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Isolated Calmness and Sharp Minima via Hölder Graphical Derivatives

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Abstract

The paper utilizes Hölder graphical derivatives for characterizing Hölder strong subregularity, isolated calmness and sharp minimum. As applications, we characterize Hölder isolated calmness in linear semi-infinite optimization and Hölder sharp minimizers of some penalty functions for constrained optimization.

Keywords Hölder subregularity · Hölder calmness · Hölder sharp minimum · Hölder graphical derivatives · Semi-infinite programming

Dedicated to the 70th birthday of our colleague and friend Miguel Goberna

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1 Introduction

This paper continues our previous work [19] and utilizes Hölder graphical derivatives (sometimes referred to as Studniarski derivatives) for characterizing certain regularity properties of set-valued mappings and real-valued functions.

In the next Section 2, we discuss q-order ($q > 0$) positively homogeneous mappings and q-order graphical (contingent) derivatives. The definitions and statements mostly follow the corresponding linear ones in [8]. Two norm-like quantities are used for quantifying Hölder graphical derivatives. One of them is a generalization of the well-known *outer norm* of a positively homogeneous mapping, while the other seems new and allows to simplify some statements (and proofs) even in the linear case.

In Section 3, Hölder graphical derivatives are used for characterizing Hölder strong subregularity, isolated calmness and sharp minimum. In particular, we give characterizations of Hölder sharp minimizers in terms of Hölder graphical derivatives of the subdifferential mapping. The characterizations from Section 3 are used in Sections 4 and 5 to characterize Hölder isolated calmness in linear semi-infinite optimization and sharp minimizers of ℓ_p penalty functions, respectively.

Our basic notation is standard, see, e.g., [8, 23]. Throughout the paper, X and Y are normed spaces. We use the same notation $\|\cdot\|$ for norms in all spaces. If not explicitly stated otherwise, products of normed spaces are assumed equipped with the maximum norms, e.g., $\|(x, y)\| := \max\{\|x\|, \|y\|\}$, $(x, y) \in X \times Y$. If X is a normed space, its topological dual is denoted by X^*, while $\langle\cdot,\cdot\rangle$ denotes the bilinear form defining the pairing between the two spaces. Symbols $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the sets of all real numbers, all nonnegative real numbers and all positive integers, respectively. For the empty subset of $\mathbb{R}_+$, we use the conventions $\sup\emptyset = 0$ and $\inf\emptyset = +\infty$. Given an $\alpha \in \mathbb{R}$, we denote $\alpha_+ := \max\{0, \alpha\}$.

For an extended-real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$, its domain and epigraph are defined, respectively, by $\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x, \alpha) \in X \times \mathbb{R} \mid f(x) \le \alpha\}$. A set-valued mapping $F : X \rightrightarrows Y$ between two sets X and Y is a mapping, which assigns to every $x \in X$ a subset (possibly empty) $F(x)$ of Y. We use the notations $\operatorname{gph} F := \{(x, y) \in X \times Y \mid y \in F(x)\}$ and $\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}$ for the graph and the domain of F, respectively, and $F^{-1} : Y \rightrightarrows X$ for the inverse of F. This inverse (which always exists with possibly empty values at some y) is defined by $F^{-1}(y) := \{x \in X \mid y \in F(x)\}$, $y \in Y$. Obviously $\operatorname{dom} F^{-1} = F(X)$.

Recall that a mapping $F : X \rightrightarrows Y$ is *outer semicontinuous* (cf., e.g., [8]) at $x \in X$ if

$$\limsup_{u \to x} F(u) \subset F(x),$$

i.e., if $\operatorname{gph} F \ni (x_k, y_k) \to (x, y)$, then $y \in F(x)$. This is always the case when $\operatorname{gph} F$ is closed.

Throughout the paper, we assume the order of all Hölder properties to be determined by a fixed number $q > 0$.

2 Hölder Graphical Derivatives

In this section, we discuss Hölder positively homogeneous mappings and Hölder versions of the ubiquitous graphical (contingent) derivatives; cf. [3, 8, 12, 23].

Definition 2.1 A mapping $H : X \rightrightarrows Y$ is q-order positively homogeneous whenever

$$0 \in H(0) \quad \text{and} \quad H(\lambda x) = \lambda^q H(x) \ \text{ for all } \ x \in X \text{ and } \lambda > 0.$$

If $q = 1$, we simply say the H is positively homogeneous. The graph of a positively homogeneous mapping is a cone. This is obviously not the case when $q \neq 1$.

The next simple fact is a direct consequence of the definition.

Proposition 2.2 *Let $H : X \rightrightarrows Y$ be q-order positively homogeneous. Then $H^{-1} : Y \rightrightarrows X$ is $\frac{1}{q}$-order positively homogeneous.*

For a q-order positively homogeneous mapping $H : X \rightrightarrows Y$, we define two norm-like quantities:

$$\|H\|_q^+ := \sup_{(x,y)\in \text{gph}\, H\setminus\{(0,0)\}} \frac{\|y\|}{\|x\|^q}, \quad \|H\|_q^{\ominus} := \inf_{(x,y)\in \text{gph}\, H\setminus\{(0,0)\}} \frac{\|y\|}{\|x\|^q}. \tag{2.1}$$

When $q = 1$, the first one reduces to the *outer norm* $\|H\|^+$ of H; cf. [23, p. 364], [8, p. 218]. Note that $\|H\|_1^{\ominus} \leq \|H\|^-$, where $\|H\|^-$ is the *inner norm* of H, and the inequality can be strict. None of the quantities in (2.1) is actually a true "norm"; see the comments in [8].

Proposition 2.3 *Let $H : X \rightrightarrows Y$ be q-order positively homogeneous.*

(i) $\|H\|_q^+ = \sup_{\|y\|=1} d(0, H^{-1}(y))^{-q}, \quad \|H\|_q^{\ominus} = \inf_{\|x\|=1} d(0, H(x))$.

(ii) $\|H\|_q^{\ominus} = \left(\|H^{-1}\|_{\frac{1}{q}}^+\right)^{-q}$.

(iii) *If gph $H \neq \{(0,0)\}$, then $\|H\|_q^{\ominus} \leq \|H\|_q^+$.*

(iv) *If gph $H = \{(0,0)\}$, then $\|H\|_q^+ = 0$ and $\|H\|_q^{\ominus} = +\infty$.*

(v) $\|H\|_q^+ = 0$ *if and only if* $H(X) = \{0\}$.

(vi) $\|H\|_q^{\ominus} = +\infty$ *if and only if* dom $H = \{0\}$.

(vii) $\|H\|_q^+ < +\infty \implies H(0) = \{0\}$. *If* $\dim Y < \infty$ *and H is outer semicontinuous at 0, the two conditions are equivalent.*

(viii) $\|H\|_q^{\ominus} > 0 \implies H^{-1}(0) = \{0\}$. *If* $\dim X < \infty$ *and H^{-1} is outer semicontinuous at 0, the two conditions are equivalent.*

Proof Assertions (i)–(vi) and the first parts of assertions (vii) and (viii) are direct consequences of (2.1) and Definition 2.1. For instance, in the case of assertion (ii) using definitions (2.1) we have:

$$\|H\|_q^{\ominus} = \left(\sup_{(x,y)\in \text{gph}\, H\setminus\{(0,0)\}} \frac{\|x\|}{\|y\|^{\frac{1}{q}}}\right)^{-q} = \left(\|H^{-1}\|_{\frac{1}{q}}^+\right)^{-q}.$$

To prove the second part of (vii), we need to show that, under the assumptions, $\|H\|_q^+ = +\infty \implies H(0) \neq \{0\}$. Let $\|H\|_q^+ = +\infty$. By (2.1), there exists a sequence $(x_k, y_k) \in \text{gph}\, H$ $(k \in \mathbb{N})$ such that $\|y_k\|/\|x_k\|^q \to +\infty$ as $k \to \infty$. Without loss of generality, $y_k \neq 0$ for all $k \in \mathbb{N}$. Set $u_k := x_k/\|y_k\|^{\frac{1}{q}}$, $v_k := y_k/\|y_k\|$ $(k \in \mathbb{N})$. Then $u_k \to 0$ as $k \to \infty$ and $\|v_k\| = 1$ $(k \in \mathbb{N})$. Without loss of generality, $v_k \to v$ as $k \to \infty$ and $\|v\| = 1$. Furthermore, by Definition 2.1, $(u_k, v_k) \in \text{gph}\, H$ $(k \in \mathbb{N})$ and, thanks to the outer semicontinuity of H, $v \in H(0)$.

