SIAM J. MATH. ANAL.
Vol. 52, No. 2, pp. 1463–1488

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO EULER EQUATIONS WITH TIME-DEPENDENT DAMPING IN CRITICAL CASE*

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Abstract. In this paper, we are concerned with the system of Euler equations with time-dependent damping like $-\frac{\mu}{(1+t)^\lambda}u$ for physical parameters $\lambda \geq 0$ and $\mu > 0$. It is well known that, when $0 \leq \lambda < 1$, the time-asymptotic-degenerate damping plays the key role which makes the damped Euler system behave like time-degenerate diffusion equations, while, when $\lambda > 1$, the damping effect becomes really weak and can be neglected, which makes the dynamic system essentially behave like a hyperbolic system, and the singularity of solutions like shock waves will form. However, in the critical case $\lambda = 1$, when $0 < \mu \leq 2$, the solutions of the system will blow up, but when $\mu > 2$, the system is expected to possess global solutions. Here, we are particularly interested in the asymptotic behavior of the solutions in the critical case. By a heuristical analysis (variable scaling technique), we realize that, in this critical case, the hyperbolicity and the damping effect both play crucial roles and cannot be neglected. We first artfully construct the asymptotic profile, a special linear wave equation with time-dependent damping, which is totally different from the case of $0 \leq \lambda < 1, \mu > 0$, whose profile is a self-similar solution to the corresponding parabolic equation. Then we rigorously prove that the solutions time-asymptotically converge to the solutions of linear wave equations with critical time-dependent damping. The convergence rates shown are optimal, by comparing with the linearized equations. The proof is based on the technical time-weighted energy method, where the time-weight is dependent on the parameter μ.

Key words. Euler equations, time-gradually-degenerate damping, time-weighted energy estimates, asymptotic profiles, convergence rates

AMS subject classifications. 35L65, 76S05, 35K65

DOI. 10.1137/19M1272846

1. Introduction.

The one-dimensional compressible Euler equations with time-dependent damping can be written, in Lagrangian coordinates, as follows:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\dfrac{\mu}{(1+t)^\lambda}u, \end{cases} \tag{1.1}$$

with the initial data

$$(v,u)(x,0) = (v_0,u_0)(x) \to (v_\pm, u_\pm) \quad \text{as} \quad x \to \pm\infty. \tag{1.2}$$

*Received by the editors July 8, 2019; accepted for publication (in revised form) February 13, 2020; published electronically March 30, 2020.
https://doi.org/10.1137/19M1272846
Funding: The work of the first author was partially supported by National Natural Science Foundation of China grant 11701489, by Natural Science Foundation of Hunan Province of China grants 2018JJ2373 and 2018JJ3481, and by Excellent Youth Project of Hunan Education Department grant 18B054. The work of the second author was partially supported by Hong Kong Special Administrative Region GRF grants 15301714 and 15327816. The work of the third author was partially supported by NSERC grant RGPIN 354724-2016 and by FRQNT grant 256440.
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Here $v > 0$ is the specific volume, u is the velocity, and the pressure $p = p(v)$ is a smooth function of v with $p(v) > 0, p'(v) < 0$. One of the typical examples is $p(v) = v^{-\gamma}$ for the polytropic gas satisfying the so-called γ-law with $\gamma > 1$. The external term $-\frac{\mu}{(1+t)^\lambda}u$ with physical parameters $\mu > 0$ and $\lambda \geq 0$ is a time-gradually-degenerate damping. In this paper, we are mainly interested in the critical case of $\lambda = 1$. $v_0(x) > 0$ and $u_0(x)$ are the given initial data, and their asymptotic states are the state-constants $v_\pm > 0$ and $u_\pm$ as $x \to \pm\infty$, respectively.

For $\mu = 0$, the system (1.1) reduces to the standard compressible Euler equations. This has been well studied [1, 2, 3, 5, 7, 18, 34], and it is well known that smooth solutions of (1.1) will in general blow up in finite time due to the formation of shocks.

For $\mu > 0, \lambda = 0$, the system (1.1) becomes the compressible system of Euler equations with damping which models the compressible flow through porous media. There is also a huge literature on the investigations of global existence and large time behaviors of smooth solutions to compressible Euler equations with damping. Among them, Hsiao and Liu [12] first showed that the solution of (1.1) tends time-asymptotically to the nonlinear diffusion waves in the form of $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(t^{-1/2}, t^{-1/2})$, where $(\bar{v}, \bar{u})(x,t) = (\bar{v}, \bar{u})(x/\sqrt{t})$ are the self-similar solutions to the corresponding porous media equations. See also the weak sense convergence first investigated by Marcati and Milani [23]. Then, by taking much finer energy estimates, Nishihara [28] succeeded in improving the convergence rates to $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(t^{-3/4}, t^{-5/4})$, when the initial perturbation belongs to $H^3 \times H^2$. Furthermore, when the initial perturbation is in $(H^3 \cap L^1) \times (H^2 \cap L^1)$, by constructing an approximate Green function and using energy methods, Nishihara, Wang, and Yang [29] obtained the convergence rates in the form of $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(t^{-1}, t^{-3/2})$. On the other hand, Mei [26] observed that the best asymptotic profile is the solution for the corresponding nonlinear diffusion equation with some particularly selected initial data and improved the convergence rates to the best asymptotic profile as $O(t^{-3/2}\ln t, t^{-2}\ln t)$ by the technique of double antiderivatives. See also [9] for the isentropic case. When $v_+ = v_-$, the global existence in the three-dimensional case was also significantly obtained by Sideris, Thomases, and Wang in [33]. When the derivatives of initial data are big, blow-up of the solutions usually occurs in finite time [37, 21, 20]. For other interesting contributions, we refer to [8, 10, 11, 13, 14, 15, 16, 17, 22, 25, 30, 41, 42] and the references therein.

For $\mu > 0$ and $\lambda > 0$, the system (1.1) reduces to compressible Euler equations with time-dependent damping. When the constant states at far fields are equal, i.e., $v_+ = v_- > 0$ and $u_+ = u_- = 0$, Pan [32, 31] (see also [35, 36]) proved that $\lambda = 1, \mu = 2$ is the critical threshold of (1.1) in Euler coordinates to separate the global existence and finite-time blow-up of solutions. More precisely, when $\lambda = 1$ with $\mu > 2$ or $0 \leq \lambda < 1$ with $\mu > 0$, (1.1) possesses global smooth solutions, while, when $\lambda = 1$ with $0 \leq \mu \leq 2$ or $\lambda > 1$ with $\mu \geq 0$, then C^1-solutions of (1.1) will blow up in finite time. Remarkably, by introducing a new energy functional to build up the maximum principle for the system (1.1) in the coordinates of Riemann invariants, Chen et al. [4] obtained the global existence with large initial data (but the derivatives of initial data still need to be small; otherwise, the singularity like shock waves will occur) for $0 \leq \lambda < 1$ and proved the blow-up of the solutions for any initial data once $\lambda > 1$, namely, such a blow-up phenomenon is determined by the mechanism of the system due to $\lambda > 1$.

For $0 \leq \lambda < 1, \mu > 0$, when the constant states at far fields are different, i.e., $v_+ \neq v_- > 0$ and $u_+ \neq u_-$, the asymptotic profiles of the solutions are no long expected to be the constant states, because of the formation of a "boundary layer."

The expected profiles should be some nontrivial functions. In this case, Cui et al. [6] (see also Li et al. [19]) further showed the asymptotic behavior of the global solutions, where the asymptotic profiles for the global solutions are the so-called diffusion waves in the form of $(\bar{v},\bar{u})(x,t) = (\bar{v},\bar{u})(x/\sqrt{(1+t)^{1+\lambda}})$ satisfying the following nonlinear diffusion equations:

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\dfrac{\mu}{(1+t)^\lambda}\bar{u}, \end{cases} \qquad \text{equivalently,} \quad \frac{\mu}{(1+t)^\lambda}\bar{v}_t + p(\bar{v})_{xx} = 0. \tag{1.3}$$

In this paper, the main purpose is to look for the asymptotic profile for the global solutions in the critical case $\lambda = 1$ with $\mu > 2$ with the so-called boundary layers (while the solutions blow up for $\lambda = 1$ and $0 < \mu \leq 2$), and further to investigate the long-time behavior of the global solutions for the system (1.1).

Asymptotic profiles. From the previous studies [6, 19], when $0 < \lambda < 1$, time-asymptotic-degenerate damping plays a key role which makes the damped Euler system behave like a diffusion equation with time-asymptotic-degenerate damping, while, from [4, 31, 36], when $\lambda > 1$, the damping effect becomes really weak and can be neglected, which makes the dynamic system essentially behave like a hyperbolic system, and the singularity like shock waves will always form. However, in the critical case $\lambda = 1$, once $\mu > 2$ the global solutions are expected, and when $0 < \mu \leq 2$ the solutions usually blow up at finite time. So, the interesting questions are what will be the asymptotic profiles of the global solutions in this critical case, and how the original solutions converge to the expected profiles.

Heuristically, we look for the asymptotic-state equations for (1.1) by taking the variables scalings. For an arbitrarily small $\varepsilon > 0$, let

$$t \to (1+\bar{t})/\varepsilon^{\theta_1}, \qquad x \to \bar{x}/\varepsilon^{\theta_2}, \qquad v \to \bar{v}, \qquad u \to \varepsilon^{\theta_3}\bar{u},$$

where $\theta_1, \theta_2, \theta_3 \geq 0$ are constants and will be determined later. We first have from $(1.1)_1$ that

$$\varepsilon^{\theta_1}\bar{v}_{\bar{t}} - \varepsilon^{\theta_2+\theta_3}\bar{u}_{\bar{x}} = 0.$$

In order to balance it, let

$$\theta_1 = \theta_2 + \theta_3, \tag{1.4}$$

then

$$\bar{v}_{\bar{t}} - \bar{u}_{\bar{x}} = 0.$$

On the other hand, it follows from $(1.1)_2$ that

$$\varepsilon^{\theta_1+\theta_3}\bar{u}_{\bar{t}} + \varepsilon^{\theta_2}p(\bar{v})_{\bar{x}} = -\frac{\mu\varepsilon^{\theta_1\lambda+\theta_3}\bar{u}}{(\varepsilon^{\theta_1}+1+\bar{t})^\lambda}. \tag{1.5}$$

When $0 \leq \lambda < 1$, we reduce (1.5) to

$$\varepsilon^{\theta_1-\theta_2+\theta_3}\bar{u}_{\bar{t}} + p(\bar{v})_{\bar{x}} = -\frac{\mu\varepsilon^{\theta_1\lambda-\theta_2+\theta_3}\bar{u}}{(\varepsilon^{\theta_1}+1+\bar{t})^\lambda}.$$

Let $\theta_1, \theta_2, \theta_3 > 0$. By setting

$$\theta_1\lambda - \theta_2 + \theta_3 = 0 \text{ for } 0 \leq \lambda < 1,$$

which, combining with (1.4), gives that

$$\theta_1 > 0, \quad \theta_2 = \frac{1+\lambda}{2}\theta_1, \quad \theta_3 = \frac{1-\lambda}{2}\theta_1.$$

So, it holds that

$$\varepsilon^{(1-\lambda)\theta_1}\bar{u}_{\bar{t}} + p(\bar{v})_{\bar{x}} = -\frac{\mu}{(1+\bar{t}\,)^\lambda}\bar{u}.$$

Thus, one can derive the state equation by neglecting the small term with ε,

$$p(\bar{v})_{\bar{x}} = -\frac{\mu}{(1+\bar{t}\,)^\lambda}\bar{u}.$$

Therefore, the expected asymptotic profile for (1.1) in the case $0 \le \lambda < 1$ is

$$\begin{cases} \bar{v}_t - \bar{u}_{\bar{x}} = 0, \\ p(\bar{v})_{\bar{x}} = -\dfrac{\mu}{(1+\bar{t}\,)^\lambda}\bar{u}, \end{cases} \quad \text{i.e.,} \quad \frac{\mu}{(1+\bar{t}\,)^\lambda}\bar{v}_t + p'(\bar{v})_{\bar{x}\bar{x}} = 0. \tag{1.6}$$

So, the asymptotic profiles of (v, u) for (1.1) are expected to be the diffusion equations with time-degenerate-damping, the so-called diffusion waves

$$(\bar{v}, \bar{u})(x, t) = (\bar{v}, \bar{u})\left(\frac{x}{\sqrt{(1+t)^{1+\lambda}}}\right).$$

This completely matches the asymptotic profiles specified in [6, 19] for the case $0 \le \lambda < 1$.

