Better Than Pre-committed Optimal Mean-Variance Policy in a Jump Diffusion Market*

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Dynamic mean-variance investment model can not be solved by dynamic programming directly due to the nonseparable structure of variance minimization problem. Instead of adopting embedding scheme, Lagrangian duality approach or mean-variance hedging approach, we transfer the model into mean field mean-variance formulation and derive the explicit pre-committed optimal mean-variance policy in a jump diffusion market. Similar to multi-period setting, the pre-committed optimal mean-variance policy is not time consistent in efficiency. When the wealth level of the investor exceeds some pre-given level, following pre-committed optimal mean-variance policy leads to irrational investment behaviours. Thus, we propose a semi-self-financing revised policy, in which the investor is allowed to withdraw partial of his wealth out of the market. And show the revised policy has a better investment performance in the sense of achieving the same mean-variance pair as pre-committed policy and receiving a nonnegative free cash flow stream.

Key Words: mean field approach, pre-committed optimal mean-variance policy, jump diffusion market, time consistency in efficiency, semi-self-financing revised policy.

1 INTRODUCTION.

Since Markowitz (1952) published his seminal paper on mean-variance model, a return-risk investment framework have been extensively investigated in financial economics. The extension of the mean-variance model to dynamic settings, however, has been unsuccessful for many years, due to an inherent nonseparable structure of the variance minimization problem in the sense of dynamic programming. There are mainly four approaches in the literature to tackle this difficulty. First approach is embedding scheme proposed by Li and Ng (2000) and Zhou and Li (2000). They considered a class of auxiliary linear quadratic stochastic control (LQSC) problems and derived the

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optimal mean-variance policy through identifying the best auxiliary parameter. Second approach is Lagrangian duality approach proposed by Li et al. (2001). They considered the Lagrangian relaxation of continuous time mean-variance model, and solved the original mean-variance model by identifying the best Lagrangian parameter. Third approach is mean-variance hedging approach proposed by Sun and Wang (2006). They proved that the optimal terminal wealth of mean-variance model takes a particular form of the terminal wealth of a mean-variance hedging problem. Then, the mean-variance portfolio selection problem is reduced into a mean-variance hedging problem, which is also a LQSC problem. All these three approaches attempt to embed the nonseparable mean-variance model into a family of tractable LQSC problems (or one particular LQSC problem). Fourth approach is mean field approach proposed by Cui et al. (2014). They considered multi-period mean-variance models and reformulated them into corresponding mean field type models, where both the wealth dynamics and the objective functional involve the wealth states as well as the expected values of the wealth states. By enlarging the state space and control space, they derived the optimal mean-variance policies by dynamic programming directly. In the first part of this paper, we extend Cui et al. (2014)'s multi-period mean field approach to consider continuous time mean-variance model in a jump diffusion market.

The optimal dynamic mean-variance policy obtained at the beginning of investment is termed by Basak and Chabakauri (2010) as *pre-committed* optimal mean-variance policy. For Bellman's principle of optimality is not applicable for dynamic mean-variance models, mean-variance investors' global and local interests are not consistent, which implies the pre-committed optimal mean-variance policy may not be optimal for a truncated mean-variance investment problem at some intermediate time t and for certain realized wealth level. Thus, the investors may have incentives to deviate from the pre-committed optimal mean-variance policy before reaching the terminal time (see Zhu et al., 2003; Basak and Chabakauri, 2010). This phenomenon is called *time inconsistency*.

To resolve the contradiction between mean-variance investors' global and local interests, Basak and Chabakauri (2010) reformulated dynamic mean-variance model as an interpersonal game, where the investor at time t optimally chooses the policy adopted at any time t, on the premise that he has already decided his policies in the future. The subgame Nash equilibrium policy of the interpersonal game is called *time consistent policy*, which is the extension of Strotz (1955-1956) and Laibson (1997)'s strategy of consistent planning in dynamic mean-variance world. Björk et al. (2014), Hu et al. (2012), Cui et al. (2014) extended Basak and Chabakauri (2010)'s work by assuming that the mean-variance investor has different forms of state-dependent risk aversion.

Different from investigating time consistent mean-variance policy, Cui et al. (2012) relaxed the concept of time consistency in the literature (see Rosazza Gianin, 2006; Artzner et al., 2007; Jobert and Rogers, 2008) to *time consistency in efficiency* (TCIE) based on a multi-objective version of the principle of optimality: The principle of optimality holds if any tail part of an efficient policy is also efficient for any realizable state at any intermediate period. TCIE is nothing, but requiring the efficiency for any truncated investment problem at every time instant during the investment horizon. Cui et al. (2012) showed that multi-period mean-variance model does not satisfy TCIE and developed a better revised mean-variance policy by relaxing the self-financing restriction to allow withdrawal of money out of the market. Dang and Forsyth (2014) termed this type of mean-variance policy as semi-self-financing mean-variance policy and studied its property when there exist portfolio constraints. In the second part of this paper, we extend Cui et al. (2012)'s analysis to construct a better semi-self-financing mean-variance policy in a jump diffusion market.

In this paper, we focus on studying the continuous time mean-variance investment model in a jump diffusion market. We derive the pre-committed optimal mean-variance policy via newly proposed mean field approach and construct a better semi-self-financing mean-variance policy, which can achieve the same mean-variance pair as pre-committed optimal mean-variance policy and receive a free cash flow stream during the investment. Our work enriches the research on the solution schemes of dynamic mean-variance models and time consistency issue of dynamic investment problems.

The organization of this paper is as follows. In Section 2, we reformulate continuous time mean-variance model in a jump diffusion market into mean field mean-variance model. In Section 3, we derive the pre-committed optimal mean-variance policy by dynamic programming. In Section 4, we demonstrate that the pre-committed optimal mean-variance policy does not satisfy TCIE and construct a better revised mean-variance policy. Finally, we conclude our paper in Section 5.

2 MEAN FIELD MEAN-VARIANCE FORMULATION IN JUMP DIFFUSION MARKETS

Throughout this paper $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is a filtered complete probability space and $\mathcal{F}_0$ is the trivial algebra over Ω. $T>0$ is given and fixed representing the terminal time of an investment. In addition, we use M' to denote the transpose of any vector or matrix M, and $L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ to denote the set of all $\mathbb{R}^n$-valued, $\mathcal{F}_t$-progressively measurable stochastic processes $f(t)$ with $\mathbb{E}\left[\int_0^T |f(t)|^2 dt\right] < +\infty$.

There is a capital market in which $m+1$ assets are traded continuously. One of the assets is a risk-free bank account whose value process $S_0(\cdot)$ is subject to the following ordinary differential equation (ODE):

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, \quad t\geq 0, \\ S_0(0) = s_0 > 0, \end{cases} \tag{1}$$

where $r(\cdot) > 0$ is the interest rate. The other m assets are risky stocks whose price processes $S_1(\cdot), S_2(\cdot), \cdots, S_m(\cdot)$ satisfy the following stochastic differential equations (SDE):

$$\begin{cases} dS_i(t) = S_i(t)\Big\{\mu_i(t)dt + \sum_{k=1}^{m} \sigma_{ik}(t)dW_k(t) + \sum_{j=1}^{\ell} \int_{\mathbb{R}} \gamma_{ij}(t,z_j)N_j(dt,dz_j)\Big\}, \quad t\geq 0, \\ S_i(0) = s_i > 0, \qquad i = 1,2,\cdots,m, \end{cases} \tag{2}$$

where $\mu_i(\cdot)$ is the appreciation rate, $\sigma_{ik}(\cdot)$ is the volatility or dispersion rate of the i-th stock, and $\gamma_{ij}(\cdot, z_j)$ is the relative change in the price $S_i(\cdot)$ given an arrival of the jth stochastic Poisson process with jump size z_j. $W(\cdot) = (W_1(\cdot), W_2(\cdot), \cdots, W_m(\cdot))'$ is m-dimensional Brownian motion, $N(dt,dz) = (N_1(dt,dz_1), \cdots, N_\ell(dt,dz_\ell))'$ is ℓ-dimensional Poisson random measure with Levy measure $(\nu_1, \cdots, \nu_\ell)$ and $\mathcal{F}_t = \sigma(W(t), N(dt,dz))$. We assume that W, N_j are independent and all the market parameters, $r(t)>0$, $\mu_i(t)$, $\sigma_{ik}(t)$ and $\gamma_{ij}(t,z_j) \geq -1$ a.s. with respect to Lévy measure ν_j, are uniformly bounded deterministic functions in $t\geq 0$ and $z_j \in \mathbb{R}$.

Stochastic differential equation (2) can be further written as

$$
(3) \qquad \begin{cases} dS_i(t) = S_i(t)\Big[\mu_i(t) + \displaystyle\sum_{j=1}^{\ell}\int_{\mathbb{R}} \gamma_{ij}(t,z_j)\nu_j(dz_j)\Big]dt \\ \qquad\qquad + S_i(t)\Big[\displaystyle\sum_{k=1}^{m}\sigma_{ik}(t)dW_k(t) + \sum_{j=1}^{\ell}\int_{\mathbb{R}}\gamma_{ij}(t,z_j)\widetilde{N}_j(dt,dz_j)\Big], \quad t \ge 0, \\ S_i(0) = s_i > 0, \qquad i = 1,2,\cdots,m, \end{cases}
$$

where

$$
\begin{aligned} \widetilde{N}(dt,dz) &= (\widetilde{N}_1(dt,dz_1),\cdots,\widetilde{N}_\ell(dt,dz_\ell))' \\ &= (N_1(dt,dz_1) - \nu_1(dz_1)dt, \cdots, N_\ell(dt,dz_\ell) - \nu_\ell(dz_\ell)dt)' \end{aligned}
$$

is the compensated Poisson random measure.