The proof of the second part of (viii) is similar. Let $\dim X < \infty$, H is outer semicontinuous at 0, and $\|H\|_q^{\ominus} = 0$. By (2.1), there exists a sequence $(x_k, y_k) \in \text{gph}\, H$ $(k \in \mathbb{N})$

such that $\|y_k\|/\|x_k\|^q \to 0$ as $k \to \infty$. Without loss of generality, $x_k \neq 0$ for all $k \in \mathbb{N}$, and $u_k := x_k/\|x_k\| \to u$ with $\|u\| = 1$, while $v_k := y_k/\|x_k\|^q \to 0$. By Definition 2.1, $(u_k, v_k) \in \operatorname{gph} H$ ($k \in \mathbb{N}$) and, thanks to the outer semicontinuity of H^{-1}, $u \in H^{-1}(0)$. Hence, $H^{-1}(0) \neq \{0\}$. □

Assertions (i), (v), (vii) and (viii) in Proposition 2.3 generalize and expand the corresponding parts of [8, Propositions 4A.6 and 5A.7, and Exercise 4A.9]. The above proof of the second part of (vii) largely follows that of the corresponding part of [8, Proposition 4A.6].

Next we briefly consider the case $Y = X^*$.

Definition 2.4 A mapping $H : X \rightrightarrows X^*$ is *q-order positively definite* if there exists a number $\lambda > 0$ such that

$$\langle x^*, x\rangle \geq \lambda \|x\|^{q+1} \quad \text{for all} \quad (x, x^*) \in \operatorname{gph} H.$$

The exact upper bound of all such $\lambda > 0$ is denoted by $\|H\|_q^*$.

In Definition 2.4, it obviously holds

$$\|H\|_q^* = \inf_{(x,x^*)\in \operatorname{gph} H,\, x\neq 0} \frac{\langle x^*, x\rangle_+}{\|x\|^{q+1}}. \tag{2.2}$$

In general, the expression in (2.2) is nonnegative, and the case $\|H\|_q^* = 0$ means that H is not q-order positively definite.

Proposition 2.5 *Let $H : X \rightrightarrows X^*$.*

(i) $\|H\|_q^* = +\infty$ *if and only if* $\operatorname{dom} H \subset \{0\}$.
(ii) *If H is q-order positively homogeneous, then* $\|H\|_q^* \leq \|H\|_q^{\ominus}$.
(iii) *If H is q-order positively homogeneous and p-order positively definite with some $p > 0$, then either* $\operatorname{dom} H = \{0\}$ *or* $p = q$.

Proof (i) is immediate from (2.2).
(ii) follows from comparing (2.2) and the second definition in (2.1).
(iii) Let H be q-order positively homogeneous and p-order positively definite with some $p > 0$. Then $0 \in \operatorname{dom} H$, $\|H\|_p^* > 0$ and, by Definition 2.1, $(x, x^*) \in \operatorname{gph} H$ if and only if $(\lambda x, \lambda^q x^*) \in \operatorname{gph} H$ for any $\lambda > 0$, and it follows from (2.2) that

$$\|H\|_p^* = \inf_{(x,x^*)\in \operatorname{gph} H,\, x\neq 0,\, \lambda>0} \frac{\langle \lambda^q x^*, \lambda x\rangle_+}{\|\lambda x\|^{p+1}} = \inf_{\lambda>0} \lambda^{q-p} \|H\|_p^*.$$

Thus, either $\|H\|_p^* = +\infty$, i.e. $\operatorname{dom} H = \{0\}$, or $p = q$.

□

Given a set-valued mapping $H : X \rightrightarrows Y$ and a function $f : X \to Y$, their sum $H + f$ is a set-valued mapping from X to Y defined by

$$(H + f)(x) := H(x) + f(x) = \{y + f(x) \mid y \in H(x)\}, \quad x \in X.$$

Note that $\operatorname{dom}(H + f) = \operatorname{dom} H \cap \operatorname{dom} f$.

The next statement characterizes perturbed positively homogeneous mappings. It generalizes [8, Theorem 5A.8] (and is accompanied by a much shorter proof).

Theorem 2.6 *Let both $H : X \rightrightarrows Y$ and $f : X \to Y$ be q-order positively homogeneous. Then $H + f$ is q-order positively homogeneous. Moreover,*

$$\|H + f\|_q^{\ominus} \geq \|H\|_q^{\ominus} - \|f\|_q^{+}. \tag{2.3}$$

Proof $H + f$ is q-order positively homogeneous by Definition 2.1. If $\operatorname{dom} H \cap \operatorname{dom} f = \{0\}$, then $\|H + f\|_q^{\ominus} = +\infty$ by Proposition 2.3(vi), and condition (2.3) is satisfied trivially. Let $(x, y) \in \operatorname{gph} H$ with $x \neq 0$ and $x \in \operatorname{dom} f$. Then $(x, y + f(x)) \in \operatorname{gph}(H + f)$ and, in view of (2.1),

$$\frac{\|y + f(x)\|}{\|x\|^q} \geq \frac{\|y\|}{\|x\|^q} - \frac{\|f(x)\|}{\|x\|^q} \geq \|H\|_q^{\ominus} - \|f\|_q^{+}.$$

Since $(x, y + f(x))$ is an arbitrary point in $\operatorname{gph}(H + f)$ with $x \neq 0$, the second representation in (2.1) yields condition (2.3). □

Given a set-valued mapping $F : X \rightrightarrows Y$, its *$q$-order graphical derivative* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is a set-valued mapping $D_q F(\bar{x}, \bar{y}) : X \rightrightarrows Y$ defined for all $x \in X$ by

$$\begin{aligned} D_q F(\bar{x}, \bar{y})(x) := \big\{ & y \in Y \mid \exists (x_k, y_k) \to (x, y),\ t_k \downarrow 0 \text{ such that} \\ & (\bar{x} + t_k x_k, \bar{y} + t_k^q y_k) \in \operatorname{gph} F,\ \forall k \in \mathbb{N} \big\}. \end{aligned} \tag{2.4}$$

$D_q F(\bar{x}, \bar{y})$ is sometimes referred to as *q-order upper Studniarski derivative* [27, Definition 3.1] of F at $(\bar{x}, \bar{y})$. When $q = 1$, it reduces to the standard graphical (contingent) derivative; cf. [2, 3, 8, 13, 17, 23]. Clearly, $D_q F(\bar{x}, \bar{y})$ is a q-order positively homogeneous mapping with closed graph, and

$$D_q F(\bar{x}, \bar{y})^{-1} = D_{\frac{1}{q}} F^{-1}(\bar{y}, \bar{x}), \tag{2.5}$$

Given a function $f : X \to Y$ and a point $\bar{x} \in \operatorname{dom} f$, we write $D_q f(\bar{x})$ instead of $D_q f(\bar{x}, f(\bar{x}))$. If $D_q f(\bar{x})$ is single-valued, i.e. the limit

$$D_q f(\bar{x})(x) = \lim_{u \to x,\, t \downarrow 0} \frac{f(\bar{x} + tu) - f(\bar{x})}{t^q}$$

exists for all $x \in X$, we say that f is *q-order Hadamard directionally differentiable* at $\bar{x}$.

The next proposition provides a sum rule for q-order graphical derivatives. It is a direct consequence of the definitions of q-order graphical derivative and q-order Hadamard directional differentiability.

Proposition 2.7 *Let $F : X \rightrightarrows Y$, $f : X \to Y$, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and $\bar{x} \in \operatorname{dom} f$. If f is q-order Hadamard directionally differentiable at $\bar{x}$, then*

$$D_q(F + f)(\bar{x}, \bar{y} + f(\bar{x})) = D_q F(\bar{x}, \bar{y}) + D_q f(\bar{x}).$$

Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, its *$q$-order Hadamard directional subderivative* [25, 26] at $\bar{x} \in \operatorname{dom} f$ is defined for all $x \in X$ by (cf. [22, Definition 1.1] and [23, Definition 8.1] for the case $q = 1$)

$$f_q'(\bar{x}; x) := \liminf_{u \to x,\, t \downarrow 0} \frac{f(\bar{x} + tu) - f(\bar{x})}{t^q}. \tag{2.6}$$

If f is Lipschitz continuous near $\bar{x}$ and $0 < q \leq 1$, the above definition takes a simpler form:

$$f_q'(\bar{x}; x) = \liminf_{t \downarrow 0} \frac{f(\bar{x} + tx) - f(\bar{x})}{t^q}.$$

Observe that the function $f'_q(\bar{x};\cdot) : X \to \mathbb{R} \cup \{\pm\infty\}$ is lower semicontinuous and *q-order positively homogeneous* in the sense that $f'_q(\bar{x};\lambda x) = \lambda^q f'_q(\bar{x};x)$ for all $x \in X$ and $\lambda > 0$. We are going to use for characterizing this function the following norm-like quantity:

$$\|f'_q(\bar{x};\cdot)\|_q := \inf_{x\in X\setminus\{0\}} \frac{(f'_q(\bar{x};x))_+}{\|x\|^q} = \inf_{\|x\|=1} (f'_q(\bar{x};x))_+. \tag{2.7}$$

The next statement is a direct consequence of the definitions. It uses the *epigraphical mapping* $x \mapsto \text{epi}\, f(x) := \{\mu \in \mathbb{R} \mid f(x) \le \mu\}$. Note that the graph of the latter mapping is the *epigraph* epi f of f. We use the same notation for the epigraph and the epigraphical mapping.