When $\lambda > 1$, (1.4) and (1.5) give us

$$\varepsilon^{\theta_2+2\theta_3}\bar{u}_{\bar{t}} + \varepsilon^{\theta_2}p(\bar{v})_{\bar{x}} = -\frac{\mu\varepsilon^{(\theta_2+\theta_3)\lambda+\theta_3}\bar{u}}{(\varepsilon^{\theta_1}+1+\bar{t}\,)^\lambda}.$$

Taking $\theta_3 = 0$, we have

$$\varepsilon^{\theta_2}\bar{u}_{\bar{t}} + \varepsilon^{\theta_2}p(\bar{v})_{\bar{x}} = -\frac{\mu\varepsilon^{\theta_2\lambda}\bar{u}}{(\varepsilon^{\theta_1}+1+\bar{t}\,)^\lambda},$$

which implies

$$\bar{u}_{\bar{t}} + p(\bar{v})_{\bar{x}} = -\frac{\mu\varepsilon^{\theta_2(\lambda-1)}\bar{u}}{(\varepsilon^{\theta_1}+1+\bar{t}\,)^\lambda}. \tag{1.7}$$

Neglecting the small term with ε, we get the asymptotic-state equations for the case $\lambda > 1$ as follows:

$$\begin{cases} \bar{v}_{\bar{t}} - \bar{u}_{\bar{x}} = 0, \\ \bar{u}_{\bar{t}} + p(\bar{v})_{\bar{x}} = 0. \end{cases}$$

This is the standard Euler system, and the smooth solutions will, in general, blow up at finite time due to the singularity formed by shock waves. The analysis presented here also matches the existing studies [4, 31, 35] for the blow-up phenomenon of the solutions to (1.1) once $\lambda > 1$.

Now we consider the critical case $\lambda = 1$ and see what will be a possible state-equation for the original system (1.1). In (1.7), since $\lambda = 1$, we get the state-equations as

$$\begin{cases} \bar{v}_{\bar{t}} - \bar{u}_{\bar{x}} = 0, \\ \bar{u}_{\bar{t}} + p(\bar{v})_{\bar{x}} = -\dfrac{\mu}{1+\bar{t}}\bar{u}. \end{cases}$$

This means, in the critical case $\lambda = 1$, the hyperbolicity and the damping effect (degenerate parabolicity) both cannot be neglected, and the asymptotic profiles should possess both hyperbolicity and degenerate parabolicity.

Now we are going to artfully construct the possible asymptotic profile. We restrict ourselves to $v_+ = v_- := \underline{v}$, but u_+ and u_- may not be equal.

Let us consider the following linear system of equations:

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ \bar{u}_t + p'(\underline{v})\bar{v}_x = -\dfrac{\mu}{1+t}\bar{u}, \\ (\bar{v}_0, \bar{u}_0)(x,0) = (\bar{v}_0, \bar{u}_0)(x) \to (\underline{v}, u_\pm) \quad \text{as} \quad x \to \pm\infty. \end{cases} \tag{1.8}$$

This will be the expected asymptotic profile with particularly selected initial data $(\bar{v}_0, \bar{u}_0)(x)$ in two different cases.

Case 1*: The general case on $v_0(x)$ and $u_\pm$, where $u_\pm$ may or may not be equal, and $v_0(x) - \underline{v}$ is Riemann integrable over* $\mathbb{R}$. Technically, in this case the initial data $\bar{v}_0(x)$ is chosen as

$$\bar{v}_0(x) = \underline{v} + \kappa\phi_0(x), \tag{1.9}$$

where $\phi_0(x)$ is a given smooth function such that

$$\phi_0(x) \in L^1(\mathbb{R}) \quad \text{and} \quad \int_{-\infty}^{+\infty} \phi_0(x)dx \neq 0,$$

and κ is a constant such that

$$\int_{-\infty}^{+\infty} [v_0(x) - \underline{v}]dx - \kappa \int_{-\infty}^{+\infty} \phi_0(x)dx = 0,$$

namely,

$$\kappa := \left(\int_{-\infty}^{+\infty} [v_0(x) - \underline{v}]dx \right) \Big/ \int_{-\infty}^{+\infty} \phi_0(x)dx, \tag{1.10}$$

which is nonzero or zero when $\int_{-\infty}^{+\infty} [v_0(x) - \underline{v}]dx \neq 0$ or $= 0$.

The problem (1.8) can be reduced to

$$\begin{cases} \bar{v}_{tt} + p'(\underline{v})\bar{v}_{xx} + \dfrac{\mu}{1+t}\bar{v}_t = 0, \\ (\bar{v}, \bar{v}_t)(x,0) = (\bar{v}_0, \bar{u}_0')(x). \end{cases} \tag{1.11}$$

This wave equation with time-degenerate damping is well studied by Wirth in [38, 39, 40]. Once the solution $\bar{v}(x,t)$ is obtained from (1.11), then $\bar{u}(x,t)$ can be solved from the second equation of (1.8) as follows:

$$\bar{u}(x,t) = (1+t)^{-\mu}\bar{u}_0(x) - (1+t)^{-\mu} \int_0^t (1+s)^{\mu} p'(\underline{v})\bar{v}_x(x,s)ds. \tag{1.12}$$

Thus, we derive the expected asymptotic profile $(\bar{v}, \bar{u})(x,t)$ for Case 1.

Case 2: The special case with $u_+ = u_-$ and $\int_{-\infty}^{\infty}[v_0(x) - \underline{v}]dx = 0$. In order to look for a better convergence to a certain asymptotic profile, let us consider this special case with $u_\pm$ and $v_0(x)$ satisfying the abovementioned conditions. We expect

$$\bar{v}(x,t) = \underline{v}. \tag{1.13}$$

Then, from (1.8), we have

$$\bar{u}_x = 0, \quad \bar{u}_t = -\frac{\mu}{1+t}\bar{u},$$

which immediately implies

$$\bar{u}(x,t) = \underline{u}(1+t)^{-\mu}. \tag{1.14}$$

So the expected asymptotic profile in this special case is

$$(\bar{v},\bar{u})(x,t) = \left(\underline{v}, \underline{u}(1+t)^{-\mu}\right),$$

and we will show a better convergence of solution for the original IVP to this asymptotic profile later.

Notation. Before stating our main results, let us introduce some notation. Throughout this paper the symbol C will be used to represent a generic constant which is independent of x and t and may vary from line to line. $\|\cdot\|_{L^p}$ and $\|\cdot\|_l$ stand for the $L^p(\mathbb{R})$-norm $(1 \le p \le \infty)$ and $H^l(\mathbb{R})$-norm, respectively. The L^2-norm on $\mathbb{R}$ is simply denoted by $\|\cdot\|$. Moreover, we also use $\int f(x)dx := \int_{\mathbb{R}} f(x)dx$, the domain $\mathbb{R}$ will often be abbreviated without confusions.

Property of asymptotic profiles. Now we are going to state the property of the asymptotic profiles in the critical case of $\lambda = 1$ with $\mu > 2$. The main point here is to derive the optimal decay rates for Case 1, because the decay rates for Case 2 are obvious. Note that by using Fourier analysis, Wirth [38] obtained the optimal decay rates only for the gradients of the solutions of (1.11) in the form

$$\|(\bar{v}_t, \bar{v}_x)(t)\|_{L^2} = O(1)(1+t)^{-1}.$$

See also the decay rates for higher order derivatives obtained in [32],

$$\|\partial_t^i \partial_x^j \bar{v}(t)\|_{L^2} = O(1)(1+t)^{-1} \quad \text{for } i+j \ge 1,$$

but these rates in the higher order case are not sufficient. Here we are going to give the optimal L^2-convergence rates for all cases, where the rates are related to the physical quantity μ as follows. The proof of the following theorem, based on the time-weighted L^2-energy estimates with trickily selected time-weights related to the physical parameter μ, will be done in the appendix later.

THEOREM 1.1 (property of asymptotic profiles for Case 1). *Suppose $(\phi_0, u_0') \in H^3(\mathbb{R}) \times H^2(\mathbb{R})$. Then, the Cauchy problem* (1.11) *admits a unique global smooth solution $\bar{v}(x,t)$, which satisfies, for $2 < \mu \le 4$,*

$$\sum_{0 \le i+j \le 1} (1+t)^{2(i+j)} \|\partial_x^i \partial_t^j (\bar{v} - \underline{v})(t)\|^2 + (1+t)^{\mu} \|(\bar{v}_{xx}, \bar{v}_{xt})(t)\|_1^2$$
$$\le C\left(\|\phi_0\|_3^2 + \|u_0'\|_2^2\right) := C_0, \tag{1.15}$$

and for $\mu > 4$,

$$\sum_{0 \le i+j \le 1} (1+t)^{2(i+j)} \|\partial_x^i \partial_t^j (\bar{v} - \underline{v})(t)\|^2 + (1+t)^4 \|(\bar{v}_{xx}, \bar{v}_{xt})(t)\|_1^2 \le C_0. \tag{1.16}$$

Using the Sobolev inequality

$$\|f\|_{L^\infty} \leq \sqrt{2}\|f\|^{\frac{1}{2}}\|f_x\|^{\frac{1}{2}},$$

one can further derive the following estimates.

