BOUNDARY LAYERS AND STABILIZATION OF THE SINGULAR KELLER-SEGEL SYSTEM

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Abstract. The original well-known Keller-Segel system proposed in [26] describing the chemotactic wave propagation remains poorly understood in many aspects due to the logarithmic singularity. As the chemical assumption rate is linear, the singular Keller-Segel model can be converted, via a Cole-Hopf type transformation, into a system of viscous conservation laws without singularity. But the chemical diffusion rate parameter ε for the original Keller-Segel system now plays a dual role in the transformed system by acting as the coefficients of both diffusion and nonlinear convection (which is newly generated by the Cole-Hopf transformation) terms. This is a new feature different from most of (if not all) other viscous conservation laws as we know, and raises new challenges in deriving the uniform estimates in ε. In this paper, we first consider the dynamics of the transformed Keller-Segel system in a bounded interval with time-dependent Dirichlet boundary conditions. By imposing some conditions on the boundary data, we show that boundary layer profiles are present as $\varepsilon \to 0$ and large-time profile of solutions will be determined by the boundary data (i.e. boundary stabilization). We employ the refined (weighted) energy estimates with the "effective viscous flux" technique to establish the uniform-in-ε estimates to show the emergence of boundary layer profiles. For asymptotic dynamics of solutions, we develop a new idea by exploring the convexity of an entropy expansion to get the basic L^1-estimate, on which our results are built up by the method of energy estimates. Finally we gain the results for the original singular Keller-Segel system by reversing the Cole-Hopf transformation. Numerical simulations are performed to interpret our analytical results and their implications.

1. Introduction

The oriented movement of species up/down to the chemical concentration gradient is termed as chemotaxis which has been a significant mechanism to interpret abundant pattern formation and biological processes such as bacteria band formation and aggregation [39, 49], slime mould formation [16], fish pigmentation patterning [42], angiogenesis in tumor progression [6–8], primitive streak formation [43], blood vessel formation [14], wound healing [45], and so on. Proposed by Keller-Segel in 1971, the chemotaxis model has two prototypes according to the chemotactic sensitivity function. One was the linear sensitivity and the other was the logarithmic sensitivity. The former was derived in [25] to model the self-aggregation of *Dictyostelium discoideum* in response to cyclic adenosine monophosphate (cAMP) and the latter in [26] to model the wave propagation of bacterial chemotaxis. Compared to massive results on the Keller-Segel (KS) model with linear sensitivity, much less is known on the KS model with logarithmic sensitivity due to its singularity nature. However logarithmic sensitivity complies with the Webber-Fecher law and has many prominent applications in biology (cf. [2, 3, 10, 22, 23]) in addition to its indispensable role to reproducing the bacterial traveling bands (cf. [50]). This paper is concerned with the original KS model proposed in [26]

$$\begin{cases} u_t = [Du_x - \chi u(\ln w)_x]_x, \\ w_t = \varepsilon w_{xx} - uw^m, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $w(x,t)$ denote the bacterial density and concentration of nutrient (chemical), respectively, at position x and time t. The parameter $D > 0$ is the diffusivity of bacterial, $\chi > 0$

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is referred to as the chemotactic coefficient measuring the intensity of chemotaxis, $\varepsilon \geq 0$ is the chemical diffusion rate and $m \geq 0$ is the consumption rate of nutrient.