Proposition 2.8 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $\bar{x} \in$ dom f.*

(i) *Either $f'_q(\bar{x};0) = 0$ or $f'_q(\bar{x};0) = -\infty$.*
(ii) *$D_q(\text{epi}\, f)(\bar{x}, f(\bar{x}))(x) = \{\mu \in \mathbb{R} \mid f'_q(\bar{x};x) \le \mu\}$ for all $x \in X$.*
(iii) *$\|D_q(\text{epi}\, f)(\bar{x}, f(\bar{x}))\|^+_q = +\infty$ and $\|D_q(\text{epi}\, f)(\bar{x}, f(\bar{x}))\|^{\ominus}_q = \|f'_q(\bar{x};\cdot)\|_q$.*
(iv) *$\|f'_q(\bar{x};\cdot)\|_q > 0 \implies f'_q(\bar{x};x) > 0$ for all $x \ne 0$. If $\dim X < \infty$, the two conditions are equivalent.*

Proof (i) By definition (2.6), $f'_q(\bar{x};0) \le 0$. Suppose that $f'_q(\bar{x};0) < 0$. Then there exist sequences $u_k \to 0$ and $t_k \downarrow 0$ such that

$$\lim_{k\to+\infty} \frac{f(\bar{x}+t_k u_k) - f(\bar{x})}{t_k^q} = \alpha < 0.$$

For any $\theta > 0$ and $k \in \mathbb{N}$, set $u'_k := \theta^{\frac{1}{q}} u_k$ and $t'_k := \theta^{-\frac{1}{q}} t_k$. We have $u'_k \to 0$ and $t'_k \downarrow 0$ and

$$\lim_{k\to+\infty} \frac{f(\bar{x}+t'_k u'_k) - f(\bar{x})}{(t'_k)^q} = \theta\alpha.$$

Hence, $f'_q(\bar{x};0) = -\infty$.

(ii) The assertion is immediate from comparing definitions (2.4) and (2.6).

(iii) By (i) and (ii), $(0,\mu) \in \text{gph}\, D_q(\text{epi}\, f)(\bar{x}, f(\bar{x}))$ for all $\mu \ge 0$, and it follows from (2.1) that $\|D_q(\text{epi}\, f)(\bar{x}, f(\bar{x}))\|^+_q = +\infty$. The second equality holds trivially when $f'_q(\bar{x};x) = +\infty$ for all $x \ne 0$. If $x \ne 0$ and $f'_q(\bar{x};x) < +\infty$, then, in view of (ii),

$$\inf_{\mu\in D_q(\text{epi}\, f)(\bar{x}, f(\bar{x}))(x)} |\mu| = \inf_{\mu \ge f'_q(\bar{x};x)} |\mu| = (f'_q(\bar{x};x))_+,$$

and the second equality follows from (2.1) and (2.7).

(iv) If $f'_q(\bar{x};x) \le 0$ for some $x \ne 0$, then $\|f'_q(\bar{x};\cdot)\|_q = 0$ by definition (2.7). This proves the implication. Let $\dim X < \infty$ and $\|f'_q(\bar{x};\cdot)\|_q = 0$. By (2.7), there is a sequence $\{x_k\}$ such that $\|x_k\| = 1$ for all $k \in \mathbb{N}$, and $(f'_q(\bar{x};x_k))_+ \to 0$ as $k \to \infty$. Without loss of generality, $x_k \to x$ as $k \to \infty$, $\|x\| = 1$ and $(f'_q(\bar{x};x))_+ = 0$ since $f'_q(\bar{x};\cdot)$ is lower semicontinuous. This proves the opposite implication.

□

The next corollary is a consequence of Propositions 2.7 and 2.8.

Corollary 2.9 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$, $g : X \to \mathbb{R}$ and $\bar{x} \in$ dom f. If g is q-order Hadamard directionally differentiable at $\bar{x}$, then*

$$(f+g)'_q(\bar{x};x) = f'_q(\bar{x};x) + D_q g(\bar{x})(x) \quad \textit{for all} \quad x \in X.$$

3 Hölder Strong Subregularity, Isolated Calmness and Sharp Minimum

In this section, Hölder graphical derivatives are used for characterizing Hölder strong subregularity, isolated calmness and sharp minimum.

Definition 3.1 (i) A mapping $F : X \rightrightarrows Y$ is q-order strongly subregular at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus $\tau > 0$ if there exist neighbourhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\tau \|x - \bar{x}\|^q \leq d(\bar{y}, F(x) \cap V) \quad \text{for all} \quad x \in U. \tag{3.1}$$

The exact upper bound of all such $\tau > 0$ is denoted by $\operatorname{srg}_{\mathrm{q}} F(\bar{x}, \bar{y})$.

(ii) A mapping $S : Y \rightrightarrows X$ possesses q-order isolated calmness property at $(\bar{y}, \bar{x}) \in \operatorname{gph} S$ with modulus $\tau > 0$ if there exist neighbourhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\tau \|x - \bar{x}\|^q \leq \|y - \bar{y}\| \quad \text{for all} \quad y \in V \text{ and } x \in S(y) \cap U. \tag{3.2}$$

The exact upper bound of all such $\tau > 0$ is denoted by $\operatorname{clm}_{\mathrm{q}} S(\bar{y}, \bar{x})$.

If F is not q-order strongly subregular at $(\bar{x}, \bar{y})$ or S does not possess q-order isolated calmness property at $(\bar{y}, \bar{x})$, we have $\operatorname{srg}_{\mathrm{q}} F(\bar{x}, \bar{y}) = 0$ or $\operatorname{clm}_{\mathrm{q}} S(\bar{y}, \bar{x}) = 0$, respectively.

The properties in the above definition are well known in the linear case $q = 1$ (see, e.g., [8]), but have also been studied in the general setting (also for not necessarily strong subregularity and not necessarily isolated calmness); cf. [7, 9, 20]. Because of the distance involved in the right-hand side of (3.1) (and also in its left-hand side in the case of the not strong version), the property in part (i) of Definition 3.1 is often referred to as q-order strong *metric* subregularity.

Remark 3.2 (i) In both parts of Definition 3.1, it suffices to take $V := Y$; cf. [8, Exercise 3H.4].

(ii) Condition (3.2) implies that $S(\bar{y}) \cap U = \{\bar{x}\}$, i.e. $\bar{x}$ is an isolated point in $S(\bar{y})$, which justifies the word 'isolated' in the name of the property in Definition 3.1(ii).

(iii) The moduli $\operatorname{srg}_{\mathrm{q}} F(\bar{x}, \bar{y})$ and $\operatorname{clm}_{\mathrm{q}} S(\bar{y}, \bar{x})$ are usually introduced to characterize the usual (not strong!) subregularity and (not isolated!) calmness. We do not consider these two weaker properties in the current paper. If a respective (strong or isolated) property in Definition 3.1 holds, then the corresponding modulus coincides with the conventional one.

(iv) When $V = Y$, condition (3.1) is obviously implied by the following q-order strong *graph* subregularity property:

$$\tau \|x - \bar{x}\|^q \leq d((x, \bar{y}), \operatorname{gph} F) \quad \text{for all} \quad x \in U$$

(with the same τ and U). It is not difficult to show that, when $q \geq 1$, q-order (strong) subregularity in part (i) of Definition 3.1 implies q-order (strong) graph subregularity (with smaller τ and U); cf. a characterization of subregularity in [12, Proposition 2.61]. A similar observation can be made about the calmness property in part (ii) of Definition 3.1; cf. the well-known characterization of q-order calmness by Kummer [20, Lemma 2.2], and the earlier result by Klatte and Kummer [18, Lemma 3.2] for the case $q = 1$.

(v) There is some inconsistency in the literature concerning whether to place the constants τ and/or q, which determine the properties in Definition 3.1, in the left or right-hand sides of the inequalities (3.1) and (3.2) (and similar inequalities involved in related definitions); cf., e.g., [17]. This applies also to our own recent paper [19], where we placed q in the right-hand sides of the inequalities. Of course, the position of the constants does not effect the properties, but it has an effect on the values of the respective moduli. Our choice in the current paper is determined by our desire to produce the simplest relations between these moduli and the quantitative characteristics of Hölder graphical derivatives and more straightforward proofs.

The next proposition is an immediate consequence of Definition 3.1.

Proposition 3.3 *Let $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in gph\, F$. Then F is q-order strongly subregular at $(\bar{x}, \bar{y})$ if and only if F^{-1} possesses q-order isolated calmness property at $(\bar{y}, \bar{x})$, and $srg_q\, F(\bar{x}, \bar{y}) = clm_q\, F^{-1}(\bar{y}, \bar{x})$.*

The next proposition and its corollaries generalize [8, Theorem 4E.1 and Corollary 4E.2].