COROLLARY 1.2. *Under the assumptions of Theorem* 1.1*, one has, for* $2 < \mu \leq 4$*,*

$$\begin{aligned} \|(\bar{v} - \underline{v})(t)\|_{L^2} &= O(1), \\ \|(\bar{v} - \underline{v})(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{1}{2}}, \\ \|(\bar{v}_x, \bar{v}_t)(t)\|_{L^2} &= O(1)(1+t)^{-1}, \\ \|(\bar{v}_x, \bar{v}_t)(t)\|_{L^\infty} &= O(1)(1+t)^{-(\frac{1}{2}+\frac{\mu}{4})}, \\ \|(\bar{v}_{xx}, \bar{v}_{xt})(t)\|_{L^2\cap L^\infty} &= O(1)(1+t)^{-\frac{\mu}{2}}, \end{aligned} \tag{1.17}$$

and for $\mu > 4$*,*

$$\begin{aligned} \|(\bar{v} - \underline{v})(t)\|_{L^2} &= O(1), \\ \|(\bar{v} - \underline{v})(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{1}{2}}, \\ \|(\bar{v}_x, \bar{v}_t)(t)\|_{L^2} &= O(1)(1+t)^{-1}, \\ \|(\bar{v}_x, \bar{v}_t)(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{3}{2}}, \\ \|(\bar{v}_{xx}, \bar{v}_{xt})(t)\|_{L^2\cap L^\infty} &= O(1)(1+t)^{-2}. \end{aligned} \tag{1.18}$$

Remark 1.3. The rates of $\|(\bar{v}_x, \bar{v}_t)(t)\|_{L^2} = O(1)t^{-1}$ shown in (1.17), (1.18) are the same as that obtained by Wirth in [38]; therefore these rates are optimal in the sense of L^2-integrable initial data. However, based on the technical time-weighted energy method, we further obtained the decays for $\|(\bar{v}_{xx}, \bar{v}_{xt})(t)\|$ and $\|(\bar{v}_x, \bar{v}_t)(t)\|_{L^\infty}$. Remarkably, these decay rates are, for $2 < \mu \leq 4$, dependent of physical parameter μ, and for $\mu > 4$, independent of μ.

Convergence to asymptotic profiles. We are going to state the convergence results in Case 1 and Case 2, respectively.

Case 1*: The general case on* $u_\pm$ *and* $v_0(x)$. Let $(\bar{v}, \bar{u})(x,t)$ be the asymptotic profiles $(\bar{v}, \bar{u})(x,t)$ constructed in (1.11) and (1.12). Subtracting the system (1.8) from the original system (1.1) for the critical case $\lambda = 1$, we see that

$$\{v(x,t) - \bar{v}(x,t)\}_t - \{u(x,t) - \bar{u}(x,t)\}_x = 0.$$

Then the integration of the above equation in x over $(-\infty, +\infty)$ yields

$$\int_{-\infty}^{+\infty} (v - \bar{v})(x,t)dx = \int_{-\infty}^{+\infty} [v_0(x) - \underline{v}]dx - \kappa \int_{-\infty}^{+\infty} \phi_0(x)dx = 0.$$

Thus, it is reasonable to introduce the following perturbations as our new variables:

$$\begin{aligned} V(x,t) &= \int_{-\infty}^{x} (v(y,t) - \bar{v}(y,t))dy, \\ z(x,t) &= u(x,t) - \bar{u}(x,t). \end{aligned} \tag{1.19}$$

Clearly, we see

$$\begin{cases} (v - \bar{v})_t - (u - \bar{u})_x = 0, \\ (u - \bar{u})_t + p(v)_x - p'(\underline{v})\bar{v}_x + \dfrac{\mu}{1+t}(u - \bar{u}) = 0. \end{cases} \tag{1.20}$$

From (1.20), we have the reformulated problem

$$\begin{cases} V_t - z = 0, \\ z_t + p(V_x + \bar{v})_x - p'(\underline{v})\bar{v}_x + \dfrac{\mu}{1+t} z = 0, \end{cases} \tag{1.21}$$

with initial data

$$(V, z)(x, 0) = (V_0, z_0)(x), \tag{1.22}$$

where

$$\begin{cases} V_0(x) := \int_{-\infty}^{x} (v_0(y) - \bar{v}(y, 0)) dy, \\ z_0(x) := u_0(x) - \bar{u}_0(x). \end{cases} \tag{1.23}$$

Substituting $(1.21)_1$ to $(1.21)_2$ gives

$$\begin{cases} V_{tt} + (p'(\bar{v})V_x)_x + \dfrac{\mu}{1+t} V_t = F_{1x} + F_2, \\ (V, V_t)(x, 0) = (V_0, z_0)(x), \end{cases} \tag{1.24}$$

where

$$\begin{cases} F_1 = -(p(V_x + \bar{v}) - p(\bar{v}) - p'(\bar{v})V_x), \\ F_2 = -(p'(\bar{v}) - p'(\underline{v}))\bar{v}_x. \end{cases} \tag{1.25}$$

The following are the first main results of the paper.

THEOREM 1.4 (convergence to asymptotic profiles for Case 1). *Let $(\bar{v}, \bar{u})(x, t)$ be the derived asymptotic profile in* (1.11) *and* (1.12), *and the initial perturbations $(V_0, z_0) \in H^3(\mathbb{R}) \times H^2(\mathbb{R})$. Then, there is a number $\delta_0 > 0$ such that if $\|V_0\|_3 + \|z_0\|_2 + C_0 \le \delta_0$, where C_0 is defined in* (1.15), *then the Cauchy problem of* (1.1) *and* (1.2) *in the critical case $\lambda = 1$ with $\mu > 2$ admits a unique global smooth solution $(v, u)(x, t)$ such that $v - \bar{v}$ and $u - \bar{u}$ both are in $C(0, \infty; H^2(\mathbb{R}))$ and satisfy, for $2 < \mu \le 4$,*

$$\begin{aligned} &(1+t)\|(v-\bar{v})(t)\|^2 + (1+t)^{\mu-1} \sum_{k=1}^{2} \|\partial_x^k (v-\bar{v})(t)\|^2 \\ &\quad + (1+t)\|(u-\bar{u})(t)\|^2 + (1+t)^{\mu-1} \sum_{k=1}^{2} \|\partial_x^k (u-\bar{u})(t)\|^2 \\ &\le C\left(\|V_0\|_3^2 + \|z_0\|_2^2 + C_0\right), \end{aligned} \tag{1.26}$$

and for $\mu > 4$

$$\begin{aligned} &(1+t)\|(v-\bar{v})(t)\|^2 + (1+t)^{3} \sum_{k=1}^{2} \|\partial_x^k (v-\bar{v})(t)\|^2 \\ &\quad + (1+t)\|(u-\bar{u})(t)\|^2 + (1+t)^{3} \sum_{k=1}^{2} \|\partial_x^k (u-\bar{u})(t)\|^2 \\ &\le C(\|V_0\|_3^2 + \|z_0\|_2^2 + C_0). \end{aligned} \tag{1.27}$$

Using the Sobolev inequality, one can further derive the following estimates.

COROLLARY 1.5. *Under the assumptions of Theorem* 1.4*, one has, for* $2 < \mu \leq 4$,

$$\begin{aligned} \|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{\mu}{4}}, \\ \|(v_x-\bar{v}_x, u_x-\bar{u}_x)(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{\mu-1}{2}}, \end{aligned} \tag{1.28}$$

and for $\mu > 4$,

$$\begin{aligned} \|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} &= O(1)(1+t)^{-1}, \\ \|(v_x-\bar{v}_x, u_x-\bar{u}_x)(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{3}{2}}. \end{aligned} \tag{1.29}$$

Remark 1.6. From the linear equation (A.1), it can be seen that the convergence rates shown in Theorem 1.1 are optimal in the sense of L^2-integrable initial data. Therefore, the convergences obtained in Theorem 1.4 as well as in Corollary 1.5 are also optimal in the L^2-sense and the L^∞-sense, respectively.

Remark 1.7. Theorem 1.4 shows that the convergent profile is the solution of the linear equations (1.11) for the case of $v_+ = v_-$. For the case of $v_+ \neq v_-$, the construction of the asymptotic profiles is more complicated and totally different from the case of $v_+ = v_-$. We will leave this case for upcoming work.

Case 2*: The special case with* $u_+ = u_-$ *and* $\int_{-\infty}^{\infty}[v_0(x) - \underline{v}]dx = 0$. In this case, the expected asymptotic profile is given in (1.13) and (1.14), which also satisfies (1.8). From (1.1) and (1.8), we have

$$(v-\underline{v})_t - (u-\bar{u})_x = 0. \tag{1.30}$$

Consider the second equation of (1.1) for the critical case $\lambda = 1$ at far fields $x = \pm\infty$, we formally have

$$\frac{d}{dt}u(\pm\infty, t) = -\frac{\mu}{1+t}u(\pm\infty, t), \quad u(\pm\infty, 0) = \underline{u},$$

which implies

$$u(\pm\infty, t) = \underline{u}(1+t)^{-\mu}.$$

On the other hand, it is clear that

$$\bar{u}(\pm\infty, t) = \underline{u}(1+t)^{-\mu}.$$

Thus, after integrating (1.30) with respect to x over $(-\infty, \infty)$, we have

$$\frac{d}{dt}\int_{-\infty}^{\infty}[v(x,t) - \underline{v}]dx = [u-\bar{u}]|_{-\infty}^{\infty} = 0. \tag{1.31}$$

Since

$$\int_{-\infty}^{\infty}[v_0(x) - \underline{v}]dx = 0,$$

then (1.31) implies

$$\int_{-\infty}^{\infty}[v(x,t) - \underline{v}]dx = \int_{-\infty}^{\infty}[v_0(x) - \underline{v}]dx = 0.$$

Thus, we reasonably set

$$\begin{aligned} V(x,t) &= \int_{-\infty}^{x}(v(y,t) - \underline{v})dy, \\ z(x,t) &= u(x,t) - \bar{u}(x,t), \end{aligned} \tag{1.32}$$

to obtain the reformulated problem

$$\begin{cases} V_t - z = 0, \\ z_t + p(V_x + \underline{v})_x + \dfrac{\mu}{1+t} z = 0, \end{cases} \tag{1.33}$$

with initial data

$$(V, z)(x, 0) = \left(\int_{-\infty}^{x} [v_0(y) - \underline{v}] dy, u_0(x) - \bar{u}(x, 0) \right) := (V_0, z_0)(x), \tag{1.34}$$

or

$$\begin{cases} V_{tt} + p'(\underline{v}) V_{xx} + \dfrac{\mu}{1+t} V_t = f_x, \\ (V, V_t)(x, 0) = (V_0, z_0)(x), \end{cases} \tag{1.35}$$

where

$$f = -(p(V_x + \underline{v}) - p(\underline{v}) - p'(\underline{v}) V_x). \tag{1.36}$$

The following are the second results of the paper.

