IMPACTS OF THE LEAST PERCEIVED TRAVEL COST ON THE WEIBIT NETWORK EQUILIBRIUM

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ABSTRACT

This study investigates the impacts of the least perceived travel cost on the stochastic user equilibrium (SUE) problem. The Weibit SUE models are considered since they have a location parameter that naturally capture the least perceived travel cost. Considering a positive location parameter enhances the behavioral reality by attaching a positive lower-bound to the perceived travel cost distributions. It reduces the perception variances route-specifically and causes route-specific coefficients of variation (CVs). The CVs reduce proportionally slower for shorter routes, thus contributing to resolving the scale insensitivity issue in the Weibit SUE models. In the meantime, the route-specific CVs cause better discrimination between short and long routes in terms of relative variability; more travelers shift to the shortest route between each origin-destination pair. Numerical results confirm the analytical results regarding the effects of the least perceived travel costs and demonstrate the efficiency and robustness of the proposed solution algorithm.

Keywords: Stochastic user equilibrium; least perceived travel cost; weibit model; location parameter; relative variability

1 INTRODUCTION

Characterization of the perceived travel cost distribution lays the foundation for modeling the stochastic user equilibrium (SUE) problem. To properly portray the perceived travel cost distributions, transportation modelers trade-off among different aspects, such as behavioral reality, modeling flexibility, analytical tractability/ computational efficiency, and extendibility, etc. In the pursuit of computational efficiency, analytical tractability, and consistency with the random utility maximization (RUM) principle, researchers assume independently and identically distributed (i.i.d.) Gumbel perception error terms; then, we have the multinomial logit (MNL) route choice model (e.g., Luce, 1959; McFadden, 1974). To account for the overlapping routes, some researchers assume generalized extreme value (GEV) distributed perception error terms; the resultant route choice models include the cross nested logit (Vovsha, 1997), paired combinatorial logit (Chu, 1989), and generalized nested logit (GNL, Wen and Koppleman, 2001), etc. Some also turn to modifying existing models by adding route-specific penalty terms (e.g., Commonality-Logit (C-logit), Cascetta *et al*., 1996; Path-Size-Logit (PSL), Ben-Akiva and Bierlaire, 1999) or scaled variances (Chen *et al*., 2012) to handle the overlapping routes. Recently, to acquire a closed-form choice probability and to capture the varied perception variances with different trip lengths, Castillo *et al*. (2008) assumed independently distributed Weibull perception errors among routes and proposed the multinomial Weibit (MNW) route choice model. To achieve better modeling flexibility by permitting different random error distributions, Natarajan *et al*. (2009), Ahipaşaoğlu *et al*. (2016) proposed the marginal distribution model (MDM) as a convex optimization problem. MDM gives the route choice probability expressions of some classic models as special cases, including the MNL, PSL, C-logit, MNW, and PSW (path-size Weibit, Kitthamkesorn and Chen, 2013). The optimal solutions of the MDM are equal to those of some specific robust optimization problems. On the other hand, some researchers pursue the traveler behavioral richness and modeling flexibility, at the cost of a closed-form choice probability expression (i.e., tractability) and perhaps computational efficiency. They adopt the multivariate normal distribution to simultaneously handle the equal variance and the overlapping routes, then obtain the multinomial probit (MNP, Daganzo and Sheffi, 1977) route choice model; or allow traveler-specific random parameters to consider the taste heterogeneity in the mixed logit model (Ben-Akiva and Lerman, 1985; McFadden and Train, 2000). Besides, others consider bounded rationality in the travel choice decisions, including the prospect maximization (Xu *et al*., 2011, Wang *et al*., 2013), regret minimization (Loomes and Sugden, 1982; Li and Huang, 2017), rank-dependent multi-attribute optimization (Wang *et al*., 2014), and budget-based risk hedging behaviors (Lo *et al*., 2006; Chen and Zhou, 2010; Xu *et al*., 2013). Interested readers are referred to Kitthamkesorn and Chen (2013), Fosgerau and Bielier (2009), and Jensen (2016) for more literature.

In most above studies, the perceived travel cost distributions are either unbounded with possible negative values (e.g., normal distribution, Gumbel distribution) or bounded below by zeros (e.g., lognormal distribution). However, due to the physical distances between OD pairs, the corresponding physical travel times (often being viewed as a core component, if not the whole, of the travel cost for a trip) are supposed to be larger than zeros. Therefore, it is reasonable to claim that the perceived travel cost distributions are bounded below by some positive values. Moreover, these positive values could form the travelers' least perceived travel costs before a trip. The information or knowledge of these values may come from a traveler's previous trip experiences, communications with other travelers, or electronic route guidance systems. Noticing this, we decide to consider the least perceived travel costs in the SUE models.

Viewing that the Weibull probability has a parameter (i.e., the location parameter) that can naturally capture the least perceived travel cost between an OD pair, we will adopt the Weibit SUE models to consider a positive location parameter. Castillo *et al*. (2008) showed that independently distributed Weibull travel costs would produce a closed-form choice probability expression. Inspired by this observation, Kitthamkesorn and Chen (2013) proposed the multinomial Weibit (MNW)-SUE model and further adopted the path-size factor to handle the overlapping routes (PSW-SUE). They enforced zero-valued location parameters for all routes to obtain a decomposable route travel cost at the link level, thereby formulating the PSW-SUE model as a constrained entropy-based mathematical programming (MP) problem. The zero-valued location parameters imply that the travelers' least perceived travel costs between each OD pair are zeros, which do not conform with the general expectations. Besides, the zero-valued location parameters cause the MNW(/PSW)-SUE model an undesirable property, namely, the Weibit route choice probabilities will not change before and after multiplying the route travel costs by a positive number (scale insensitivity for short). This issue is rooted in the multiplicative route travel disutility of the Weibit choice probabilities. It also exists in the unconstrained Weibit SUE model with zero-valued location parameters (Kitthamkesorn and Chen, 2014). Efforts have been paid to resolve the scale insensitivity issue (Yao *et al.*, 2014; Xu *et al.*, 2015; Ahipaşaoğlu *et al*., 2016).

However, except for the MDM SUE models that permit a positive location parameter, the other models presuppose zero-valued location parameters to build up equivalent MP formulations. Little is known about how the least perceived travel costs (i.e., the positive location parameters) affect the perceived travel cost distributions and the corresponding SUEs. In this study, we use a positive location parameter to characterize a traveler's least perceived travel cost between an OD pair, and take the Weibit SUE models to unveil the impacts of the least perceived travel cost on the perceived travel cost distributions and the corresponding SUEs. The study contributes to the literature in the following aspects:

(1) consider the travelers' least perceived travel cost in the SUE problem.

(2) theoretically analyses the impacts of the least perceived travel cost (i.e., a positive location parameter) on the perception variances, coefficients of variation of the Weibull route travel costs, and on the resultant route choice probabilities.

(3) unveil the mechanism for resolving the scale insensitivity issue by considering a positive location parameter in the Weibit models.

(4) provide extensive numerical examples to demonstrate the impacts of the least perceived travel cost (i.e., a positive location parameter) on the network equilibrium states.

2 LITERATURE REVIEW

The importance of a location parameter has been widely recognized in different probability distributions and applications. Termed also as a shift, a threshold, or an origin in the literature, a location parameter is often used to depict the natural lower bound of a given distribution. For example, it may represent the minimum time headway (Zhang *et al*., 2007; Ha *et al*., 2012); the minimum travel time on a link or a route (Carey and Kwieciński, 1995; Castillo *et al*., 2008; Srinivasan *et al*., 2014); the minimum train schedule deviation (Corman *et al*., 2017); the minimum flight travel demand for a fare class for a flight leg (Kenan *et al*., 2018); the minimum lead time of supply from a vendor (Tyworth, 2018; Chiang and Benton, 1994); the minimum response time toward information/signals (Anders *et al*., 2016); the minimum time to deterioration for a product (Chakraborty *et al*., 2018; Yang, 2012); the base functioning duration (Bain and Englehardt, 1991; Zeng *et al*., 2016); or the minimum precipitation level (Baran and Nemoda, 2016). Recently, Watling *et al*. (2018) proposed a bounded SUE model, in which they used an exogenously-defined bound to distinguish the routes as unused according to the utility differences relative to a reference route. The bound can be either fixed or proportional to the minimum travel cost in each OD pair, it specifies the least acceptable utility (or maximum utility difference relative to the reference route) that makes a route an effective candidate. More reviews are referred to Johnson *et al.* (1995), Pham (2006), and Li and Chen (2017). Table 1 gives a summary.

Table 1. Applications of a location parameter in different topics

Reference	Application topic	Distribution type
Zhang *et al*. (2007)	Time headway	Shifted lognormal
Ha *et al*. (2012)	Time headway	Shifted hyper-Lognormal
Carey and Kwieciński (1995)	Link travel time	Shifted negative exponential
Srinivasan *et al*. (2014)	Link and route travel time	Shifted lognormal
Castillo *et al*. (2008)	Route travel time	Shifted Weibull
Watling *et al*. (2018)	Difference in route travel utilities	I.I.D. Gumbel
Corman *et al*. (2017)	Train schedule deviation	Shifted Weibull
Kenan *et al*. (2018)	Flight travel demand for a fare class	Truncated normal
Tyworth (2018)	Supply lead-time	Truncated normal
Chiang and Benton (1994)	Supply lead-time	Shifted exponential

Chakraborty *et al*. (2018); Yang (2012)	Goods deterioration rate	Shifted Weibull
Anders *et al*. (2016)	Response time	Shifted Wald
Bain and Englehardt (1991)	Lifetime reliability	Shifted Weibull
Zeng *et al*. (2016)	Lifetime reliability	Perks5[†]; Perks4
Baran and Nemoda (2016)	Precipitation	Shifted gamma

[†] Perks5 refers to a 5-parameter probability distribution derived from the Perks' (1932) hazard rate function. The cumulative density function and probability density function of the Perks5 hazard rate distribution are respectively defined by $F(t)=1-\exp\left(-\int_0^t h(t)dt\right)$ and $f(t)=h(t)\times\exp\left(-\int_0^t h(t)dt\right)$, where $h(t)=\dfrac{k_1+\exp(k_2t+k_3)}{1-\exp(-k_2t+k_4)+\exp(k_2t+k_5)}$ is the Perks hazard rate function, $k_i(i=1,2,\cdots,5)$ are the 5 parameters satisfying $k_1,k_2>0;-\infty<k_3,k_4,k_5<+\infty$. Perks4 is a special case of Perks5 by dropping $\exp(k_2t+k_5)$ in $h(t)$ via setting $k_5\to-\infty$. More information about the Perks5 or Perks4 distribution is referred to Perks (1932), Zeng *et al*. (2016) and the references therein.

In these applications, a location parameter represents the prior knowledge or information of the attributes under investigation. Take the example of the Weibull travel time distribution, a location parameter represents a traveler's perception/ knowledge of the minimum probabilistic travel time between an OD pair (Castillo *et al*., 2008). Considering this knowledge enhances the behavioral reality of the Weibit models and contributes to resolving the scale insensitivity property. By considering a positive location parameter, we can restate the travel disutility on a route as a function of the average route travel cost and the corresponding coefficient of variation (CV). The CVs, being route-specific with route-specifically reduced perception variances, change variously on different routes when the average route travel costs are multiplied by a scale (i.e., a positive number). Hence, the route travel disutilities are subject to unproportionate changes when scaling the average route travel costs, i.e., the scale insensitivity property being resolved. We formulate the Weibit SUE models as a probability-based equivalent variational inequality (VI) problem at the route level, then develop a route-based self-adaptive gradient projection (SAGP) algorithm. Numerical examples are provided to demonstrate the impacts of a positive location parameter on the perceived travel cost distributions and the corresponding network equilibria.

The study proceeds as follows. In Section 3, we briefly review the Weibit route choice models, then theoretically investigate the impacts of a positive location parameter on the perceived Weibull travel cost distributions and corresponding route choice probabilities. Section 4 provides an equivalent VI formulation for the Weibit SUE problems, a convex optimization formulation for the MDM SUE model, and develops the SAGP algorithm for solving the VI formulation. Two numerical examples are given in Section 5 to show the impacts of a positive location parameter and the efficiency and robustness of the algorithm. The study concludes with a short discussion in Section 6.

3 BACKGROUND

Before introducing the Weibit route choice models, a complete list of notations used in the study is presented in Table 2.