Consider an investor, with an initial wealth x_0 and an investment horizon $[0,T]$, whose total wealth at time $t \in [0,T]$ is denoted by $X(t)$. Assume that the trading of shares is self-financed and takes place continuously, and that transaction cost and consumptions are not considered. Then $X(\cdot)$ satisfies

$$
(4) \quad \begin{cases} dX(t) = \Big\{r(t)X(t) + \displaystyle\sum_{i=1}^{m}\big[\mu_i(t) + \sum_{j=1}^{\ell}\int_{\mathbb{R}}\gamma_{ij}(t,z_j)\nu_j(dz_j) - r(t)\big]\pi_i(t)\Big\}dt \\ \qquad + \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{m}\pi_i(t)\sigma_{ik}(t)dW_k(t) + \sum_{i=1}^{m}\sum_{j=1}^{\ell}\pi_i(t)\int_{\mathbb{R}}\gamma_{ij}(t,z_j)\widetilde{N}_j(dt,dz_j), \quad 0 \le t \le T, \\ X(0) = x_0, \end{cases}
$$

where $\pi_i(t),\ i = 1,2\cdots,m$, denotes the market value of the investor's wealth in the i-th stock at time t. We call the process $\pi(t) = (\pi_1(t),\cdots,\pi_m(t))'$, $0 \le t \le T$, a *portfolio* of the investor.

To simplify our analysis, we introduce the following notation

$$
\begin{aligned} &\sigma(\cdot) := (\sigma_{ik}(\cdot))_{m\times m}, \\ &\gamma_i(\cdot,z_i) := (\gamma_{1i}(\cdot,z_i),\cdots,\gamma_{mi}(\cdot,z_i))', \quad i = 1,\cdots,\ell, \\ &\gamma(\cdot,z) := (\gamma_1(\cdot,z_1),\cdots,\gamma_\ell(\cdot,z_\ell)) := (\gamma_{ij}(\cdot,z_j))_{m\times\ell}, \\ &B(\cdot) := \Big(\mu_1(\cdot) + \sum_{j=1}^{\ell}\int_{\mathbb{R}}\gamma_{1j}(\cdot,z_j)\nu_j(dz_j) - r(\cdot),\cdots,\mu_m(\cdot) + \sum_{j=1}^{\ell}\int_{\mathbb{R}}\gamma_{mj}(\cdot,z_j)\nu_j(dz_j) - r(\cdot)\Big)', \\ &\Gamma(\cdot) := \Big(\sum_{j=1}^{\ell}\int_{\mathbb{R}}\gamma_{ij}(\cdot,z_j)\gamma_{kj}(\cdot,z_j)\nu_j(dz_j)\Big)_{m\times m}. \end{aligned}
$$

Then, the wealth dynamic system (4) can be rewritten as

$$
(5) \quad \begin{cases} dX(t) = [r(t)X(t) + B(t)'\pi(t)]dt + \pi(t)'\sigma(t)dW(t) + \pi(t)'\displaystyle\int_{\mathbb{R}^\ell}\gamma(t,z)\widetilde{N}(dt,dz), \quad 0 \le t \le T, \\ X(0) = x_0, \end{cases}
$$

where $\int_{\mathbb{R}^\ell}\gamma(t,z)\widetilde{N}(dt,dz)$ is the notation of vector

$$
\Big(\sum_{j=1}^{\ell}\int_{\mathbb{R}}\gamma_{1j}(t,z_j)\widetilde{N}_j(dt,dz_j),\cdots,\sum_{j=1}^{\ell}\int_{\mathbb{R}}\gamma_{mj}(t,z_j)\widetilde{N}_j(dt,dz_j)\Big)'.
$$

Definition 2.1 *A portfolio $\pi(\cdot)$ is said to be admissible if $\pi(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ and the SDE (4) (or SDE (5)) has a unique solution $X(\cdot)$ corresponding to $\pi(\cdot)$.*

Along the admissible portfolio $\{\pi(\cdot)\}$, the corresponding wealth process $\{X(\cdot)\}$ is a Markov process and $\mathcal{F}_t = \sigma(X(\cdot))$. The investor's objective is to find an optimal portfolio $\pi^*(\cdot)$ such that the expected terminal wealth $\mathbb{E}_0[X(T)] = \mathbb{E}[X(T)|\mathcal{F}_0]$ is maximized and the variance of the terminal wealth $\mathrm{Var}_0(X(T)) = \mathrm{Var}(X(T)|\mathcal{F}_0) = \mathbb{E}\big[X(T) - \mathbb{E}[X(T)|\mathcal{F}_0]\big|\mathcal{F}_0\big]^2$ is minimized. The problem of choosing such a portfolio $\pi^*(\cdot)$ is referred to as the mean-variance portfolio selection problem. Mathematically, we have the following formulation,

$$(MV_\lambda) \quad \begin{cases} \displaystyle\min_{\pi(\cdot)} & \mathrm{Var}_0(X(T)) - \lambda\mathbb{E}_0[X(T)], \\ \text{subject to} & (X(\cdot),\pi(\cdot)) \text{ satisfies (5)}, \end{cases}$$

where $\lambda \geq 0$ is the trade-off of two conflict objectives and called the risk aversion parameter. The larger λ, the less risk aversion of the investor. To ensure problem (MV_λ) is well posed, we need two regular assumptions:

Assumption 1 $\Sigma(t) := \sigma(t)\sigma(t)' + \Gamma(t) \succeq \delta I, \quad \forall t \in [0,T]$ *for some* $\delta > 0$.

Assumption 2 *There is no arbitrage opportunity in the market.*

Next, we derive the mean-field formulation for problem (MV_λ). The $\mathcal{F}_0$-expected equation of the wealth system can be written as the following ODE

$$\begin{cases} d\mathbb{E}_0[X(t)] = \big\{r(t)\mathbb{E}_0[X(t)] + B(t)'\mathbb{E}_0[\pi(t)]\big\}dt, \quad 0 \leq t \leq T, \\ \mathbb{E}_0[X(0)] = x_0. \end{cases} \tag{6}$$

Then, we have

$$\begin{cases} d\big(X(t) - \mathbb{E}_0[X(t)]\big) &= \big\{r(t)\big(X(t) - \mathbb{E}_0[X(t)]\big) + B(t)'\big(\pi(t) - \mathbb{E}_0[\pi(t)]\big)\big\}dt \\ & \quad +\pi(t)'\sigma(t)dW(t) + \pi(t)'\displaystyle\int_{\mathbb{R}^\ell}\gamma(t,z)\widetilde{N}(dt,dz) \\ &= \big\{r(t)\big(X(t) - \mathbb{E}_0[X(t)]\big) + B(t)'\big(\pi(t) - \mathbb{E}_0[\pi(t)]\big)\big\}dt \\ & \quad +\big\{\big(\pi(t) - \mathbb{E}_0[\pi(t)]\big)'\sigma(t) + \mathbb{E}_0[\pi(t)]'\sigma(t)\big\}dW(t) \\ & \quad +\big\{\big(\pi(t) - \mathbb{E}_0[\pi(t)]\big)' + \mathbb{E}_0[\pi(t)]'\big\}\displaystyle\int_{\mathbb{R}^\ell}\gamma(t,z)\widetilde{N}(dt,dz), \quad 0 \leq t \leq T, \\ X(0) - \mathbb{E}_0[X(0)] &= 0. \end{cases} \tag{7}$$

What we are actually doing here is to enlarge the state space $(X(t))$ into $\big(X(t)-\mathbb{E}_0[X(t)], \mathbb{E}_0[X(t)]\big)$ and the control space $(\pi(t))$ into $\big(\pi(t) - \mathbb{E}_0[\pi(t)], \mathbb{E}_0[\pi(t)]\big)$. Although the two control vectors $\mathbb{E}_0[\pi(t)]$ and $\pi(t)-\mathbb{E}_0[\pi(t)]$ can be derived independently at time t, there exists an intrinsic property that

$$\mathbb{E}_0\big[\pi(t) - \mathbb{E}_0[\pi(t)]\big] = \mathbf{0}, \quad t = 0,1,\cdots,T-1,$$

where $\mathbf{0}$ is m-dimensional vector of all zeros.

Definition 2.2 *A portfolio* $\big(\pi(t)-\mathbb{E}_0[\pi(t)],\mathbb{E}_0[\pi(t)]\big)$ *is said to be admissible if* $\pi(t)-\mathbb{E}_0[\pi(t)]\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$, $\mathbb{E}_0[\pi(t)]\in\mathbb{R}^m$, $\mathbb{E}_0\big(\pi(t)-\mathbb{E}_0[\pi(t)]\big)=\mathbf{0}$, *ODE (6) and SDE (7) have unique solutions corresponding to* $\big(\pi(t)-\mathbb{E}_0[\pi(t)],\mathbb{E}_0[\pi(t)]\big)$, *respectively.*

We see that $\big(X(t)-\mathbb{E}_0[X(t)],\mathbb{E}_0[X(t)]\big)$ is a Markov process and $\mathcal{F}_t=\sigma(X(t)-\mathbb{E}_0[X(t)],\mathbb{E}_0[X(t)])$. Therefore, the mean-variance portfolio selection problem (MV_λ) can be then reformulated as the following mean field mean-variance problem

$$
(MV-MF)\quad\left\{\begin{array}{cl}
\min\limits_{(\pi(\cdot)-\mathbb{E}_0[\pi(\cdot)],\mathbb{E}_0[\pi(\cdot)])} & \mathbb{E}_0\big[\big(X(T)-\mathbb{E}_0[X(T)]\big)^2\big]-\lambda\mathbb{E}_0[X(T)],\\
\text{subject to} & (\mathbb{E}_0[X(\cdot)],\mathbb{E}_0[\pi(\cdot)])\text{ follows (6)},\\
 & (X(\cdot)-\mathbb{E}_0[X(\cdot)],\pi(\cdot)-\mathbb{E}_0[\pi(\cdot)],\mathbb{E}_0[\pi(\cdot)])\text{ follows (7)},\\
 & \mathbb{E}_0\big[\pi(t)-\mathbb{E}_0[\pi(t)]\big]=\mathbf{0},\quad 0\le t\le T,
\end{array}\right.
$$

which is a linear quadratic optimal control problem and can be solved by dynamic programming.