THEOREM 1.8 (convergence to asymptotic profile for Case 2). *Let* $(\underline{v}, \bar{u}(x,t))$ *be the expected asymptotic profile specified in* (1.13) *and* (1.14)*, and the initial perturbations* $(V_0, z_0) \in H^3(\mathbb{R}) \times H^2(\mathbb{R})$*. Then, there is a number* $\delta_0 > 0$ *such that if* $\|V_0\|_3 + \|z_0\|_2 \le \delta_0$*, then the Cauchy problem of* (1.1) *and* (1.2) *in the critical case* $\lambda = 1$ *with* $\mu > 2$ *admits a unique global smooth solution* $(v, u)(x, t)$ *satisfying* $v - \underline{v}$*,* $u - \bar{u} \in C(0, \infty; H^2(\mathbb{R}))$*, and, for* $2 < \mu \le 4$*,*

$$\begin{aligned} &(1+t)^2 \|(v - \underline{v})(t)\|^2 + (1+t)^{\mu} \sum_{k=1}^{2} \left\| \partial_x^k (v - \underline{v})(t) \right\|^2 \\ &\quad + (1+t)^2 \|(u - \bar{u})(t)\|^2 + (1+t)^{\mu} \sum_{k=1}^{2} \left\| \partial_x^k (u - \bar{u})(t) \right\|^2 \\ &\le C \left(\|V_0\|_3^2 + \|z_0\|_2^2 \right), \end{aligned} \tag{1.37}$$

and for $\mu > 4$*,*

$$\begin{aligned} &(1+t)^2 \|(v - \underline{v})(t)\|^2 + (1+t)^{4} \sum_{k=1}^{2} \left\| \partial_x^k (v - \underline{v})(t) \right\|^2 \\ &\quad + (1+t)^2 \|(u - \bar{u})(t)\|^2 + (1+t)^{4} \sum_{k=1}^{2} \left\| \partial_x^k (u - \bar{u})(t) \right\|^2 \\ &\le C \left(\|V_0\|_3^2 + \|z_0\|_2^2 \right). \end{aligned} \tag{1.38}$$

Remark 1.9. It is clear that the convergence rates obtained in the special Case 2 are better than for Case 1. In fact, in Case 2, we don't have a term like F_2 in Case 1 for the perturbation equation, because the term F_2 usually causes the decay estimates a bit slow.

We can further derive the following estimates.

COROLLARY 1.10. *Under the assumptions of Theorem* 1.8*, one has, for* $2 < \mu \le 4$*,*

$$\begin{aligned} \|(v-\underline{v},u-\bar{u})(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{1}{2}-\frac{\mu}{4}}, \\ \|(v_x,u_x)(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{\mu}{2}}, \end{aligned} \tag{1.39}$$

and for $\mu > 4$,

$$\begin{aligned} \|(v-\underline{v},u-\bar{u})(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{3}{2}}, \\ \|(v_x,u_x)(t)\|_{L^\infty} &= O(1)(1+t)^{-2}. \end{aligned} \tag{1.40}$$

Remark 1.11. The rates shown in (1.39) and (1.40) are much better than the previous studies [4, 32, 35, 38] for the decay rates around the constant states.

The rest of this paper is organized as follows. In section 2, we will prove the convergence of the solution (v,u) for the Cauchy problem (1.1), (1.2) in the critical case of $\lambda = 1$ with $\mu > 2$ to the solution of the problem (1.8). The convergence rates (1.15), (1.16) of the problem (1.8) will be established in the appendix.

2. Proof of Theorem 1.4. In this section, we will prove the global existence of smooth solutions to Cauchy problems (1.21) and (1.33). As in [24, 27], the local existence of solutions can be obtained by the iteration method; thus, we only need to establish the a priori estimates under the following a priori assumption.

Let $T \in (0,\infty]$; we define the solution space for

$$X(T) = \left\{ V(x,t) \middle| \partial_t^j V \in C\left(0,T;H^{3-j}(\mathbb{R})\right), j=0,1, \quad 0 \le t \le T \right\}$$

with the norm for $2 < \mu \le 4$,

$$N(T)^2 = \sup_{0\le t\le T} \left\{ \sum_{0\le k+l\le 1} (1+t)^{2(k+l)-1} \left\|\partial_x^k\partial_t^l V(t)\right\|^2 + (1+t)^{\mu-1}\|(V_{xx},V_{xt})(t)\|_1^2 \right\}, \tag{2.1}$$

and for $\mu > 4$,

$$N(T)^2 = \sup_{0\le t\le T} \left\{ \sum_{0\le k+l\le 1} (1+t)^{2(k+l)-1} \left\|\partial_x^k\partial_t^l V(t)\right\|^2 + (1+t)^{3}\|(V_{xx},V_{xt})(t)\|_1^2 \right\}. \tag{2.2}$$

LEMMA 2.1. *Under the assumptions of Theorem* 1.4*, it holds that*

$$\begin{aligned} &\|V(t)\|^2 + (1+t)^2\|(V_x,V_t)(t)\|^2 + \int_0^t (1+\tau)\|(V_x,V_t)(\tau)\|^2 d\tau \\ &\quad \le C\left(\|V_0\|_1^2 + \|V_1\|^2 + C_0\right)(1+t) := C_1(1+t), \end{aligned} \tag{2.3}$$

provided that $N(T) \ll 1$.

Proof. Multiplying (1.24) by $(1+t)^2 V_t$ and integrating it over $\mathbb{R}$ and using integration by parts give

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int (1+t)^2\left(V_t^2 - p'(\bar{v})V_x^2\right)dx + (\mu-1)\int (1+t)V_t^2 dx + \int (1+t)p'(\bar{v})V_x^2 dx \\ &\quad = -\frac{1}{2}\int (1+t)^2 p''(\bar{v})\bar{v}_t V_x^2 dx + \int (1+t)^2(F_{1x}+F_2)V_t dx. \end{aligned} \tag{2.4}$$

Also multiplying (1.24) by $(1+t)V$ and integrating it over $\mathbb{R}$, we then have

$$\frac{1}{2}\frac{d}{dt}\int\left[2(1+t)VV_t+(\mu-1)V^2\right]dx-\int(1+t)p'(\bar{v})V_x^2dx-\int(1+t)V_t^2dx \\ =\int(1+t)(F_{1x}+F_2)V\,dx. \tag{2.5}$$

Adding (2.5) and $\eta\cdot$(2.4) with η to be determined later, and integrating the resultant equation over $[0,t]$, we then get

$$\begin{aligned} &\frac{1}{2}\int\left[\eta(1+t)^2\left(V_t^2-p'(\bar{v})V_x^2\right)+2(1+t)VV_t+(\mu-1)V^2\right]dx \\ &\quad+(\eta(\mu-1)-1)\int_0^t\int(1+\tau)V_t^2dxd\tau-(1-\eta)\int_0^t\int(1+\tau)p'(\bar{v})V_x^2dxd\tau \\ &\quad\le C\left(\|V_0\|_1^2+\|V_1\|^2\right)-\frac{1}{2}\eta\int_0^t\int(1+\tau)^2p''(\bar{v})\bar{v}_tV_x^2dxd\tau \\ &\quad+\eta\int_0^t\int(1+\tau)^2(F_{1x}+F_2)V_tdxd\tau+\int_0^t\int(1+\tau)(F_{1x}+F_2)V\,dxd\tau. \end{aligned} \tag{2.6}$$

Using the Cauchy–Schwarz inequality, we have

$$2\int(1+t)VV_tdx\ge-\frac{4(\mu-1)}{\mu+2}\int V^2dx-\frac{\mu+2}{4(\mu-1)}\int(1+t)^2V_t^2dx. \tag{2.7}$$

From (2.6) and (2.7) by choosing $\eta=\frac{\mu}{2(\mu-1)}$, we get

$$\begin{aligned} &\frac{\mu-2}{8(\mu-1)}\int(1+t)^2V_t^2-\frac{\mu}{4(\mu-1)}\int(1+t)^2p'(\bar{v})V_x^2dx+\frac{(\mu-1)(\mu-2)}{2(\mu+2)}\int V^2dx \\ &\quad+\frac{\mu-2}{2}\int_0^t\int(1+\tau)V_t^2dxd\tau-\frac{\mu-2}{2(\mu-1)}\int_0^t\int(1+\tau)p'(\bar{v})V_x^2dxd\tau \\ &\quad\le C\left(\|V_0\|_1^2+\|V_1\|^2\right)-\frac{\mu}{4(\mu-1)}\int_0^t\int(1+\tau)^2p''(\bar{v})\bar{v}_tV_x^2dxd\tau \\ &\quad+\frac{\mu}{2(\mu-1)}\int_0^t\int(1+\tau)^2(F_{1x}+F_2)V_tdxd\tau \\ &\quad+\int_0^t\int(1+\tau)(F_{1x}+F_2)V\,dxd\tau. \end{aligned} \tag{2.8}$$

So we have

$$\int V^2dx+\int(1+t)^2\left(V_x^2+V_t^2\right)dx+\int_0^t\int(1+\tau)\left(V_x^2+V_t^2\right)dxd\tau \\ \le C\left(\|V_0\|_1^2+\|V_1\|^2\right)+CI, \tag{2.9}$$

where C depends on μ and

$$I=\left|\int_0^t\int(1+\tau)^2p''(\bar{v})\bar{v}_tV_x^2dxd\tau\right|+\left|\int_0^t\int(1+\tau)F_2V\,dxd\tau\right| \\ +\left|\int_0^t\int(1+\tau)^2F_2V_tdxd\tau\right|+\left|\int_0^t\int(1+\tau)F_{1x}V\,dxd\tau\right|$$

$$+\left|\int_0^t\int(1+\tau)^2F_{1x}V_tdxd\tau\right|$$

$$=:\sum_{1=1}^{5}I_i. \tag{2.10}$$

From (1.17) and (1.18), we get

$$\begin{aligned} I_1 &\le C\int_0^t\int(1+\tau)^2|\bar v_t|V_x^2dxd\tau \\ &\le C_0\int_0^t\int(1+\tau)^{2-\gamma}V_x^2dxd\tau, \end{aligned} \tag{2.11}$$

where

$$\gamma=\begin{cases}\frac{1}{2}+\frac{\mu}{4} & \text{for } \ 2<\mu\le 4,\\ \frac{3}{2} & \text{for } \ \mu>4.\end{cases}$$

Noting $\mu>2$, we have $\gamma>1$. By using Theorem 1.1 and a priori assumptions (2.1) and (2.2), one gets

$$\begin{aligned} I_2 &= \int_0^t\int(1+\tau)(p'(\bar v)-p'(\underline{v}))\bar v_xV\,dxd\tau \\ &\le C\int_0^t(1+\tau)\|\bar v-\underline{v}\|_{L^\infty}\|\bar v_x\|\|V\|d\tau \\ &\le C_0(1+t), \end{aligned} \tag{2.12}$$

$$\begin{aligned} I_3 &= \int_0^t\int(1+\tau)^2\left(p'(\bar v)-p'(\underline{v})\right)\bar v_xV_tdxd\tau \\ &= \left|\int_0^t\int\Big\{\partial_\tau\left[(1+\tau)^2(p'(\bar v)-p'(\underline{v}))\bar v_xV\right]-2(1+\tau)\left(p'(\bar v)-p'(\underline{v})\right)\bar v_xV\right. \\ &\qquad\left.-(1+\tau)^2\left[p''(\bar v)\bar v_t\bar v_x+\left(p'(\bar v)-p'(\underline{v})\right)\bar v_{xt}\right]V\Big\}dxd\tau\right| \\ &\le\left|\int(1+t)^2\left(p'(\bar v)-p'(\underline{v})\right)\bar v_xV\,dx\right|+C_0 \\ &\quad+2\left|\int_0^t\int(1+\tau)\left(p'(\bar v)-p'(\underline{v})\right)\bar v_xV\,dxd\tau\right| \\ &\quad+\left|\int_0^t\int(1+\tau)^2\left[p''(\bar v)\bar v_t\bar v_x+\left(p'(\bar v)-p'(\underline{v})\right)\bar v_{xt}\right]V\,dxd\tau\right| \\ &\le C(1+t)^2\|\bar v-\underline{v}\|_{L^\infty}\|\bar v-\underline{v}\|\|V\|+C_0 \\ &\quad+C\int_0^t(1+\tau)\|\bar v-\underline{v}\|_{L^\infty}\|\bar v_x\|\|V\|d\tau \\ &\quad+C\int_0^t\int(1+\tau)^2\left(\|\bar v_t\|\|\bar v_x\|+\|\bar v-\underline{v}\|\|\bar v_{xt}\|\right)\|V\|_{L^\infty}d\tau \\ &\le C_0(1+t), \end{aligned} \tag{2.13}$$