Table 2. Notations

Sets	
IJ	Set of OD pairs
R^{ij}	Set of routes connecting OD pair $ij \in IJ$
A	Set of links
Ω	Set of feasible flow defined by $\Omega = \left\{ \mathbf{f} = (f_r^{ij})_{\sum_{ij}\lvert R^{ij}\rvert} \mid \sum_{r \in R^{ij}} f_r^{ij} = q^{ij}, f_r^{ij} \geq 0, \forall r \in R^{ij}, ij \in IJ \right\}$
Indices	
a	Index of link $a \in A$
r	Index of route $r \in R^{ij}$
ij	Index of OD pair $ij \in IJ$
Decision Variables	
$\mathbf{f}$	Vector of route flow defined by $\mathbf{f} = \left(f_r^{ij} \right)_{\sum_{ij}\lvert R^{ij}\rvert}$ where $\lvert\rvert$ is the cardinality operator
f_r^{ij}	Total flow on route $r \in R^{ij}$ between OD pair $ij \in IJ$
v_a	Total flow on link $a \in A$
Intermediate Variables	
$\tau_a(v_a)$	Average travel cost on link $a \in A$ as a function of the total link flow v_a
g_r^{ij}	Average travel cost on route $r \in R^{ij}$ between OD pair $ij \in IJ$
G_r^{ij}	Random travel cost on route $r \in R^{ij}$ between OD pair $ij \in IJ$
$\psi_{G_r^{ij}}$	Probability distribution of random route travel cost G_r^{ij} on route $r \in R^{ij}$ between OD pair $ij \in IJ$
U_r^{ij}	Random travel disutility on route $r \in R^{ij}$ between OD pair $ij \in IJ$
ε_r^{ij}	Random error term of the travel disutility on route $r \in R^{ij}$ between OD pair $ij \in IJ$
σ_r^{ij}	Standard perception variance of the travel cost on route $r \in R^{ij}$ between OD pair $ij \in IJ$
ϑ_r^{ij}	Coefficient of variation of the travel cost on route $r \in R^{ij}$ between OD pair $ij \in IJ$
$\Gamma()$	Gamma function
P_r^{ij}	Choice probability of route $r \in R^{ij}$ between OD pair $ij \in IJ$
$F(\mathbf{f})$	General mapping function from the feasible flow set Ω to R^n at point $\mathbf{f}$
Parameters	
l_a	Length of link $a \in A$
L_r^{ij}	Length of route $r \in R^{ij}$ between OD pair $ij \in IJ$
C_a	Capacity of link $a \in A$
q^{ij}	Travel demand between OD pair $ij \in IJ$
θ^{ij}	Dispersion parameter of the MNL route choice model for OD pair $ij \in IJ$

$\varphi_r^{ij}, \beta_r^{ij}, \zeta_r^{ij}$	Scale, shape, and location parameters of the Weibull-type travel cost G_r^{ij} on route $r \in R^{ij}$ between OD pair $ij \in IJ$; ζ_r^{ij} also refers to the least perceived travel cost on a route
κ, η	Scale parameters of the route travel cost and the location parameter, respectively
ϖ_r^{ij}	Path-size factor of route $r \in R^{ij}$ between OD pair $ij \in IJ$
δ_{ar}^{ij}	Link-route incidence parameter, δ_{ar}^{ij} equals 1 if route $r \in R^{ij}$ between OD pair $ij \in IJ$ uses link $a \in A$ and 0 otherwise
ρ_a, ϕ_a	Parameters of the Bureau of Public Road average link travel time function
α, u, γ, ξ	Parameters of the self-adaptive gradient projection (SAGP) algorithm

To relax the assumption of identical perception variances in the MNL model, Castillo *et al*. (2008) assumed a Weibull-type route travel cost G_r^{ij} with a probability density function $\psi_{G_r^{ij}}$ defined by

$$\psi_{G_r^{ij}}\left(t;\varphi_r^{ij},\beta_r^{ij},\zeta_r^{ij}\right)=\begin{cases}\dfrac{\beta_r^{ij}}{\varphi_r^{ij}}\left(\dfrac{t-\zeta_r^{ij}}{\varphi_r^{ij}}\right)^{\beta_r^{ij}-1}\exp\left[-\left(\dfrac{t-\zeta_r^{ij}}{\varphi_r^{ij}}\right)^{\beta_r^{ij}}\right], \ t\geq\zeta_r^{ij} \\ 0, \quad t<\zeta_r^{ij}\end{cases}, \quad \forall r\in R^{ij}, ij\in IJ \tag{1}$$

where $\varphi_r^{ij}\in(0,\infty)$, $\beta_r^{ij}\in(0,\infty)$ and $\zeta_r^{ij}\in\left[0,G_r^{ij}\right)$ are the scale, shape, and location parameters for route r between OD pair ij, respectively. The average route travel cost g_r^{ij} is specified by

$$g_r^{ij}=\zeta_r^{ij}+\varphi_r^{ij}\Gamma\left(1+1/\beta_r^{ij}\right), \quad \forall r\in R^{ij}, ij\in IJ \tag{2}$$

where Γ() is the gamma function. In the Weibull-type route travel cost distributions, we have route-specific perception variances $(\sigma_r^{ij})^2$ defined as a function of the average route travel cost g_r^{ij}, shape parameter β_r^{ij}, and location parameter ζ_r^{ij},

$$(\sigma_r^{ij})^2=\left(g_r^{ij}-\zeta_r^{ij}\right)^2\left[\Gamma\left(1+2/\beta_r^{ij}\right)\Big/\left(\Gamma\left(1+1/\beta_r^{ij}\right)\right)^2-1\right], \quad \forall r\in R^{ij}, ij\in IJ . \tag{3}$$

Based on Eqs. (2) and (3), the route coefficient of variation (CV) ϑ_r^{ij} can be computed as

$$\vartheta_r^{ij}=\frac{g_r^{ij}-\zeta_r^{ij}}{g_r^{ij}}\sqrt{\Gamma\left(1+2/\beta_r^{ij}\right)\Big/\left(\Gamma\left(1+1/\beta_r^{ij}\right)\right)^2-1}, \quad \forall r\in R^{ij}, ij\in IJ . \tag{4}$$

3.1 MNW and PSW Models

Castillo *et al.* (2008) developed a closed-form route choice probability by assuming independently distributed Weibull-type travel costs G_r^{ij}

disutility as a multiplicative function of the observable (i.e., average) route travel cost and the unobservable random error term ε_r^{ij} (Fosgerau and Bierlaire, 2009),

$$U_r^{ij} = \left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}} \cdot \varepsilon_r^{ij}, \quad \forall r \in R^{ij}, ij \in IJ, \tag{5}$$

where $\zeta^{ij} = \min\left\{\zeta_r^{ij}, r \in R^{ij}\right\}$ is the lower bound of the perceived travel cost distributions between OD pair ij. Hence, we have $\zeta^{ij} \le \zeta_r^{ij} \le g_r^{ij}$, $\forall r \in R^{ij}, ij \in IJ$. In the meantime, the shape parameter are assumed to be identical for the routes between the same OD pair, i.e., $\beta_r^{ij} = \beta^{ij}, \forall r \in R^{ij}, ij \in IJ$. Based on the route travel disutility in Eq. (5), we have the MNW route choice probability as follows,

$$P_r^{ij} = \frac{\left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R^{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}, \quad \forall r \in R^{ij}, ij \in IJ. \tag{6}$$

The MNW model relaxes the assumption of identical perception variances but holds the assumption of independent route travel disutilities. We can use a path-size factor ϖ_r^{ij} to handle the overlapping routes (Ben-Akiva and Bierlaire, 1999),

$$\varpi_r^{ij} = \sum_{a \in A_r} \frac{l_a}{L_r^{ij}} \frac{1}{\sum_{k \in R^{ij}} \delta_{ak}^{ij}}, \quad \forall r \in R^{ij}, ij \in IJ, \tag{7}$$

where l_a and L_r^{ij} are the lengths of link a and route r, respectively. A_r is the set of links comprising route r, δ_{ak}^{ij} is the link-route incidence indicator that equals 1 when route k uses link a and 0 otherwise. A small ϖ_r^{ij} indicates a strong overlap between route r and other routes in OD pair ij. By including the path-size factor, we can rewrite the route travel disutility in Eq. (5) as

$$U_r^{ij} = \frac{\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}}}{\varpi_r^{ij}} \varepsilon_r^{ij}, \quad \forall r \in R^{ij}, ij \in IJ. \tag{8}$$

Based on Eq. (8), we have the PSW route choice probability as follows:

$$P_r^{ij} = \frac{\varpi_r^{ij}\left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R^{ij}} \varpi_k^{ij}\left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}, \quad \forall r \in R^{ij}, ij \in IJ. \tag{9}$$

3.2 Impacts of a positive location parameter on the perceived travel cost distributions

To build an entropy-based MP formulation with a decomposable travel cost at the link level, Kitthamkesorn and Chen (2013, 2014) assumed zero-valued location parameters $\zeta^{ij} \quad \forall ij \in IJ$

$\Delta^{ij} = \sqrt{\Gamma\left(1+2/\beta^{ij}\right) \Big/ \left(\Gamma\left(1+1/\beta^{ij}\right)\right)^2 - 1}$ and restate the PSW route choice probability in Eq. (9) as a function of the average route travel cost g_r^{ij}, CV ϑ_r^{ij}, path-size factor ϖ_r^{ij}, and shape parameter β^{ij},

$$P_r^{ij} = \frac{\varpi_r^{ij}\left(\vartheta_r^{ij} g_r^{ij} / \Delta^{ij}\right)^{-\beta^{ij}}}{\sum_{k\in R^{ij}} \varpi_k^{ij}\left(\vartheta_k^{ij} g_k^{ij} / \Delta^{ij}\right)^{-\beta^{ij}}}, \quad \forall r \in R^{ij}, ij \in IJ, \tag{10}$$

where $g_r^{ij} - \zeta^{ij} = \vartheta_r^{ij} g_r^{ij} \big/ \Delta^{ij}$ is derived from Eqs. (3) and (4), Δ^{ij} is constant given the shape parameter β^{ij} . The reformulation provides a perspective to unveil the impacts of CV ϑ_r^{ij} on the route choice probabilities.

When $\zeta^{ij}=0$, we have route-specific perception variances $(\sigma_r^{ij})^2=\left(g_r^{ij}\cdot\Delta^{ij}\right)^2$ and an identical CV $\vartheta_r^{ij}=\Delta^{ij}$ between each OD pair. The route choice probability in Eq. (10) degenerates into a function of g_r^{ij},

$$P_r^{ij} = \frac{\varpi_r^{ij}\left(\Delta^{ij} g_r^{ij} / \Delta^{ij}\right)^{-\beta^{ij}}}{\sum_{k\in R^{ij}} \varpi_k^{ij}\left(\Delta^{ij} g_k^{ij} / \Delta^{ij}\right)^{-\beta^{ij}}} = \frac{\varpi_r^{ij}\left(g_r^{ij}\right)^{-\beta^{ij}}}{\sum_{k\in R^{ij}} \varpi_k^{ij}\left(g_k^{ij}\right)^{-\beta^{ij}}}, \quad \forall r \in R^{ij}, ij \in IJ. \tag{11}$$

It is easy to tell that the route choice probability in Eq. (11) is insensitive to the arbitrarily scaled route travel costs via κ:

$$P_r^{ij}(\kappa) = \frac{\varpi_r^{ij}\left(\kappa\cdot g_r^{ij}\right)^{-\beta^{ij}}}{\sum_{k\in R^{ij}} \varpi_k^{ij}\left(\kappa\cdot g_k^{ij}\right)^{-\beta^{ij}}} = \frac{\varpi_r^{ij}\left(g_r^{ij}\right)^{-\beta^{ij}}\kappa^{-\beta^{ij}}}{\sum_{k\in R^{ij}} \varpi_k^{ij}\left(g_k^{ij}\right)^{-\beta^{ij}}\kappa^{-\beta^{ij}}} = \frac{\varpi_r^{ij}\left(g_r^{ij}\right)^{-\beta^{ij}}}{\sum_{k\in R^{ij}} \varpi_k^{ij}\left(g_k^{ij}\right)^{-\beta^{ij}}}, \quad \forall r \in R^{ij}, ij \in IJ. \tag{12}$$

Comparatively, when $\zeta^{ij}>0$, the route travel cost is specified by the average route travel cost g_r^{ij} and CV ϑ_r^{ij}; the route choice probability is then affected by both values of g_r^{ij} and ϑ_r^{ij}.

Proposition 1. For the Weibull route travel cost distributions, given $0 < g_r^{ij} \le g_s^{ij}$, $0 \le \zeta_1^{ij} \le \zeta_2^{ij} < g_r^{ij}$, and the parameters $\varphi_r^{ij} > 0$ and $\beta^{ij} > 0$, the following conditions hold:

(1) $$\begin{cases} (\sigma_{r|\zeta_1^{ij}}^{ij})^2 \ge (\sigma_{r|\zeta_2^{ij}}^{ij})^2 & (13) \\ \vartheta_{r|\zeta_1^{ij}}^{ij} \ge \vartheta_{r|\zeta_2^{ij}}^{ij} & (14) \end{cases}, \forall r \in R_{ij}, ij \in IJ;$$

(2) $$\begin{cases} (\sigma^{ij}_{r|\zeta^{ij}_1})^2 - (\sigma^{ij}_{r|\zeta^{ij}_2})^2 \le (\sigma^{ij}_{s|\zeta^{ij}_1})^2 - (\sigma^{ij}_{s|\zeta^{ij}_2})^2 & (15) \\ \vartheta^{ij}_{r|\zeta^{ij}_1} - \vartheta^{ij}_{r|\zeta^{ij}_2} \ge \vartheta^{ij}_{s|\zeta^{ij}_1} - \vartheta^{ij}_{s|\zeta^{ij}_2} & (16) \end{cases}, \quad \forall r,s \in R^{ij}, ij \in IJ;$$

(3) $$\begin{cases} \dfrac{(\sigma^{ij}_{r|\zeta^{ij}_1})^2 - (\sigma^{ij}_{r|\zeta^{ij}_2})^2}{(\sigma^{ij}_{r|\zeta^{ij}_1})^2} \ge \dfrac{(\sigma^{ij}_{s|\zeta^{ij}_1})^2 - (\sigma^{ij}_{s|\zeta^{ij}_2})^2}{(\sigma^{ij}_{s|\zeta^{ij}_1})^2} & (17) \\ \dfrac{\vartheta^{ij}_{r|\zeta^{ij}_1} - \vartheta^{ij}_{r|\zeta^{ij}_2}}{\vartheta^{ij}_{r|\zeta^{ij}_1}} \ge \dfrac{\vartheta^{ij}_{s|\zeta^{ij}_1} - \vartheta^{ij}_{s|\zeta^{ij}_2}}{\vartheta^{ij}_{s|\zeta^{ij}_1}} & (18) \end{cases}, \quad \forall r,s \in R^{ij}, ij \in IJ,$$

Proof. Refer to Appendix A.1.

Remark 1. Proposition 1 depicts the changing features of the route perception variances $(\sigma^{ij}_r)^2$ and CV ϑ^{ij}_r from different perspectives. Eqs. (13) and (14) state that $(\sigma^{ij}_r)^2$ and ϑ^{ij}_r decrease with an increasing location parameter. Eqs. (15) and (16) show that while the *absolute* change of $(\sigma^{ij}_r)^2$ is smaller for a shorter route, that of ϑ^{ij}_r is larger. Comparatively, Eqs. (17) and (18) demonstrate that either $(\sigma^{ij}_r)^2$ or ϑ^{ij}_r decreases *proportionally* (*or relatively*) faster for a shorter route. In fact, the results in Eqs. (13) and (14) can be explained from the behavioral perspective: *A location parameter indicates the lower bound of the perceived travel cost distributions between an OD pair,* c*onsidering this lower bound helps to better characterize the travelers' perception of the travel cost distributions, i.e., the travelers will naturally filter out the implausible smaller-than-the-lower-bound travel cost distributions, thus a larger location parameter leads to smaller* $(\sigma^{ij}_r)^2$ *and* ϑ^{ij}_r.