Proposition 3.4 *Let $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in gph\, F$. Then*

$$srg_q\, F(\bar{x}, \bar{y}) \le \|D_q F(\bar{x}, \bar{y})\|_q^{\ominus}. \tag{3.3}$$

If $\dim X < +\infty$ *and* $\dim Y < +\infty$*, then* (3.3) *holds as equality.*

Proof If $\mathrm{srg_q}\, F(\bar{x}, \bar{y}) = 0$ or $\|D_q F(\bar{x}, \bar{y})\|_q^{\ominus} = +\infty$, inequality (3.3) holds trivially. Let $\tau \in (0, \mathrm{srg_q}\, F(\bar{x}, \bar{y}))$. Let $u \in X \setminus \{0\}$ and $v \in D_q F(\bar{x}, \bar{y})(u)$, i.e. there exist sequences $(u_k, v_k) \to (u, v)$ and $t_k \downarrow 0$ such that $\bar{y} + t_k^q v_k \in F(\bar{x} + t_k u_k)$ for all $k \in \mathbb{N}$. By Definition 3.1(i), $\tau\|u_k\|^q \le \|v_k\|$ for all sufficiently large $k \in \mathbb{N}$, and consequently, $\tau \le \|v\|/\|u\|^q$. In view of definition (2.1), we have $\tau \le \|D_q F(\bar{x}, \bar{y})\|_q^{\ominus}$. Inequality (3.3) follows. Let $\dim X < +\infty$, $\dim Y < +\infty$, and $\tau > \mathrm{srg_q}\, F(\bar{x}, \bar{y})$. By Definition 3.1(i), there exists a sequence $(x_k, y_k) \to (\bar{x}, \bar{y})$ such that $(x_k, y_k) \in \mathrm{gph}\, F$ and $\tau\|x_k - \bar{x}\|^q > \|y_k - \bar{y}\|$ for all $k \in \mathbb{N}$. Then $t_k := \|x_k - \bar{x}\| \downarrow 0$. Set $u_k := (x_k - \bar{x})/t_k$ and $v_k := (y_k - \bar{y})/t_k^q$ ($k \in \mathbb{N}$). Without loss of generality, $u_k \to u \in X$, $\|u\| = 1$, and $v_k \to v \in Y$, $\|v\| \le \tau$. Thus, $v \in D_q F(\bar{x}, \bar{y})(u)$ and $\|D_q F(\bar{x}, \bar{y})\|_q^{\ominus} \le \|v\|/\|u\|^q \le \tau$. Hence, (3.3) holds as equality. □

The following statement provides a characterization of q-order strong subregularity of a mapping in terms of its q-order graphical derivative.

Corollary 3.5 *Let $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in gph\, F$. Consider the following conditions:*

(i) *F is q-order strongly subregular at $(\bar{x}, \bar{y})$;*
(ii) $\|D_q F(\bar{x}, \bar{y})\|_q^{\ominus} > 0$*;*
(iii) $D_q F(\bar{x}, \bar{y})^{-1}(0) = \{0\}$.

Then (i) ⇒ (ii) ⇒ (iii). *If* $\dim X < +\infty$ *and* $\dim Y < +\infty$*, then* (i) ⇔ (ii) ⇔ (iii).

Proof Thanks to Proposition 3.4, we have the implication (i) ⇒ (ii) in general, and the equivalence (i) ⇔ (ii) when $\dim X < +\infty$ and $\dim Y < +\infty$. The implication (ii) ⇒ (iii) is an immediate consequence of Proposition 2.3(viii). The graph of $D_q F(\bar{x}, \bar{y})$ is closed by definition, hence, $D_q F(\bar{x}, \bar{y})^{-1}$ is outer semicontinuous at 0. Employing Proposition 2.3(viii) again, we conclude that (ii) ⇔ (iii) when $\dim X < +\infty$. □

Remark 3.6 A coderivative analogue (employing a special kind of limiting coderivative) of the equality in Corollary 3.5(iii) is used in [29, Theorem 5.2] to characterize nonlinear subregularity.

The next example illustrates application of Corollary 3.5 for checking Hölder strong subregularity as well as computation of the Hölder graphical derivative and relevant norm-like quantity.

Example 3.7 Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be the epigraphical mapping: $F(x) := [x^2, +\infty)$ for all $x \in \mathbb{R}$, and $\bar{x} = \bar{y} = 0$. By definition (2.4), $y \in D_q F(0,0)(x)$ if and only if there exist sequences $(x_k, y_k) \to (x, y)$ and $t_k \downarrow 0$ such that $y_k \geq t_k^{2-q} x_k^2$ for all $k \in \mathbb{N}$, or equivalently, $y \geq \lim_{k\to\infty} t_k^{2-q} x^2$. Thus, there are three distinct possibilities.

<u>$0 < q < 2$</u>. $D_q F(0,0)(x) = \mathbb{R}_+$ for all $x \in \mathbb{R}$. Thus, $D_q F(0,0)^{-1}(0) = \mathbb{R}$ and, by (2.1), $\|D_q F(0,0)\|_q^{\ominus} = 0$. Each of the conditions (ii) and (iii) in Corollary 3.5 yields that F is not q-order strongly subregular at $(0,0)$.

<u>$q = 2$</u>. $D_q F(0,0)(x) = [x^2, +\infty)$ for all $x \in \mathbb{R}$. Thus, $D_q F(0,0)^{-1}(0) = \{0\}$ and, by (2.1), $\|D_q F(0,0)\|_q^{\ominus} = 1$. Each of the conditions (ii) and (iii) in Corollary 3.5 yields that F is q-order strongly subregular at $(0,0)$.

<u>$q > 2$</u>. $D_q F(0,0)(0) = \{0\}$, and $D_q F(0,0)(x) = \varnothing$ for all $x \neq 0$. Thus, $D_q F(0,0)^{-1}(0) = \{0\}$ and, by (2.1), $\|D_q F(0,0)\|_q^{\ominus} = +\infty$. Each of the conditions (ii) and (iii) in Corollary 3.5 yields that F is q-order strongly subregular at $(0,0)$.

Of course, in this simple example, the same conclusions can be obtained directly from Definition 3.1(i).

Corollary 3.8 *Let $S : Y \rightrightarrows X$ and $(\bar{y}, \bar{x}) \in \text{gph}\, S$. Then*

$$clm_q\, S(\bar{y}, \bar{x}) \leq \left(\|D_{\frac{1}{q}} S(\bar{y}, \bar{x})\|_{\frac{1}{q}}^{+}\right)^{-q}. \tag{3.4}$$

If $\dim X < +\infty$ *and* $\dim Y < +\infty$*, then* (3.4) *holds as equality.*

Proof By Proposition 2.3(ii) and (2.5), we have

$$\|D_q F(\bar{x}, \bar{y})\|_q^{\ominus} = \left(\|D_q F(\bar{x}, \bar{y})^{-1}\|_{\frac{1}{q}}^{+}\right)^{-q} = \left(\|D_{\frac{1}{q}} F^{-1}(\bar{y}, \bar{x})\|_{\frac{1}{q}}^{+}\right)^{-q}.$$

The assertion follows from Propositions 3.3 and 3.4. □

Corollary 3.9 *Let $S : Y \rightrightarrows X$ and $(\bar{y}, \bar{x}) \in \text{gph}\, S$. Consider the following conditions:*

(i) *S possesses q-order isolated calmness property at $(\bar{y}, \bar{x})$;*
(ii) $\|D_{\frac{1}{q}} S(\bar{y}, \bar{x})\|_{\frac{1}{q}}^{+} < +\infty$*;*
(iii) $D_{\frac{1}{q}} S(\bar{y}, \bar{x})(0) = \{0\}$.

Then (i) ⇒ (ii) ⇒ (iii). *If* $\dim X < +\infty$ *and* $\dim Y < +\infty$*, then* (i) ⇔ (ii) ⇔ (iii).

Proof Thanks to Corollary 3.8, we have the implication (i) ⇒ (ii) in general, and the equivalence (i) ⇔ (ii) when $\dim X < +\infty$ and $\dim Y < +\infty$. The implication (ii) ⇒ (iii) is an immediate consequence of Proposition 2.3(vii). The graph of $D_{\frac{1}{q}} S(\bar{y}, \bar{x})$ is closed by definition, hence, $D_{\frac{1}{q}} S(\bar{y}, \bar{x})$ is outer semicontinuous at 0. Employing Proposition 2.3(vii) again, we conclude that (ii) $\Leftrightarrow$ (iii) when $\dim Y < +\infty$. □

The following theorem shows that q-order strong subregularity enjoys stability under perturbations by functions with small q-order Hadamard directional derivatives.