$$\begin{aligned} I_4 &= \left|\int_0^t\int(1+\tau)\left(p(V_x+\bar v)-p(\underline{v})-p'(\underline{v})V_x\right)V_xdxd\tau\right| \\ &\le CN(T)\int_0^t\int(1+\tau)V_x^2dxd\tau, \end{aligned} \tag{2.14}$$

and

$$
\begin{aligned}
I_5 &= \left| \int_0^t \int (1+\tau)^2 \left(p(V_x+\bar v) - p(\bar v) - p'(\underline{v})V_x\right) V_{xt} dx d\tau \right| \\
&= \left| \int_0^t \int \left\{ \partial_\tau \left[(1+\tau)^2 \left(\int_{\bar v}^{V_x+\bar v} p(s)ds - p(\bar v)V_x - \frac{1}{2}p'(\bar v)V_x^2 \right) \right] \right.\right. \\
&\quad - 2(1+\tau) \left(\int_{\bar v}^{V_x+\bar v} p(s)ds - p(\bar v)V_x - \frac{1}{2}p'(\bar v)V_x^2 \right) \\
&\quad \left.\left. - \bar v_t (1+\tau)^2 \left(p(V_x+\bar v) - p(\bar v) - p'(\bar v)V_x - \frac{1}{2}p''(\bar v)V_x^2 \right) \right\} dx d\tau \right| \\
&\le (1+t)^2 \left| \int \left(\int_{\bar v}^{V_x+\bar v} p(s)ds - p(\bar v)V_x - \frac{1}{2}p'(\bar v)V_x^2 \right) dx \right| + C\|V_0\|_1^2 \\
&\quad + \left| \int_0^t \int (1+\tau) \left(\int_{\bar v}^{V_x+\bar v} p(s)ds - p(\bar v)V_x - \frac{1}{2}p'(\bar v)V_x^2 \right) dx d\tau \right| \\
&\quad + \left| \int_0^t \int (1+\tau)^2 \left(p(V_x+\bar v) - p(\bar v) - p(\bar v)V_x - \frac{1}{2}p''(\bar v)V_x^2 \right) \bar v_t dx d\tau \right| \\
&\le C(1+t)^2 \|V_x\|_{L^\infty} \|V_x\|^2 + C\|V_0\|_1^2 + C\int_0^t (1+\tau)\|V_x\|_{L^\infty}\|V_x\|^2 d\tau \\
&\quad + C\int_0^t (1+\tau)^2 \|\bar v_t\|_{L^\infty} \|V_x\|_{L^\infty} \|V_x\|^2 d\tau \\
&\le C(1+t)^2 N(T)\|V_x\|^2 + C\|V_0\|_1^2 + CN(T)\int_0^t (1+\tau)\|V_x\|^2 d\tau. \qquad (2.15)
\end{aligned}
$$

Inserting (2.10)–(2.15) into (2.9), using the smallness of $N(T)$, we have

$$
\begin{aligned}
&\|V(t)\|^2 + (1+t)^2 \left(\|V_x\|^2 + \|V_t\|^2\right) + \int_0^t (1+\tau)\left(\|V_x\|^2 + \|V_t\|^2\right) d\tau \\
&\quad \le C_0 \int_0^t \int (1+\tau)^{2-\gamma} V_x^2 dx d\tau + C\left(\|V_0\|_1^2 + \|V_1\|^2 + C_0\right)(1+t),
\end{aligned}
$$

and the Gronwall's inequality implies (2.3), since $\gamma > 1$. This completes the proof of Lemma 2.1. ∎

LEMMA 2.2. *Assume that $N(T) \ll 1$; then for $2 < \mu \le 4$*

$$
\begin{aligned}
&(1+t)^{\mu-\epsilon_0} \|(V_{xx}, V_{xt})(t)\|_1^2 + \epsilon_0 \int_0^t (1+\tau)^{\mu-1-\epsilon_0} \|(V_{xx}, V_{xt})(\tau)\|_1^2 d\tau \\
&\quad \le C\left(\|V_0\|_2^2 + \|V_1\|_1^2 + C_0\right)(1+t)^{1-\epsilon_0},
\end{aligned}
$$

and for $\mu > 4$,

$$
\begin{aligned}
&(1+t)^{4-\epsilon_0} \|(V_{xx}, V_{xt})(t)\|_1^2 + \epsilon_0 \int_0^t (1+\tau)^{3-\epsilon_0} \|(V_{xx}, V_{xt})(\tau)\|_1^2 d\tau \\
&\quad \le C(\|V_0\|_2^2 + \|V_1\|_1^2 + C_0)(1+t)^{1-\epsilon_0}.
\end{aligned}
$$

Proof. For $0 < \epsilon_0 < \min\{1, \mu-2\}$, multiplying $\partial_x(1.24)$ by $(1+t)^{\mu-\epsilon_0}V_{xt}$ and integrating it over $\mathbb{R}$, one then has

$$\begin{aligned}(2.16)\qquad &\frac{1}{2}\frac{d}{dt}\int(1+t)^{\mu-\epsilon_0}\left(V_{xt}^2 - p'(\underline{v})V_{xx}^2\right)dx + \frac{1}{2}(\mu+\epsilon_0)\int(1+t)^{\mu-1-\epsilon_0}V_{xt}^2dx\\ &+\frac{1}{2}(\mu-\epsilon_0)\int(1+t)^{\mu-1-\epsilon_0}p'(\bar{v})V_{xx}^2dx\\ &= -\int(1+t)^{\mu-\epsilon_0}\left(p''(\bar{v})\bar{v}_xV_x\right)_x V_{xt}dx - \frac{1}{2}\int(1+t)^{\mu-\epsilon_0}p''(\bar{v})\bar{v}_tV_{xx}^2dx\\ &+\int(1+t)^{\mu-\epsilon_0}(F_{1x}+F_2)_xV_{xt}dx.\end{aligned}$$

Also multiplying $\partial_x(1.24)$ by $(1+t)^{\mu-1-\epsilon_0}V_x$ and integrating it over $\mathbb{R}$ yields

$$\begin{aligned}(2.17)\qquad &\frac{1}{2}\frac{d}{dt}\int\left[2(1+t)^{\mu-1-\epsilon_0}V_xV_{xt}dx + (1+\epsilon_0)(1+t)^{\mu-2-\epsilon_0}V_x^2\right]dx\\ &-\int(1+t)^{\mu-1-\epsilon_0}V_{xt}^2dx - \int(1+t)^{\mu-1-\epsilon_0}p'(\bar{v})V_{xx}^2dx\\ &= \frac{1}{2}(1+\epsilon_0)(\mu-2-\epsilon_0)\int(1+t)^{\mu-3-\epsilon_0}V_x^2dx + \int(1+t)^{\mu-1-\epsilon_0}p''(\bar{v})\bar{v}_xV_xV_{xx}dx\\ &-\int(1+t)^{\mu-1-\epsilon_0}(F_{1x}+F_2)V_{xx}dx.\end{aligned}$$

Adding (2.17) and $\eta_1\cdot(2.16)$ with η_1 to be determined later, and integrating the resultant equation over $[0,t]$, it then follows that

$$\begin{aligned}&\frac{1}{2}\int\Big[(1+t)^{\mu-\epsilon_0}\eta_1\left(V_{xt}^2 - p'(\bar{v})V_{xx}^2\right) + 2(1+t)^{\mu-1-\epsilon_0}V_xV_{xt}\\ &\qquad + (1+\epsilon_0)(1+t)^{\mu-2-\epsilon_0}V_x^2\Big]dx\\ &+\left[\frac{1}{2}\eta_1(\mu+\epsilon_0)-1\right]\int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}V_{xt}^2dxd\tau\\ &-\left[1-\frac{1}{2}\eta_1(\mu-\epsilon_0)\right]\int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}p'(\bar{v})V_{xx}^2dxd\tau\\ &\le C\left(\|V_0\|_2^2+\|V_1\|_1^2\right) + \frac{1}{2}(1+\epsilon_0)(\mu-2-\epsilon_0)\int_0^t\int(1+\tau)^{\mu-3-\epsilon_0}V_x^2dxd\tau\\ &-\eta_1\int_0^t\int(1+\tau)^{\mu-\epsilon_0}\left[(p''(\bar{v})\bar{v}_xV_x)_xV_{xt} + \frac{1}{2}p''(\bar{v})\bar{v}_tV_{xx}^2\right]dxd\tau\\ &+\eta_1\int_0^t\int(1+\tau)^{\mu-\epsilon_0}(F_{1x}+F_2)_xV_{xt}dxd\tau\\ (2.18)\qquad &+\int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}\left[p''(\bar{v})\bar{v}_xV_x - (F_{1x}+F_2)\right]V_{xx}dxd\tau.\end{aligned}$$

Letting $\eta_1 = \frac{2}{\mu}$, using the Cauchy–Schwarz inequality, we have for $2 < \mu \le 4$

$$\begin{aligned}2\int(1+t)^{\mu-1-\epsilon_0}V_xV_{xt}dx &\ge -\frac{1}{\mu}\int(1+t)^{\mu-\epsilon_0}V_{xt}^2dx - \mu\int(1+t)^{\mu-2-\epsilon_0}V_x^2dx\\ (2.19)\qquad &\ge -\frac{1}{\mu}\int(1+t)^{\mu-\epsilon_0}V_{xt}^2dx - C_1(1+t)^{1-\epsilon_0}\end{aligned}$$

and

$$
\begin{aligned}
&\frac{1}{2}(1+\epsilon_0)(\mu-2-\epsilon_0)\int_0^t\int(1+\tau)^{\mu-3-\epsilon_0}V_x^2dxd\tau\\
&\quad\le C\int_0^t(1+\tau)^{\mu-4-\epsilon_0}d\tau\\
&\quad\le C_1(1+t)^{1-\epsilon_0}. \qquad (2.20)
\end{aligned}
$$

