line

Port	Arrive	Depart	Travel Time (d)	Distribution of fuel prices (USD)
Qingdao		MON		U(170,270)
Shanghai	TUE	WED	1	U(170,260)
Ningbo	WED	THU	0.5	U(160,260)
Busan	SAT	SAT	2	U(160,270)
Manzanillo	SUN	SUN	15	U(150,290)
Lazaro Cardenas	MON	MON	1	U(150,290)
Balboa	FRI	SUN	4	U(140,240)
Buenaventura	MON	TUE	1	U(140,290)
Lazaro Cardenas	MON	MON	6	U(150,290)
Qingdao	SAT		26	

Based on the above liner itinerary, comparative experiments are conducted between the proposed optimal refueling policy and some other decision rules. Specifically, we have consulted a manager with 16 years' experience in shipping industry about the refueling policies in practice. In practice, shipping companies will propose several possible refueling decisions based on experience considering factors such as cargo load, weather conditions, fuel prices, detour for bunkering ports. These refueling decisions will be compared taking into account a few scenarios of uncertainty in fuel prices and fuel consumption. The decision with the lowest average cost will be chosen. Therefore, it is hard to compare our solutions directly with the ones in practice, because different managers will make different decisions and even the same manager may make different decisions under two identical situations. We therefore proposed several rules to mimic the practical decision-making process. As our paper proposed the optimal policy that minimizes the expected cost, no other policy, either in the literature or in practice, can outperform our policy. The rules we propose are:

Rule 1: Refuel the least required amount of fuel each time.

Rule 2: If the fuel price at a port is higher than the expected price at the next port, add the least required amount of fuel at this port; otherwise, refuel till the oil tanker is full.

Rule 3: Calculate the average value of all the ports' expected fuel prices. If a port's actual fuel price is higher than that value, add the least required amount of fuel at this port; otherwise, refuel till the oil tanker is full.

Rule 4: When deciding the optimal amount of fuel added at each port, we assume the fuel consumption during each leg in the remaining voyage is the expected value of the fuel consumption.

Rule 5: When deciding the optimal amount of fuel added at each port, we assume the fuel price of each port in the remaining voyage is the expected value of the fuel price.

Note that in all of the above five rules, we further impose that when the ship leaves a port, the amount of fuel in its tank is sufficient to ensure the ship can sail to the next port even if the fuel consumption takes its maximum value. Table 2 shows the comparative result between the optimal refueling policy proposed in this study and the five other decision rules in terms of the total expected cost. The results validate the outperformance of our method is evident by comparing to these simple but practical decision rules. Moreover, we change the parameter setting on the standard deviation (S.D.) of the fuel consumption in the cases by increasing or decreasing the S.D. values on the basis of the baseline case. From the results in each column of Table 2, we cannot observe an evident trend of changing with respect to the objective values when the variance of the fuel consumption increases. The outperformance degree (reflected by the 'gap' values) also does not show a uniform trend of changing along each column for all the five comparisons between our method and other rules. For the comparison between our method and Rule 2 (and Rule 3), the results show the outperformance degree of our method is decreasing when the variance of fuel consumption is growing. Although the result in Table 2 reflects neither a positive nor a negative influence of the fuel consumption's variance on the final cost as well as our method's outperformance degree, it could further validate the effective of our proposed refueling policy.

As aforementioned, Rule 5 is to decide the optimal amount of fuel added at each port by assuming that the fuel price of each port in the remaining voyage is the expected value of the fuel price. The comparative result is shown in the last column of Table 2. The gap value is about 1%, which seems not a large number but can bring significant benefit (or saving) to liner companies. For example, the Emma Maersk is one of the largest ship in service throughout the world, whose total annual fuel consumption is approximately 143,400 tons and total annual fuel costs are about \$64.5 million (http://www.instructables.com/community/Fuel-economy-of-the-worlds-longest-in-service-shi/). For this ship, saving 1% of the total annual fuel costs means saving more than half a million dollars. Therefore, compared with five other decision rules, the method in this study can help shipping companies save a large amount of cost.