3 THE PRE-COMMITTED OPTIMAL MEAN-VARIANCE POLICY

In this section, we solve mean field mean-variance problem $(MV-MF)$ via dynamic programming and derive the pre-committed optimal mean-variance policy of original problem (MV_λ).

To simplify our analysis, we denote the portfolio policy and state, respectively, as

$$
\begin{aligned}
u(\cdot)&=(u^1(\cdot),u^2(\cdot)):=(\pi(\cdot)-\mathbb{E}_0[\pi(\cdot)],\mathbb{E}_0[\pi(\cdot)]),\\
Y(\cdot)&=(Y_1(\cdot),Y_2(\cdot)):=(X(\cdot)-\mathbb{E}_0[X(\cdot)],\mathbb{E}_0[X(\cdot)]),
\end{aligned}
$$

and the set of admissible portfolio policies between time s and t as $\mathcal{U}[s,t]$. Then, the cost function and the value function of problem $(MV-MF)$ at time t are defined as

$$
\begin{aligned}
J(t,y_1,y_2;u(\cdot))&=\mathbb{E}_t\left[(Y_1(T))^2-\lambda(Y_2(T))\right],\\
V(t,y_1,y_2)&=\min_{u(\cdot)\in\mathcal{U}[t,T]}J(t,y_1,y_2;u(\cdot)),
\end{aligned}
$$

respectively, where $\mathbb{E}_t[\cdot]=\mathbb{E}[\cdot|\mathcal{F}_t]$. Obviously, $V(T,y_1,y_2)=(y_1)^2-\lambda(y_2)$. We assume that both $J(t,y_1,y_2;u)\in\mathcal{C}^{1,2,2}$ and $V(t,y_1,y_2)\in\mathcal{C}^{1,2,2}$ (once continuously differentiable on t, twice continuously differentiable on y_1 and y_2).

Define the infinitesimal generator for any function $F(t,y_1,y_2)\in\mathcal{C}^{1,2,2}$ as

$$
\begin{aligned}
&\mathcal{A}^uF(t,y_1,y_2)\\
&=F_t+F_{y_2}[r(t)y_2+B(t)'u_2]+F_{y_1}\big[r(t)y_1+B(t)'u_1\big]+\frac{1}{2}F_{y_1y_1}\big|u_1'\sigma(t)+u_2'\sigma(t)\big|^2\\
&\quad+\sum_{j=1}^{\ell}\int_{\mathbb{R}}\Big\{F\big(t,y_1+\big[u_1'+u_2'\big]\gamma_j(t,z_j),y_2\big)-F(t,y_1,y_2)-F_{y_1}\big[u_1'+u_2'\big]\gamma_j(t,z_j)\Big\}v_j(dz_j),
\end{aligned}
$$

where F_t, F_{y_2}, F_{y_1}, $F_{y_1y_1}$ represent corresponding derivatives. We can always write $\mathcal{A}^uF(t,y_1,y_2)$ into summation of two parts,

$$
\mathcal{A}^uF(t,y_1,y_2)=\mathcal{A}_1^uF(t,y_1,y_2)+\mathcal{A}_2^uF(t,y_1,y_2),
$$

where $\mathcal{A}_2^u F(t, y_1, y_2)$ is a deterministic function such that

$$\mathbb{E}\left[\int_0^T |\mathcal{A}_2^u F(t, Y_1(t), Y_2(t))| dt\right] < +\infty,$$
$$\mathbb{E}_0\left[\mathcal{A}_2^u F(t, Y_1(t), Y_2(t))\right] = 0,$$

hold for all admissible $u(\cdot) \in \mathcal{U}[0, T]$.

Then, we have the following important proposition according to the value function $V(0, y_1, y_2)$.

Proposition 3.1 *The value function of problem $(MV - MF)$ at time 0 is*

$$V(0, y_1, y_2) = \bar{V}(0, y_1, y_2),$$

where $\bar{V}(t, y_1, y_2)$ is the solution of following HJB equation,

$$\text{(8)} \qquad \begin{cases} \mathcal{A}^u v(t, y_1, y_2) = \mathcal{A}_1^u v(t, y_1, y_2) + \mathcal{A}_2^u v(t, y_1, y_2), \\ \mathbb{E}_0\left[\int_0^T |\mathcal{A}_2^u v(t, Y_1(t), Y_2(t))| dt\right] < +\infty, \quad \forall u(\cdot) \in \mathcal{U}[0, T], \\ \mathbb{E}_0[\mathcal{A}_2^u v(t, Y_1(t), Y_2(t))] = 0, \quad \forall u(\cdot) \in \mathcal{U}[0, T], \\ \min\limits_u \mathcal{A}_1^u v(t, y_1, y_2) = 0, \\ v(T, y_1, y_2) = (y_1)^2 - \lambda y_2. \end{cases}$$

Moreover, the optimal portfolio policy of problem $(MV - MF)$ at time t can be derived through

$$\text{(9)} \qquad u^*(t, y_1, y_2) = \arg\min_u \left\{\mathcal{A}_1^u \bar{V}(t, y_1, y_2)\right\}.$$

Proof. See Appendix A. □

Remark 3.1 The HJB equation in Proposition 3.1 is a little different from classical HJB equation. Actually, Proposition 3.1 suggests that if unconditional expectation of one part of the value function is not influenced by any admissible portfolio policy and equals to zero, we can just eliminate that part in the dynamic programming.

Theorem 3.1 *The pre-committed optimal mean-variance policy of problem (MV_λ) can be uniquely represented as a feedback strategy*

$$\text{(10)} \qquad \pi^*(t, X^*(t)) = -\big(\Sigma(t)\big)^{-1} B(t)\Big(X^*(t) - \beta e^{-\int_t^T r(s)ds}\Big),$$

where the risk attitude parameter d is given by

$$\beta = x_0 e^{\int_0^T r(s)ds} + \frac{\lambda}{2} e^{\int_0^T B(s)'(\Sigma(s))^{-1} B(s)ds}.$$

The mean and variance of the optimal terminal wealth process $X^*(T)$ are

$$\text{(11)} \qquad \mathbb{E}_0[X^*(T)] = x_0 e^{\int_0^T r(s)ds} - \frac{\lambda}{2}\Big(1 - e^{\int_0^T B(s)'(\Sigma(s))^{-1} B(s)ds}\Big)$$

and

$$\text{(12)} \qquad \mathrm{Var}_0(X^*(T)) = \frac{\lambda^2}{4}\Big(e^{\int_0^T B(s)'(\Sigma(s))^{-1} B(s)ds} - 1\Big),$$

respectively. Moreover, the efficient frontier can be expressed as

$$\mathrm{Var}_0(X^*(T)) = \frac{1}{e^{\int_0^T B(s)'(\Sigma(s))^{-1}B(s)ds} - 1}\Big(\mathbb{E}_0[X^*(T)] - x_0 e^{\int_0^T r(s)ds}\Big)^2, \tag{13}$$
$$\textit{for } \mathbb{E}_0[X^*(T)] \geq x_0 e^{\int_0^T r(s)ds}.$$

Proof. See Appendix B. □

The advantage of adopting mean field formulation instead of other approaches is that the optimal portfolio policy can be derived by dynamic programming directly.

4 A BETTER REVISED MEAN-VARIANCE POLICY

Consider the following truncated mean-variance portfolio selection problem faced by the investor at time t,

$$(MV_{\lambda_t}) \quad \begin{cases} \min\limits_{\pi(\cdot)} & \mathrm{Var}_t(X(T)) - \lambda(t, X(t))\mathbb{E}_t[X(T)], \\ \text{subject to} & (X(\cdot), \pi(\cdot)) \text{ satisfies (5)}, \end{cases}$$

where the trade-off parameter is denoted by $\lambda(t, X(t))$. Based on Theorem 3.1, we can derive the optimal policy of this truncated mean-variance model as, for $t \leq s \leq T$,

$$\hat{\pi}^*(s, \widehat{X}^*(s)) = -\big(\Sigma(s)\big)^{-1} B(s)\Big(\widehat{X}^*(s) - \beta(t, X(t))e^{-\int_s^T r(\tau)d\tau}\Big),$$

where the risk attitude parameter is

$$\beta(t, X(t)) = X(t)e^{\int_t^T r(s)ds} + \frac{1}{2}\lambda(t, X(t))e^{\int_t^T B(s)'(\Sigma(s))^{-1}B(s)ds}.$$

Then, the mean and variance of the corresponding terminal wealth $\widehat{X}^*(T)$ are as follows,

$$\mathbb{E}_t\big[\widehat{X}^*(T)\big] = X(t)e^{\int_t^T r(s)ds} - \frac{1}{2}\lambda(t, X(t))\Big(1 - e^{\int_t^T B(s)'(\Sigma(s))^{-1}B(s)ds}\Big) \tag{14}$$

and

$$\mathrm{Var}_t\big(\widehat{X}^*(T)\big) = \frac{1}{4}[\lambda(t, X(t))]^2\Big(e^{\int_t^T B(s)'(\Sigma(s))^{-1}B(s)ds} - 1\Big), \tag{15}$$

respectively. Comparing the optimal policy of truncated mean-variance portfolio selection problem $\hat{\pi}^*(\cdot)$ and the pre-committed optimal mean-variance policy $\pi^*(\cdot)$, we can get the following proposition.