From (2.18), (2.19), and (2.20), we have

(2.21)

$$
\begin{aligned}
&\frac{1}{2\mu}\int\left[(1+t)^{\mu-\epsilon_0}\left(V_{xt}^2-2p'(\bar v)V_{xx}^2\right)+\mu(1+\epsilon_0)(1+t)^{\mu-2-\epsilon_0}V_x^2\right]dx\\
&\quad+\frac{\epsilon_0}{\mu}\int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}V_{xt}^2dxd\tau-\frac{\epsilon_0}{\mu}\int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}p'(\bar v)V_{xx}^2dxd\tau\\
&\le C\left(\|V_0\|_2^2+\|V_1\|_1^2\right)-\frac{2}{\mu}\int_0^t\int(1+\tau)^{\mu-\epsilon_0}\left[(p''(\bar v)\bar v_xV_x)_xV_{xt}+\frac{1}{2}p''(\bar v)\bar v_tV_{xx}^2\right]dxd\tau\\
&\quad+\frac{2}{\mu}\int_0^t\int(1+\tau)^{\mu-\epsilon_0}(F_{1x}+F_2)_xV_{xt}dxd\tau+C_1(1+t)^{1-\epsilon_0}\\
&\quad+\int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}\left[p''(\bar v)\bar v_xV_x-(F_{1x}+F_2)\right]V_{xx}dxd\tau.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\int(1+t)^{\mu-\epsilon_0}\left(V_{xx}^2+V_{xt}^2\right)dx+\int(1+t)^{\mu-2-\epsilon_0}V_x^2dx\\
&\quad+\epsilon_0\int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}\left(V_{xx}^2+V_{xt}^2\right)dxd\tau\\
&\le C\left(\|V_0\|_2^2+\|V_1\|_1^2+C_0\right)(1+t)^{1-\epsilon_0}+CJ, \qquad (2.22)
\end{aligned}
$$

where C depends on μ and

$$
\begin{aligned}
J&=\left|\int_0^t\int(1+\tau)^{\mu-\epsilon_0}\left[(p''(\bar v)\bar v_xV_x)_xV_{xt}+\frac{1}{2}p''(\bar v)\bar v_tV_{xx}^2\right]dxd\tau\right|\\
&\quad+\left|\int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}\left[p''(\bar v)\bar v_xV_x-F_{1x}-F_2\right]V_{xx}dxd\tau\right|\\
&\quad+\left|\int_0^t\int(1+\tau)^{\mu-\epsilon_0}(F_{1xx}+F_{2x})V_{xt}dxd\tau\right|\\
&=\left(J_1^1+J_1^2\right)+\left(J_2^1+J_2^2+J_2^3\right)+\left(J_3^1+J_3^2\right). \qquad (2.23)
\end{aligned}
$$

Similar to (2.11), we have

$$
\begin{aligned}
J_1^2&=\left|\int_0^t\int(1+\tau)^{\mu-\epsilon_0}p''(\bar v)\bar v_tV_{xx}^2dxd\tau\right| \qquad (2.24)\\
&\le C_0\int_0^t\int(1+\tau)^{\mu-\gamma-\epsilon_0}V_{xx}^2dxd\tau.
\end{aligned}
$$

For $2 < \mu \leq 4$, we estimate the other terms of the right-hand sides of (2.23) as follows:

$$\begin{aligned} J_1^1 &= \left| \int_0^t \int (1+\tau)^{\mu-\epsilon_0} (p''(\bar{v})\bar{v}_x V_x)_x V_{xt} dx d\tau \right| \\ &\leq C \int_0^t (1+\tau)^{\mu-\epsilon_0} [(\|\bar{v}_x\|_{L^\infty}^2 + \|\bar{v}_{xx}\|_{L^\infty})\|V_x\| + \|\bar{v}_x\|_{L^\infty}\|V_{xx}\|]\|V_{xt}\| d\tau \\ &\leq C \int_0^t (1+\tau)^{\mu-\gamma-\epsilon_0} (\|V_{xx}\|^2 + \|V_{xt}\|^2) d\tau + C_0(1+t)^{1-\epsilon_0}, \end{aligned} \tag{2.25}$$

$$\begin{aligned} J_2^1 &= \left| \int_0^t \int (1+\tau)^{\mu-1-\epsilon_0} p''(\bar{v})\bar{v}_x V_x V_{xx} dx d\tau \right| \\ &= \frac{1}{2} \left| \int_0^t \int (1+\tau)^{\mu-1-\epsilon_0} \left[p'''(\bar{v})\bar{v}_x^2 + p''(\bar{v})\bar{v}_{xx} \right] V_x^2 dx d\tau \right| \\ &\leq C \int_0^t (1+\tau)^{\mu-1-\epsilon_0} \left[\|\bar{v}_x\|_{L^\infty}^2 + \|\bar{v}_{xx}\|_{L^\infty} \right] \|V_x\|^2 d\tau \\ &\leq C_0(1+t)^{1-\epsilon_0}, \end{aligned} \tag{2.26}$$

$$\begin{aligned} J_2^2 &= \left| \int_0^t \int (1+\tau)^{\mu-1-\epsilon_0} (p(V_x+\bar{v}) - p(\bar{v}) - p'(\bar{v})V_x)_x V_{xx} dx d\tau \right| \\ &= \left| \int_0^t \int (1+\tau)^{\mu-1-\epsilon_0} (p'(V_x+\bar{v}) - p'(\bar{v})) V_{xx}^2 dx d\tau \right. \\ &\quad \left. + \int_0^t \int (1+\tau)^{\mu-1-\epsilon_0} \bar{v}_x (p'(V_x+\bar{v}) - p'(\bar{v}) - p''(\bar{v})V_x) V_{xx} dx d\tau \right| \\ &\leq \int_0^t (1+\tau)^{\mu-1-\epsilon_0} \left(\|V_x\|_{L^\infty}\|V_{xx}\|^2 + \|\bar{v}_x\|\|V_x\|_{L^\infty}^2\|V_{xx}\| \right) d\tau \\ &\leq C \int_0^t (1+\tau)^{\mu-\gamma-\epsilon_0} \|V_{xx}\|^2 d\tau + C_0, \end{aligned} \tag{2.27}$$

$$\begin{aligned} J_2^3 &= \left| \int_0^t \int (1+\tau)^{\mu-1-\epsilon_0} (p'(\bar{v}) - p'(\underline{v}))\bar{v}_x V_{xx} dx d\tau \right| \\ &\leq \int_0^t \int (1+\tau)^{\mu-1-\epsilon_0} \|\bar{v} - \underline{v}\|_{L^\infty} \|\bar{v}_x\| \|V_{xx}\| d\tau \\ &\leq C_0(1+t)^{1-\epsilon_0}, \end{aligned} \tag{2.28}$$

$$\begin{aligned} J_3^1 &= \left| \int_0^t \int (1+\tau)^{\mu-\epsilon_0} \left(p(V_x+\bar{v}) - p(\bar{v}) - p'(\bar{v})V_x \right)_x V_{xxt} dx d\tau \right| \\ &\leq \frac{1}{2} \left| \int (1+t)^{\mu-\epsilon_0} \left(p'(V_x+\bar{v}) - p'(\bar{v}) \right) V_{xx}^2 dx \right| + C\|V_0\|_2^2 \\ &\quad + \frac{\mu-\epsilon_0}{2} \left| \int_0^t \int (1+\tau)^{\mu-1-\epsilon_0} (p'(V_x+\bar{v}) - p'(\bar{v})) V_{xx}^2 dx d\tau \right| \\ &\quad + \frac{1}{2} \left| \int_0^t \int (1+\tau)^{\mu-\epsilon_0} p'(V_x+\bar{v}) V_{xt} V_{xx}^2 dx d\tau \right| \\ &\quad + \frac{1}{2} \left| \int_0^t \int (1+\tau)^{\mu-\epsilon_0} \left(p'(V_x+\bar{v}) - p'(\bar{v}) \right) \bar{v}_t V_{xx}^2 dx d\tau \right| \\ &\quad + \left| \int_0^t \int (1+\tau)^{\mu-\epsilon_0} \left[(p'(V_x+\bar{v}) - p'(\bar{v}) - p''(\bar{v})V_x)\bar{v}_x \right]_x V_{xt} dx d\tau \right| \end{aligned}$$

$$
\begin{aligned}
&\le C(1+t)^{\mu-\epsilon_0}\|V_x\|_{L^\infty}\|V_{xx}\|^2 + C\|V_0\|_2^2\\
&\quad + C\int_0^t (1+\tau)^{\mu-1-\epsilon_0}\|V_x\|_{L^\infty}\|V_{xx}\|^2 d\tau\\
&\quad + C\int_0^t (1+\tau)^{\mu-\epsilon_0}\left(\|V_{xt}\|_{L^\infty} + \|V_x\|_{L^\infty}\|\bar v_t\|_{L^\infty}\right)\|V_{xx}\|^2 d\tau\\
&\quad + C\int_0^t (1+\tau)^{\mu-\epsilon_0}\left(\|\bar v_{xx}\|_{L^\infty}\|V_x\| + \|\bar v_x\|_{L^\infty}\|V_{xx}\|\right.\\
&\qquad\quad \left.+\|\bar v_x\|_{L^\infty}^2\|V_x\|\right)\|V_x\|_{L^\infty}\|V_{xt}\| d\tau\\
&\le CN(T)(1+t)^{\mu-\epsilon_0}\|V_{xx}\|^2 + C\|V_0\|_2^2\\
&\quad + C\int_0^t (1+\tau)^{\mu-\gamma-\epsilon_0}\left(\|V_{xx}\|^2 + \|V_{xt}\|^2\right) d\tau + C_0,
\end{aligned}
\tag{2.29}
$$

and

$$
\begin{aligned}
J_3^2 &= \left|\int_0^t \int (1+\tau)^{\mu-\epsilon_0} F_{2x} V_{xt} dx d\tau\right|\\
&\le \left|\int (1+t)^{\mu-\epsilon_0} F_2 V_{xx} dx\right| + (\mu-\epsilon_0)\left|\int_0^t \int (1+\tau)^{\mu-1-\epsilon_0} F_2 V_{xx} dx d\tau\right|\\
&\quad + \left|\int (1+t)^{\mu-\epsilon_0} F_{2t} V_{xx} dx\right| + C\|V_0\|_2^2 + C_0\\
&\le C(1+t)^{\mu-\epsilon_0}\|\bar v - \underline{v}\|_{L^\infty}\|\bar v_x\|\|V_{xx}\| + C\|V_0\|_2^2 + C_0\\
&\quad + C\int_0^t (1+\tau)^{\mu-1-\epsilon_0}\|\bar v - \underline{v}\|_{L^\infty}\|\bar v_x\|\|V_{xx}\| d\tau\\
&\quad + C\int_0^t (1+\tau)^{\mu-\epsilon_0}[\|\bar v - \underline{v}\|_{L^\infty}\|\bar v_{xt}\| + \|\bar v_t\|_{L^\infty}\|\bar v_x\|\\
&\qquad\quad + \|\bar v_t\|_{L^\infty}\|\bar v_x\|\|\bar v_t\|]\|V_{xt}\| d\tau\\
&\le \frac{1}{4}(1+t)^{\mu-\epsilon_0}\|V_{xx}\|^2 + +C\|V_0\|_2^2 + C_0(1+t)^{1-\epsilon_0}.
\end{aligned}
\tag{2.30}
$$

Inserting (2.23)–(2.30) into (2.22), using the smallness of $N(T)$, we have

$$
\begin{aligned}
&(1+t)^{\mu-\epsilon_0}\|(V_{xx}, V_{xt})(t)\|_1^2 + \epsilon_0\int_0^t (1+\tau)^{\mu-1-\epsilon_0}\|(V_{xx}, V_{xt})(\tau)\|^2 d\tau\\
&\quad \le C\int_0^t (1+\tau)^{\mu-\gamma-\epsilon_0}\|(V_{xx}, V_{xt})(\tau)\|^2 d\tau\\
&\qquad + C\left(\|V_0\|_2^2 + \|V_1\|_1^2 + C_0\right)(1+t)^{1-\epsilon_0}.
\end{aligned}
\tag{2.31}
$$