Remark 2. Restate Eqs. (15) to (18), we have

$$\begin{cases} (\sigma^{ij}_{s|\zeta^{ij}_1})^2 - (\sigma^{ij}_{r|\zeta^{ij}_1})^2 \ge (\sigma^{ij}_{s|\zeta^{ij}_2})^2 - (\sigma^{ij}_{r|\zeta^{ij}_2})^2 \ge 0 & (19) \\ 0 \le \vartheta^{ij}_{s|\zeta^{ij}_1} - \vartheta^{ij}_{r|\zeta^{ij}_1} \le \vartheta^{ij}_{s|\zeta^{ij}_2} - \vartheta^{ij}_{r|\zeta^{ij}_2} & (20) \end{cases}, \quad \forall r,s \in R^{ij}, ij \in IJ.$$

$$\begin{cases} \dfrac{(\sigma^{ij}_{s|\zeta^{ij}_1})^2 - (\sigma^{ij}_{r|\zeta^{ij}_1})^2}{(\sigma^{ij}_{r|\zeta^{ij}_1})^2} \le \dfrac{(\sigma^{ij}_{s|\zeta^{ij}_2})^2 - (\sigma^{ij}_{r|\zeta^{ij}_2})^2}{(\sigma^{ij}_{r|\zeta^{ij}_2})^2} & (21) \\ \dfrac{\vartheta^{ij}_{s|\zeta^{ij}_1} - \vartheta^{ij}_{r|\zeta^{ij}_1}}{\vartheta^{ij}_{r|\zeta^{ij}_1}} \le \dfrac{\vartheta^{ij}_{s|\zeta^{ij}_2} - \vartheta^{ij}_{r|\zeta^{ij}_2}}{\vartheta^{ij}_{r|\zeta^{ij}_2}} & (22) \end{cases}, \quad \forall r,s \in R^{ij}, ij \in IJ.$$

As indicated by Eqs. (19) and (20), while the absolute differences of $(\sigma^{ij}_r)^2$ between routes decrease with an increasing location parameter, those of ϑ^{ij}_r between routes tend to increase. Eqs. (21) and (22) display that the relative differences of both $(\sigma^{ij}_r)^2$ and ϑ^{ij}_r decrease with an increasing location parameter.

Based on the special case of Proposition 1 where $\zeta_1^{ij}=0$ and $\zeta_2^{ij}>0$, we can derive that considering a positive location parameter leads to smaller perception variances and CVs. Particularly, while the *absolute* decrease of perception variances is smaller for a shorter route, that of CV is larger. In the meantime, the *proportional* decrease of perception variance and that of CV are larger for a shorter route. As a result, both the perception variances and the CVs are route-specific after considering a positive location parameter. In the following, we will show that the route-specific CVs contribute to resolving the scale insensitivity issue in the Weibit models.

Lemma 1. Given that two ratio series $\{a_0/a_i\}$ and $\{b_0/b_i\}$ $(i\in I)$ satisfy $0<\dfrac{a_0}{a_i}\le\dfrac{b_0}{b_i}$ $(\forall i\in I)$ with $a_0>0$ and $b_0>0$, we have $\dfrac{a_0}{\sum_{i\in I}a_i}\le\dfrac{b_0}{\sum_{i\in I}b_i}$.

It is easy to reach Lemma 1: inverting $0<\dfrac{a_0}{a_i}\le\dfrac{b_0}{b_i}$ $(\forall i\in I)$ gives $\dfrac{a_i}{a_0}\ge\dfrac{b_i}{b_0}>0$ $(\forall i\in I)$; adding up the two inverted ratio series leads to $\dfrac{\sum_{i\in I}a_i}{a_0}\ge\dfrac{\sum_{i\in I}b_i}{b_0}>0$; taking the inverse of both sides brings about Lemma 1.

Corollary 1. Routes with the lowest average travel cost in each OD pair will attract more flow after considering a positive location parameter, i.e., $P^{ij}_{r|\zeta_2^{ij}}>P^{ij}_{r|\zeta_1^{ij}}$ $(\forall r\in R^{ij},ij\in IJ)$ when $g_r^{ij}\le g_s^{ij}$ $(\forall s\in R^{ij},ij\in IJ)$ and $0\le\zeta_1^{ij}\le\zeta_2^{ij}<g_r^{ij}$ $(\forall r\in R^{ij},ij\in IJ)$.

Proof. Refer to Appendix A.2.

Corollary 2. Considering a positive location parameter can alleviate the scale insensitivity issue in the Weibit models.

Proof. Refer to Appendix A.3.

Remark 3. For the Weibull route travel cost distribution, considering a positive location parameter leads to route-specific CVs. The CVs increase with the average route travel costs; at the same time, the CV ratio $\vartheta_r^{ij}(\kappa)/\vartheta_s^{ij}(\kappa)$ increases with the scale κ when $g_r^{ij}\le g_s^{ij}$ ($\forall r,s\in R^{ij},ij\in IJ$). As a result, the CVs increase faster for shorter routes when scaling up the average route travel costs (κ >1), leading to larger choice probabilities for the routes with the *lowest* average travel cost between each OD pair. In other words, considering a positive location parameter resolves the scale insensitivity issue in the Weibit models.

3.3 An Illustrative Numerical Example

In the following, we use a simple network in Fig. 1 to illustrate the impacts of the least perceived travel cost (i.e., a positive location parameter) on the perceived travel cost distributions. We consider three indexes, including the route perception variances, coefficients of variation, and the resultant route choice probabilities. The network consists of two OD pairs. In the short OD pair (O, A), the average travel costs for Routes R1 and R2 are set to 4 and 2 minutes, respectively; in the long OD pair (O, B), those for Routes R3 and R4 are scaled two times, being equal to 8 and 4 minutes, respectively. In the meantime, the shape parameter β^{ij} is set to 1.2 and the location parameter is set to 1.5 minutes for both OD pairs.

As shown in Fig. 2, the perceived travel costs of the lower routes originate from zero when $\zeta^{ij}=0$ minutes and from 1.5 minutes when $\zeta^{ij}=1.5$ minutes. The cumulative probabilities of the travel cost interval [0, 1.5 minutes] are positive for the lower routes, being as large as 0.4821 and 0.2490 for the short and long OD pairs, respectively. Hence, assuming zero-valued location parameters may cause an undesirable consequence that travelers have perceptions of smaller-than-the-lower-bound trip times. Comparatively, by assuming a positive location parameter between each OD pair (e.g., $\zeta^{ij}=1.5$ minutes), the perceived travel cost distributions shift rightward and are bounded below by the positive value. The adjusted distributions have higher peak and lower perception variances $(\sigma_r^{ij})^2$. In the meanwhile, the CVs ϑ_r^{ij} decrease, in a route-specific manner, faster for the lower routes. As a result, we have route-specific CVs ϑ_r^{ij} between each OD pair; moreover, a smaller CV for route R2 than that for route R4, as stated in Proposition 1.

Given that the location parameter is positive, the route travel cost can be specified by both g_r^{ij} and ϑ_r^{ij}; at the same time, the perceived CVs ϑ_r^{ij} change at route-specific speeds when scaling the route travel costs. Then, we have different route choice probabilities in the long OD pair from those in the short OD pair, i.e., resolving the scale insensitivity issue in the Weibit models (see the last row of the two tables in Fig. 2), which confirms Corollary 2.

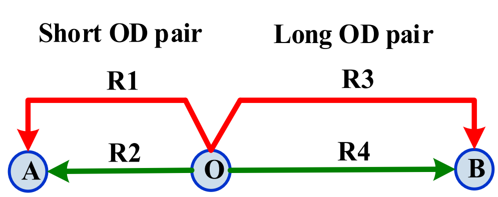


Fig. 1 A small network

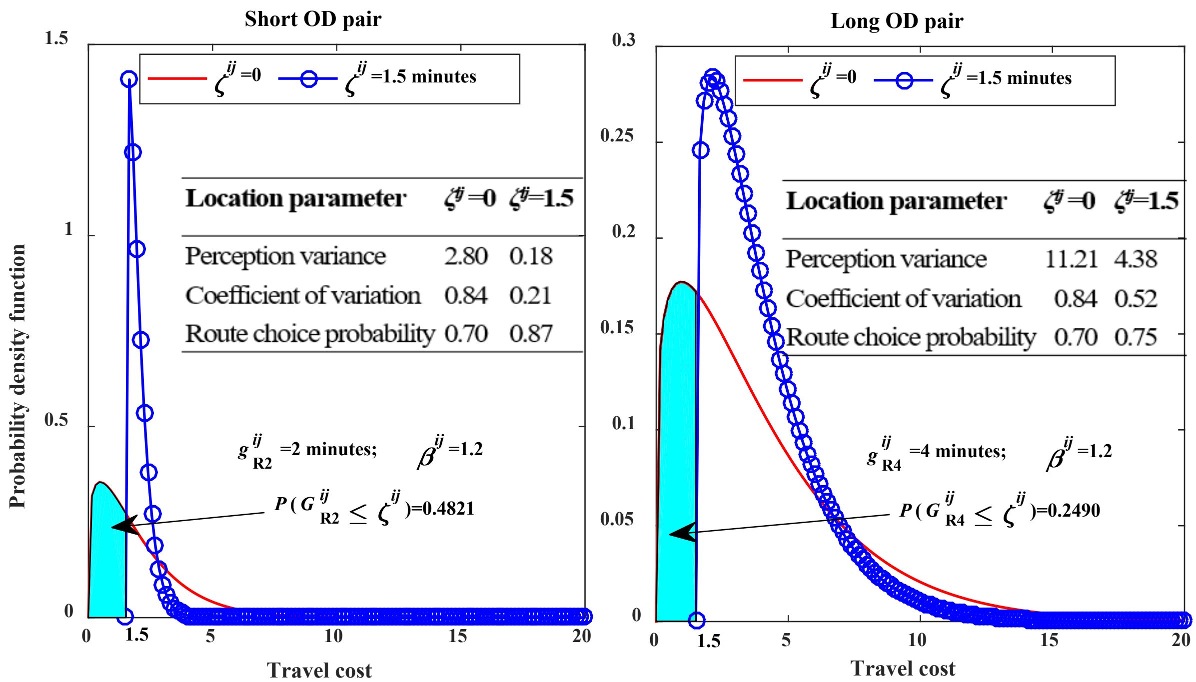


Fig. 2 Weibull travel cost distributions on the lower routes

4 WEIBIT-SUE MODEL WITH A LOCATION PARAMETER

In this section, we provide two different formulations of the Weibit SUE problems that are capable to handle the least perceived travel cost (i.e., a positive location parameter), including a variational inequality (VI) formulation and a marginal distribution model (MDM) formulation.

4.1 Model Formulation

After considering a positive location parameter, it becomes challenging to decompose the route-based travel costs in Eq. (8) at the link level or to build the entropy-based MP formulations for the Weibit SUE problems. In this section, we provide an equivalent VI formulation for the Weibit SUE models with a positive location parameter (i.e., the MNW*l*(/PSW*l*)-SUE problem). Following Nagurney (1999) and Zhou *et al.* (2012), the general VI formulation is presented as:

$$F(\mathbf{f}^*)^{\mathrm{T}}(\mathbf{f}-\mathbf{f}^*)\geq 0,\quad \forall \mathbf{f}=\left(f_r^{ij}\right)_{\sum_{ij}|R^{ij}|}\in\Omega, \tag{23}$$

where $\mathbf{f}^*$ is the optimal flow pattern, $F(\mathbf{f})$ is a general mapping from the feasible flow set Ω to R^n at point $\mathbf{f}$; the feasible flow set is defined by $\Omega=\left\{\mathbf{f}=(f_r^{ij})_{\sum_{ij}|R^{ij}|}\mid \sum_{r\in R^{ij}} f_r^{ij}=q^{ij}, f_r^{ij}\geq 0, \forall r\in R^{ij}, ij\in IJ\right\}$ where $|\ |$ is the cardinality operator. The general mapping $F(\mathbf{f})$ can take different forms according to the interpretations of SUE conditions. For example, based on the SUE conditions related to the perceived travel costs: *No user* *believes she can lower her travel cost by unilaterally changing routes* (e.g., Daganzo and Sheffi, 1977), we can define $F(\mathbf{f})$ as the *generalized perceived travel cost*. The corresponding solution will guarantee an equal and minimum *generalized perceived travel cost* on the used routes in each OD pair. When interpreting from the SUE conditions related to the flow assignments (e.g., Sheffi, 1985),

$$f_r^{ij} = P_r^{ij} \cdot q^{ij}, \ \forall r \in R^{ij}, ij \in IJ, \tag{24}$$

we can define $F(\mathbf{f})$ as a *gap function* between the current and the auxiliary flow patterns, i.e.,

$$F(\mathbf{f}) = \mathbf{f} - \mathbf{q}^{\mathrm{T}} \cdot P(\mathbf{f}), \tag{25}$$

where $\mathbf{q}^{\mathrm{T}} \cdot P(\mathbf{f})$ is the auxiliary flow pattern, $P(\mathbf{f})$ is the route choice probability under current flow pattern $\mathbf{f}$, and $\mathbf{q}$ is the vector of OD demands. Then, we have the VI formulation for the MNW*l*(/PSW*l*)-SUE problem as:

$$(\mathbf{f}^* - \mathbf{q}^{\mathrm{T}} \cdot P(\mathbf{f}^*))^{\mathrm{T}} (\mathbf{f} - \mathbf{f}^*) \geq 0, \quad \forall \mathbf{f} \in \Omega. \tag{26}$$

The VI formulation in Eq. (26) states an equivalent complementary condition $0 \leq \mathbf{f}^* \perp (\mathbf{f}^* - \mathbf{q}^{\mathrm{T}} \cdot P(\mathbf{f}^*))$ (e.g., Lo *et al.*, 1999, Zhou *et al.*, 2012); the corresponding solution will ensure a zero gap between the current flow pattern and the auxiliary one, i.e., the SUE conditions defined in Eq. (24).