Proposition 4.1 *When the trade-off parameter, $\lambda(t, X(t))$, takes the following linear form of $X(t)$,*

$$\lambda(t, X(t)) = 2\Big(\beta - X(t)e^{\int_t^T r(s)ds}\Big)e^{-\int_t^T B(s)(\Sigma(s))^{-1}B(s)ds}, \tag{16}$$

the optimal policy of truncated mean-variance portfolio selection problem and the pre-committed optimal mean-variance policy are the same, i.e., the truncated pre-committed optimal mean-variance policy is the solution of truncated problem (MV_{λ_t}).

Remark 4.1 Similar to multi-period setting in Cui et al. (2012), we call the trade-off parameter in Proposition 4.1 as *trade-off parameter induced by pre-committed policy* (or induced trade-off parameter for short). If the induced trade-off parameter is negative for some state $X(t)$, i.e., wealth level at time t, $X(t)$, exceeds a deterministic threshold $\beta e^{-\int_t^T r(s)ds}$, the investor aims to minimize the variance and the expected value of terminal wealth level. In this situation, the investor changes his (her) risk attitude for the remaining investment horizon and the truncated pre-committed optimal mean-variance policy is no longer mean-variance efficient policy for the truncated mean-variance problem. Thus, the pre-committed optimal mean-variance policy does not satisfy TCIE in general.

An important question is when $X(t) > \beta e^{-\int_t^T r(s)ds}$ happens. Consider the dynamics of $\widetilde{X}(t) := X(t) - \beta e^{-\int_t^T r(s)ds}$ along the pre-committed optimal policy $\pi^*(t, X(t))$. Applying Itô Lemma, we can show that $\widetilde{X}(\cdot)$ is a geometric Lévy process and follows

$$\begin{cases} d\widetilde{X}(t) = \widetilde{X}(t)\Big\{\Big[r(t) - B(t)'(\Sigma(t))^{-1}B(t)\Big]dt - B(t)'(\Sigma(t))^{-1}\sigma(t)dW(t) \\ \qquad\qquad -\displaystyle\sum_{j=1}^{\ell} B(t)'(\Sigma(t))^{-1}\gamma_j(t,z_j)\widetilde{N}_j(dt,dz_j)\Big\}, \qquad\qquad 0 \le t \le T, \\ \widetilde{X}(0) = -\frac{1}{2}\lambda e^{\int_0^T [B(s)'(\Sigma(s))^{-1}B(s) - r(s)]ds} \le 0. \end{cases}$$

It is easy to see that a necessary and sufficient condition of $Pr\Big(X(t) \le de^{-\int_t^T r(s)ds} \text{ for all } t\Big) = 1$ is the following Condition 1.

Condition 1 *The inequality $B(t)'(\Sigma(t))^{-1}\gamma_j(t,z_j) \le 1$ almost surely holds with respect to Lévy measure ν_j, for all $j = 1, 2, \cdots, \ell$ and $t \in [0,T]$.*

In other words, under Condition 1, the pre-committed optimal mean-variance policy is always efficient for any possible wealth state $X(t)$ at time t. There are two very important types of market settings such that Condition 1 holds. One is pure diffusion market and another is complete market with only jumps. For pure diffusion market, $\widetilde{X}(t)$ is a geometric diffusion process, which is always non-positive. For complete market with only jumps, we have the following proposition.

Proposition 4.2 *If the market is a complete market with only jumps,*

$$B(t)'(\Sigma(t))^{-1}\gamma_j(t,z_j) \le 1, \quad t \in [0,T], \tag{17}$$

holds for all $j = 1, 2, \cdots, \ell$.

Proof. See Appendix C. ☐

If Condition 1 does not hold, we relax the self-financing into semi-self-financing, under which the investor is allowed to withdraw money out of the market. And then introduce a better revised semi-self-financing mean-variance policy.

Assume that the Poisson jumps happen at a series of time instances denoted by a sequence of stopping times $\{\tilde{\tau}_i\}$, and the truncated stopping times are denoted as $\{\tau_i = \tilde{\tau}_i \wedge T\}$. It is easy to see that $\widetilde{X}(\cdot)$ is a geometric diffusion process in time intervals $[0, \tau_1)$ or $[\tau_i, \tau_{i+1})$. Therefore, for $\tau_i \leq t < \tau_{i+1}$, $\widetilde{X}(t)$ has the same sign of $\widetilde{X}(\tau_i)$ after i-th jump. In other words, $\widetilde{X}(\cdot)$ may only become positive at stopping times $\{\tau_1, \tau_2, \cdots\}$. We propose the following semi-self-financing revised mean-variance policy:

For $0 \leq t < \tau_1$, an investor implements pre-committed optimal mean-variance policy,

$$\bar{\pi}(t, \bar{X}(t)) = -\big(\Sigma(t)\big)^{-1} B(t)\Big(\bar{X}(t) - \beta_0 e^{-\int_t^T r(s)ds}\Big),$$

where the wealth process follows the following SDE

$$\begin{cases} d\bar{X}(t) \;=\; [r(t)\bar{X}(t) + B(t)'\bar{\pi}(t, \bar{X}(t))]dt + \bar{\pi}(t, \bar{X}(t))'\sigma(t)dW(t) + \bar{\pi}(t, \bar{X}(t))'\displaystyle\int_{\mathbb{R}^\ell} \gamma(t,z)\widetilde{N}(dt, dz), \\ \hfill 0 \leq t \leq \tau_1, \\ \bar{X}(0) \;=\; \bar{x}_0, \end{cases}$$

the initial wealth invested in the market $\bar{x}_0$ and the risk attitude parameter β_0 are given as $\bar{x}_0 = x_0$ and $\beta_0 = \beta = x_0 e^{\int_0^T r(s)ds} + \frac{\lambda}{2} e^{\int_0^T B(s)'(\Sigma(s))^{-1}B(s)ds}$, respectively. Denote the wealth process level on or after the first jump by $\bar{X}(\tau_1)$.

Then, similarly, denote the wealth process level on or after the i-th jump by $\bar{X}(\tau_i)$. Set

(18)
$$\bar{x}_i = \begin{cases} \bar{X}(\tau_i), & \text{if } \bar{X}(\tau_i) \leq \beta_{i-1} e^{-\int_{\tau_i}^T r(s)ds}, \\ \bar{X}(\tau_i) + 2\Big(\beta_{i-1} e^{-\int_{\tau_i}^T r(s)ds} - \bar{X}(\tau_i)\Big)\Big(1 - e^{-\int_{\tau_i}^T B(s)'(\Sigma(s))^{-1}B(s)ds}\Big), & \text{if } \bar{X}(\tau_i) > \beta_{i-1} e^{-\int_{\tau_i}^T r(s)ds}, \end{cases}$$

(19)
$$\beta_i = \begin{cases} \beta_{i-1}, & \text{if } \bar{X}(\tau_i) \leq \beta_{i-1} e^{-\int_{\tau_i}^T r(s)ds}, \\ 2\bar{X}(\tau_i) e^{\int_{\tau_i}^T [r(s) - B(s)'(\Sigma(s))^{-1}B(s)]ds} + \beta_{i-1}\Big(1 - 2e^{-\int_{\tau_i}^T B(s)'(\Sigma(s))^{-1}B(s)ds}\Big), & \text{if } \bar{X}(\tau_i) > \beta_{i-1} e^{-\int_{\tau_i}^T r(s)ds}. \end{cases}$$

For $\tau_i \leq t < \tau_{i+1}$, the investor implements the linear feedback policy

$$\bar{\pi}(t, \bar{X}(t)) = -\big(\Sigma(t)\big)^{-1} B(t)\Big(\bar{X}(t) - \beta_i e^{-\int_t^T r(s)ds}\Big), \tag{20}$$

where the wealth process follows the following SDE

$$\begin{cases} d\bar{X}(t) \;=\; [r(t)\bar{X}(t) + B(t)'\bar{\pi}(t, \bar{X}(t))]dt + \bar{\pi}(t, \bar{X}(t))'\sigma(t)dW(t) + \bar{\pi}(t, \bar{X}(t))'\displaystyle\int_{\mathbb{R}^\ell} \gamma(t,z)\widetilde{N}(dt, dz), \\ \hfill \tau_i \leq t \leq \tau_{i+1}, \\ \bar{X}(\tau_i) \;=\; \bar{x}_i. \end{cases}$$

Note that both $\bar{x}_i$ and β_i ($i = 1, 2, \cdots$) are path-dependent. One major feature of this revised mean-variance policy is that, when the wealth process exceeds some threshold after i-th jump, i.e., $\bar{X}(\tau_i) > \beta_{i-1} e^{-\int_{\tau_i}^T r(s)ds}$, the investor withdraws a positive cash flow,

$$\bar{X}(\tau_i) - \bar{x}_i = 2\Big(1 - e^{-\int_{\tau_i}^T B(s)'(\Sigma(s))^{-1}B(s)ds}\Big)\Big(\bar{X}(\tau_i) - \beta_{i-1} e^{-\int_{\tau_i}^T r(s)ds}\Big) > 0, \tag{21}$$

out of the market and takes another efficient mean-variance policy with the risk attitude parameter β_i for the remaining amount in the market, $\bar{x}_i$, until the $(i+1)$-th jump. The efficiency of newly adopted policy $\bar{\pi}(t, \bar{X}(t))$ can be proved by checking $\bar{x}_i \leq \beta_i e^{-\int_{\tau_i}^T r(s)ds}$.