Noting $\gamma > 1$, using the Gronwall's inequality, we have

$$
\begin{aligned}
&(1+t)^{\mu-\epsilon_0}\|(V_{xx}, V_{xt})(t)\|_1^2 + \epsilon_0\int_0^t (1+\tau)^{\mu-1-\epsilon_0}\|(V_{xx}, V_{xt})(\tau)\|^2 d\tau\\
&\quad \le C\left(\|V_0\|_2^2 + \|V_1\|_1^2 + C_0\right)(1+t)^{1-\epsilon_0}.
\end{aligned}
\tag{2.32}
$$

If $\mu > 4$, similar to the proof of (2.32), multiplying $\partial_x(1.24)$ by $[\frac{1}{4}(1+t)^{4-\epsilon_0}V_{xt} + (1+t)^{3-\epsilon_0}V_x]$ and integrating it, we can obtain

$$(1+t)^{4-\epsilon_0}\|(V_{xx},V_{xt})(t)\|_1^2+\epsilon_0\int_0^t(1+\tau)^{3-\epsilon_0}\|(V_{xx},V_{xt})(\tau)\|_1^2d\tau$$
$$\le C\left(\|V_0\|_2^2+\|V_1\|_1^2+C_0\right)(1+t)^{1-\epsilon_0}. \tag{2.33}$$

A similar calculation to (2.31) and (2.33) then yields the higher order estimate for $\|V_{xxx}(t)\|$ and $\|V_{xxt}(t)\|$. This completes the proof of Lemma 2.2. ∎

From Lemmas 2.1 and 2.2, we have for $2<\mu\le 4$,

$$(1+t)^{\mu-1}\|(V_{xx},V_{xt})(t)\|_1^2\le C(\|V_0\|_3^2+\|V_2\|_1^2+C_0), \tag{2.34}$$

and for $\mu>4$,

$$(1+t)^{3}\|(V_{xx},V_{xt})(t)\|_1^2\le C(\|V_0\|_3^2+\|V_1\|_2^2+C_0). \tag{2.35}$$

Thus, by using the continuity technique, we can verify that the a priori assumptions (2.1) and (2.2) are true, provided with $\|V_0\|_3^2+\|V_1\|_2^2+C_0$ sufficiently small. Hence, we have obtained the global existence to the problem (1.24). The proof of Theorem 1.4 is completed.

Proof of Theorem 1.8. Letting $T\in(0,\infty]$, we define the solution space for

$$X_1(T)=\left\{V(x,t)\Big|\partial_t^jV\in C\left(0,T;H^{3-j}(\mathbb{R})\right),j=0,1,\quad 0\le t\le T\right\}$$

with the norm for $2<\mu\le 4$,

$$N_1(T)^2=\sup_{0\le t\le T}\left\{\sum_{0\le k+l\le 1}(1+t)^{2(k+l)}\|\partial_x^k\partial_t^lV(t)\|^2+(1+t)^{\mu}\|(V_{xx},V_{xt})(t)\|_1^2\right\}, \tag{2.36}$$

and for $\mu>4$,

$$N_1(T)^2=\sup_{0\le t\le T}\left\{\sum_{0\le k+l\le 1}(1+t)^{2(k+l)}\|\partial_x^k\partial_t^lV(t)\|^2+(1+t)^{\mu}\|(V_{xx},V_{xt})(t)\|_1^2\right\}. \tag{2.37}$$

Similar to the proof of (2.8), multiplying (1.35) by $[\frac{\mu}{2(\mu-1)}(1+t)^2V_t+(1+t)V]$ and integrating it over $\mathbb{R}\times[0,t]$ and using integration by parts give

$$\begin{aligned}&\frac{\mu-2}{8(\mu-1)}\int(1+t)^2V_t^2-\frac{\mu}{4(\mu-1)}\int(1+t)^2p'(\underline{v})V_x^2dx+\frac{(\mu-1)(\mu-2)}{2(\mu+2)}\int V^2dx\\&\quad+\frac{\mu-2}{2}\int_0^t\int(1+\tau)V_t^2dxd\tau-\frac{\mu-2}{2(\mu-1)}\int_0^t\int(1+\tau)p'(\underline{v})V_x^2dxd\tau\\&\le C\left(\|V_0\|_1^2+\|V_1\|^2\right)+\frac{\mu}{2(\mu-1)}\int_0^t\int(1+\tau)^2f_xV_tdxd\tau+\int_0^t\int(1+\tau)f_xVdxd\tau.\end{aligned} \tag{2.38}$$

Using (2.36) and (2.37), we get

$$\begin{aligned}\left|\int_0^t\int(1+\tau)f_xVdxd\tau\right|&=\left|\int_0^t\int(1+\tau)(p(V_x+\underline{v})-p(\underline{v})-p'(\underline{v})V_x)V_xdxd\tau\right|\\&\le CN_1(T)\int_0^t\int(1+\tau)V_x^2dxd\tau\end{aligned} \tag{2.39}$$

and

$$
\begin{aligned}
&\left|\int_0^t\int(1+\tau)^2 f_x V_t dx d\tau\right| \\
&\quad = \left|\int_0^t\int(1+\tau)^2(p(V_x+\underline{v})-p(\underline{v})-p'(\underline{v})V_x)V_{xt}dxd\tau\right| \\
&\quad = \left|\int_0^t\int\left\{\partial_\tau\left[(1+\tau)^2\left(\int_{\underline{v}}^{V_x+\underline{v}}p(s)ds-p(\underline{v})V_x-\frac{1}{2}p'(\underline{v})V_x^2\right)\right]\right.\right. \\
&\qquad - 2(1+\tau)\left(\int_{\underline{v}}^{V_x+\underline{v}}p(s)ds-p(\underline{v})V_x-\frac{1}{2}p'(\underline{v})V_x^2\right) \\
&\quad \le (1+t)^2\left|\int\left(\int_{\underline{v}}^{V_x+\underline{v}}p(s)ds-p(\underline{v})V_x-\frac{1}{2}p'(\underline{v})V_x^2\right)dx\right| + C\|V_0\|_1^2 \\
&\qquad + \left|\int_0^t\int(1+\tau)\left(\int_{\underline{v}}^{V_x+\underline{v}}p(s)ds-p(\underline{v})V_x-\frac{1}{2}p'(\underline{v})V_x^2\right)dxd\tau\right| \\
&\quad \le C(1+t)^2\|V_x\|_{L^\infty}\|V_x\|^2 + C\|V_0\|_1^2 + C\int_0^t(1+\tau)\|V_x\|_{L^\infty}\|V_x\|^2 d\tau \\
&\quad \le C(1+t)^2 N_1(T)\|V_x\|^2 + C\|V_0\|_1^2 + CN_1(T)\int_0^t(1+\tau)\|V_x\|^2 d\tau. \qquad (2.40)
\end{aligned}
$$

Inserting (2.39) and (2.40) into (2.38), using the smallness of $N_1(T)$, we have

(2.41)

$$
\|V(t)\|^2 + (1+t)^2\|(V_x,V_t)(t)\|^2 + \int_0^t(1+\tau)\|(V_x,V_t)(\tau)\|^2 d\tau \le C(\|V_0\|_1^2 + \|V_1\|^2).
$$

For $0<\epsilon_0<\min\{1,\mu-2\}$ and $2<\mu\le 4$, multiplying $\partial_x(1.35)$ by $[\frac{2}{\mu}(1+t)^{\mu-\epsilon_0}V_{xt} + (1+t)^{\mu-1-\epsilon_0}V_x]$ and integrating it over $\mathbb{R}\times[0,t]$, one then has

$$
\begin{aligned}
&\frac{1}{2\mu}\int\left[(1+t)^{\mu-\epsilon_0}(V_{xt}^2-2p'(\underline{v})V_{xx}^2)+\mu(1+\epsilon_0)(1+t)^{\mu-2-\epsilon_0}V_x^2\right]dx \qquad (2.42)\\
&\qquad + \frac{\epsilon_0}{\mu}\int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}V_{xt}^2 dxd\tau - \frac{\epsilon_0}{\mu}\int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}p'(\underline{v})V_{xx}^2 dxd\tau \\
&\quad \le C\left(\|V_0\|_2^2+\|V_1\|_1^2\right) + \frac{2}{\mu}\int_0^t\int(1+\tau)^{\mu-\epsilon_0}f_{xx}V_{xt}dxd\tau \\
&\qquad + \int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}f_x V_{xx}dxd\tau.
\end{aligned}
$$

From (2.36), (2.37), and (2.41), we have

(2.43)

$$
\begin{aligned}
\left|\int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}f_x V_{xx}dxd\tau\right| &= \left|\int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}(p'(V_x+\underline{v})-p'(\underline{v}))V_{xx}^2 dxd\tau\right| \\
&\le \int_0^t(1+\tau)^{\mu-1-\epsilon_0}\|V_x\|_{L^\infty}\|V_{xx}\|^2 d\tau \\
&\le C\int_0^t(1+\tau)^{\mu-1-\gamma-\epsilon_0}\|V_{xx}\|^2 d\tau
\end{aligned}
$$

and

(2.44)
$$\begin{aligned}
&\left|\int_0^t\int(1+\tau)^{\mu-\epsilon_0}f_xV_{xt}dxd\tau\right| \\
&\quad=\left|\int_0^t\int(1+\tau)^{\mu-\epsilon_0}(p'(V_x+\underline{v})-p'(\underline{v}))V_{xx}V_{xxt}dxd\tau\right| \\
&\quad\le\frac{1}{2}\left|\int(1+t)^{\mu-\epsilon_0}(p'(V_x+\underline{v})-p'(\underline{v}))V_{xx}^2dx\right|+C\|V_0\|_2^2 \\
&\qquad+\frac{\mu-\epsilon_0}{2}\left|\int_0^t\int(1+\tau)^{\mu-1-\epsilon_0}(p'(V_x+\underline{v})-p'(\underline{v}))V_{xx}^2dxd\tau\right| \\
&\qquad+\frac{1}{2}\left|\int_0^t\int(1+\tau)^{\mu-\epsilon_0}p'(V_x+\underline{v})V_{xt}V_{xx}^2dxd\tau\right| \\
&\quad\le C(1+t)^{\mu-\epsilon_0}\|V_x\|_{L^\infty}\|V_{xx}\|^2+C\|V_0\|_2^2+C\int_0^t(1+\tau)^{\mu-1-\epsilon_0}\|V_x\|_{L^\infty}\|V_{xx}\|^2d\tau \\
&\quad\le CN_1(T)(1+t)^{\mu-\epsilon_0}\|V_{xx}\|^2+C\int_0^t(1+\tau)^{\mu-1-\gamma-\epsilon_0}\|V_{xx}\|^2d\tau+C\|V_0\|_2^2.
\end{aligned}$$