Theorem 3.10 *Let* $\dim X < +\infty$, $\dim Y < +\infty$, $F : X \rightrightarrows Y$, $g : X \to Y$, $(\bar{x}, \bar{y}) \in gph\, F$, $\bar{x} \in dom\, g$, *and* g *be* q*-order Hadamard directionally differentiable at* $\bar{x}$. *Then*

$$srg_q\,(F + g)(\bar{x}, \bar{y} + g(\bar{x})) \geq srg_q\, F(\bar{x}, \bar{y}) - \|D_q g(\bar{x})\|_q^+.$$

If $\|D_q F(\bar{x}, \bar{y})\|_q^{\ominus} > \|D_q g(\bar{x})\|_q^+$, *then* $F+g$ *is* q*-order strongly subregular at* $(\bar{x}, \bar{y}+g(\bar{x}))$.

Proof By Proposition 2.7 and Theorem 2.6,

$$\begin{aligned} \|D_q(F + g)(\bar{x}, \bar{y} + g(\bar{x}))\|_q^{\ominus} &= \|D_q F(\bar{x}, \bar{y}) + D_q g(\bar{x})\|_q^{\ominus} \\ &\geq \|D_q F(\bar{x}, \bar{y})\|_q^{\ominus} - \|D_q g(\bar{x})\|_q^+. \end{aligned}$$

The assertion follows from Proposition 3.4 and Corollary 3.5. □

The next proposition is a consequence of Propositions 2.5(ii) and 3.4.

Proposition 3.11 *Let* $\dim X < +\infty$, $F : X \rightrightarrows X^*$ *and* $(\bar{x}, x^*) \in gph\, F$. *Then* $\|D_q F(\bar{x}, x^*)\|_q^* \leq srg_q\, F(\bar{x}, x^*)$. *As a consequence, if* $D_q F(\bar{x}, x^*)$ *is* q*-order positively definite with modulus* $\lambda > 0$, *then* F *is* q*-order strongly subregular at* $(\bar{x}, x^*)$ *with any modulus* $\tau \in (0, \lambda)$.

Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, it is easy to check (taking into account Remark 3.2(i)) that the q-order strong subregularity of its epigraphical mapping at $(\bar{x}, f(\bar{x}))$ reduces to the property in the next definition.

Definition 3.12 Let $f : X \to \mathbb{R} \cup \{+\infty\}$. A point $\bar{x} \in \operatorname{dom} f$ is a q-order sharp minimizer of f with modulus $\tau > 0$ if there exists a neighbourhood U of $\bar{x}$ such that

$$\tau \|x - \bar{x}\|^q \leq f(x) - f(\bar{x}) \quad \text{for all} \quad x \in U. \tag{3.5}$$

The exact upper bound of all such $\tau > 0$ is denoted by $\mathrm{shrp_q}\, f(\bar{x})$.

If $\bar{x}$ is not a q-order sharp minimizer of f, we have $\mathrm{shrp_q}\, f(\bar{x}) = 0$.

Remark 3.13 (i) The property in Definition 3.12 is also known as *isolated local minimum with order* q; cf. [25].

(ii) If $f(\bar{x}) = 0$ and $\bar{x}$ is a q-order sharp minimizer of f, then $\mathrm{shrp_q}\, f(\bar{x})$ coincides with the q-order error bound modulus $\mathrm{er_q}\, f(\bar{x})$ of f at $\bar{x}$.

The next proposition is a consequence of Propositions 3.4 and 2.8(iii).

Proposition 3.14 *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *and* $\bar{x} \in dom\, f$. *Then*

$$shrp_q\, f(\bar{x}) \leq \|f'_q(\bar{x}; \cdot)\|_q. \tag{3.6}$$

If $\dim X < +\infty$, *then* (3.6) *holds as equality.*

The following lemma describing Hölder sharp minimizers in terms of the Hölder strong subregularity of the subdifferential mappings is a reformulation of [28, Theorem 4.1] in the convex setting.

Lemma 3.15 *Let X be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ lower semicontinuous and convex, and $\bar{x} \in \operatorname{dom} f$ be a local minimizer of f. Consider the following assertions:*

(i) *$\bar{x}$ is a $(q+1)$-order sharp minimizer of f with modulus $\rho > 0$;*
(ii) *∂f is q-order strongly subregular at $(\bar{x}, 0)$ with modulus $\tau > 0$.*

Then (i) $\Rightarrow$ (ii) *with $\tau := \rho$, and* (ii) $\Rightarrow$ (i) *with $\rho := \frac{q^q}{(q+1)^{q+1}}\tau$. As a consequence, $\bar{x}$ is a $(q+1)$-order sharp minimizer of f if and only if ∂f is q-order strongly subregular at $(\bar{x}, 0)$.*

Next we give characterizations of Hölder sharp minimizers in terms of Hölder graphical derivatives of the subdifferential mapping. The theorem below is partially motivated by [1, Corollary 3.7], which provides a characterization of the strong subregularity in terms of the positive-definiteness of the graphical derivative. The modulus estimate in the following theorem is inspired by [21, Theorem 3.6], where a characterization of tilt stability of local minimizers for extended-real-valued functions is derived via the second-order subdifferential.

Theorem 3.16 *Let X be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ lower semicontinuous and convex, and $\bar{x} \in \operatorname{dom} f$ be a local minimizer of f. Consider the following assertions:*

(i) *$\bar{x}$ is a $(q+1)$-order sharp minimizer of f with modulus $\rho > 0$;*
(ii) *$D_q\partial f(\bar{x}, 0)$ is q-order positively definite with modulus $\lambda > 0$.*

Then (i) $\Rightarrow$ (ii) *with $\lambda := \rho$. If $\dim X < +\infty$, then, for any $\tau \in (0, \lambda)$,* (ii) $\Rightarrow$ (i) *with $\rho := \frac{q^q}{(q+1)^{q+1}}\tau$. As a consequence, if $\dim X < +\infty$, then $\bar{x}$ is a $(q+1)$-order sharp minimizer of f if and only if $D_q\partial f(\bar{x}, 0)$ is q-order positively definite, and*

$$\frac{q^q}{(q+1)^{q+1}}\|D_q\partial f(\bar{x}, 0)\|_q^* \le shrp_{q+1}\, f(\bar{x}) \le \|D_q\partial f(\bar{x}, 0)\|_q^*. \tag{3.7}$$

Proof Let (i) hold. Let $u \in X$ and $u^* \in D_q\partial f(\bar{x}, 0)(u)$, i.e. there are sequences $t_k \downarrow 0$ and $(u_k, u_k^*) \to (u, u^*)$ such that $t_k^q u_k^* \in \partial f(\bar{x} + t_k u_k)$ for all $k \in \mathbb{N}$. For all sufficiently large k, we have $\rho(t_k\|u_k\|)^{q+1} \le f(\bar{x} + t_k u_k) - f(\bar{x}) \le \langle t_k^q u_k^*, t_k u_k\rangle$, and consequently, $\rho\|u\|^{q+1} \le \langle u^*, u\rangle$, i.e. $D_q\partial f(\bar{x}, 0)$ is q-order positively definite with modulus $\lambda := \rho$. Let (ii) hold, $\dim X < +\infty$, and $\tau \in (0, \lambda)$. By Proposition 3.11, ∂f is q-order strongly subregular at $(\bar{x}, 0)$ with modulus τ. By Lemma 3.15, $\bar{x}$ is a $(q+1)$-order sharp minimizer of f with modulus $\frac{q^q}{(q+1)^{q+1}}\tau$. □

The next example illustrates application of Theorem 3.16 for checking the order of sharp minimizers.

Example 3.17 Let $f(x) = x^{2n}$ for some integer $n > 0$ and all $x \in \mathbb{R}$, and $\bar{x} = 0$. Thus, $f'(x) = x^{2n-1}$ for all $x \in \mathbb{R}$ and, by definition (2.4), $y \in D_q\partial f(0, 0)(x)$ if and only if there exist sequences $(x_k, y_k) \to (x, y)$ and $t_k \downarrow 0$ such that $y_k = t_k^{2n-1-q}x_k^{2n-1}$ for all $k \in \mathbb{N}$, or equivalently, $y = \lim_{k\to\infty} t_k^{2n-1-q}x^{2n-1}$. Thus, there are three distinct possibilities.

$\underline{0 < q < 2n-1}$. $D_q\partial f(0,0)(x) = \{0\}$ for all $x \in \mathbb{R}$. Thus, $yx = 0$ for all $(x,y) \in \operatorname{gph} D_q\partial f(0,0)$, i.e. $D_q\partial f(0,0)$ is not q-order positively definite. By Theorem 3.16, 0 is not a $(q+1)$-order sharp minimizer of f.

$\underline{q = 2n-1}$. $D_q\partial f(0,0)(x) = \{x^{2n-1}\}$ for all $x \in \mathbb{R}$. Thus, $yx = x^{2n} = |x|^{2n}$ for all $(x,y) \in \operatorname{gph} D_q\partial f(0,0)$, i.e. $D_q\partial f(0,0)$ is $(2n-1)$-order positively definite with any modulus $\lambda \in (0,1]$, and $\|D_q\partial f(0,0)\|_q^* = 1$. By Theorem 3.16, 0 is a $2n$-order sharp minimizer of f, and $\frac{(2n-1)^{2n-1}}{(2n)^{2n}} \le \operatorname{shrp}_{2n} f(0) \le 1$.