Theorem 4.1 *The revised mean-variance policy achieves the same mean-variance pair of the terminal wealth as does the original pre-committed optimal mean-variance policy, while having possibility to take a positive cash flow stream out of the market during the investment horizon.*

Proof. See Appendix D. □

Remark 4.2 Theorem 4.1 is an extension of Cui et al. (2012)'s main result in a jump diffusion market. The current revised mean-variance policy satisfies time consistent in efficiency and achieves better investment performance than pre-committed optimal mean-variance policy. Furthermore, the positive cash flow stream may only occur at time instances when there is a Poisson jump.

5 CONCLUSION

By formulating continuous-time mean-variance model in a jump diffusion market into a mean-field type model, we can derive the pre-committed optimal mean-variance policy easily and directly by dynamic programming. Thus, it has been proved again that mean field approach is a powerful tool to tackle dynamic mean-variance models and an efficient method to solve other nonseparable stochastic control problems.

After studying the induced trade-off parameter, we show that pre-committed optimal mean-variance policy in a jump diffusion market is not TCIE. By relaxing the self-financing restriction to allow withdrawal of money out of the market, we construct a better semi-self-financing mean-variance policy, achieving the same mean-variance pair as the pre-committed optimal mean-variance policy and receiving a free cash flow stream during the investment horizon. This finding motivates us to study similar semi-self-financing policies for other general return-risk portfolio selection models in the future.

APPENDIX

Appendix A: The Proof of Proposition 3.1

Proof. Consider the following auxiliary optimal control problem

$$\min_{u(\cdot)\in\mathcal{U}[0,T]}\left\{\mathbb{E}_0\left[J(T,Y_1(T),Y_2(T);u(\cdot))-\int_0^T\mathcal{A}_2^u\bar{V}(\tau,Y_1(\tau),Y_2(\tau))d\tau\right]\right\},$$

where

$$\bar{V}(s,y_1,y_2)=\min_{u(\cdot)\in\mathcal{U}[s,T]}\left\{\mathbb{E}_s\left[J(T,Y_1(T),Y_2(T);u(\cdot))-\int_s^T\mathcal{A}_2^u\bar{V}(\tau,Y_1(s),Y_2(\tau))d\tau\right]\right\}$$

is the value function of the problem.

Due to assumptions made on $\mathcal{A}_2^u\bar{V}(t,Y_1(t),Y_2(t))$ and Fubini's theorem, we have

$$\begin{aligned}\bar{V}(0,y_1,y_2)&=\min_{u(\cdot)\in\mathcal{U}[0,T]}\left\{\mathbb{E}_0\left[J(T,Y_1(T),Y_2(T);u(\cdot))-\int_0^T\mathcal{A}_2^u\bar{V}(\tau,Y_1(\tau),Y_2(\tau))d\tau\right]\right\}\\&=\min_{u(\cdot)\in\mathcal{U}[0,T]}\left\{\mathbb{E}_0\left[J(T,Y_1(T),Y_2(T);u(\cdot))\right]\right\}\\&=V(0,y_1,y_2).\end{aligned}$$

This means that the value function of auxiliary optimal control problem is the same as the value function of problem $(MV - MF)$ at time 0.

Furthermore, we have

$$
\begin{aligned}
&\arg\min_{u(\cdot)\in\mathcal{U}[0,T]} \left\{ \mathbb{E}_0 \left[J(T, Y_1(T), Y_2(T); u(\cdot)) - \int_0^T \mathcal{A}_2^u \bar{V}(\tau, Y_1(\tau), Y_2(\tau); u) d\tau \right] \right\} \\
=&\arg\min_{u(\cdot)\in\mathcal{U}[0,T]} \left\{ \mathbb{E}_0 \big[J(T, Y_1(T), Y_2(T); u(\cdot)) \big] \right\},
\end{aligned}
$$

which implies the optimal control set of auxiliary control problem is the same as the optimal control set of problem $(MV - MF)$.

The HJB equation of auxiliary optimal control problem reads,

$$
\begin{cases}
\min\limits_u \{-\mathcal{A}_2^u v(t, y_1, y_2) + \mathcal{A}^u v(t, y_1, y_2)\} = 0, \\
v(T, y_1, y_2) = y_1^2 - \lambda y_2,
\end{cases}
$$

with assumptions on $\mathcal{A}_2^u \bar{V}(t, Y_1(t), Y_2(t))$ holding. It can be reduced to

$$
\begin{cases}
\mathcal{A}^u v(t, y_1, y_2) = \mathcal{A}_1^u v(t, y_1, y_2) + \mathcal{A}_2^u v(t, y_1, y_2), \\
\mathbb{E}_0 \left[\int_0^T |\mathcal{A}_2^u v(t, Y_1(t), Y_2(t)) \Big| dt \right] < +\infty, \quad \forall u(\cdot) \in \mathcal{U}[0,T], \\
\mathbb{E}_0 [\mathcal{A}_2^u v(t, Y_1(t), Y_2(t))] = 0, \quad \forall u(\cdot) \in \mathcal{U}[0,T], \\
\min\limits_u \mathcal{A}_1^u v(t, y_1, y_2) = 0, \\
v(T, y_1, y_2) = (y_1)^2 - \lambda y_2.
\end{cases}
$$

And the optimal control is given by

$$
u^*(t, y_1, y_2) = \arg\min_u \big\{ \mathcal{A}_1^u \bar{V}(t, y_1, y_2) \big\}.
$$

□

Appendix B: The Proof of Theorem 3.1

Proof. We conjecture the solution of HJB (8) as the quadratic form

$$
\bar{V}(t, y_1, y_2) = P(t)(y_1)^2 + \Pi(t) y_2 + Q(t).
$$

It is easy to see

$$
\begin{aligned}
&\mathcal{A}^u \bar{V}(t, y_1, y_2) \\
=& \dot{P}(t) y_1^2 + \dot{\Pi}(t) y_2 + \dot{Q}(t) + \Pi(t) \big[r(t) y_2 + B(t)' u_2 \big] + 2P(t) y_1 \big[r(t) y_1 + B(t)' u_1 \big] \\
&+ P(t) \big| u_1' \sigma(t) + u_2' \sigma(t) \big|^2 + P(t) \big[u_1' + u_2' \big] \Gamma(t) \big[u_1 + u_2 \big] \\
=& \big[\dot{P}(t) + 2r(t) P(t) \big] y_1^2 + 2P(t) y_1 B(t)' u_1 + P(t) u_1' \Sigma(t) u_1 \\
&+ P(t) u_2' \Sigma(t) u_2 + \Pi(t) B(t)' u_2 + \dot{\Pi}(t) y_2 + r(t) \Pi(t) y_2 + \dot{Q}(t) + 2P(t) u_1' \Sigma(t) u_2 \\
=& \mathcal{A}_1^u \bar{V}(t, y_1, y_2) + \mathcal{A}_2^u \bar{V}(t, y_1, y_2),
\end{aligned}
$$

where $\dot{P}(t) = \frac{dP(t)}{dt}$, $\dot{\Pi}(t) = \frac{d\Pi(t)}{dt}$, $\dot{Q}(t) = \frac{dQ(t)}{dt}$ and

$$\begin{aligned}
\mathcal{A}_1^u \bar{V}(t, y_1, y_2) = {} & \big[\dot{P}(t) + 2r(t)P(t)\big] y_1^2 + 2P(t)y_1 B(t)' u_1 + P(t)u_1' \Sigma(t) u_1 \\
& + P(t)u_2'\Sigma(t)u_2 + \Pi(t)B(t)'u_2 + \dot{\Pi}(t)y_2 + r(t)\Pi(t)y_2 + \dot{Q}(t), \\
\mathcal{A}_2^u \bar{V}(t, y_1, y_2) = {} & 2P(t)u_1'\Sigma(t)u_2.
\end{aligned}$$

Due to Definition 2.2 of admissible portfolio, it is also easy to check that

$$\begin{aligned}
& \mathbb{E}_0\Big[\int_0^T |\mathcal{A}_2^u \bar{V}(t, Y_1(t), Y_2(t))| dt\Big] < +\infty, \quad \forall u(\cdot) \in \mathcal{U}[0, T], \\
& \mathbb{E}_0\big[\mathcal{A}_2^u \bar{V}(t, Y_1(t), Y_2(t))\big] = 0.
\end{aligned}$$

Furthermore, taking completion of the square yields

$$\begin{aligned}
& \mathcal{A}_1^u \bar{V}(t, y_1, y_2) \\
= & \big[\dot{P}(t) + 2r(t)P(t) - P(t)B(t)'(\Sigma(t))^{-1}B(t)\big] y_1^2 \\
& + P(t)\big[u_1 + \big(\Sigma(t)\big)^{-1} B(t) y_1\big]' \Sigma(t) \big[u_1 + \big(\Sigma(t)\big)^{-1} B(t) y_1\big] \\
& + \big[\dot{\Pi}(t) + r(t)\Pi(t)\big] y_2 - \frac{1}{4} B(t)'\big(P(t)\Sigma(t)\big)^{-1} B(t)\Pi(t)^2 + \dot{Q}(t) \\
& + P(t)\Big[u_2 + \frac{1}{2}\Pi(t)\big(P(t)\Sigma(t)\big)^{-1}B(t)\Big]' \Sigma(t) \Big[u_2 + \frac{1}{2}\Pi(t)\big(P(t)\Sigma(t)\big)^{-1}B(t)\Big].
\end{aligned}$$