Inserting (2.43) and (2.44) into (2.42), using the smallness of $N_1(T)$, we have

$$\begin{aligned}
&(1+t)^{\mu-\epsilon_0}\|(V_{xx},V_{xt})(t)\|_1^2+\epsilon_0\int_0^t(1+\tau)^{\mu-1-\epsilon_0}\|(V_{xx},V_{xt})(\tau)\|^2d\tau \\
&\quad\le C(\|V_0\|_2^2+\|V_1\|_1^2)++C\int_0^t(1+\tau)^{\mu-1-\gamma-\epsilon_0}\|V_{xx}\|^2d\tau. \qquad (2.45)
\end{aligned}$$

Noting $\gamma>1$, the Gronwall's inequality implies

$$\begin{aligned}
&(1+t)^{\mu-\epsilon_0}\|(V_{xx},V_{xt})(t)\|_1^2+\epsilon_0\int_0^t(1+\tau)^{\mu-1-\epsilon_0}\|(V_{xx},V_{xt})(\tau)\|^2d\tau \\
&\quad\le C(\|V_0\|_2^2+\|V_1\|_1^2). \qquad (2.46)
\end{aligned}$$

If $\mu>4$, similar to the proof of (2.46), multiplying $\partial_x(1.35)$ by $[\frac{1}{4}(1+t)^{4-\epsilon_0}V_{xt}+(1+t)^{3-\epsilon_0}V_x]$ and integrating it, we can obtain

$$\begin{aligned}
&(1+t)^{4-\epsilon_0}\|(V_{xx},V_{xt})(t)\|_1^2+\epsilon_0\int_0^t(1+\tau)^{3-\epsilon_0}\|(V_{xx},V_{xt})(\tau)\|_1^2d\tau \\
&\quad\le C(\|V_0\|_2^2+\|V_1\|_1^2). \qquad (2.47)
\end{aligned}$$

From (2.46) and (2.47), we have, letting $\epsilon_0\to0$, for $2<\mu\le4$,

$$(1+t)^{\mu}\|(V_{xx},V_{xt})(t)\|_1^2\le C\left(\|V_0\|_2^2+\|V_1\|_1^2\right), \qquad (2.48)$$

and for $\mu>4$,

$$(1+t)^{4}\|(V_{xx},V_{xt})(t)\|_1^2\le C(\|V_0\|_2^2+\|V_1\|_1^2). \qquad (2.49)$$

A similar calculation to (2.48) and (2.49) then yields the higher order estimate for $\|V_{xxx}(t)\|$ and $\|V_{xxt}(t)\|$. Thus, we have for $2<\mu\le4$

$$(1+t)^{\mu}\|(V_{xx},V_{xt})(t)\|_1^2\le C\left(\|V_0\|_2^2+\|V_1\|_1^2\right), \qquad (2.50)$$

and for $\mu > 4$,

$$(1+t)^4\|(V_{xx},V_{xt})(t)\|_1^2 \le C\left(\|V_0\|_2^2+\|V_1\|_1^2\right). \tag{2.51}$$

From (2.50) and (2.51), by using the continuity technique, we can obtain the global existence to the problem (1.35) for Case 2. The proof of Theorem 1.8 is completed. ∎

Appendix A. The proof of Theorem 1.1. The existence of the global solution for problem (1.8) has been shown by Wirth in [38]. Set $W=\bar{v}-\underline{v}$; from (1.11) then W satisfies

$$\begin{cases} W_{tt}+p'(\underline{v})W_{xx}+\frac{\mu}{1+t}W_t=0, \\ (W,W_t)(x,0)=(W_0,W_1)(x), \end{cases} \tag{A.1}$$

where

$$(W_0,W_1)(x)=(\kappa\phi_0(x),\bar{u}_0'(x)).$$

What we will do is obtain the convergence rates (1.15), (1.16). First, let us multiply (A.1) by $(1+t)^2W_t$; then by integrating it over $\mathbb{R}$, it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int(1+t)^2(W_t^2-p'(\underline{v})W_x^2)dx+(\mu-1)\int(1+t)W_t^2dx \\ &\quad+\int(1+t)p'(\underline{v})W_x^2dx=0. \end{aligned} \tag{A.2}$$

Also multiplying (A.1) by $(1+t)W$ and integrating it over $\mathbb{R}$ give

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\left[2(1+t)WW_t+(\mu-1)W^2\right]dx-\int(1+t)p'(\underline{v})W_x^2dx \\ &\quad-\int(1+t)W_t^2dx=0. \end{aligned} \tag{A.3}$$

Similar to the proof of (2.8), adding (A.3) and $\frac{\mu}{2(\mu-1)}\cdot$ (A.2), and integrating the resultant equation over $[0,t]$ yield

$$\begin{aligned} &\frac{\mu-2}{8(\mu-1)}\int(1+t)^2W_t^2-\frac{\mu}{4(\mu-1)}\int(1+t)^2p'(\underline{v})W_x^2dx+\frac{(\mu-1)(\mu-2)}{2(\mu+2)}\int W^2dx \\ &\quad+\frac{\mu-2}{2}\int_0^t\int(1+\tau)W_t^2dxd\tau-\frac{\mu-2}{2(\mu-1)}\int_0^t\int(1+\tau)p'(\underline{v})W_x^2dxd\tau \\ &\le C\left(\|W_0\|_1^2+\|W_1\|^2\right), \end{aligned}$$

where we have used

$$2\int(1+t)WW_tdx \ge -\frac{4(\mu-1)}{\mu+2}\int W^2dx-\frac{\mu+2}{4(\mu-1)}\int(1+t)^2W_t^2dx.$$

So we have

$$\begin{aligned} &\|W(t)\|^2+(1+t)^2\left(\|W_x\|^2+\|W_t\|^2\right)+\int_0^t(1+\tau)\left(\|W_x\|^2+\|W_t\|^2\right)d\tau \\ &\le C(\|W_0\|_1^2+\|W_1\|^2). \end{aligned} \tag{A.4}$$

For $0 < \epsilon_0 < \min\{1, \mu - 2\}$, multiplying ∂_x(A.1) by $(1+t)^{\mu-\epsilon_0} W_{xt}$ and integrating it over $\mathbb{R}$ give

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int (1+t)^{\mu-\epsilon_0}(W_{xt}^2 - p'(\underline{v})W_{xx}^2)dx + \frac{1}{2}(\mu+\epsilon_0)\int (1+t)^{\mu-1-\epsilon_0}W_{xt}^2 dx \\ &\quad + \frac{1}{2}(\mu-\epsilon_0)\int (1+t)^{\mu-1-\epsilon_0}p'(\underline{v})W_{xx}^2 dx = 0. \end{aligned} \tag{A.5}$$

Also, multiplying ∂_x(A.1) by $(1+t)^{\mu-1-\epsilon_0} W_x$ and integrating it over $\mathbb{R}$, we then have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int [2(1+t)^{\mu-1-\epsilon_0}W_x W_{xt} + (1+\epsilon_0)(1+t)^{\mu-2-\epsilon_0}W_x^2]dx \\ &\quad - \int (1+t)^{\mu-1-\epsilon_0}W_{xt}^2 dx - \int (1+t)^{\mu-1-\epsilon_0}p'(\underline{v})W_{xx}^2 dx \\ &\quad = \frac{1}{2}(1+\epsilon_0)(\mu-2-\epsilon_0)\int (1+t)^{\mu-3-\epsilon_0}W_x^2 dx. \end{aligned} \tag{A.6}$$

Similar to the proof of (2.21), adding (A.6) and $\frac{2}{\mu}\cdot$ (A.5), integrating the resultant equation over $[0,t]$ yields

$$\begin{aligned} &\frac{1}{2\mu}\int \left[2(1+t)^{\mu-\epsilon_0}(W_{xt}^2 - 2p'(\underline{v})W_{xx}^2) + \mu(1+\epsilon_0)(1+t)^{\mu-2-\epsilon_0}W_x^2\right]dx \\ &\quad + \frac{\epsilon_0}{\mu}\int_0^t\int (1+\tau)^{\mu-1-\epsilon_0}W_{xt}^2 dx d\tau - \frac{\epsilon_0}{\mu}\int_0^t\int (1+\tau)^{\mu-1-\epsilon_0}p'(\underline{v})W_{xx}^2 dx d\tau \\ &\le C\left(\|W_0\|_2^2 + \|W_1\|_1^2\right), \end{aligned} \tag{A.7}$$

where we have used for $2 < \mu \le 4$,

$$\begin{aligned} &2\int (1+t)^{\mu-1-\epsilon_0}W_x W_{xt} dx \\ &\quad \ge -\frac{1}{\mu}\int (1+t)^{\mu-\epsilon_0}W_{xt}^2 dx - \mu\int (1+t)^{\mu-2-\epsilon_0}W_x^2 dx \\ &\quad \ge -\frac{1}{\mu}\int (1+t)^{\mu-\epsilon_0}W_{xt}^2 dx - C\left(\|W_0\|_1^2 + \|W_1\|^2\right) \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2}(1+\epsilon_0)(\mu-2-\epsilon_0)\int_0^t\int (1+t)^{\mu-3-\epsilon_0}W_x^2 dx d\tau \\ &\quad \le C\int_0^t\int (1+\tau)W_x^2 dx d\tau \\ &\quad \le C\left(\|W_0\|_1^2 + \|W_1\|^2\right). \end{aligned}$$

So we have

$$\begin{aligned} &\int (1+t)^{\mu-\epsilon_0}(W_{xx}^2 + W_{xt}^2)dx + \epsilon_0\int_0^t\int (1+\tau)^{\mu-1-\epsilon_0}(W_{xx}^2 + W_{xt}^2)dx d\tau \\ &\quad \le C\left(\|W_0\|_2^2 + \|W_1\|_1^2\right). \end{aligned} \tag{A.8}$$

For $\mu > 4$, Multiplying $(A.1)_x$ by $[\frac{1}{4}(1+t)^{4-\epsilon_0}W_{xt} + (1+t)^{3-\epsilon_0}W_x]$ and integrating it, we can obtain

$$(1+t)^{4-\epsilon_0}\|(W_{xx}, W_{xt})(t)\|^2 + \epsilon_0\int_0^t (1+\tau)^{3-\epsilon_0}\|(W_{xx}, W_{xt})(\tau)\|^2 d\tau \\ \leq C\left(\|W_0\|_2^2 + \|W_1\|_1^2\right). \tag{A.9}$$

From (A.8) and (A.9), we have, letting $\epsilon_0 \to 0$, for $2 < \mu \leq 4$,

$$\int (1+t)^{\mu}(W_{xx}^2 + W_{xt}^2)dx \leq C\left(\|W_0\|_2^2 + \|W_1\|_1^2\right), \tag{A.10}$$

and for $\mu > 4$,

$$\int (1+t)^4\left(W_{xx}^2 + W_{xt}^2\right)dx \leq C\left(\|W_0\|_2^2 + \|W_1\|_1^2\right). \tag{A.11}$$

By a similar calculation to (A.10) and (A.11), we can get the high order estimate for $\|W_{xxx}(t)\|$ and $\|W_{xxt}(t)\|$. This completes the proof of Theorem 1.1.

Acknowledgments. The work was initiated when S. Geng visited McGill University and was finalized when M. Mei visited Xiangtan University and Hong Kong Polytechnic University. Both authors would like to express their sincere thanks to the host universities for great hospitality.

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