Remark 4. Apart from the gap function defined in Eq. (25), we can also define $F(\mathbf{f})$ as the generalized perceived travel cost $F(\mathbf{f}) = \ln(\mathbf{g} - \boldsymbol{\zeta}) + (\mathbf{1}/\boldsymbol{\beta})^{\mathrm{T}} \cdot \ln \mathbf{f} - (\mathbf{1}/\boldsymbol{\beta})^{\mathrm{T}} \cdot \ln \varpi$, where $\mathbf{f} \in \Omega$ is the equilibrium flow pattern, $\varpi = (\varpi_r^{ij})_{|IJ| \times |R|}$ is the vector of route path-size factors, and $\boldsymbol{\zeta}$ and $\boldsymbol{\beta}$ are the vectors of location and shape parameters, respectively. Then, we can construct the PSW*l*-SUE model as $\left(\ln(\mathbf{g}^* - \boldsymbol{\zeta}) + (\mathbf{1}/\boldsymbol{\beta})^{\mathrm{T}} \cdot \ln \mathbf{f}^* - (\mathbf{1}/\boldsymbol{\beta})^{\mathrm{T}} \cdot \ln \varpi\right)(\mathbf{f} - \mathbf{f}^*) \geq 0$, $\forall \mathbf{f} \in \Omega$. When setting the path-size factor ϖ_r^{ij} to one for every route, the PSW*l*-SUE model degenerates to the MNW*l*-SUE model.

Besides, according to Aghassi *et al*. (2006), each VI formulation has its convex optimization equivalent(s). Hence, we may transform the VI formulation in Eq. (26) into an equivalent programming problem.

4.2 Qualitative Properties

In the following, we give out some qualitative properties of the MNW*l*(/PSW*l*)-SUE model concerning the solution equivalence and existence. Two assumptions are made as follows:

Assumption 1. The average link travel cost τ_a is a monotonically increasing function of the total link flow v_a.

Assumption 2. The average route travel cost g_r^{ij} is continuous with respect to (w.r.t.) the route flow pattern $\mathbf{f}$.

Based on Assumptions 1 and 2, we have the following propositions:

Proposition 2. $\mathbf{f}^*$ *is a solution to the MNWl(/PSWl)-SUE model if and only if it is a solution to the VI problem in Eq. (26).*

Proof. Refer to Appendix A.4.

Proposition 3. *The VI problem in Eq. (26) admits at least one solution.*

Proof. Refer to Appendix A.5.

Note that the uniqueness of the solution to the VI formulation in Eq. (26) relies on the properties of $F(\mathbf{f})$. A strongly monotone $F(\mathbf{f})$ will ensure a unique solution (Nagurney, 1999). Due to the nonlinearities of the average link travel cost function and the route choice probability function, the monotonicity of the mapping function in Eq. (25) may not be established. Thus, the uniqueness of the solution to the VI formulation in Eq. (26) might not be guaranteed at the route level.

4.3 MDM SUE Model

The marginal distribution model (MDM) by Ahipaşaoğlu *et al*. (2016) is another model that could handle the least perceived travel costs (i.e., positive location parameters) in the Weibit SUE problems. Being constructed as a convex optimization problem, MDM gives the MNW or PSW choice probabilities by assuming a multiplicative disutility function (e.g., Eq. (5)) and uniform marginal random error terms. The corresponding MDM-SUE flow pattern can be identified as a solution to a robust optimization problem that minimizes the worst-case objective over the cost variables; whereas the worst-case objective *"maximizes over the probability distributions of the random utilities with given marginal uniform distributions"* (Ahipaşaoğlu *et al*., 2016). The formulation of the MDM-SUE problem is given as follows:

$$\min_{P,\lambda} Z = \sum_{ij\in IJ}\sum_{r\in R^{ij}} (g_r^{ij}-\zeta^{ij})^{\beta_r^{ij}} \int_0^{P_r^{ij}} F_{r,ij}^{-1}(v)dv, \tag{27}$$

$$s.t. \sum_{r\in R^{ij}} P_r^{ij} = 1, \forall ij \in IJ, \tag{28}$$

$$P_r^{ij} = \Phi_r^{ij}(\lambda^{ij}) = F_r^{ij}(\lambda^{ij} / (g_r^{ij}-\zeta^{ij})^{\beta^{ij}}), \tag{29}$$

$$P_r^{ij} \ge 0, \forall r \in R^{ij}, ij \in IJ, \tag{30}$$

$$\lambda^{ij} \ge 0, \forall ij \in IJ, \tag{31}$$

where $P = (P_r^{ij})_{\sum_{ij}|R^{ij}|\times 1}$ $\lambda = (\lambda^{ij})_{|IJ|\times 1}$ is a vector of OD-specific variables, $F_r^{ij}(t)$ and $\Phi_r^{ij}(t)$ ε_r^{ij}

G_r^{ij} for route r between OD pair ij, respectively. Given uniform marginal perception error terms $\varepsilon_r^{ij} \sim \mathrm{U}[0, B_r^{ij}]$ $(\forall r \in R^{ij}, ij \in IJ)$ and the multiplicative disutility function in Eq. (5), i.e., $U_r^{ij} = \left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}} \cdot \varepsilon_r^{ij}$ ($\forall r \in R^{ij}, ij \in IJ$), we have the uniform route travel disutilities $U_r^{ij} \sim \mathrm{U}[0, B_r^{ij}\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}}]$ $(\forall r \in R^{ij}, ij \in IJ)$, where B_r^{ij} is a route-specific parameter, g_r^{ij}, ζ^{ij}, and β^{ij} are input parameters being numerically equal to the Weibull-type average route travel costs, OD-specific location parameters, and shape parameters, respectively. Constraints (28) and (30) ensure that the sum of nonnegative route choice probabilities is equal to 1 for each OD pair, constraints (29) and (31) guarantee that the route choice probabilities are assigned in a regulated manner.

By varying the settings of B_r^{ij} and ζ^{ij}, the MDM-SUE formulation admits different Weibit network equilibria. The MDM-SUE model admits the MNW-SUE (denoted by MWM-SUE) when setting $B_r^{ij} = 1$ and $\zeta^{ij} = 0$ $(\forall r \in R^{ij}, ij \in IJ)$, or the PSW-SUE (denoted by PSWM-SUE) when setting $B_r^{ij} = (\varpi_r^{ij})^{-1}$ and $\zeta^{ij} = 0$ $(\forall r \in R^{ij}, ij \in IJ)$ where ϖ_r^{ij} is the path-size factor defined by Eq. (7). Furthermore, by setting $\zeta^{ij} > 0$ $(\forall ij \in IJ)$ for the MWM-SUE model and the PSWM-SUE model, we have the MWM*l*-SUE model and the PSWM*l*-SUE model, respectively. The corresponding perception variances $(\sigma_{r,}^{ij})^2$ and CVs ϑ_r^{ij} for the four MDM SUE models are presented in Table3.

Table 3. Characteristics of the perceived travel costs for the MDM SUE models

Model	Travel cost distribution	Average travel cost	$(\sigma_{r,}^{ij})^2$	ϑ_r^{ij}
MWM-SUE	$G_r^{ij} \sim \mathrm{U}\left[0, \left(g_r^{ij}\right)^{\beta^{ij}}\right]$	$\frac{\left(g_r^{ij}\right)^{\beta^{ij}}}{2}$	$\frac{1}{12}\left[\left(g_r^{ij}\right)^{\beta^{ij}}\right]^2$	$\frac{\sqrt{3}}{3} \approx 0.57$
PSWM-SUE	$G_r^{ij} \sim \mathrm{U}\left[0, \left(\varpi_r^{ij}\right)^{-1}\left(g_r^{ij}\right)^{\beta^{ij}}\right]$	$\frac{\left(\varpi_r^{ij}\right)^{-1}\left(g_r^{ij}\right)^{\beta^{ij}}}{2}$	$\frac{\left[\left(\varpi_r^{ij}\right)^{-1}\left(g_r^{ij}\right)^{\beta^{ij}}\right]^2}{12}$	$\frac{\sqrt{3}}{3} \approx 0.57$
MWM*l*-SUE	$G_r^{ij} \sim \mathrm{U}\left[0, \left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}}\right]$	$\frac{\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}}}{2}$	$\frac{\left[\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}}\right]^2}{12}$	$\frac{\sqrt{3}}{3} \approx 0.57$
PSWM*l*-SUE	$G_r^{ij} \sim \mathrm{U}\left[0, \left(\varpi_r^{ij}\right)^{-1}\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}}\right]$	$\frac{\left(\varpi_r^{ij}\right)^{-1}\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}}}{2}$	$\frac{\left[\left(\varpi_r^{ij}\right)^{-1}\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}}\right]^2}{12}$	$\frac{\sqrt{3}}{3} \approx 0.57$

From Table 3, we can expect that, given the same settings of g_r^{ij}, ϖ_r^{ij}, β^{ij}, ζ^{ij} ($\forall r \in R^{ij}, ij \in IJ$), the perception variances $(\sigma_{r,}^{ij})^2$ of the MDM SUE models are much larger than those of the Weibit SUE models. In the meantime, the perception variances $(\sigma_{r,}^{ij})^2$ of the MWM-SUE model and the PSWM-SUE model are larger than those of the MWM*l*-SUE model and the PSWM*l*-SUE model, respectively.

4.4 Solution Algorithm

The VI formulation in Eq. (26) belongs to a class of nonadditive traffic equilibrium problems (NaTEP). It has route-based perceived travel costs; hence, the link-based loading algorithms may not work out in this context. In the following, we provide a route-based gradient projection algorithm with a self-adaptive step-size (SAGP) for solving the VI problem.

The SAGP algorithm was proposed by Chen *et al.* (2012) as an integration of an ingenious gradient projection method and a self-adaptive step-size scheme. The gradient projection operation is equivalent to solving a quadratic programming problem when the feasible set Ω is a general polyhedron set, or even more complicated when Ω is a general convex set. To avoid these situations and make it easier for implementation, Jayakrishnan *et al.* (1994) embedded the flow conservation constraints in the projection operations by exempting the shortest paths from the general mapping operation in Eq. (25). Merely simple projections on a nonnegative orthant are required (Chen *et al.*, 2002, 2012). We redefine the projection direction as the differences of the general mappings in Eq. (25) between the non-shortest routes and the shortest route in each OD pair.

On the other side, the self-adaptive step-size scheme determines the step-size automatically by utilizing the convergence information on previous iterations. It automatically guarantees the Lipschitz condition and the strong monotonicity assumption without solving the time-consuming quadratic programs (refer to Chen *et al.* (2012) for detailed proofs). Besides, it allows non-monotone step-size sequences; hence, the step-size may increase or decrease. Fig. 3 presents the procedure of the SAGP algorithm.

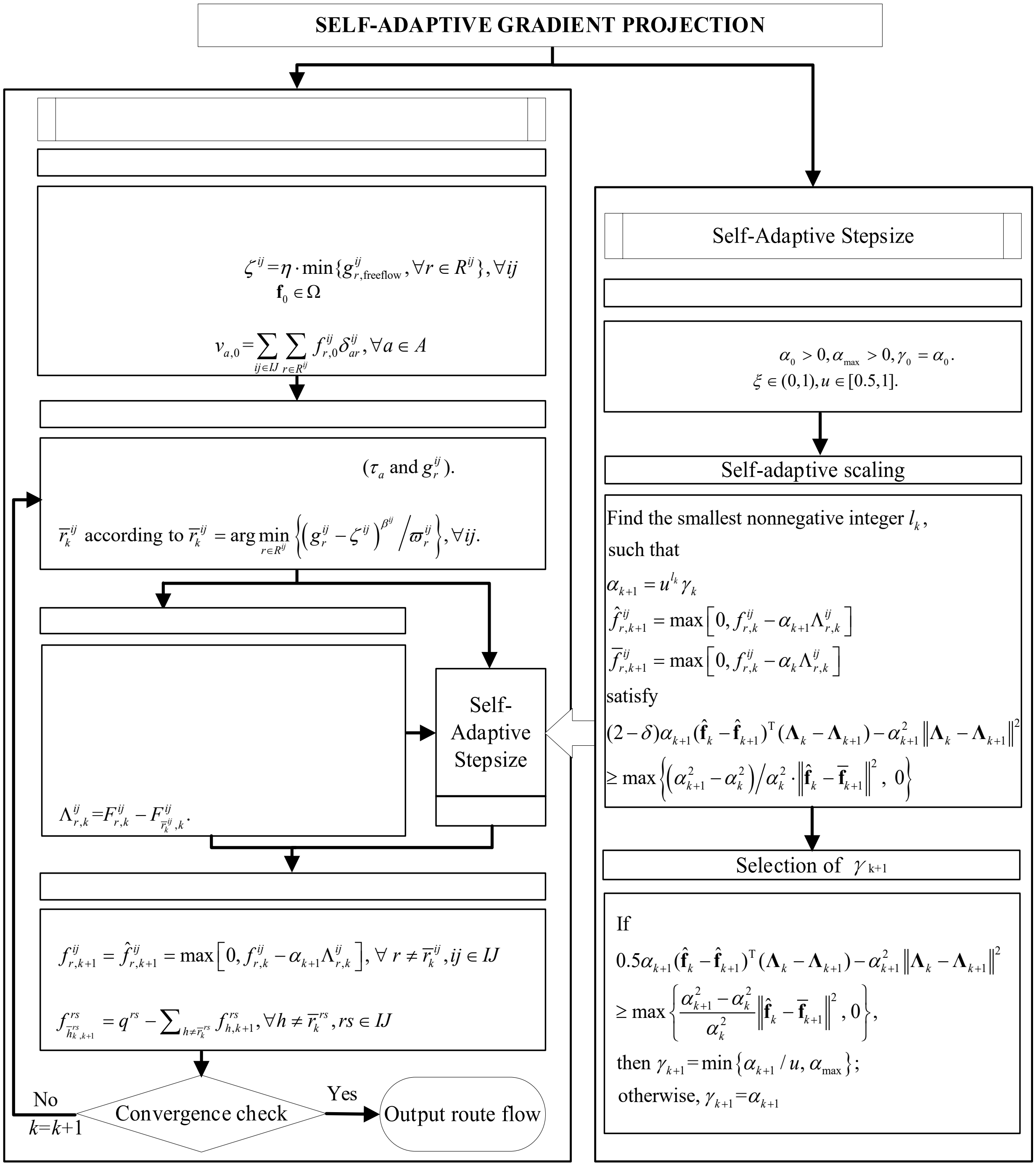


Fig. 3. Flow chart of the self-adaptive gradient projection (SAGP) algorithm

Note that, since the SAGP algorithm is route-based, a route set generation procedure might be required for real-world implementations. Then, we may incorporate a greedy heuristic algorithm or a column generation procedure (Dantzig, 1963) for generating an effective route set.