We first identify the candidate optimal control without considering the linear constraint $\mathbb{E}_0\big[\pi(t) - \mathbb{E}_0[\pi(t)]\big] = \mathbf{0}$ and verify then the derived optimal policy satisfies the constraint automatically. The candidate optimal control is given as

$$u_1^*(t) = -\big(\Sigma(t)\big)^{-1} B(t) y_1, \tag{22}$$

$$u_2^*(t) = -\frac{1}{2}\big(P(t)\Sigma(t)\big)^{-1} B(t)\Pi(t). \tag{23}$$

Then, according to

$$\begin{cases} \min\limits_u \mathcal{A}_1^u \bar{V}(t, y_1, y_2) = 0, \\ \bar{V}(T, y_1, y_2) = (y_1)^2 - \lambda y_2, \end{cases}$$

we have

$$\begin{cases} \dot{P}(t) + 2r(t)P(t) - P(t)B(t)'(\Sigma(t))^{-1}B(t) = 0, \\ P(T) = 1, \end{cases}$$
$$\begin{cases} \dot{\Pi}(t) + r(t)\Pi(t) = 0, \\ \Pi(T) = -\lambda, \end{cases}$$
$$\begin{cases} \dot{Q}(t) - \frac{1}{4}B(t)'\big(P(t)\Sigma(t)\big)^{-1}B(t)\Pi(t)^2 = 0, \\ Q(T) = 0. \end{cases}$$

Solving the above ODEs, we obtain

$$\begin{aligned}
P(t) &= e^{\int_t^T [2r(s) - B(s)'(\Sigma(s))^{-1}B(s)] ds}, \\
\Pi(t) &= -\lambda e^{\int_t^T r(s) ds}, \\
Q(t) &= \frac{\lambda^2}{4}\left(1 - e^{\int_t^T B(s)'(\Sigma(s))^{-1}B(s) ds}\right).
\end{aligned}$$

Thus,

$$
\begin{aligned}
\mathbb{E}_0[\pi^*(t)] &= -\frac{1}{2}\big(P(t)\Sigma(t)\big)^{-1}B(t)\Pi(t)\\
&= \frac{\lambda}{2}\big(\Sigma(t)\big)^{-1}B(t)e^{-\int_t^T[2r(s)-B(s)'(\Sigma(s))^{-1}B(s)]ds}e^{\int_t^T r(s)ds}\\
&= \frac{\lambda}{2}\big(\Sigma(t)\big)^{-1}B(t)e^{-\int_t^T[r(s)-B(s)'(\Sigma(s))^{-1}B(s)]ds}.
\end{aligned}
\tag{24}
$$

Solving the ODE equation (6) yields the solution

$$
\mathbb{E}_0[X^*(t)] = \Big(x_0 + \int_0^t B(s)\mathbb{E}_0[\pi^*(s)]e^{-\int_0^s r(v)dv}ds\Big)e^{\int_0^t r(s)ds}.
\tag{25}
$$

Substituting (24) into (25) yields

$$
\begin{aligned}
\mathbb{E}_0[X^*(t)] &= \Big(x_0 + \int_0^t B(s)'\mathbb{E}_0[\pi^*(s)]e^{-\int_0^s r(v)dv}ds\Big)e^{\int_0^t r(s)ds}\\
&= \Big(x_0 + \frac{\lambda}{2}\int_0^t B(s)'(\Sigma(s))^{-1}B(s)e^{-\int_s^T[r(v)-B(v)'(\Sigma(v))^{-1}B(v)]dv}e^{-\int_0^s r(v)dv}ds\Big)e^{\int_0^t r(s)ds}\\
&= \Big(x_0e^{\int_0^T r(s)ds} + \frac{\lambda}{2}\int_0^t B(s)'(\Sigma(s))^{-1}B(s)e^{\int_s^T B(v)'(\Sigma(v))^{-1}B(v)dv}ds\Big)e^{-\int_0^T r(v)dv}e^{\int_0^t r(s)ds}\\
&= \Big[x_0e^{\int_0^T r(s)ds} - \frac{\lambda}{2}\Big(e^{\int_t^T B(s)'(\Sigma(s))^{-1}B(s)ds} - e^{\int_0^T B(s)'(\Sigma(s))^{-1}B(s)ds}\Big)\Big]e^{-\int_t^T r(s)ds}\\
&= \Big(x_0e^{\int_0^T r(s)ds} + \frac{\lambda}{2}e^{\int_0^T B(s)'(\Sigma(s))^{-1}B(s)ds}\Big)e^{-\int_t^T r(s)ds} - \frac{\lambda}{2}e^{-\int_t^T[r(s)-B(s)'(\Sigma(s))^{-1}B(s)]ds}.
\end{aligned}
$$

Hence,

$$
\mathbb{E}_0[X^*(T)] = x_0e^{\int_0^T r(s)ds} - \frac{\lambda}{2}\Big(1 - e^{\int_0^T B(s)'(\Sigma(s))^{-1}B(s)ds}\Big).
\tag{26}
$$

Also, based on (22) and (23), we have

$$
\begin{aligned}
\pi^*(t,X(t)) &= -\big(\Sigma(t)\big)^{-1}B(t)\big(X(t)-\mathbb{E}_0[X(t)]|_{\pi^*}\big) + \mathbb{E}_0[\pi^*(t)]\\
&= -\big(\Sigma(t)\big)^{-1}B(t)\Big[X(t) - \Big(x_0e^{\int_0^T r(s)ds} + \frac{\lambda}{2}e^{\int_0^T B(s)'(\Sigma(s))^{-1}B(s)ds}\Big)e^{-\int_t^T r(s)ds}\\
&\qquad + \frac{\lambda}{2}e^{-\int_t^T(r(s)-B(s)'(\Sigma(s))^{-1}B(s))ds}\Big] + \mathbb{E}_0[\pi^*(t)]\\
&= -\big(\Sigma(t)\big)^{-1}B(t)\Big[X(t) - \Big(x_0e^{\int_0^T r(s)ds} + \frac{\lambda}{2}e^{\int_0^T B(s)'(\Sigma(s))^{-1}B(s)ds}\Big)e^{-\int_t^T r(s)ds}\Big]\\
&= -\big(\Sigma(t)\big)^{-1}B(t)\Big(X(t) - \beta e^{-\int_t^T r(s)ds}\Big),
\end{aligned}
$$

where

$$
\beta = x_0e^{\int_0^T r(s)ds} + \frac{\lambda}{2}e^{\int_0^T B(s)'(\Sigma(s))^{-1}B(s)ds}.
$$

Substituting $\mathbb{E}_0[\pi^*(t)]$ and $\pi^*(t)-\mathbb{E}_0[\pi^*(t)]$ into (7) yields the optimal process $X^*(\cdot)-\mathbb{E}_0[X^*(\cdot)]$ and

$$
\left\{
\begin{aligned}
d\big(X^*(t)-\mathbb{E}_0[X^*(t)]\big) &= \big[\big(r(t)-B(t)'\big(\Sigma(t)\big)^{-1}B(t)\big]\big(X^*(t)-\mathbb{E}_0[X^*(t)]\big)dt\\
&\quad + \Big[\frac{\lambda}{2}e^{-\int_t^T[r(s)-B(s)'(\Sigma(s))^{-1}B(s)]ds} - \big(X^*(t)-\mathbb{E}_0[X^*(t)]\big)\Big]\\
&\quad \cdot\big(\Sigma(t)\big)^{-1}B(t)'\Big[\sigma(t)dW(t) + \textstyle\int_{\mathbb{R}^\ell}\gamma(t,z)\widetilde{N}(dt,dz)\Big],\\
X^*(0)-\mathbb{E}_0[X^*(0)] &= 0.
\end{aligned}
\right.
\tag{27}
$$

Applying Itô Lemma and taking expectation, we get

$$\begin{cases} d\Big(\mathbb{E}_0\big[(X^*(t)-\mathbb{E}_0[X^*(t)])^2\big]\Big) &= \Big\{\big[2r(t)-B(t)'\big(\Sigma(t)\big)^{-1}B(t)\big]\mathbb{E}_0\big[(X^*(t)-\mathbb{E}_0[X^*(t)])^2\big] \\ &\quad +\dfrac{\lambda^2}{4}B(t)'\big(\Sigma(t)\big)^{-1}B(t)e^{-\int_t^T 2[r(s)-B(s)'(\Sigma(s))^{-1}B(s)]ds}\Big\}dt, \\ \mathbb{E}_0\big[(X^*(0)-\mathbb{E}_0[X^*(0)])^2\big] &= 0. \end{cases}$$

Hence,

$$\begin{aligned} &\mathbb{E}_0\big[(X^*(t)-\mathbb{E}_0[X^*(t)])^2\big] \\ &= -\frac{\lambda^2}{4}e^{-\int_t^T[2r(s)-B(s)'(\Sigma(s))^{-1}B(s)]ds}\left(e^{\int_t^T B(s)'(\Sigma(s))^{-1}B(s)ds}-e^{\int_0^T B(s)'(\Sigma(s))^{-1}B(s)ds}\right). \end{aligned}$$

Therefore, we have the variance at the terminal

$$\begin{aligned} \mathrm{Var}_0(X^*(T)) =& \frac{\lambda^2}{4}\Big(e^{\int_0^T B(s)'(\Sigma(s))^{-1}B(s)ds}-1\Big) \\ =& \frac{1}{e^{\int_0^T B(s)'(\Sigma(s))^{-1}B(s)ds}-1}\Big(\mathbb{E}_0[X^*(T)]-x_0e^{\int_0^T r(s)ds}\Big)^2. \end{aligned}$$

Finally, we show the optimal portfolio policy satisfies the linear constraint $\mathbb{E}_0\big[\pi^*(t)-\mathbb{E}_0[\pi^*(t)]\big]=\mathbf{0}$. According to (27), we have $\mathbb{E}_0\big[X(t)-\mathbb{E}_0[X(t)]\big]=0$ for all t, which implies

$$\mathbb{E}_0\big[\pi^*(t)-\mathbb{E}_0[\pi^*(t)]\big]=-(\Sigma(t))^{-1}B(t)\mathbb{E}_0\big[X^*(t)-\mathbb{E}_0[X^*(t)]\big]=\mathbf{0},\quad \forall t\in[0,T].$$

□

Appendix C: The Proof of Proposition 4.2

Proof. Let us recall the first and second fundamental theories of asset pricing before we prove the theorem. The first fundamental theory of asset pricing states that there is no arbitrage opportunity in the market if and only if there exists at least one martingale measure under which the discounted price processes of all risky assets in the market are martingales. While the second fundamental theory of asset pricing states that the market is complete if and only if the martingale measure is unique (See Delbaen and Schachermayer, 1994; Levental and Skorohod, 1995).