It has been shown (cf. [24, 47, 50]) that the KS model (1.1) will produce traveling bands (pulsating waves) if $0 \leq m < 1$, and fronts if $m = 1$ and no traveling waves if $m > 1$, where the logarithmic sensitivity is indispensable to generate traveling waves. In the case of $0 \leq m < 1$, the KS model (1.1) was employed by Keller and Segel to interpret the bacterial traveling band formation observed in the experiment by Adler [1]. When $m = 1$, (1.1) was first used by Nossal [41] to model the boundary movement of bacterial and later by Levine *et al* [28] to model the dynamics between vascular endothelial growth factor (VEGF) and vascular endothelial cells (VECs) in the initiation of tumor angiogenesis. Except the existence of traveling waves, the understanding of (1.1) with $m \neq 1$ is very poor due to the singularity of logarithm $\ln w$ (at $w = 0$), where in particular the stability of traveling waves remains an outstanding open question to date except some instability results [11, 40]. However for the linear consumption case $m = 1$, the model can be understood to some extend since the logarithmic singularity can be resolved by a Cole-Hopf type transformation ([27, 36])

$$v = -(\ln w)_x = -\frac{w_x}{w}, \tag{1.2}$$

which converts the KS model(1.1) into a non-singular system of conservation laws as follows

$$\begin{cases} u_t - (\chi u v)_x = D u_{xx}, \\ v_t + \varepsilon (v^2)_x - u_x = \varepsilon v_{xx}. \end{cases} \tag{1.3}$$

Though the singularity no longer exists in (1.3), a quadratic nonlinear convection is generated. In multi-dimensions, v is a gradient vector and the curl of v is intrinsic required to be zero, namely curl$v = \nabla \times v = 0$. A characteristic feature of the transformed system (1.3) distinct from other system of conservation laws (e.g. see [4, 9, 48]) is that the parameter ε plays a dual role: coefficient of viscosity (diffusion) and nonlinear convection. Hence it is hard to justify the parameter $\varepsilon > 0$ is "good" or "bad" for analysis, and how to find a balance between the nonlinear convection and viscosity with the curl-free condition becomes an art of analysis. Indeed the transformed chemotaxis model (1.3) has been well understood in one-dimension for both $\varepsilon = 0$ and $\varepsilon > 0$ from various aspects such as the traveling wave solutions (cf. [5, 21, 31, 33–36]), global dynamics of large-data solutions in $\mathbb{R}$ (cf. [29, 38]) or in the bounded interval subject to various boundary conditions (cf. [30, 51, 53]). However it still remains poorly understood in multi-dimensions except few results on the small-data solutions (cf. [12, 15, 44, 52]) or radial solutions (cf. [54]). In addition to these works, there was another class of results by considering singular limits of solutions to (1.3) as $\varepsilon \to 0$. Such a topic is of particular interest since the vanishing as $\varepsilon \to 0$ occurs concurrently to both viscosity and quadratic nonlinear convection in the transformed system (1.3). It is also of relevance since the chemical diffusion rate $\varepsilon > 0$ was assumed to be zero in the analysis of many early works (cf. [24, 26, 28]) on the grounds of simplicity and hence it is desirable to reveal the role of ε. Next we shall first recall existing results connecting the limit problem of $\varepsilon \to 0$ and then propose our new questions.

If the spatial domain is unbounded (i.e. $x \in \mathbb{R}^N, N \geq 1$), it has been shown in [44, 50, 52] that both traveling wave solutions (see [50]) and global solutions of the Cauchy problem (see [44, 52]) are uniformly convergent in ε, namely the solutions with $\varepsilon > 0$ converges to those with $\varepsilon = 0$ as $\varepsilon \to 0$ in L^∞-norm. If the domain is an interval say $(0, 1)$, and zero mixed Neumann-Dirichlet (ND) boundary conditions are prescribed:

$$u_x|_{x=0,1} = 0, \quad v|_{x=0,1} = 0, \ \varepsilon \geq 0$$

it was shown in [53] that the solution is still uniformly convergent in ε. However if the Dirichlet boundary conditions are imposed, one cannot impose the boundary conditions for v with $\varepsilon = 0$ since otherwise the problem may be over-determined. In this circumstance, boundary layers may arise due to the possible mismatch of boundary conditions. This was first observed and numerically verified in a recent work by Li and Zhao in [30], and later was justified in [18]. Considering that the boundary conditions are dynamic in vivo environment for tumor angiogenesis, in this paper we consider the system (1.3) with time-dependent Dirichlet boundary values, and for simplicity hereafter we assume $\chi = D = 1$ since their specific values are not important for our analysis. Hence precisely we shall consider the initial-boundary value problem (1.3) for $(x,t) \in [0,1] \times [0,\infty)$ as follows:

$$\begin{cases} u_t - (uv)_x = u_{xx}, & x \in (0,1) \\ v_t + \varepsilon (v^2)_x - u_x = \varepsilon v_{xx}, & x \in (0,1) \\ (u,v)(x,0) = (u_0,v_0)(x),\ u_0 \geq 0, & x \in [0,1] \\ u(0,t) = u(1,t) = \alpha(t) \geq 0,\ v(0,t) = v(1,t) = \beta(t), \end{cases} \tag{1.4}$$

where $\alpha(t)$ and $\beta(t)$ are known functions dependent on t. In (1.4) we always assume $\varepsilon > 0$. The non-diffusive initial-boundary value problem associated with (1.4) is

$$\begin{cases} u_t - \chi(uv)_x = D u_{xx}, & x \in (0,1) \\ v_t - u_x = 0, & x \in (0,1) \\ (u,v)(x,0) = (u_0,v_0)(x), u_0 \geq 0, & x \in [0,1], \\ u(0,t) = u(1,t) = \alpha(t) \geq 0. \end{cases} \tag{1.5}$$

Since now the boundary conditions are time-dependent, the global existence and asymptotic behavior of solutions may become elusive due to time-variable boundary data. Whether the boundary layer profiles for constant Dirichlet boundary data can be destructed by time-varying boundary data is also concerned. Hence we set two goals to this paper. First we show that the global strong solutions of the initial-boundary value problem (1.4) and (1.5) exists and boundary layer profile will arise as $\varepsilon \to 0$ under mild conditions on boundary data $\alpha(t)$ and $\beta(t)$, where the solution component u converges in L^∞, v converges in L^2 while diverges in L^∞. Second, we prove under certain constraints, the time-dependent boundary data $\alpha(t)$ and $\beta(t)$ will act as the asymptotic profiles of solutions to (1.4) approaching some constant states. We remark that the approaches and estimates developed in previous works [18, 30] for constant boundary conditions are not adequate for our current problem with time-dependent boundary data and various delicate boundary estimates and uniform-in-ε estimates are desired. In this paper we shall introduce the so called "effective viscous flux" technique employed in the study of the Navier-Stokes equations (see [17, 37]) to gain the desired estimates to achieve our first goal. For the second goal, we develop a new entropy-like energy framework and fully explore the convexity of the entropy expansion to establish a basic L^1 energy estimate, on which the results of the asymptotic behavior of solutions are built up. We shall state our main results in the next section.

2. Statement of main results

To proceed, we first specify some notations for clarity. In the sequel, $H^k[0,1]$ denotes the usual k-th order Sobolev space on $[0,1]$ with norm $\|f\|_{H^k[0,1]} := \left(\sum_{j=0}^k \|\partial_x^j f\|^2\right)^{1/2}$, where we simply denote $\|\cdot\| := \|\cdot\|_{L^2[0,1]}$. We also use $\|\cdot\|_{L^\infty}$ to denote $\|\cdot\|_{L^\infty[0,1]}$. Unless otherwise specified, we use C to denote a generic positive constant and $C(t)$ denotes a generic positive constant which depends on t. The values of the constants may vary line by line according to the context.

The first result of this paper on the existence and uniform-in-ε boundedness of global solutions to (1.4) is stated as follows.