5 NUMERICAL RESULTS

In this section, we provide two numerical examples, of which the first one shows the impacts of the least perceived travel cost (i.e., a positive location parameter) on the network equilibria, on the perception variances, and on the coefficients of variation; the second one displays the robustness and applicability of the solution algorithm for resolving real-network problems.

5.1 Numerical Example I: A Simple Network

In the first numerical example, we use a small network with 10 routes (see Fig. 4) to show the equilibrium state in each model. Table 4 presents the network settings. Particularly, we adopt the MNP-SUE model as a benchmark and take the MNL-SUE model for reference. The MNP-SUE model is chosen because it can flexibly handle any valid correlation matrix among routes and permit varied perception variances, and the assumption of multivariate normal link/route flow and travel cost distributions are justifiable according to the Central Limit Theorem (Castillo *et al*., 2014; Watling, 2006). We investigate the Weibit SUE models based on whether the positive location parameters and/or the overlapping routes are considered. Combining the two conditions comes to four cases of the Weibit SUE models, namely, the MNW-SUE model, the MNW*l*-SUE model, the PSW-SUE model, and the PSW*l*-SUE model. Besides, we take four MDM SUE models for comparison, including the MWM-SUE model, the MWM*l*-SUE model, the PSWM-SUE model, and the PSWM*l*-SUE model.

For the VI formulation in Eq. (26), both additive (Castillo *et al*., 2008; Fosgerau and Bierlaire, 2009) and multiplicative (Kitthamkesorn and Chen, 2013) route travel cost functions are applicable. In this study, we adopt the commonly accepted additive route travel cost function

$$g_r^{ij} = \sum_{a \in A_r} \tau_a \ , \quad \forall r \in R^{ij}, ij \in IJ \ , \tag{32}$$

where τ_a is the average travel cost (/time) of link a, defined by the Bureau of Public Road (BPR) function

$$\tau_a = \tau_{a,0}\left[1 + \rho_a \left(v_a / C_a\right)^{\phi_a}\right], \ \forall a \in A . \tag{33}$$

The parameters ρ_a and ϕ_a are set to 0.15 and 4.0 for all the links, respectively. The link travel cost is assumed to equal the link travel time. As for the other parameters, we set the CV parameter ϑ_r^{ij}

$\overline{\vartheta_{r,\text{freeflow}}^{ij}}$ =0.26. Specifically, we set the dispersion parameter ϑ_r^{ij} to 0.5 for the MNL-SUE model, the shape parameter β^{ij}

ζ^{ij} as a linear function of the minimum free-flow travel cost between each OD pair, i.e., $\zeta^{ij} = \eta \cdot \min_{r \in R^{ij}} \{g^{ij}_{r,\text{freeflow}}\}$, where $0 \le \eta < \min_{r \in R^{ij}} \{g^{ij}_r\} \Big/ \min_{s \in R^{ij}} \{g^{ij}_{s,\text{freeflow}}\}$ is a scale to capture the public familiarity with the transportation network. In practical implementations, the location parameters can be roughly set as the minimum travel time (cost) between each OD pair, or be estimated indirectly by calibrating the scale η.

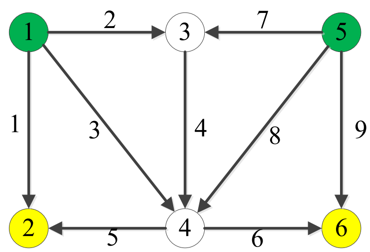


Fig. 4. A testing network

Table 4. Link parameter setting

Link#	Capacity (veh/min)	Length (km)	Free-flow speed (km/min)	Link#	Capacity (veh/min)	Length (km)	Free-flow speed (km/min)
1	200	3.5	0.5	6	300	4.5	0.9
2	300	2.0	0.8	7	250	8.4	1.2
3	300	4.0	1.0	8	300	6.6	0.6
4	350	1.8	1.2	9	300	11.4	0.6
5	300	2.5	1.0				

5.1.1 Comparison of the equilibrium state in each model

We consider three indexes at equilibrium, including the flow assignment pattern, the perception variances $(\sigma^{ij}_r)^2$, and the corresponding CVs ϑ^{ij}_r.

Table 5 presents the equilibrium flow assignments in each model. Comparing with the MNP-SUE model, the MNL-SUE model assigns more flow onto the heavily overlapped routes (e.g., routes 2, 4, 7, and 9) by assuming independence among routes. Comparatively, the MNW-SUE model, assuming length-based perception variances, assigns more flow to longer routes than the MNL-SUE model. In the meantime, the PSW-SUE model, considering the overlapping routes, assigns less flow to the heavily overlapped routes (e.g., routes 2, 3, 8, and 9) than the MNW-SUE model. Furthermore, after considering the least perceived travel cost (i.e., a positive location parameter), the MNW*l*-SUE model and the PSW*l*-SUE model distribute more flow to the shortest route between each OD pair (e.g., routes 1, 4, 6, and 10). This phenomenon can be explained by Proposition 1 and Corollary 2. Given positive location parameters, we can restate the route choice probability as a function of the average route travel cost and the associated CV; the CV, in the meanwhile, reduces route-specifically and proportionally faster for shorter routes. Hence, the shortest route between each OD pair has smaller travel disutility and accordingly a larger choice probability. As for the four MDM SUE models, they produce the same Weibit SUEs under the given network settings. Phenomena like those of the Weibit SUE models are observed. Note that, it is worth remarking that the PSW*l*-SUE, the MNW*l*-SUE and the corresponding MDM SUEs (i.e., PSWM*l*-SUE and MWM*l*-SUE) show the smallest mean square errors (MSEs, the last row in Table 5) relative to the MNP-SUE; namely, they present the best approximations of the MNP-SUE. Therefore, we may infer that considering the travelers' least perceived travel cost between each OD pair would enhance the capability of the Weibit models in picturing the individuals' travel choice decisions.

In Table 6, we take OD pair 1 for instance to analyze the features of $(\sigma_{r,}^{ij})^2$ in each model. In the MNP-SUE model, by assuming a constant ϑ_r^{ij}, we have larger route-specific $(\sigma_{r,}^{ij})^2$ for longer routes. In contrast, we have a uniform $(\sigma_{r,}^{ij})^2$ in the MNL-SUE model. In the meantime, the MDM SUE models demonstrate a different picture. All the MDM SUE models have route-specific $(\sigma_{r,}^{ij})^2$. Among them, the PSWM-SUE model has larger $(\sigma_r^{ij})^2$ than the MWM-SUE model; so does the PSWM*l*-SUE model relative to the MWM*l*-SUE model. The reason is obvious: after considering the overlapping routes, the upper bounds of the uniform perceived travel costs in the PSWM-SUE model and the PSWM*l*-SUE model are larger than those in the MWM-SUE and MWM*l*-SUE models, i.e., $\left(\varpi_r^{ij}\right)^{-1}\left(g_r^{ij}\right)^{\beta^{ij}} \geq \left(g_r^{ij}\right)^{\beta^{ij}}$ and $\left(\varpi_r^{ij}\right)^{-1}\left(g_r^{ij}-\zeta^{ij}\right)^{\beta^{ij}}$ $\geq \left(g_r^{ij}-\zeta^{ij}\right)^{\beta^{ij}}$. Moreover, the perception variances $(\sigma_{r,}^{ij})^2$ of the MWM*l*-SUE model and the PSWM*l*-SUE model decrease sharply due to the decreased upper bounds of the uniformly distributed perceived travel costs, i.e., $\left(g_r^{ij}-\zeta^{ij}\right)^{\beta^{ij}} < \left(g_r^{ij}\right)^{\beta^{ij}}$ and $\left(\varpi_r^{ij}\right)^{-1}\left(g_r^{ij}-\zeta^{ij}\right)^{\beta^{ij}} < \left(\varpi_r^{ij}\right)^{-1}\left(g_r^{ij}\right)^{\beta^{ij}}$. For the Weibit SUE models, the MNW-SUE model and the PSW-SUE model have route-specific $(\sigma_{r,}^{ij})^2$ that are larger for longer routes. By considering a positive location parameter, the route-specific $(\sigma_{r,}^{ij})^2$ decrease sharply for the MNW*l*-SUE model and the PSW*l*-SUE model, however, being larger for longer routes.

Table 5. Equilibrium flow assignments (β^{ij}=4.3 for the Weibit-related models)

OD# (Origin, Destination)	OD Demand	Route #	Link seq.	Route FFTT	PSF*	MNP-SUE (ϑ_r^{ij}=0.26)	MNL-SUE (θ=0.5)	MWM-SUE† (η=0)	MWM*l*-SUE (η=0.6)	PSWM-SUE (η=0)	PSWM*l*-SUE (η=0.6)	MNW-SUE (η=0)	MNW*l*-SUE (η=0.6)	PSW-SUE (η=0)	PSW*l*-SUE (η=0.6)
1		1	1	7	1.00	149.50	121.97	123.32	134.58	132.64	140.86	123.32	134.58	132.91	141.04
(1, 2)	300	2	2-4-5	6.5	0.80	51.67	80.10	79.51	67.11	75.17	64.29	79.51	67.11	75.29	64.37
		3	3-5	6.5	0.81	98.83	97.94	97.17	98.31	92.19	94.85	97.17	98.31	91.80	94.58
2		4	2-4-6	9	0.73	65.08	89.98	93.54	89.41	93.20	88.89	93.54	89.41	93.58	89.23
(1, 6)	200	5	3-6	9	0.74	134.92	110.02	106.46	110.59	106.80	111.11	106.46	110.59	106.42	110.77
3		6	7-4-5	11	0.90	134.15	123.25	119.75	128.06	121.78	129.74	119.75	128.06	122.14	130.14
(5, 2)	200	7	8-5	13.5	0.86	65.85	76.75	80.25	71.94	78.22	70.26	80.25	71.94	77.86	69.86
4		8	7-4-6	13.5	0.85	157.47	134.48	122.64	130.33	120.79	129.24	122.64	130.33	120.18	128.65
(5, 6)	300	9	8-6	16	0.80	64.80	83.75	90.26	86.44	83.67	80.81	90.26	86.44	84.35	81.46
		10	9	19	1.00	77.73	81.77	87.10	83.24	95.53	89.95	87.10	83.24	95.46	89.89
					MSE‡	-	394.86	544.70	295.46	478.64	262.51	544.70	295.46	488.26	270.55

* Route path-size factor, defined by Eq. (7) as an indicator to show the correlation or overlapping index among routes between an OD pair.

† The MWM-SUE model is solved via the MDM-MSA algorithm by Ahipaşaoğlu *et al*. (2016), so do the MWM*l*-SUE model, the PSWM-SUE model, and the PSWM*l*-SUE model. The source codes are available at request.

‡ We take the MNP-SUE model as a benchmark and calculate the MSEs of other SUE flow assignments relative to the MNP-SUE.

Fig. 5 demonstrates the coefficients of variation ϑ_r^{ij} in each model. The MNP-SUE model presumes a common ϑ_r^{ij} for all routes; the MNL-SUE model has route-specific ϑ_r^{ij} that is smaller for routes with a larger average travel cost, meaning smaller relative variabilities for longer routes. Comparatively, all the MDM SUE models have the same and constant ϑ_r^{ij} that is significantly larger than those of the other SUE models. The results are expectable given that the uniform perceived travel costs are bounded below by zeros in the MDM SUE models, i.e., the second column in Table 3. Meanwhile, the MNW-SUE model and the PSW-SUE model have the same ϑ_r^{ij} for all routes, inferring the same relative variability for all routes. Different from the MDM SUE models, the MNW*l*-SUE model and the PSW*l*-SUE model have smaller and route-specific ϑ_r^{ij} after considering the least perceived travel cost (i.e., a positive location parameter), and present consistent change patterns with the perception variances $(\sigma_{r,}^{ij})^2$. Table 7 summarizes the change characteristics of $(\sigma_{r,}^{ij})^2$ and ϑ_r^{ij} in the 10 SUE models.