$\underline{q > 2n-1}$. $D_q\partial f(0,0)(0) = \{0\}$, and $D_q\partial f(0,0)(x) = \emptyset$ for all $x \neq 0$. Thus, $yx = 0$ for all $(x,y) \in \operatorname{gph} D_q\partial f(0,0)$, i.e. $D_q\partial f(0,0)$ is not q-order positively definite. By Theorem 3.16, 0 is not a $(q+1)$-order sharp minimizer of f.

Of course, in this simple example, the same conclusions can be obtained directly from Definition 3.12. Moreover, $\operatorname{shrp}_{2n} f(0) = 1$, i.e. the lower estimate in (3.7) is not sharp.

Comparing the statements of Proposition 3.11, Lemma 3.15 and Theorem 3.16, we arrive at the following corollary, which provides an important special case when the implication in Proposition 3.11 holds as equivalence.

Corollary 3.18 *Let* $\dim X < +\infty$, $f : X \to \mathbb{R} \cup \{+\infty\}$ *be lower semicontinuous and convex, and* $\bar{x} \in \operatorname{dom} f$ *be a local minimizer of* f. *Then* $D_q\partial f(\bar{x},0)$ *is* q*-order positively definite if and only if* ∂f *is* q*-order strongly subregular at* $(\bar{x},0)$, *and*

$$\|D_q\partial f(\bar{x},0)\|_q^* \le srg_q\, \partial f(\bar{x},0) \le \frac{(q+1)^{q+1}}{q^q}\|D_q\partial f(\bar{x},0)\|_q^*.$$

4 *q*-Order Isolated Calmness in Linear Semi-infinite Optimization

In this section, we consider a canonically perturbed linear semi-infinite optimization problem:

$$\begin{aligned} P(c,b):\quad & \text{minimize} && \langle c, x\rangle \\ & \text{subject to} && \langle a_t, x\rangle \le b_t,\ t \in T, \end{aligned}$$

where $x \in \mathbb{R}^n$ is the vector of variables, $c \in \mathbb{R}^n$, $\langle\cdot,\cdot\rangle$ represents the usual inner product in $\mathbb{R}^n$, T is a compact Hausdorff space, and the function $t \mapsto (a_t, b_t)$ is continuous on T. In this setting, the pair $(c,b) \in \mathbb{R}^n \times C(T,\mathbb{R})$ is regarded as the perturbation parameter. The parameter space $\mathbb{R}^n \times C(T,\mathbb{R})$ is endowed with the uniform convergence topology through the maximum norm $\|(c,b)\| := \max\{\|c\|, \|b\|_\infty\}$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$ and $\|b\|_\infty := \max_{t\in T} |b_t|$.

The *feasible set* and *solution* mappings corresponding to the above problem are defined, respectively, by

$$\mathcal{F}(b) := \{x \in \mathbb{R}^n \mid \langle a_t, x\rangle \le b_t,\ t \in T\}, \quad b \in C(T,\mathbb{R}), \tag{4.1}$$

$$\mathcal{S}(c,b) := \{x \in \mathcal{F}(b) \mid x \text{ solves } P(c,b)\}, \quad (c,b) \in \mathbb{R}^n \times C(T,\mathbb{R}). \tag{4.2}$$

From now on, we assume a point $((\bar{c},\bar{b}),\bar{x}) \in \operatorname{gph}\mathcal{S}$ to be given. We are going to consider also the partial solution mapping $\mathcal{S}_{\bar{c}} : C(T,\mathbb{R}) \rightrightarrows \mathbb{R}^n$ given by $\mathcal{S}_{\bar{c}}(b) = \mathcal{S}(\bar{c},b)$ and the *level set* mapping

$$\mathcal{L}(\alpha,b) := \{x \in \mathcal{F}(b) \mid \langle\bar{c},x\rangle \le \alpha\}, \quad (\alpha,b) \in \mathbb{R} \times C(T,\mathbb{R}),$$

and employ the following convex and continuous function:

$$f(x) := \max\{\langle \bar{c}, x - \bar{x}\rangle,\ \max_{t\in T}(\langle a_t, x\rangle - \bar{b}_t)\}, \quad x \in \mathbb{R}^n. \tag{4.3}$$

Observe that $f(\bar{x}) = 0$, and

$$\mathcal{S}(\bar{c}, \bar{b}) = [f = 0] = [f \leq 0] = \mathcal{L}\langle \bar{c}, \bar{x}\rangle, \bar{b}). \tag{4.4}$$

The problem $P(c, b)$ satisfies the *Slater condition* if there exists a point $\hat{x} \in \mathbb{R}^n$ such that $\langle a_t, \hat{x}\rangle < b_t$ for all $t \in T$. The set of *active indices* at $x \in \mathcal{F}(b)$ is defined by $T_b(x) := \{t \in T \mid \langle a_t, x\rangle = b_t\}$.

The following lemma is an analogue of [19, Proposition 4.5].

Lemma 4.1 *Suppose that $P(\bar{c}, \bar{b})$ satisfies the Slater condition, and $\mathcal{S}(\bar{c}, \bar{b}) = \{\bar{x}\}$. If $\mathcal{L}$ does not possess q-order isolated calmness property at $((\langle \bar{c}, \bar{x}\rangle, \bar{b}), \bar{x}) \in \operatorname{gph}(\mathcal{L})$, then there exist a sequence $\{(b_k, x_k)\} \subset \operatorname{gph}\mathcal{F}$ such that $x_k \neq \bar{x}$ for all $k \in \mathbb{N}$, and*

$$\lim_{k\to+\infty}(b_k, x_k) = (\bar{b}, \bar{x}), \qquad \lim_{k\to+\infty}\frac{\|b_k - \bar{b}\|_\infty}{\|x_k - \bar{x}\|^q} = 0,$$

a finite subset $T_0 \subset \cap_{k\in\mathbb{N}} T_{b_k}(x_k)$, and positive scalars $\gamma_t,\ t \in T_0$, satisfying

$$-\bar{c} \in \sum_{t\in T_0}\gamma_t a_t. \tag{4.5}$$

Theorem 4.2 *Suppose that $P(\bar{c}, \bar{b})$ satisfies the Slater condition. Consider the following assertions:*

(i) *$\mathcal{S}$ possesses q-order isolated calmness property at $((\bar{c}, \bar{b}), \bar{x})$;*
(ii) *$\mathcal{S}_{\bar{c}}$ possesses q-order isolated calmness property at $(\bar{b}, \bar{x})$;*
(iii) *$\mathcal{L}$ possesses q-order isolated calmness property at $((\langle \bar{c}, \bar{x}\rangle, \bar{b}), \bar{x})$;*
(iv) *$\bar{x}$ is a q-order sharp minimizer of f;*
(v) $\|f'_q(\bar{x}; \cdot)\|_q > 0$;
(vi) $\|D_{\frac{1}{q}} S_{\bar{c}}(\bar{b}, \bar{x})\|^+_{\frac{1}{q}} < +\infty$.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Rightarrow$ (vi). *If T is finite, then all the assertions are equivalent, and*

$$clm_q\ \mathcal{S}_{\bar{c}}(\bar{b}, \bar{x}) = \left(\left\| D_{\frac{1}{q}}\mathcal{S}_{\bar{c}}(\bar{b}, \bar{x})\right\|^+_{\frac{1}{q}}\right)^{-q}. \tag{4.6}$$

If $q > 1$, then assertions (i)–(v) *are equivalent to the next one:*

(vii) *$D_{q-1}\partial f(\bar{x}, 0)$ is $(q-1)$-order positively definite.*

Proof (i) $\Rightarrow$ (ii) is immediate from Definition 3.1(ii) in view of the definition of $\mathcal{S}_{\bar{c}}$.

(ii) $\Rightarrow$ (iii). Suppose that $\mathcal{L}$ does not possess q-order isolated calmness property at $((\langle \bar{c}, \bar{x}\rangle, \bar{b}), \bar{x})$. To reach a contradiction with (ii), it suffices to show that, for the sequence $\{(b_k, x_k)\} \subset \operatorname{gph}\mathcal{F}$ in Lemma 4.1, it holds $x_k \in \mathcal{S}_{\bar{c}}(b_k)$, $k \in \mathbb{N}$, which readily follows from the KKT conditions (4.5) (by continuity, it is not restrictive to assume that $P(\bar{c}, b_k)$ satisfies the Slater condition).

(iii) $\Leftrightarrow$ (iv) follows from comparing Definitions 3.1(ii) and 3.12 in view of (4.3) and (4.4).