Theorem 2.1. *Assume that the initial and boundary data satisfy*

$$(u_0, v_0) \in H^2[0,1],\ u_0 \geq 0,\ \alpha(t) \geq 0,\ (\alpha,\beta)(t) \in C^2([0,\infty)),\ |\alpha(t)| \leq c_0, \tag{2.1}$$

where c_0 is a positive constant. Then for any $\varepsilon \geq 0$, the initial boundary value problem (1.4) has a unique global solution (u,v), such that for any $T > 0$, there hold that $(u,v) \in L^\infty(0,T;H^2(0,1)) \cap L^2(0,T;H^2(0,1))$, $u \geq 0$ and

$$\begin{aligned} &\|u\|_{H^1}^2 + \|u_t\|^2 + \|u_x\|_{L^\infty}^2 + \|v\|^2 + \|v\|_{L^\infty}^2 + \|v_t\|^2 + \varepsilon^{\frac{1}{2}} \|v_x\|^2 \\ &+ \int_0^T \left(\varepsilon^{\frac{1}{2}} \|u_{xx}\|^2 + \varepsilon^{\frac{3}{2}} \|v_{xx}\|^2 + \|u_{xt}\|^2 + \varepsilon \|v_{xt}\|^2 \right) d\tau \leq C(T), \end{aligned}$$

where $C(T)$ is a positive constant dependent on T but independent of ε.

The second result is concerned with the zero chemical diffusion limit of solutions of (1.4) and boundary layer emergence as $\varepsilon \to 0$. Before stating the results, we first define boundary layer solutions of the problem (1.4) (cf. [13, 20, 46, 55]).

Definition 2.1. *Let $(u^\varepsilon, v^\varepsilon)$ and (u^0, v^0) denote the solutions of the initial-boundary value problems (1.4) and (1.5), respectively. If there exists a non-negative function $\delta(\varepsilon)$ satisfying $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that*

$$\lim_{\varepsilon\to 0} \|(u^\varepsilon - u^0, v^\varepsilon - v^0)\|_{L^\infty([0,T);\, C[\delta,1-\delta])} = 0,$$
$$\liminf_{\varepsilon\to 0} \|(u^\varepsilon - u^0, v^\varepsilon - v^0)\|_{L^\infty([0,T);\, C[0,1])} > 0,$$

then the initial-boundary value problem (1.4) is said to have a boundary layer solution as $\varepsilon \to 0$ and $\delta(\varepsilon)$ is called a BL-thickness, where $\|(f,g)\|_{\mathcal{X}} = \|f\|_{\mathcal{X}} + \|g\|_{\mathcal{X}}$, $\mathcal{X} = L^\infty([0,T);\, C[0,1])$.

Remark 2.1. As mentioned in [13], the definition 2.1 does not determine the BL-thickness uniquely since any function $\delta_*(\varepsilon)$ satisfying $\delta_*(\varepsilon) > \delta(\varepsilon)$ for $0 < \varepsilon \ll 1$ is also a BL-thickness.

Then our second main result is the following.

Theorem 2.2. *Let the assumptions in Theorem 2.1 hold. Let $(u^\varepsilon, v^\varepsilon)$ and (u^0, v^0) be the solutions of the initial boundary value problems (1.4) and (1.5) respectively. Then*

(i) As $\varepsilon \to 0$, the following convergence holds:

$$\begin{cases} (u^\varepsilon, u^\varepsilon_x, v^\varepsilon, \varepsilon v^\varepsilon_x) \to (u^0, u^0_x, v^0, 0) & \text{strongly in } \ L^\infty\left([0,T); L^2(0,1)\right), \\ (u^\varepsilon_t, v^\varepsilon_t) \to (u^0_t, v^0_t) & \text{strongly in } \ L^2\left([0,T); L^2(0,1)\right). \end{cases}$$

(ii) There exists a function $\delta(\varepsilon)$ satisfying

$$\delta(\varepsilon) \to 0 \ \text{ and } \ \frac{\varepsilon^{\frac{1}{2}}}{\delta(\varepsilon)} \to 0, \ \text{ as } \ \varepsilon \to 0, \tag{2.2}$$

such that the initial-boundary value problem (1.4) has a boundary layer solution satisfying

$$\lim_{\varepsilon\to 0} \|v^\varepsilon - v^0\|_{L^\infty([0,T);\, C[\delta,1-\delta])} = 0, \tag{2.3}$$

$$\liminf_{\varepsilon\to 0} \|v^\varepsilon - v^0\|_{L^\infty([0,T);\, C[0,1])} > 0, \tag{2.4}$$

provided that $\beta(t) \neq \int_0^t u^0_x(0,s)ds + v_0(0)$.