Table 6. Route perception variances between OD pair 1 (ζ^1=3.9 minutes)

SUE Models	Perception Variances			SUE Models	Perception Variances		
	Route 1	Route 2	Route 3		Route 1	Route 2	Route 3
MNP-SUE	3.63	3.93	3.72	PSWM*l*-SUE	2.79E+3	1.34E+4	6.27E+3
MNL-SUE	6.58	6.58	6.58	MNW-SUE	3.53	4.33	3.95
MWM-SUE	1.86E+6	4.47E+6	2.99E+6	MNW*l*-SUE	0.76	1.05	0.88
MWM*l*-SUE	2.50E+3	1.00E+4	4.68E+3	PSW-SUE	3.58	4.21	3.84
PSWM-SUE	1.97E+6	6.17E+6	4.17E+6	PSW*l*-SUE	0.78	1.01	0.85

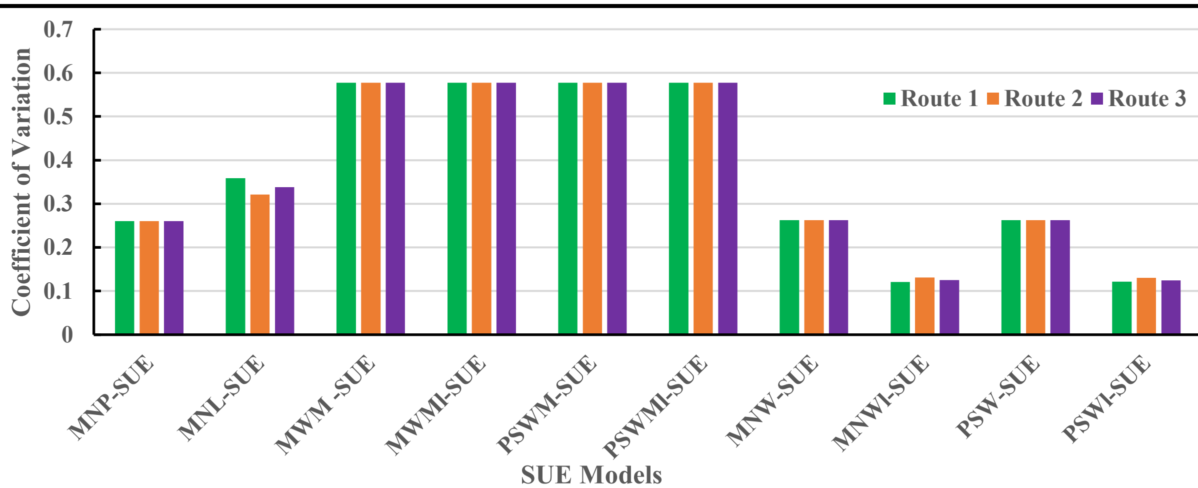


Fig. 5. Route coefficients of variation between OD pair 1 (ζ^1=3.9 minutes)

Table 7. Change characteristics of the perception variances and coefficient of variation in each model

	Perception variance (*Absolute* perceived variability)	**Coefficient of variation** (*Relative* perceived variability)
MNP-SUE	•**Route-specific** •Increase with the average route travel cost	•Pre-assumed or other specified
MNL-SUE	•Identical within each OD pair •Determined by the dispersion parameter	•**Route-specific** •Decrease with the average route travel cost
MWM (/PSWM)-SUE	•**Route-specific** •Increase with the average route travel cost	•Identical within each OD pair
MWM*l* (/PSWM*l*)-SUE (η>0)	•**Route-specific** •Increase with the average route travel cost •Decrease with the location parameter	•Identical within each OD pair
MNW (/PSW)-SUE	•**Route-specific** •Increase with the average route travel cost	•Identical within each OD pair •Determined by the shape parameter
MNW*l* (/PSW*l*)-SUE (η>0)	•**Route-specific** •Increase with the average route travel cost •Decrease with the location parameter	•**Route-specific** •Increase with the average route travel cost •Decrease with the location parameter

5.1.2 Sensitivity analysis

In this section, we adopt two Weibit SUE models (i.e., MNW*l*-SUE and PSW*l*-SUE) to examine the impacts of the least perceived travel cost ζ^{ij} on two variability indexes, namely the absolute perceived variability $(\sigma_r^{ij})^2$ and the relative perceived variability ϑ_r^{ij}. Particularly, we consider the values of both indexes on routes 1 and 2, and the differences of each index between the two routes, i.e., $(\sigma_2^1)^2 - (\sigma_1^1)^2$ and $\vartheta_2^1 - \vartheta_1^1$.

As shown in Fig. 6, the perception variances $(\sigma_r^{ij})^2$ decrease with η for both routes in both models. Particularly, the value of $(\sigma_r^{ij})^2$ on the long route (i.e., route 2) decreases faster than that on the short one (i.e., route 1). As a result, $(\sigma_2^1)^2 - (\sigma_1^1)^2$ decrease with η (Fig. 7), which confirms Eq. (19). In the meantime, the perceived CVs ϑ_r^{ij} decrease with η (Fig. 8), however, *slower* on route 2 than on route 1. The resultant differences in ϑ_r^{ij} between the two routes (i.e., $\vartheta_2^1 - \vartheta_1^1$) increase with η, leading to a larger ϑ_r^{ij} for route 2 (Fig. 9). The results confirm Eq. (20).

To sum up, by increasing the location parameter ζ^{ij} (/η) between an OD pair, shorter routes experience *slower* decreases in $(\sigma_r^{ij})^2$, however, *faster* decreases in ϑ_r^{ij}. These changes enlarge the proportional differences of $(\sigma_r^{ij})^2$ (or the absolute differences of ϑ_r^{ij}) between routes. The synthetic behavioral effects are that travelers can better differentiate the routes in terms of $(\sigma_r^{ij})^2$ and ϑ_r^{ij}, and place more probabilities on taking the shorter routes (e.g., route 1 in Fig. 10). As a result, both the average travel time on the shortest route between an OD pair (e.g., route 1 in Fig. 11) and the network total travel times (Fig. 12) increase with η. These observations conform to Proposition 1.

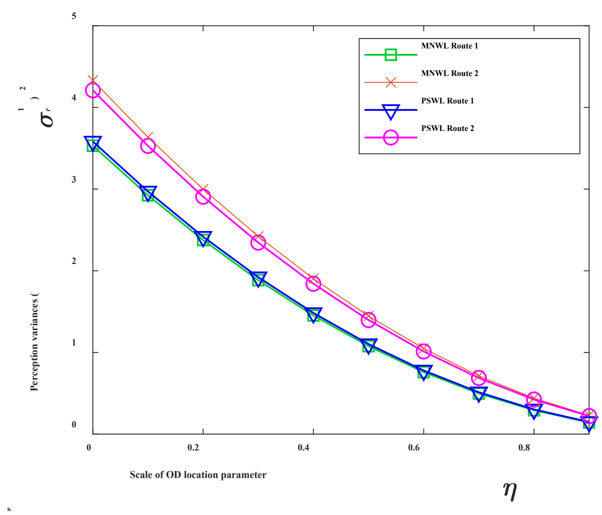


Fig. 6. Perception variances $(\sigma_r^{ij})^2$ with η (β^{ij}=4.3)

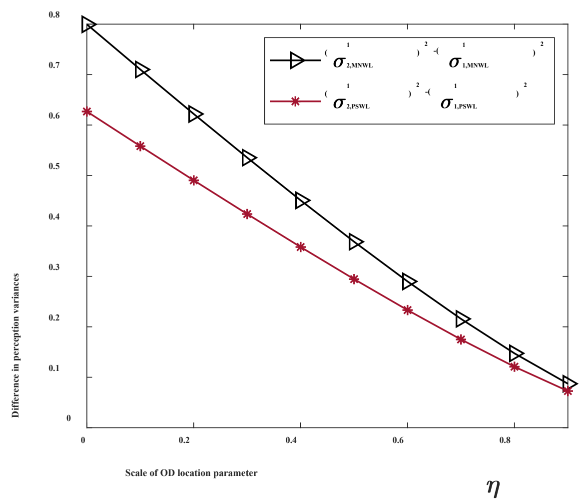


Fig. 7. Differences in $(\sigma_r^{ij})^2$ with η (β^{ij}=4.3)

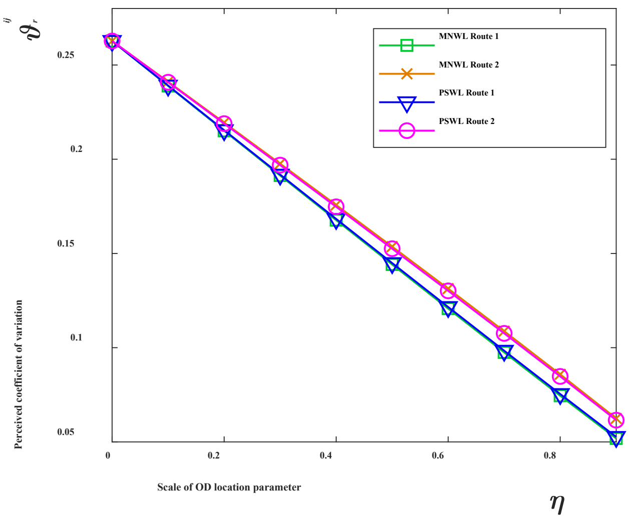


Fig. 8. Perceived CVs ϑ_r^{ij} with η (β^{ij}=4.3)

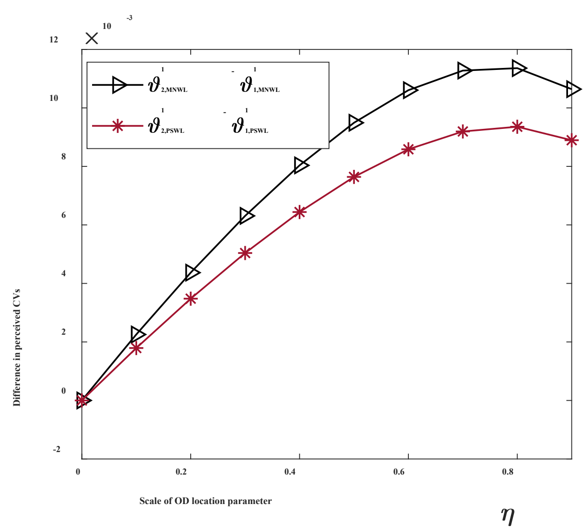


Fig. 9. Differences in ϑ_r^{ij} with η (β^{ij}=4.3)

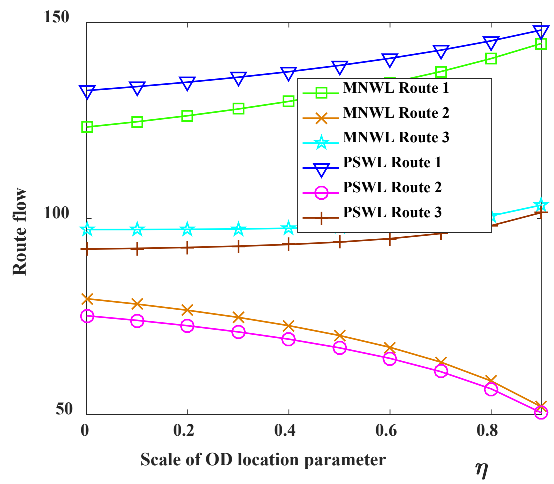


Fig. 10. Route flow with η (β^{ij}=4.3)

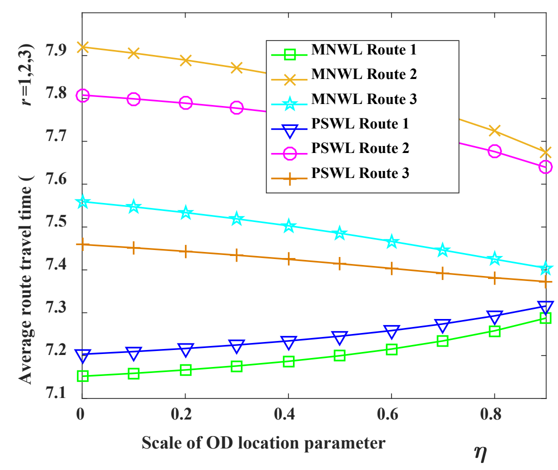


Fig. 11. Route travel time with η (β^{ij}=4.3)

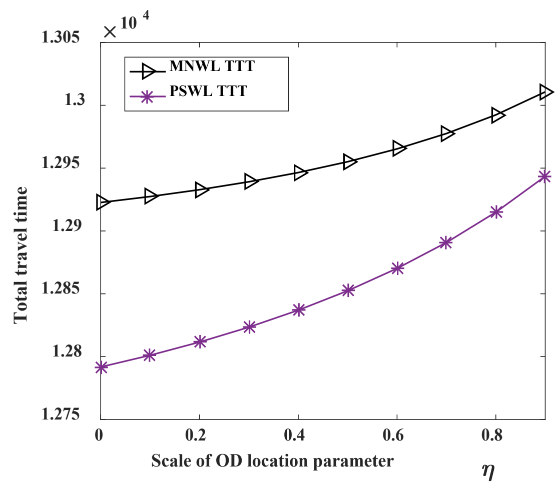


Fig. 12. Total travel time with η (β^{ij}=4.3)

5.2. Numerical Example II: The Winnipeg Network

In the second numerical example, we use the Winnipeg network to show the effects of the least perceived travel cost ζ^{ij} on the convergence of the solution algorithm and the equilibrium flow assignments. The Winnipeg network consists of 154 zones, 1,067 nodes, 2,535 links, and 4,345 OD pairs. The network structure, OD demands, and link performance parameters are borrowed from the Emme/2 software (INRO Consultants, 1999). We set the SAGP algorithm parameters δ and u to 0.5 and 0.9, respectively. In this study, we use the working route-set from Bekhor *et al.* (2008), which consists of 174,491 routes with an average of 40.16 routes per OD pair. This behavioral route-set has been adopted in Chen et al. (2012), Kitthamkesorn and Chen (2013, 2014), and Bekhor et al. (2008) to name a few, to analyze the route choice models and path-based traffic assignment algorithms. We take the root mean square error (RMSE) as the convergence criterion: $RMSE = \sqrt{\frac{1}{|\sum_{ij\in IJ} R^{ij}|} \sum_{ij\in IJ} \sum_{r\in R^{ij}} \left(f_r^{ij}(k+1) - f_r^{ij}(k) \right)^2}$ where $|\sum_{ij\in IJ} R^{ij}|$ is the number of routes between all the OD pairs.

5.2.1 Sensitivity of the algorithm convergence speed to the scale of the least perceived travel cost

We take the PSW*l*-SUE model for instance to examine the effects of the scale of the least perceived travel costs on the convergence speed of the algorithm. We set the shape parameters β^{ij} to 4.3 for all OD pairs, and vary η from 0.1 to 0.9 at an interval of 0.2 to examine the number of iterations required to reach different convergence accuracies (denoted as the logarithm to base 10 of the accuracies).