(iv) ⇒ (i). By [19, Lemma 4.2] (with $f \equiv 0$), there exist a number $M > 0$ and neighbourhoods U of $\bar{x}$ and V of $(\bar{c}, \bar{b})$ such that

$$-c \in [0, M]\,\mathrm{co}\,\{a_t,\ t \in T_b(x)\} \quad \text{for all } (c, b) \in V \text{ and } x \in \mathcal{S}(c, b) \cap U, \tag{4.7}$$

where 'co' stands for the convex hull. Let $\bar{x}$ be a q-order sharp minimizer of f. By Definition 3.12, condition (3.5) holds with some number $\tau > 0$ and a smaller neighbourhood U if necessary. Without loss of generality, we assume that $M > 1$, and U is bounded: $\|x - \bar{x}\| < \delta$ for some $\delta > 0$ and all $x \in U$. Let $(c, b) \in V$ and $x \in \mathcal{S}(c, b) \cap U$. By (4.7),

$$-c = \sum_{t \in T_b(x)} \eta_t a_t, \tag{4.8}$$

for some $\eta_t \geq 0,\ t \in T_b(x)$, satisfying $\sum_{t \in T_b(x)} \eta_t \leq M$ and only finitely many being positive. Hence, in view of representation (4.8), and definitions (4.1) and (4.2),

$$\langle c, x - \bar{x}\rangle = -\sum_{t \in T_b(x)} \eta_t \langle a_t, x - \bar{x}\rangle \leq \sum_{t \in T_b(x)} \eta_t(\bar{b}_t - b_t) \leq M\|b - \bar{b}\|_\infty,$$

Recalling definition (4.3) and the fact that $f(\bar{x}) = 0$, we have

$$\begin{aligned}\tau\,\|x - \bar{x}\|^q \leq f(x) &\leq \max\{\langle \bar{c}, x - \bar{x}\rangle,\ \max_{t \in T}(b_t - \bar{b}_t)\}\\ &\leq \max\{\langle c, x - \bar{x}\rangle + \|c - \bar{c}\|\|x - \bar{x}\|,\ \|b - \bar{b}\|_\infty\}\\ &\leq \max\{M\|b - \bar{b}\|_\infty + \delta\|c - \bar{c}\|,\ \|b - \bar{b}\|_\infty\}\\ &\leq (M + \delta)\|(c, b) - (\bar{c}, \bar{b})\|.\end{aligned}$$

By Definition 3.1(ii), $\mathcal{S}$ possesses q-order isolated calmness property at $((\bar{c}, \bar{b}), \bar{x})$. (iv) ⇔ (v) is immediate from Proposition 3.14. (ii) ⇒ (vi) and the opposite implication when T is finite, together with the equality (4.6) follow from Corollaries 3.8 and 3.9. It suffices to notice that, when T is finite, the parameter space $C(T, \mathbb{R})$ is finite-dimensional. When $q > 1$, the equivalence (iv) ⇔ (vii) is a consequence of Theorem 3.16.

□

Remark 4.3 Implication (iii) ⇒ (i) in Theorem 4.2 is a consequence of [16, Corollary 3] and the fact that, under the Slater condition, $\mathcal{F}$ is calm and Lipschitz lower semicontinuous.

In the case $q \geq 1$, implication (iv) ⇒ (i) is a special case of [14, Theorem 2.2]. For the semi-infinite optimization model $P(c, b)$, this implication was explicitly given, e.g., in [15, Proposition 4.2]. Indeed, $\bar{x}$ is a q-order sharp minimizer of f, then, using the notation of Definition 3.1, one has in particular

$$\tau\,\|x - \bar{x}\|^q \leq \langle c, x - \bar{x}\rangle, \quad \text{for all } x \in \mathcal{F}(\bar{b}) \cap U,$$

i.e., $\bar{x}$ is a strict local minimizer of $P(\bar{c}, \bar{b})$ in the sense of [15]. Since the Slater condition is equivalent to the extended Mangasarian-Fromovitz CQ (for this equivalence in relation to the linear SIP problem $P(c, b)$ see, e.g., [10, Theorem 6.1] and [4, Theorem 2.1]), [15, Proposition 4.2] applies and gives (in particular) that $\mathcal{S}$ possesses the q-order isolated calmness property at $((\bar{c}, \bar{b}), \bar{x})$.

Next we recall the *Extended Nürnberger Condition* (*ENC,* in brief) [6, Definition 2.1].

Definition 4.4 ENC is satisfied at $((\bar{c},\bar{b}),\bar{x})$ when $P(\bar{c},\bar{b})$ satisfies the Slater condition, and there is no subset $D \subset T_{\bar{b}}(\bar{x})$ with $|D| < n$ such that $-\bar{c} \in \operatorname{cone}\{a_t, t \in D\}$.

The following lemma is [6, Theorem 2.1 and Lemma 3.1]).

Lemma 4.5 *Suppose that ENC is satisfied at* $((\bar{c},\bar{b}),\bar{x})$. *Then*

(i) $\mathcal{S}$ *is single valued and Lipschitz continuous in a neighbourhood of* $(\bar{c},\bar{b})$*;*
(ii) *if a sequence* $\{((c_k,b_k),x_k)\} \subset \operatorname{gph}\mathcal{S}$ *converges to* $((\bar{c},\bar{b}),\bar{x})$*, then* $(b_k,x_k) \in \operatorname{gph}\mathcal{S}_{\bar{c}}$ *for all k large enough.*

Thanks to Lemma 4.5, we can show that the parameter c can be considered fixed in our analysis, provided that ENC holds at $((\bar{c},\bar{b}),\bar{x})$.

Theorem 4.6 *If ENC is satisfied at* $((\bar{c},\bar{b}),\bar{x})$*, then* $clm_q\,\mathcal{S}((\bar{c},\bar{b}),\bar{x}) = clm_q\,\mathcal{S}_{\bar{c}}(\bar{b},\bar{x})$.

Proof It obviously holds $\operatorname{clm_q}\mathcal{S}_{\bar{c}}(\bar{b},\bar{x}) \geq \operatorname{clm_q}\mathcal{S}((\bar{c},\bar{b}),\bar{x})$, and we need to show the opposite inequality. If $\operatorname{clm_q}\mathcal{S}((\bar{c},\bar{b}),\bar{x}) = +\infty$, there is nothing to prove. Let $\operatorname{clm_q}\mathcal{S}((\bar{c},\bar{b}),\bar{x}) < +\infty$. Then

$$\operatorname{clm_q}\mathcal{S}((\bar{c},\bar{b}),\bar{x}) = \lim_{k\to+\infty}\frac{\|(c_k,b_k)-(\bar{c},\bar{b})\|}{\|x_k-\bar{x}\|^q}$$

for some sequence $\{((c_k,b_k),x_k)\} \subset \operatorname{gph}\mathcal{S}$ such that $((c_k,b_k),x_k) \to ((\bar{c},\bar{b}),\bar{x})$ and $x_k \neq \bar{x}$ for all $k \in \mathbb{N}$. If ENC is satisfied at $((\bar{c},\bar{b}),\bar{x})$, then, by Lemma 4.5, $(b_k,x_k) \in \operatorname{gph}\mathcal{S}_{\bar{c}}$ for all k large enough. Hence,

$$\operatorname{clm_q}\mathcal{S}((\bar{c},\bar{b}),\bar{x}) \geq \liminf_{k\to+\infty}\frac{\|b_k-\bar{b}\|_\infty}{\|x_k-\bar{x}\|^q} \geq \operatorname{clm_q}\mathcal{S}_{\bar{c}}(\bar{b},\bar{x}).$$

This completes the proof. □

Example 4.7 Consider the linear semi-infinite optimization problem in $\mathbb{R}^2$:

$$\begin{aligned} P(c,b): \quad & \text{minimize} \quad c_1x_1 + c_2x_2 \\ & \text{subject to} \quad (\cos t)\,x_1 + (\sin t)\,x_2 \leq b_t,\ \ t \in [0,2\pi]. \end{aligned}$$

Let $\bar{c} := (1,0)$ and $\bar{b}_t := 1$, for all $t \in [0,2\pi]$. It is easy to check that $\bar{x} := (-1,0)$ is the unique solution of $P\left(\bar{c},\bar{b}\right)$. It obviously satisfies the Slater condition. We are going to use condition (v) in Theorem 4.2 to check the isolated calmness property of the solution mapping $\mathcal{S}$ of $P\left(\bar{c},\bar{b}\right)$. The function (4.3) takes the form

$$f(x) = \max\left\{x_1+1, \sqrt{x_1^2+x_2^2}-1\right\}, \quad x = (x_1,x_2) \in \mathbb{R}^2.$$

Obviously, $f(\bar{x}) = 0$ and, for any $x = (x_1,x_2) \in \mathbb{R}^2$ and $t > 0$, we have

$$\begin{aligned} f(\bar{x}+tx) &= \max\left\{tx_1, \sqrt{(tx_1-1)^2+(tx_2)^2}-1\right\} \\ &= t\max\left\{x_1, \frac{-2x_1+t(x_1^2+x_2^2)}{\sqrt{(tx_1-1)^2+(tx_2)^2}+1}\right\}. \end{aligned}$$