The result in Theorem 2.2 (i) yields that $\lim_{\varepsilon\to 0} \|u^\varepsilon - u^0\|_{L^\infty([0,T);\, C[0,1])} = 0$. This implies that u^ε does not have boundary layer profile, and only v^ε has as given in Theorem 2.2 (ii).

Next we shall state the result on the asymptotic behavior of solutions to (1.4).

Theorem 2.3. *Consider the initial-boundary value problem (1.4). Suppose that the initial data $(u_0, v_0) \in H^1[0,1]$ are compatible with the boundary conditions. Assume that*

- *there exist constants $\underline{\alpha}, \overline{\alpha}, \overline{\beta}$, such that $0 < \underline{\alpha} = \inf \alpha(t) \le \sup \alpha(t) = \overline{\alpha} < \infty$ and $\sup |\beta(t)| = \overline{\beta} < \infty$, for all $t \ge 0$,*
- *$(\alpha_t, \beta_t) \in L^1(0,\infty) \cap L^2(0,\infty)$.*

Then for any $\varepsilon > 0$ there exists a unique global-in-time solution (u,v) to (1.4), such that $(u - \alpha(t), v - \beta(t)) \in L^\infty(0,\infty; H^1(0,1)) \cap L^2(0,\infty; H^2(0,1))$ and satisfies

$$\lim_{t\to\infty} \left(\|u(\cdot,t) - \alpha(t)\|^2_{H^1} + \|v(\cdot,t) - \beta(t)\|^2_{H^1}\right) = 0.$$

We have the following remark regarding Theorem 2.3.

Remark 2.2. *The conditions on the time-dependent boundary data admit a family of functions approaching constant states with certain decaying/growth rates, such as algebraic or exponential, as time goes to infinity. Since the temporal integrability of the boundary data is not required, boundary functions which approach constant states with slow decaying/growth rates, such* *as* $\alpha(t) = 2 \pm \frac{1}{(1+t)^\epsilon}$ *for* $0 < \epsilon \ll 1$*, are permitted. The long-time behavior result indicates that the solution decays asymptotically and its decay profile is determined by the boundary data. In addition, we require* $\alpha(t)$ *to be bounded from below and above away from zero, which is consistent with and generalizes the previous result [30] wherein the boundary data are constants.*

Finally we reverse the results of the transformed system to the pre-transformed chemotaxis model (1.1) with $m = 1$. The counterpart of the initial-boundary value problem of (1.1) with $m = 1$ corresponding to (1.3) reads as

$$\begin{cases} u_t = [Du_x - \chi u(\ln w)_x]_x, \\ w_t = \varepsilon w_{xx} - uw, \\ (u,w)(x,0) = (u_0,w_0)(x), \quad x \in [0,1], \\ u(0,t) = u(1,t) = \alpha(t) \geq 0, \ (\ln w)_x|_{x=0} = (\ln w)_x|_{x=1} = -\beta(t), \ \text{if } \varepsilon > 0, \\ u(0,t) = u(1,t) = \alpha(t) \geq 0, \quad \text{if} \quad \varepsilon = 0. \end{cases} \tag{2.5}$$

Then we have following results for (2.5).