Based on the price process (2) and our assumption of only jumps, the discounted price of the i-th risky assets, $\hat{S}_i(t)=S_i(t)e^{-\int_0^t r(s)ds}$, follows,

$$\begin{cases} d\hat{S}_i(t)=\hat{S}_i(t)\big[\mu_i(t)+\sum_{j=1}^{\ell}\int_{\mathbb{R}}\gamma_{ij}(t,z_j)\nu_j(dz_j)-r(t)\big]dt+\hat{S}_i(t)\sum_{j=1}^{\ell}\int_{\mathbb{R}}\gamma_{ij}(t,z_j)\widetilde{N}_j(dt,dz_j), \\ \hat{S}_i(0)=s_i>0. \end{cases}$$

The Girsanov theorem for jump processes (Theorem 1.35 in Øsendal and Sulem , 2005) states that if the equivalent (local) martingale measure Q exists, it can be expressed by

$$dQ(\omega)=Z(T)dP(\omega),$$

where

$$Z(t) := \exp\Big(\sum_{j=1}^{\ell}\int_0^t\int_{\mathbb{R}}[\ln(1-\theta_j(s,z_j))N_j(ds,dz_j)+\theta_j(s,z_j)\nu_j(dz_j)ds]\Big),$$

and $\theta_j(t,z_j)$ satisfies

$$\theta_j(t,z_j)\le 1,\quad 0\le t\le T, \tag{28}$$

$$\sum_{j=1}^{\ell}\int_{\mathbb{R}}\gamma_{ij}(t,z_j)\theta_j(t,z_j)\nu_j(dz_j)=B_i(t),\quad 0\le t\le T, \tag{29}$$

for $i=1,2,\cdots,m$, $j=1,2,\cdots,\ell$.

It is easy to check that

$$\theta_j(t,z_j)=(\gamma_j(t,z_j))'(\Sigma(t))^{-1}B(t)$$

is the solution of equation (29). Furthermore, the complete market assumption ensures there exists a unique solution for equation (29), and the unique solution also satisfies (28). Therefore, we have

$$B(t)'(\Sigma(t))^{-1}\gamma_j(t,z_j)=(\gamma_j(t,z_j))'(\Sigma(t))^{-1}B(t)\le 1.$$

□

Appendix D: The Proof of Theorem 4.1

Proof. We adopt the mathematical induction to prove the main result. Before the proof, we introduce the following notations throughout this proof procedure. Suppose that $X(s;t,x,\{u\}_t^s)$ is the wealth level at time s depending on the initial pair (t,x) and self-financing policy $\{u(\cdot)\}_t^s$, while $\bar{X}(s;t,x,\{v\}_t^s)$ is the wealth level at time s depending on the initial pair (t,x) and semi-self-financing policy $\{v(\cdot)\}_t^s$.

First, assume that the revised semi-self-financing policy only makes adjustment on or after the first jump, which is denoted by $\bar{\pi}^1(\cdot)$. This means that the revised policy $\bar{\pi}^1(\cdot)$ is the same as the pre-committed optimal policy $\pi^*(\cdot)$ during $[0,\tau_1)$. Thus, the wealth levels on or after the first jump are the same, i.e., $X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1})=\bar{X}(\tau_1;0,x_0,\{\bar{\pi}^1\}_0^{\tau_1})$. The conditional mean and variance achieved by the truncated pre-committed optimal policy is

$$\begin{aligned}
&\mathbb{E}\big[X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big]\\
=&X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1})e^{\int_{\tau_1}^T r(s)ds}-\frac{1}{2}\lambda(\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}))\Big(1-e^{\int_{\tau_1}^T B(s)'(\Sigma(s))^{-1}B(s)ds}\Big),\\
&\mathrm{Var}\big(X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big)\\
=&\frac{1}{4}[\lambda(\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}))]^2\Big(e^{\int_{\tau_1}^T B(s)'(\Sigma(s))^{-1}B(s)ds}-1\Big),
\end{aligned}$$

where the induced trade-off parameter is

$$\lambda(\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}))=2\Big(\beta_0-X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1})e^{\int_{\tau_1}^T r(s)ds}\Big)e^{-\int_{\tau_1}^T B(s)(\Sigma(s))^{-1}B(s)ds}.$$

If $\bar{X}(\tau_1; 0, x_0, \{\bar{\pi}^1\}_0^{\tau_1}) \le \beta_0 e^{-\int_{\tau_1}^T r(s)ds}$, i.e. $X(\tau_1; 0, x_0, \{\pi^*\}_0^{\tau_1}) \le \beta_0 e^{-\int_{\tau_1}^T r(s)ds}$, the truncated pre-committed optimal policy is also efficient after the first jump. Thus, we need not adjust the policy, i.e., $\bar{\pi}^1(s) = \pi^*(s)$ for $s \in [\tau_1, T]$, and we have

$$\begin{cases} \mathbb{E}\big[X\big(T; \tau_1, X(\tau_1; 0, x_0, \{\pi^*\}_0^{\tau_1}), \{\pi^*\}_{\tau_1}^T\big)\big] = \mathbb{E}\big[\bar{X}\big(T; \tau_1, \bar{x}_1, \{\bar{\pi}^1\}_{\tau_1}^T\big)\big], \\ \mathrm{Var}\big(X\big(T; \tau_1, X(\tau_1; 0, x_0, \{\pi^*\}_0^{\tau_1}), \{\pi^*\}_{\tau_1}^T\big)\big) = \mathrm{Var}\big(\bar{X}\big(T; \tau_1, \bar{x}_1, \{\bar{\pi}^1\}_{\tau_1}^T\big)\big). \end{cases}$$

While $\bar{X}(\tau_1; 0, x_0, \{\bar{\pi}^1\}_0^{\tau_1}) > \beta_0 e^{-\int_{\tau_1}^T r(s)ds}$, i.e. $X(\tau_1; 0, x_0, \{\pi^*\}_0^{\tau_1}) > \beta_0 e^{-\int_{\tau_1}^T r(s)ds}$, the truncated pre-committed optimal policy becomes inefficient on or after the first jump. Therefore, we replace it by another efficient policy with a new starting wealth level $\bar{x}_1$ and a new risk attitude parameter β_1,

$$\bar{\pi}^1(t, \bar{X}(t)) = -\big(\Sigma(t)\big)^{-1} B(t) \Big(\bar{X}(t) - \beta_1 e^{-\int_t^T r(s)ds}\Big), \quad \tau_1 \le t \le T,$$

where the new wealth process invested in the market $\bar{X}(\cdot)$ follows

$$\begin{cases} d\bar{X}(t) = [r(t)\bar{X}(t) + B(t)'\bar{\pi}^1(t, \bar{X}(t))]dt + \bar{\pi}^1(t, \bar{X}(t))'\sigma(t)dW(t) + \bar{\pi}^1(t, \bar{X}(t))'\int_{\mathbb{R}^\ell} \gamma(t, z)\widetilde{N}(dt, dz), \\ \bar{X}(\tau_1) = \bar{x}_1. \end{cases}$$

The conditional mean and variance of the terminal wealth determined by this new policy $\bar{\pi}^1(\cdot)$ are as follows

$$\begin{aligned} \mathbb{E}\big[\bar{X}\big(T; \tau_1, \bar{x}_1, \{\bar{\pi}^1\}_{\tau_1}^T\big)\big] &= \bar{x}_1 e^{\int_{\tau_1}^T r(s)ds} - \frac{1}{2}\lambda(\tau_1, \bar{x}_1)\Big(1 - e^{\int_{\tau_1}^T B(s)'(\Sigma(s))^{-1}B(s)ds}\Big) \\ \mathrm{Var}\big(\bar{X}\big(T; \tau_1, \bar{x}_1, \{\bar{\pi}^1\}_{\tau_1}^T\big)\big) &= \frac{1}{4}[\lambda(\tau_1, \bar{x}_1)]^2 \Big(e^{\int_{\tau_1}^T B(s)'(\Sigma(s))^{-1}B(s)ds} - 1\Big) \end{aligned}$$

with

$$\lambda(\tau_1, \bar{x}_1) = 2\Big(\beta_1 - \bar{x}_1 e^{\int_{\tau_1}^T r(s)ds}\Big) e^{-\int_{\tau_1}^T B(s)(\Sigma(s))^{-1}B(s)ds}.$$

Equalizing two conditional mean-variance pairs,

$$\begin{cases} \mathbb{E}\big[X\big(T; \tau_1, X(\tau_1; 0, x_0, \{\pi^*\}_0^{\tau_1}), \{\pi^*\}_{\tau_1}^T\big)\big] = \mathbb{E}\big[\bar{X}\big(T; \tau_1, \bar{x}_1, \{\bar{\pi}^1\}_{\tau_1}^T\big)\big], \\ \mathrm{Var}\big(X\big(T; \tau_1, X(\tau_1; 0, x_0, \{\pi^*\}_0^{\tau_1}), \{\pi^*\}_{\tau_1}^T\big)\big) = \mathrm{Var}\big(\bar{X}\big(T; \tau_1, \bar{x}_1, \{\bar{\pi}^1\}_{\tau_1}^T\big)\big), \end{cases}$$

we get $\bar{x}_1$ and β_1 given in (18) and (19), respectively.