Thus,

$$f_q'(\bar{x}; x) = \liminf_{(u_1,u_2)\to(x_1,x_2),\, t\downarrow 0} t^{1-q} \max\left\{u_1, -u_1 + \frac{t(u_1^2+u_2^2)}{2}\right\} \geq 0. \tag{4.9}$$

If $x_1 = 0$ and $q < 2$, then, by (4.9),

$$f_q'(\bar{x}; x) = \liminf_{u_1\to 0,\, t\downarrow 0} t^{1-q}|u_1| = 0,$$

and consequently, $\|f_q'(\bar{x};\cdot)\|_q = 0$. Thus, by Theorem 4.2, $\mathcal{S}$ does not possess q-order isolated calmness property at $((\bar{c}, \bar{b}), \bar{x})$ when $q < 2$. With $q = 1$, this fact was established in [5]. Similarly, if $x_1 = 0$ and $q = 2$, then

$$f_2'(\bar{x}; x) = \liminf_{u_1\to 0,\, t\downarrow 0} \max\left\{\frac{u_1}{t}, -\frac{u_1}{t} + \frac{x_2^2}{2}\right\} = \inf_{\alpha\in\mathbb{R}} \max\left\{\alpha, -\alpha + \frac{x_2^2}{2}\right\} = \frac{x_2^2}{4}.$$

Finally, if $x_1 \neq 0$, it follows from (4.9) that $f_2'(\bar{x}; x) \geq \lim_{t\downarrow 0} t^{-1}|x_1| = +\infty$. Hence, $\|f_2'(\bar{x};\cdot)\|_2 = \frac{1}{4} > 0$ and, by Theorem 4.2, $\mathcal{S}$ possesses 2-order isolated calmness property at $((\bar{c}, \bar{b}), \bar{x})$.

5 q-Order Sharp Minimizers of ℓ_p Penalty Functions

In this section, we consider an inequality constrained optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0,\ i \in I := \{1, \ldots, m\}, \end{array} \tag{5.1}$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i \in I$. Given numbers $p > 0$ and $r > 0$, the l_p penalty optimization problem corresponding to (5.1) can be defined as follows:

$$\text{minimize} \quad \ell_p(x) := f(x) + r\sum_{i=1}^m (g_i^+)^p(x), \tag{5.2}$$

where $g_i^+(x) := \max\{0, g_i(x)\}$, $i \in I$.

By virtue of the optimal value function, relations between local minimizers of (5.1) and (5.2) were given in [11, 24]. Below q-order Hadamard directional subderivatives are used to identify q-order sharp minimizers of the penalty problem (5.2).

By Propositions 3.14 and 2.8(iv), a point $\bar{x} \in \cap_{i=1}^m \operatorname{dom} g_i \cap \operatorname{dom} f$ is a q-order sharp minimizer of (5.2) if and only if

$$(\ell_p)_q'(\bar{x}; x) > 0 \quad \text{for all} \quad x \neq 0. \tag{5.3}$$

Define

$$\begin{aligned} I(\bar{x}) &= \{i \in I \mid g_i(\bar{x}) = 0\}, \\ K^*(\bar{x}) &= \{x \in \mathbb{R}^n \mid \|x\| = 1,\ f_q'(\bar{x}; x) \leq 0\}, \\ a(\bar{x}) &= \min\left\{\sum_{i\in I(\bar{x})} \left[(g_i^+)_{q/p}'(\bar{x}; x)\right]^p \mid x \in K^*(\bar{x})\right\}, \\ b(\bar{x}) &= \min\{f_q'(\bar{x}; x) \mid \|x\| = 1\}. \end{aligned}$$

Theorem 5.1 (i) *Suppose that $f_q'(\bar{x};\cdot)$ is proper and*

$$f_q'(\bar{x};x)>0 \quad \textit{for all} \quad x\neq 0 \textit{ with } (g_i^+)'_{q/p}(\bar{x};x)=0,\ i\in I(\bar{x}). \tag{5.4}$$

Then $\bar{x}$ is a q-order sharp minimizer of (5.2) for all $r>\rho_0$, where

$$\rho_0 := \begin{cases} -b(\bar{x})/a(\bar{x}), & \textit{if } K^*(\bar{x})\neq\emptyset,\\ 0, & \textit{otherwise}. \end{cases}$$

(ii) *Suppose that f is q-order Hadamard directionally differentiable at $\bar{x}$, $m=1$ and $\bar{x}$ is a q-order sharp minimizer of (5.2) for some $r>0$. Then (5.4) holds.*

Proof (i) If $K^*(\bar{x})\neq\emptyset$, then $a(\bar{x})>0$ and $b(\bar{x})\leq 0$. Therefore ρ_0 is well defined and nonnegative. Let $r>\rho_0$. Let $x\neq 0$. As $(\ell_p)_q'(\bar{x};\cdot)$ is positively homogeneous, without loss of generality, we assume that $\|x\|=1$. Obviously, $(g_i^+)'_{q/p}(\bar{x};x)\geq 0$ for all $i\in I$. It is easy to show that

$$([g_i^+]^p)_q'(\bar{x};x)=\left[(g_i^+)'_{q/p}(\bar{x};x)\right]^p \quad \text{for all} \quad i\in I(\bar{x}).$$

Since $f_q'(\bar{x};\cdot)$ is proper, it follows that

$$(\ell_p)_q'(\bar{x};x)\geq f_q'(\bar{x};x)+r\sum_{i\in I(\bar{x})}\left[(g_i^+)'_{q/p}(\bar{x};x)\right]^p. \tag{5.5}$$

If $x\notin K^*(\bar{x})$, we have $f_q'(\bar{x};x)>0$, and consequently, $(\ell_p)_q'(\bar{x};x)>0$. If $x\in K^*(\bar{x})$, then $b(\bar{x})\leq 0$. So, by definitions of ρ_0 and $a(\bar{x})$, we have

$$r\sum_{i\in I(\bar{x})}\left[(g_i^+)'_{q/p}(\bar{x};x)\right]^p>-b(\bar{x}),$$

and thus it follows from (5.5) that

$$(\ell_p)_q'(\bar{x};x)>f_q'(\bar{x};x)-b(\bar{x})\geq 0.$$

So, by (5.3), $\bar{x}$ is a q-order sharp minimizer for (5.2).

(ii) It follows from Corollary 2.9 that (5.5) holds as equality. The conclusion is verified by Propositions 3.14 and 2.8(iv).

□

Condition (5.4) in Theorem 5.1 uses $(g_i^+)'_{q/p}(\bar{x};\cdot)$, which allows the treatment of q-order sharp minimizers of rather general penalty functions. When $p=q=1$, Theorem 5.1(i) is a consequence of [26, Theorem 4.1].

Furthermore, for all $i\in I(\bar{x})$, $u\in\mathbb{R}^n$ and $t>0$, we obviously have

$$\frac{g_i^+(\bar{x}+tu)-g_i^+(\bar{x})}{t^{q/p}}=\frac{g_i^+(\bar{x}+tu)}{t^{q/p}}=\frac{\max\{0,g_i(\bar{x}+tu)\}}{t^{q/p}}$$
$$=\max\left\{0,\frac{g_i(\bar{x}+tu)}{t^{q/p}}\right\}=\max\left\{0,\frac{g_i(\bar{x}+tu)-g_i(\bar{x})}{t^{q/p}}\right\}.$$

Hence, if g_i $(i\in I(\bar{x}))$ is (q/p)-order Hadamard directionally differentiable at $\bar{x}$, then so is g_i^+, and

$$(g_i^+)'_{q/p}(\bar{x};x)=\max\{0,(g_i)'_{q/p}(\bar{x};x)\}$$

for all $x\in\mathbb{R}^n$. Therefore, if all g_i $(i\in I(\bar{x}))$ are (q/p)-order Hadamard directionally differentiable at $\bar{x}$, then (5.4) is equivalent to the following condition:

$$f_q'(\bar{x};x)>0 \quad \text{for all} \quad x\neq 0 \text{ with } (g_i)'_{q/p}(\bar{x};x)\leq 0,\ i\in I(\bar{x}).$$

The following simple example shows the calculation of the exact upper bound of the 1-order sharp minimizer of the penalty problem (5.2).

Example 5.2 Consider the following problem on $\mathbb{R}$:

$$\text{minimize } x \quad \text{subject to } x^{2s} \leq 0,$$

where $s > 0$. Obviously $\bar{x} = 0$ is a minimizer of this problem. With any $p > 0$, we have

$$\text{shrp}_1 \ell_p(0) = \begin{cases} +\infty, & \text{if } sp < 1/2, \\ r+1, & \text{if } sp = 1/2, \\ 0, & \text{otherwise.} \end{cases}$$

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