Theorem 2.4. *Consider the problem (2.5).*

(i) Assume that the initial and boundary data satisfy

$$u_0 \in H^2, \ (\ln w_0)_x \in H^2, \ u_0(x) \geq 0, \ w_0(x) > 0, \quad (\alpha(t), \beta(t)) \in C^2([0,\infty)), \ |\alpha(t)| \leq c_0.$$

Then for any $\varepsilon \geq 0$*, the IVBP* (2.5) *has a unique global solution* (u,w)*, such that* $u \geq 0$ *and for any* $T > 0$*,*

$$\begin{cases} u \in L^\infty([0,T]; H^2(0,1)) \cap L^2([0,T]; H^2(0,1)), \\ w \in L^\infty([0,T]; H^3(0,1)) \cap L^2([0,T]; H^3(0,1)). \end{cases} \tag{2.6}$$

Let $(u^\varepsilon, w^\varepsilon)$ *and* (u^0, w^0) *be the solutions to* (2.5) *with* $\varepsilon > 0$ *and* $\varepsilon = 0$*, respectively. Then for any* $t > 0$*, as the the chemical diffusion coefficient* ε *tends to zero, there is a positive constant* $C(t)$ *independent of* ε *such that*

$$\left\|(u^\varepsilon - u^0)(\cdot,t)\right\|^2_{C[0,1]} + \left\|(w^\varepsilon - w^0)(\cdot,t)\right\|^2_{C[0,1]} \leq C(t)\varepsilon^{\frac{1}{2}}. \tag{2.7}$$

Moreover, there is a function $\delta(\varepsilon)$ *satisfying* $\delta(\varepsilon) \to 0$ and $\frac{\varepsilon^{\frac{1}{2}}}{\delta(\varepsilon)} \to 0$, as $\varepsilon \to 0$, *such that*

$$\lim_{\varepsilon \to 0} \|w^\varepsilon_x - w^0_x\|_{L^\infty([0,T];\, C[\delta, 1-\delta])} = 0, \tag{2.8}$$

$$\liminf_{\varepsilon \to 0} \|w^\varepsilon_x - w^0_x\|_{L^\infty([0,T];\, C[0,1])} > 0. \tag{2.9}$$

(ii) Let the initial data satisfy $(u_0, (\ln w_0)_x) \in H^1(0,1)$*, and let the boundary data satisfy the conditions in Theorem 2.3. Then for any* $\varepsilon > 0$ *there exists a unique global-in-time solution* (u,w) *to* (2.5)*, such that* $(u - \alpha(t), (\ln w)_x + \beta(t)) \in L^\infty(0,\infty; H^1(0,1)) \cap L^2(0,\infty; H^2(0,1))$ *and*

$$\lim_{t\to\infty} \|u(\cdot,t) - \alpha(t)\|_{L^\infty} = 0, \quad \|w(\cdot,t)\|_{L^\infty} \leq Ce^{-\frac{1}{2}(\underline{\alpha} - \varepsilon\overline{\beta}^2)t}.$$

Remark 2.3. *Although the result* (2.7) *shows that the solutions of the original Keller-Segel model (1.1) with* $m = 1$ *do not have boundary layer profiles, the results* (2.8) *and* (2.9) *indicate that the derivative of* w *will have boundary layer profiles.*

Remark 2.4. *The result in Theorem 2.4 (ii) means that when* $\underline{\alpha} > \varepsilon\overline{\beta}^2$ *(this condition is satisfied naturally in the case of* $\beta(t) = 0$*), the* L^∞*-norm of* w *will exponentially decay to zero as time goes to infinity. However the result for the case* $\underline{\alpha} < \varepsilon\overline{\beta}^2$ *is unclear. But our result implies that if the solutions diverge in this case, the divergence rate is not faster than an exponential rate.*

The rest of this paper is organized as follows. In section 3, we shall establish the global existence of solutions of (1.4) and prove Theorem 2.1. In section 4, we explore the vanishing limits as $\varepsilon \to 0$ of solutions (boundary layer solutions) and prove Theorem 2.2. The results on the asymptotic behavior of solutions (Theorem 2.3) will be shown in section 5, and the proof of Theorem 2.4 will be given in section 6. Finally we show the numerical simulations to illustrate boundary layer profiles and interpret our analytical results in section 7.