As presented in Fig. 13, while the iterations required to reach each accuracy level increase with η, the iterations between two consecutive accuracy levels remain almost the same under each η. In the meantime, the iterations between two consecutive accuracy levels increase as η gets larger. For example, it requires 210 (or 211) iterations to reach the accuracy level 1E-4 from 1E-3 (or 1E-7 from 1E-6) when η=0.1, or 286 or 287 iterations to reach the accuracy level 1E-5 from 1E-4 (or from 1E-8 from 1E-7) when η=0.7. The results demonstrate the robustness and efficiency of the SAGP algorithm.

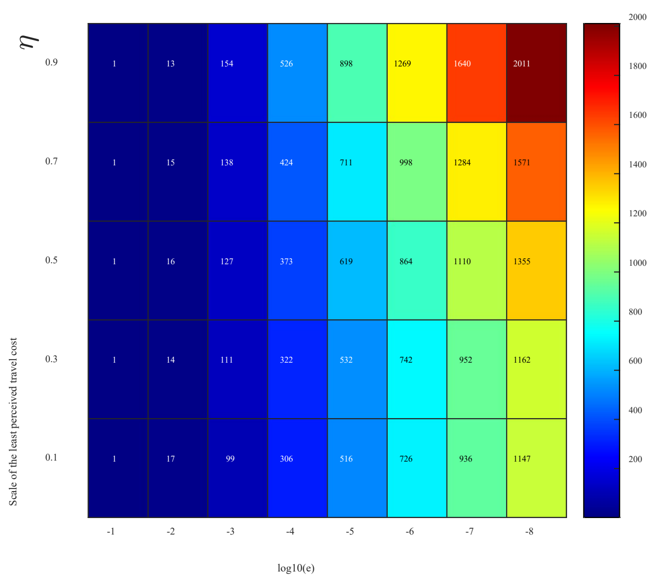


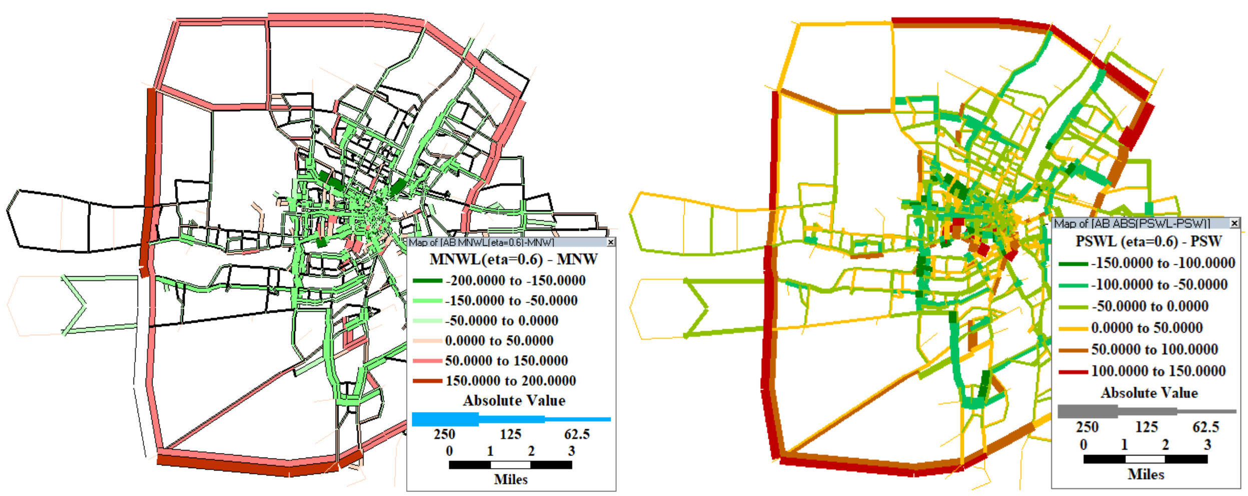
Fig. 13. Number of iterations required for each accuracy level under each η

5.2.2 Effects of the least perceived travel cost on the equilibrium states

In the following, we investigate the impacts of the least perceived travel cost on two types of network indexes, namely, the link volume/capacity (V/C) ratio and the link flow differences between models. We consider two Weibit-SUE model pairs, including the MNW*l*-SUE (η=0.6) model versus the MNW-SUE model and the PSWl-SUE (η=0.6) model versus the PSW-SUE model. The shape parameters β^{ij} are set to 4.3 for all OD pairs.

Fig. 14 displays the link flow differences in the two model pairs. Most link flow differences fall in the interval of [-50, 50]; some even are larger than 100 or smaller than -100. Fig. 15 summarizes the distribution of the link flow differences in the two model pairs. After considering the least perceived travel cost between each OD pair, about 70 percent (1789/2535) of links attract less flow in the MNWl-SUE model than in the MNW-SUE model. Similar situations happen to about 66 percent (1688/2535) of links in the PSWl-SUE model versus in the PSW-SUE model.

Correspondingly, the lowest V/C ratio interval [0, 0.60) embraces more links in the MNWl-SUE model and the PSWl-SUE model (Table 8); the network average link V/C ratio decreases from 0.43 in the MNW-SUE model to 0.40 in the MNWl-SUE model and from 0.46 in the PSW-SUE model to 0.43 in the PSWl-SUE model. The decreased network average link V/C ratios indicate mitigated congestion situations. We may explain the results by referring to the physical meaning of a location parameter. A location parameter represents the individuals' experience and knowledge of the least potential travel cost (i.e., the certain part) between an OD pair. Eliminating the certain part adds to more advantages of shorter routes in terms of absolute and relative variability, i.e., shorter routes have smaller perception variances and CVs. More travelers shift to the shortest route between each OD pair, somehow alleviate the network congestion situation under the given network settings. However, attention should be paid that considering the least perceived travel cost does not necessarily mitigate the congestion situation at the network level. Numerical example I has posed a counter-example (see Fig. 12): the network TTT increases for the MNWl-SUE model and the PSWl-SUE model when the location parameters increase from 0s to above.



(1) Link flow differences between the MNWl-SUE (η=0.6) model and the MNW-SUE model

(2) Link flow differences between the PSWl-SUE (η=0.6) model and the PSW-SUE model

Fig. 14. Link flow differences between different models

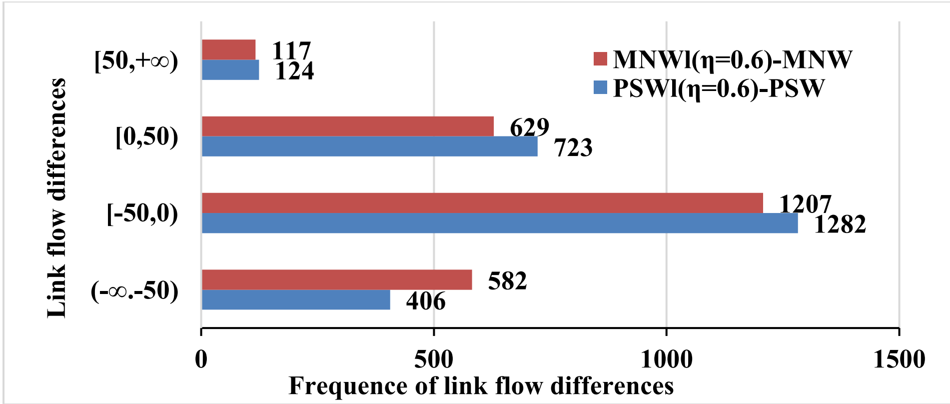


Fig. 15. Distribution of link flow differences

Table 8. Link V/C ratio distribution in each model

Link V/C Ratio Interval	MNW-SUE	MNW*l* (η=0.6)-SUE	PSW-SUE	PSW*l* (η=0.6)-SUE
[0,0.6)	1762	1824	1700	1767
[0.6,1.0)	480	446	524	480
[1.0,1.5)	245	222	261	248
[1.5, +∞)	48	43	51	41
Mean V/C	0.43	0.40	0.46	0.43

6 CONCLUSIONS

In this study, we investigate the impacts of a special kind of travelers' pre-trip information or knowledge, i.e., the least perceived travel cost between each OD pair, on the network SUE problems. Particularly, we conduct the investigation with the Weibit SUE models, since they have a location parameter that can naturally characterize the least perceived travel cost between each OD pair. The impact patterns and mechanisms are analyzed. By considering a positive location parameter,

(1) the Weibull-type perception variances decrease, in a route-specific manner, faster (however proportionally slower) for longer routes;

(2) the varied changes of the perception variances lead to route-specific coefficients of variations (CVs);

(3) the route travel cost can be specified as a function of both the average travel cost and the route-specific CV, thus resolving the scale insensitivity issue of the weibit-based SUE models; as a result,

(4) travelers can better distinguish the routes in terms of absolute and relative variability; more travelers shift to the routes with the lowest average travel cost between each OD pair.

Numerical examples are provided to show the impacts of the least perceived travel cost on the Weibit route choice probabilities and the network SUEs, and also to show the efficiency and robustness of the proposed solution algorithm.

We consider the Weibit SUE models in this study; for the other SUE models that do not have a location parameter to naturally capture the least perceived travel cost, we may apply a simulation-based approach to carry out the investigation. We may take the least perceived travel cost as a lower bound to reframe (e.g., to truncate or right shift) the perceived travel cost distributions, and observe the resultant route choice distributions to unveil the impacts of the least perceived travel cost on the general SUE problems.

ACKNOWLEDGEMENTS

This research was supported by the National Natural Science Foundation of China (Grant No. 72101085, 71801106, 71701078), the Science Foundation of Ministry of Education of China (Grant No. 17YJC630150), the Hubei Provincial Natural Science Foundation of China (Grant No. 2020CFB264), the Research Committee of the Hong Kong Polytechnic University (Grant No. 1-ZE5T), and the Research Grants Council of the Hong Kong Special Administrative Region (Grant No. 15212217). These supports are gratefully acknowledged.

APPENDIX

A.1. Proof of Proposition 1

Proof. The proof includes three parts. The first part is equivalent to proving that $(\sigma_r^{ij})^2$ and ϑ_r^{ij} are decreasing with the location parameter ζ^{ij}. Because $0 \le \zeta_1^{ij} \le \zeta_2^{ij}$, we have

$$\begin{cases} (\sigma_{r|\zeta_2^{ij}}^{ij})^2 = \left(g_r^{ij} - \zeta_2^{ij}\right)^2 \left(\Delta^{ij}\right)^2 \le \left(g_r^{ij} - \zeta_1^{ij}\right)^2 \left(\Delta^{ij}\right)^2 = (\sigma_{r|\zeta_1^{ij}}^{ij})^2 \\ \vartheta_{r|\zeta_2^{ij}}^{ij} = \left(1 - \zeta_2^{ij} / g_r^{ij}\right)\Delta^{ij} \le \left(1 - \zeta_1^{ij} / g_r^{ij}\right)\Delta^{ij} = \vartheta_{r|\zeta_1^{ij}}^{ij} \end{cases}, \forall r \in R^{ij}, ij \in IJ .$$

The strict inequalities hold when $\zeta_1^{ij} < \zeta_2^{ij}$.

For the second part, given $g_r^{ij} \le g_s^{ij}$ and $0 \le \zeta_1^{ij} \le \zeta_2^{ij} < g_r^{ij}$, we have

$$\begin{aligned} &\left[(g_r^{ij} - \zeta_1^{ij}) + (g_r^{ij} - \zeta_2^{ij})\right](\zeta_2^{ij} - \zeta_1^{ij}) \le \left[(g_s^{ij} - \zeta_1^{ij}) + (g_s^{ij} - \zeta_2^{ij})\right](\zeta_2^{ij} - \zeta_1^{ij}) \\ &\Rightarrow (g_r^{ij})^2 - 2g_r^{ij}\zeta_1^{ij} + (\zeta_1^{ij})^2 - \left[(g_r^{ij})^2 - 2g_r^{ij}\zeta_2^{ij} + (\zeta_2^{ij})^2\right] \\ &\quad \le (g_s^{ij})^2 - 2g_s^{ij}\zeta_1^{ij} + (\zeta_1^{ij})^2 - \left[(g_s^{ij})^2 - 2g_s^{ij}\zeta_2^{ij} + (\zeta_2^{ij})^2\right] \\ &\Rightarrow (g_r^{ij} - \zeta_1^{ij})^2 - (g_r^{ij} - \zeta_2^{ij})^2 \le (g_s^{ij} - \zeta_1^{ij})^2 - (g_s^{ij} - \zeta_2^{ij})^2, \forall r,s \in R^{ij}, ij \in IJ. \end{aligned} \tag{A.1}$$

Multiplying each side by $(\Delta^{ij})^2$ gives $(\sigma_{r|\zeta_1^{ij}}^{ij})^2 - (\sigma_{r|\zeta_2^{ij}}^{ij})^2 \ge (\sigma_{s|\zeta_1^{ij}}^{ij})^2 - (\sigma_{s|\zeta_2^{ij}}^{ij})^2$, $\forall r,s \in R^{ij}, ij \in IJ$.

Similarly, given $0 \le \zeta_1^{ij} \le \zeta_2^{ij}$, computing the CV difference between routes in the same OD pair gives

$$\vartheta^{ij}_{r|\zeta^{ij}_1} - \vartheta^{ij}_{r|\zeta^{ij}_2} = (1-\frac{\zeta^{ij}_1}{g^{ij}_r})\Delta^{ij} - (1-\frac{\zeta^{ij}_2}{g^{ij}_r})\Delta^{ij} = \frac{\zeta^{ij}_2 - \zeta^{ij}_1}{g^{ij}_r}\Delta^{ij}, \ \forall r,s \in R^{ij}, ij \in IJ, \tag{A.2}$$

which is decreasing with the average route travel cost g^{ij}_r. Then we have $\vartheta^{ij}_{r|\zeta^{ij}_1} - \vartheta^{ij}_{r|\zeta^{ij}_2} \geq \vartheta^{ij}_{s|\zeta^{ij}_1} - \vartheta^{ij}_{s|\zeta^{ij}_2}$ when $g^{ij}_r \leq g^{ij}_s$.