Table 2: Comparison between the proposed method and other decision rules under different setting on the S.D. (standard deviation) of the fuel consumption

Ratio of the S.D. in the case to the S.D. in the baseline case	The method in this study	Rule1	Gap	Rule2	Gap	Rule3	Gap	Rule4	Gap	Rule5	Gap
0.3	1010867	1160825	14.83%	1294492	28.06%	1295757	28.18%	1019124	0.82%	1023723	1.27%
0.7	1084160	1253386	15.61%	1310038	20.83%	1310050	20.84%	1100390	1.50%	1094554	0.96%
Baseline case	**1053212**	**1330272**	**26.31%**	**1244111**	**18.13%**	**1246761**	**18.38%**	**1055743**	**0.24%**	**1065465**	**1.16%**
1.3	1177212	1403549	19.23%	1314198	11.64%	1313281	11.56%	1198589	1.82%	1186064	0.75%
1.7	1139827	1502231	31.79%	1239253	8.72%	1241503	8.92%	1142808	0.26%	1151838	1.05%
Avg.			21.55%		17.48%		17.58%		0.93%		1.04%

Note: Gap = (the objective value of a rule – the objective value of our method) / the objective value of our method.

In the previous experiments, the fuel consumption and fuel price are assumed to follow the uniform distribution. To further investigate the performance of the proposed method under different settings of the probability distributions with respect to the fuel consumption and fuel price, some more experiments are conducted and the results are shown in Table 3. Since the proposed method is an exact solution method, which outputs the optimal result, Table 3 just lists the computation time of the method under different settings of the probability distributions.

We consider three types of probability distributions, i.e., Normal distribution, Poisson distribution, and Triangular distribution. Then we have nine combinations of the 'fuel price distribution – fuel consumption distribution'. Moreover, we also consider three different problem scales, i.e., 10 ports, 20 ports, and 30 ports. It should be noted that each value in Table 3 represents the average computation time of five different cases under the same combination of 'fuel price distribution – fuel consumption distribution – number of ports'. These extensive experiments demonstrate that it takes less than three minutes to solve instances with 30 ports, which are larger than the scales of problems encountered in reality. Therefore, our proposed optimal decision method is applicable for various settings of probability distributions as well as the realistic scale of problem instances.

Table 3: The average computation time of different scales under different distribution settings on fuel price and fuel consumption

Fuel price	Fuel consumption	10 Ports	20 Ports	30 Ports
Normal Distribution	Normal Distribution	14.8 s	31.8 s	48.8 s
	Poisson Distribution	5.8 s	10.4 s	16.4 s
	Triangular Distribution	5.4 s	10.6 s	16.4 s
Poisson Distribution	Normal Distribution	47.8 s	98.2 s	148.4 s
	Poisson Distribution	16.0 s	32.4 s	49.4 s
	Triangular Distribution	16.0 s	34.8 s	52.2 s
Triangular Distribution	Normal Distribution	46.0 s	95.8 s	165.2 s
	Poisson Distribution	14.8 s	32.4 s	54.6 s
	Triangular Distribution	16.4 s	33.8 s	52.2 s

7. Conclusions

This paper studies a general decision problem on ship refueling, and proposes an optimal control policy for a liner ship to decide at which ports and how much fuel the liner ship should be refueled. Facing stochastic fuel consumption during each leg in its voyage as well as stochastic fuel price at each port, a liner ship can use the proposed optimal control policy to minimize the expected total fuel cost of the whole voyage under stochastic context with respect to fuel consumption and fuel price. Several properties are proved in this study. On the basis of these properties, a dynamic programming algorithm is then proposed to obtain some important threshold values, which are used in the optimal control policy for ship refueling decision. In addition, numerical experiments are conducted to validate the effectiveness of the proposed method. The result of the comparative experiments shows that the proposed optimal decision policy can save at least 8% fuel consumption cost by comparing with some relatively simple rules (Rules 1, 2, 3) and save about 1% cost on average by comparing with some brilliantly-designed rules (Rules 4 and 5). In addition, extensive experiments also show that the proposed method can obtain the optimal decision within a reasonable time (about 170 seconds) for various scales of problem instances (up to 30 ports) as well as various settings of probability distributions (e.g., Uniform, Normal, Poisson, and Triangular distributions) with respect to the fuel consumption and fuel price.

The proposed properties in this study can act as the basis for possible extensions of this problem to consider more realistic factors such as weather routing (Du et al. 2011 and 2015). For considering some complex factors in the refueling decision problems, the dynamic programming may need adjustment so as to reduce the complexity, which also requires decision makers to relax their demand for optimality.

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