Furthermore, we have

$$\begin{aligned} &\mathbb{E}\big[X\big(T; 0, x_0, \{\pi^*\}_0^T\big)\big] \\ =&\mathbb{E}\Big[\mathbb{E}\big[X\big(T; \tau_1, X(\tau_1; 0, x_0, \{\pi^*\}_0^{\tau_1}), \{\pi^*\}_{\tau_1}^T\big)\big]\Big|(0, x_0, \{\pi^*\}_0^{\tau_1})\Big] \\ =&\mathbb{E}\Big[\mathbb{E}\big[\bar{X}\big(T; \tau_1, \bar{x}_1, \{\bar{\pi}^1\}_{\tau_1}^T\big)\big]\Big|(0, x_0, \{\pi^*\}_0^{\tau_1})\Big] \\ =&\mathbb{E}\Big[\mathbb{E}\big[\bar{X}\big(T; \tau_1, \bar{x}_1, \{\bar{\pi}^1\}_{\tau_1}^T\big)\big]\Big|(0, x_0, \{\bar{\pi}^1\}_0^{\tau_1})\Big] \\ =&\mathbb{E}\big[\bar{X}\big(T; 0, x_0, \{\bar{\pi}^1\}_0^T\big)\big] \end{aligned}$$

and

$$
\begin{aligned}
&\mathrm{Var}\big(X\big(T;0,x_0,\{\pi^*\}_0^T\big)\big)\\
=&\mathbb{E}\Big[\mathrm{Var}\big(X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big)\Big|(0,x_0,\{\pi^*\}_0^{\tau_1})\Big]\\
&+\mathrm{Var}\Big(E\big[X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big]\Big|(0,x_0,\{\pi^*\}_0^{\tau_1})\Big)\\
=&\mathbb{E}\Big[\mathrm{Var}\big(\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^1\}_{\tau_1}^T\big)\big)\Big|(0,x_0,\{\pi^*\}_0^{\tau_1})\Big]+\mathrm{Var}\Big(\mathbb{E}\big[\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^1\}_{\tau_1}^T\big)\big]\Big|(0,x_0,\{\pi^*\}_0^{\tau_1})\Big)\\
=&\mathbb{E}\Big[\mathrm{Var}\big(\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^1\}_{\tau_1}^T\big)\big)\Big|(0,x_0,\{\bar{\pi}^1\}_0^{\tau_1})\Big]+\mathrm{Var}\Big(\mathbb{E}\big[\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^1\}_{\tau_1}^T\big)\big]\Big|(0,x_0,\{\bar{\pi}^1\}_0^{\tau_1})\Big)\\
=&\mathrm{Var}\big(\bar{X}\big(T;0,x_0,\{\bar{\pi}^1\}_0^T\big)\big).
\end{aligned}
$$

This means that mean and variance of the terminal wealth are equivalent under the above different policies.

Second, assume that the result of the theorem holds when the revised policy makes adjustment for k jumps, which is denoted by $\bar{\pi}^k(\cdot)$. We now proceed to prove that the result of the theorem still holds when the revised policy makes adjustment for $(k+1)$ jumps.

At time τ_1, we have $X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1})=\bar{X}(\tau_1;0,x_0,\{\bar{\pi}^{k+1}\}_0^{\tau_1})$. When $\bar{X}(\tau_1;0,x_0,\{\bar{\pi}^{k+1}\}_0^{\tau_1})\leq \beta_0 e^{-\int_{\tau_1}^T r(s)ds}$, the truncated pre-committed optimal policy with the risk attitude parameter β_0 and the initial wealth $X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1})$ is efficient policy, which yields the efficient conditional mean-variance pair

$$
\Big(\mathbb{E}\big[X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big],\mathrm{Var}\big(X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big)\Big).
$$

For time interval $[\tau_1,T]$, the revised policy $\bar{\pi}^{k+1}(\cdot)$ is the same as the revised policy $\bar{\pi}^k(\cdot)$ with β_0 and initial pair $\big(\tau_1,\bar{X}(\tau_1;0,x_0,\{\bar{\pi}^{k+1}\}_0^{\tau_1})\big)$. From the assumption of the mathematical induction, we have the following

$$
\begin{cases}
\mathbb{E}\big[X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big]=\mathbb{E}\big[\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^k\}_{\tau_1}^T\big)\big]=\mathbb{E}\big[\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^{k+1}\}_{\tau_1}^T\big)\big],\\
\mathrm{Var}\big(X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big)=\mathrm{Var}\big(\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^k\}_{\tau_1}^T\big)\big)=\mathrm{Var}\big(\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^{k+1}\}_{\tau_1}^T\big)\big).
\end{cases}
$$

While $\bar{X}(\tau_1;0,x_0,\{\bar{\pi}^{k+1}\}_0^{\tau_1})>\beta_0 e^{-\int_{\tau_1}^T r(s)ds}$, the truncated pre-committed optimal policy with the risk attitude parameter β_0 and the initial wealth $X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1})$ is no longer mean-variance efficient policy, which yields the inefficient conditional mean-variance pair

$$
\Big(\mathbb{E}\big[X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big],\mathrm{Var}\big(X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big)\Big).
$$

We have shown that the revised policy $\bar{\pi}^1(\cdot)$ achieves an efficient conditional mean-variance pair such that

$$
\begin{cases}
\mathbb{E}\big[X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big]=\mathbb{E}\big[\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^1\}_{\tau_1}^T\big)\big],\\
\mathrm{Var}\big(X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big)=\mathrm{Var}\big(\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^1\}_{\tau_1}^T\big)\big),
\end{cases}
$$

On the other hand, for time interval $[\tau_1,T]$, the revised policy $\bar{\pi}^{k+1}(\cdot)$ is the same as revised policy $\bar{\pi}^k(\cdot)$ with initial condition β_1 and $(\tau_1,\bar{x}_1)$. From the assumption of the mathematical induction, we have the following,

$$
\begin{cases}
\mathbb{E}\big[\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^1\}_{\tau_1}^T\big)\big]=\mathbb{E}\big[\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^k\}_{\tau_1}^T\big)\big]=\mathbb{E}\big[\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^{k+1}\}_{\tau_1}^T\big)\big],\\
\mathrm{Var}\big(\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^1\}_{\tau_1}^T\big)\big)=\mathrm{Var}\big(\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^k\}_{\tau_1}^T\big)\big)=\mathrm{Var}\big(\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^{k+1}\}_{\tau_1}^T\big)\big).
\end{cases}
$$

which implies

$$\begin{cases} \mathbb{E}\big[X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big]=\mathbb{E}\big[\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^{k+1}\}_{\tau_1}^T\big)\big], \\ \mathrm{Var}\big(X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big)=\mathrm{Var}\big(\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^{k+1}\}_{\tau_1}^T\big)\big). \end{cases}$$

Finally, we have

$$\begin{aligned} &\mathbb{E}\big[X\big(T;0,x_0,\{\pi^*\}_0^T\big)\big] \\ =&\mathbb{E}\Big[\mathbb{E}\big[X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big]\Big|(0,x_0,\{\pi^*\}_0^{\tau_1})\Big] \\ =&\mathbb{E}\Big[\mathbb{E}\big[\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^{k+1}\}_{\tau_1}^T\big)\big]\Big|(0,x_0,\{\pi^*\}_0^{\tau_1})\Big] \\ =&\mathbb{E}\Big[\mathbb{E}\big[\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^{k+1}\}_{\tau_1}^T\big)\big]\Big|(0,x_0,\{\bar{\pi}^{k+1}\}_0^{\tau_1})\Big] \\ =&\mathbb{E}\big[\bar{X}\big(T;0,x_0,\{\bar{\pi}^{k+1}\}_0^T\big)\big] \end{aligned}$$

and

$$\begin{aligned} &\mathrm{Var}\big(X\big(T;0,x_0,\{\pi^*\}_0^T\big)\big) \\ =&\mathbb{E}\Big[\mathrm{Var}\big(X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big)\Big|(0,x_0,\{\pi^*\}_0^{\tau_1})\Big] \\ &+\mathrm{Var}\Big(E\big[X\big(T;\tau_1,X(\tau_1;0,x_0,\{\pi^*\}_0^{\tau_1}),\{\pi^*\}_{\tau_1}^T\big)\big]\Big|(0,x_0,\{\pi^*\}_0^{\tau_1})\Big) \\ =&\mathbb{E}\Big[\mathrm{Var}\big(\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^{k+1}\}_{\tau_1}^T\big)\big)\Big|(0,x_0,\{\pi^*\}_0^{\tau_1})\Big]+\mathrm{Var}\Big(\mathbb{E}\big[\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^{k+1}\}_{\tau_1}^T\big)\big]\Big|(0,x_0,\{\pi^*\}_0^{\tau_1})\Big) \\ =&\mathbb{E}\Big[\mathrm{Var}\big(\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^{k+1}\}_{\tau_1}^T\big)\big)\Big|(0,x_0,\{\bar{\pi}^{k+1}\}_0^{\tau_1})\Big]+\mathrm{Var}\Big(\mathbb{E}\big[\bar{X}\big(T;\tau_1,\bar{x}_1,\{\bar{\pi}^{k+1}\}_{\tau_1}^T\big)\big]\Big|(0,x_0,\{\bar{\pi}^{k+1}\}_0^{\tau_1})\Big) \\ =&\mathrm{Var}\big(\bar{X}\big(T;0,x_0,\{\bar{\pi}^{k+1}\}_0^T\big)\big), \end{aligned}$$

which completes our proof. □

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