As for the third part, given $g^{ij}_r \leq g^{ij}_s$ and $0 \leq \zeta^{ij}_1 \leq \zeta^{ij}_2 < g^{ij}_r$, with simple manipulations we have

$$1-\frac{\zeta^{ij}_2 - \zeta^{ij}_1}{g^{ij}_r - \zeta^{ij}_1} \leq 1-\frac{\zeta^{ij}_2 - \zeta^{ij}_1}{g^{ij}_s - \zeta^{ij}_1} \Rightarrow \left(\frac{g^{ij}_r - \zeta^{ij}_2}{g^{ij}_r - \zeta^{ij}_1}\cdot\frac{\Delta^{ij}}{\Delta^{ij}}\right)^2 \leq \left(\frac{g^{ij}_s - \zeta^{ij}_2}{g^{ij}_s - \zeta^{ij}_1}\cdot\frac{\Delta^{ij}}{\Delta^{ij}}\right)^2, \forall r,s \in R^{ij}, ij \in IJ, \tag{A.3}$$

$$\Rightarrow \frac{(\sigma^{ij}_{r|\zeta^{ij}_2})^2}{(\sigma^{ij}_{r|\zeta^{ij}_1})^2} \leq \frac{(\sigma^{ij}_{s|\zeta^{ij}_2})^2}{(\sigma^{ij}_{s|\zeta^{ij}_1})^2} \tag{A.4}$$

$$\Rightarrow \frac{(\sigma^{ij}_{r|\zeta^{ij}_1})^2 - (\sigma^{ij}_{r|\zeta^{ij}_2})^2}{(\sigma^{ij}_{r|\zeta^{ij}_1})^2} \leq \frac{(\sigma^{ij}_{s|\zeta^{ij}_1})^2 - (\sigma^{ij}_{s|\zeta^{ij}_2})^2}{(\sigma^{ij}_{s|\zeta^{ij}_1})^2}, \forall r,s \in R^{ij}, ij \in IJ. \tag{A.5}$$

Strict inequality holds when either $g^{ij}_r < g^{ij}_s$ or $\zeta^{ij}_1 < \zeta^{ij}_2$ establishes.

Similarly, from Eq. (A.4) and the relationship between $(\sigma^{ij}_r)^2$ and ϑ^{ij}_r, we have

$$\frac{\sigma^{ij}_{r|\zeta^{ij}_2} \big/ g^{ij}_r}{\sigma^{ij}_{r|\zeta^{ij}_1} \big/ g^{ij}_r} \leq \frac{\sigma^{ij}_{s|\zeta^{ij}_2} \big/ g^{ij}_s}{\sigma^{ij}_{s|\zeta^{ij}_1} \big/ g^{ij}_s} \tag{A.6}$$

$$\Rightarrow \frac{\vartheta^{ij}_{r|\zeta^{ij}_2}}{\vartheta^{ij}_{r|\zeta^{ij}_1}} \leq \frac{\vartheta^{ij}_{s|\zeta^{ij}_2}}{\vartheta^{ij}_{s|\zeta^{ij}_1}} \tag{A.7}$$

$$\Rightarrow \frac{\vartheta^{ij}_{r|\zeta^{ij}_1} - \vartheta^{ij}_{r|\zeta^{ij}_2}}{\vartheta^{ij}_{r|\zeta^{ij}_1}} \geq \frac{\vartheta^{ij}_{s|\zeta^{ij}_1} - \vartheta^{ij}_{s|\zeta^{ij}_2}}{\vartheta^{ij}_{s|\zeta^{ij}_1}}, \forall r,s \in R^{ij}, ij \in IJ. \tag{A.8}$$

This completes the proof. □

A.2. Proof of Corollary 1

Proof. Arrange the average route travel costs g^{ij}_r in an ascending order $g^{ij}_r \leq \cdots \leq g^{ij}_s \leq \cdots$, $\forall s \neq r, s \in R^{ij}, ij \in IJ$, then, restating Eq. (A.7) gives

$$\frac{\vartheta^{ij}_{r|\zeta^{ij}_2}}{\vartheta^{ij}_{s|\zeta^{ij}_2}} \leq \frac{\vartheta^{ij}_{r|\zeta^{ij}_1}}{\vartheta^{ij}_{s|\zeta^{ij}_1}}, \forall s \neq r, s \in R^{ij}, ij \in IJ. \tag{A.9}$$

Multiplying each side by $\frac{g^{ij}_r / \Delta^{ij}}{g^{ij}_s / \Delta^{ij}}$, we have

$$\frac{\vartheta^{ij}_{r|\zeta_2^{ij}} g_r^{ij}/\Delta^{ij}}{\vartheta^{ij}_{s|\zeta_2^{ij}} g_s^{ij}/\Delta^{ij}} \le \frac{\vartheta^{ij}_{r|\zeta_1^{ij}} g_r^{ij}/\Delta^{ij}}{\vartheta^{ij}_{s|\zeta_1^{ij}} g_s^{ij}/\Delta^{ij}} \Rightarrow \left(\frac{\vartheta^{ij}_{r|\zeta_2^{ij}} g_r^{ij}/\Delta^{ij}}{\vartheta^{ij}_{s|\zeta_2^{ij}} g_s^{ij}/\Delta^{ij}}\right)^{-\beta^{ij}} \ge \left(\frac{\vartheta^{ij}_{r|\zeta_1^{ij}} g_r^{ij}/\Delta^{ij}}{\vartheta^{ij}_{s|\zeta_1^{ij}} g_s^{ij}/\Delta^{ij}}\right)^{-\beta^{ij}}, \forall s \in R^{ij}, ij \in IJ \tag{A.10}$$

From Lemma 1, we have

$$\frac{\left(\vartheta^{ij}_{r|\zeta_2^{ij}} g_r^{ij}/\Delta^{ij}\right)^{-\beta^{ij}}}{\sum_{s\in R^{ij}}\left(\vartheta^{ij}_{s|\zeta_2^{ij}} g_s^{ij}/\Delta^{ij}\right)^{-\beta^{ij}}} \ge \frac{\left(\vartheta^{ij}_{r|\zeta_1^{ij}} g_r^{ij}/\Delta^{ij}\right)^{-\beta^{ij}}}{\sum_{s\in R^{ij}}\left(\vartheta^{ij}_{s|\zeta_1^{ij}} g_s^{ij}/\Delta^{ij}\right)^{-\beta^{ij}}}, \tag{A.11}$$

$$i.e., P^{ij}_{r|\zeta_2^{ij}} \ge P^{ij}_{r|\zeta_1^{ij}}, \forall ij \in IJ. \tag{A.12}$$

This finishes the proof. □

A.3. Proof of Corollary 2

Proof. The proof is equivalent to showing that, given an ascending sequence $\mathbf{g}=\{g_r^{ij} \le \cdots \le g_s^{ij} \le \cdots, \forall s \ne r, s \in R^{ij}, ij \in IJ\}$ and the OD-specific positive location parameters ζ^{ij}, imposing different scales $\kappa(\neq 1)$ onto **g** would produce different route choice probabilities, i.e., there exists at least one route satisfying $P_r^{ij}(\kappa_1) \ne P_r^{ij}(\kappa_2)$ when $\kappa_1 \ne \kappa_2$.

When $\zeta^{ij}>0$, the scaled route choice probability in Eq. (10) reduces to

$$P_r^{ij}(\kappa)=\frac{\varpi_r^{ij}\left(\vartheta_r^{ij}(\kappa)\cdot\kappa g_r^{ij}\right)^{-\beta^{ij}}}{\sum_{k\in R^{ij}}\varpi_k^{ij}\left(\vartheta_k^{ij}(\kappa)\cdot\kappa g_k^{ij}\right)^{-\beta^{ij}}} = \frac{\varpi_r^{ij}\left(\vartheta_r^{ij}(\kappa)\cdot g_r^{ij}\right)^{-\beta^{ij}}}{\sum_{k\in R^{ij}}\varpi_k^{ij}\left(\vartheta_k^{ij}(\kappa)\cdot g_k^{ij}\right)^{-\beta^{ij}}},\ \forall r \in R^{ij}, ij \in IJ\ . \tag{A.13}$$

Considering the critical role of $\vartheta_r^{ij}(\kappa)$ in shaping the PSW route choice probability, we take the ratios of $\vartheta_r^{ij}(\kappa)$ between route *r* and the other routes, i.e., $\vartheta_r^{ij}(\kappa)/\vartheta_s^{ij}(\kappa)$, $\forall s \in R^{ij}, ij \in IJ$.

Take the derivative of $\vartheta_r^{ij}(\kappa)/\vartheta_s^{ij}(\kappa)$ w.r.t. *κ*, we have

$$\frac{d}{d\kappa}\frac{\vartheta_r^{ij}(\kappa)}{\vartheta_s^{ij}(\kappa)}=\frac{d}{d\kappa}\frac{(\kappa g_r^{ij}-\zeta^{ij})\Delta^{ij}\cdot\kappa g_s^{ij}}{\kappa g_r^{ij}\cdot(\kappa g_s^{ij}-\zeta^{ij})\Delta^{ij}}=\frac{g_s^{ij}}{g_r^{ij}}\frac{(g_s^{ij}-g_r^{ij})\zeta^{ij}}{(\kappa g_s^{ij}-\zeta^{ij})^2}\ge 0,\ \forall s \in R^{ij}, ij \in IJ\ . \tag{A.14}$$

Eq. (A.14) shows that the CV ratio $\vartheta_r^{ij}(\kappa)/\vartheta_s^{ij}(\kappa)$ ($\forall s \in R^{ij}, ij \in IJ$) is an increasing function of the scale *κ*. Then, we have

$$\frac{\vartheta_r^{ij}(\kappa_1)}{\vartheta_s^{ij}(\kappa_1)} \le \frac{\vartheta_r^{ij}(\kappa_2)}{\vartheta_s^{ij}(\kappa_2)},\ \forall s \in R^{ij}, ij \in IJ \tag{A.15}$$

when $\kappa_1 \le \kappa_2$. After some manipulations, we can transform Eq. (A.15) into

$$\Rightarrow \frac{\vartheta_r^{ij}(\kappa_1)}{\vartheta_s^{ij}(\kappa_1)}\frac{\kappa_1 g_r^{ij}}{\kappa_1 g_s^{ij}} \le \frac{\vartheta_r^{ij}(\kappa_2)}{\vartheta_s^{ij}(\kappa_2)}\frac{\kappa_2 g_r^{ij}}{\kappa_2 g_s^{ij}}, \forall s \in R^{ij}, ij \in IJ \tag{A.16}$$

$$\Rightarrow \frac{\varpi_r^{ij}\left(\vartheta_r^{ij}(\kappa_1)\cdot\kappa_1 g_r^{ij}\right)^{-\beta^{ij}}}{\varpi_s^{ij}\left(\vartheta_s^{ij}(\kappa_1)\cdot\kappa_1 g_s^{ij}\right)^{-\beta^{ij}}} \ge \frac{\varpi_r^{ij}\left(\vartheta_r^{ij}(\kappa_2)\cdot\kappa_2 g_r^{ij}\right)^{-\beta^{ij}}}{\varpi_s^{ij}\left(\vartheta_s^{ij}(\kappa_2)\cdot\kappa_2 g_s^{ij}\right)^{-\beta^{ij}}}, \forall \beta^{ij} > 0, s \in R^{ij}, ij \in IJ \tag{A.17}$$

From Lemma 1, we have

$$\frac{\varpi_r^{ij}\left(\vartheta_r^{ij}(\kappa_1)\cdot\kappa_1 g_r^{ij}\right)^{-\beta^{ij}}}{\sum_{s\in R^{ij}}\varpi_s^{ij}\left(\vartheta_s^{ij}(\kappa_1)\cdot\kappa_1 g_s^{ij}\right)^{-\beta^{ij}}} \le \frac{\varpi_r^{ij}\left(\vartheta_r^{ij}(\kappa_2)\cdot\kappa_2 g_r^{ij}\right)^{-\beta^{ij}}}{\sum_{s\in R^{ij}}\varpi_s^{ij}\left(\vartheta_s^{ij}(\kappa_2)\cdot\kappa_2 g_s^{ij}\right)^{-\beta^{ij}}}, \forall \beta^{ij} > 0, ij \in IJ\ , \tag{A.18}$$

i.e., $P_r^{ij}(\kappa_1) \le P_r^{ij}(\kappa_2)$. The strict inequality establishes when $g_r^{ij} < g_s^{ij}$ for at least one route $s \in R^{ij}, ij \in IJ$. This completes the proof. □

A.4. Proof of Proposition 2

Proof. Firstly, given that $\mathbf{f}^*$ is a solution to the PSW*l*-SUE problem, the SUE conditions in Eq. (24) will naturally ensure Eq. (26). That is, any solution to the PSW*l*-SUE model is also a solution to the VI problem in Eq. (26).

Secondly, suppose $\mathbf{f}^*$ is a solution to the VI problem in Eq. (26), $\mathbf{f}$ is a feasible flow that differs from $\mathbf{f}^*$, i.e., for some route $r \in R_{ij}, ij \in IJ$, $f_r^{ij} \ne f_r^{ij*}$. Substituting $\mathbf{f}$ and $\mathbf{f}^*$ into the VI formulation in Eq. (26) gives $(f_r^{ij*} - q^{ij}\cdot P_r^{ij}(\mathbf{f}^*))^{\mathrm{T}}(f_r^{ij} - f_r^{ij*}) \ge 0$. Since $f_r^{ij} - f_r^{ij*} \ne 0$ can be either positive or negative, we have $f_r^{ij*} - q^{ij}\cdot P_r^{ij}(\mathbf{f}^*) = 0$, i.e., the SUE condition in Eq. (24) is satisfied. Hence, a solution to the VI problem in Eq. (26) is also a solution to the PSW*l*-SUE problem.

This completes the proof. □

A.5. Proof of Proposition 3

Proof. Based on the assumption of continuity, the general mapping $F(\mathbf{f})$=$\mathbf{f}$- $\mathbf{q}^{\mathrm{T}}$ $P(\mathbf{f})$ is continuous w.r.t. $\mathbf{f} \in \Omega$. Since Ω is a nonempty, convex, and compact set, a solution is guaranteed for the VI formulation in Eq. (26) (e.g., Theorem 1.4, Nagurney, 1999).

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