3. Proof of Theorem 2.1

In this section, we will prove Theorem 2.1. First, using the standard arguments (e.g. see [51]), one can show the local existence of solutions to (1.4).

Lemma 3.1 (Local existence). *Suppose that the assumptions in Theorem* 2.1 *hold. For any* $\varepsilon \geq 0$, *there exists a positive constant* T_0 *such that* (1.4) *has a unique solution* $(u, v) \in L^\infty\left([0, T_0); H^2(0,1)\right) \cap L^2\left([0, T_0); H^2(0,1)\right)$ *satisfying* $u \geq 0$ *in* $(x,t) \in [0,1] \times [0, T_0)$.

Next we derive some *a priori* uniform-in-ε estimates of solutions, which not only extend the local solutions to global ones, but also play important parts in investigating the vanishing diffusion limit. We depart from the following boundary estimates on (u_x, v_x).

Lemma 3.2. *Let the assumptions in Theorem* 2.1 *hold. Then it holds that*

$$\begin{cases} u_x(0,t) = -\dfrac{d}{dt}\left(\displaystyle\int_0^1 \int_0^x u(\xi,t) d\xi dx\right) + \displaystyle\int_0^1 uv dx - \alpha(t)\beta(t), \\ \varepsilon v_x(0,t) = -\dfrac{d}{dt}\left(\displaystyle\int_0^1 \int_0^x v(\xi,t) d\xi dx\right) + \displaystyle\int_0^1 u dx - \alpha(t) - \varepsilon \int_0^1 (v^2 - \beta^2(t)) dx, \end{cases} \tag{3.1}$$

and

$$\begin{cases} u_x(1,t) = \dfrac{d}{dt}\left(\displaystyle\int_0^1 \int_x^1 u(\xi,t) d\xi dx\right) + \displaystyle\int_0^1 uv dx - \alpha(t)\beta(t), \\ \varepsilon v_x(1,t) = \dfrac{d}{dt}\left(\displaystyle\int_0^1 \int_x^1 v(\xi,t) d\xi dx\right) + \displaystyle\int_0^1 u dx - \alpha(t) - \varepsilon \int_0^1 (v^2 - \beta^2(t)) dx. \end{cases} \tag{3.2}$$

Proof. By integrating (1.3) over $(0,x)$ and using the boundary condition in (1.4), we have

$$\begin{cases} u_x(0,t) = u_x - \dfrac{d}{dt}\left(\displaystyle\int_0^x u dx\right) + uv - \alpha(t)\beta(t), \\ \varepsilon v_x(0,t) = \varepsilon v_x - \dfrac{d}{dt}\left(\displaystyle\int_0^x v dx\right) + u - \alpha(t) - \varepsilon(v^2 - \beta^2(t)). \end{cases} \tag{3.3}$$

Then, integrating (3.3) with respect to x over $(0,1)$, yields (3.1). Similarly, (3.2) is obtained. □

Lemma 3.3. *Let the assumptions in Theorem* 2.1 *hold. Then for any* $t > 0$, *there exists a positive constant* $C(t)$ *which is dependent on* t *but independent of* ε, *such that*

$$\int_0^1 u(x,t) dx + \|v(\cdot,t)\|^2 + \int_0^t \left(\int_0^1 \frac{(u_x)^2}{u+1} dx + \varepsilon \|v_x\|^2\right) d\tau \leq C(t). \tag{3.4}$$

Proof. To resolve the logarithmic singularity in the following estimates, inspired by [30], we make a technical treatment by introducing a change of variable $\tilde{u} = u + 1$. Thus, problem (1.4) turns into

$$\begin{cases} \tilde{u}_t - \tilde{u}_{xx} = (\tilde{u}v)_x - v_x, \\ v_t - \varepsilon v_{xx} = (\tilde{u} - \varepsilon v^2)_x, \\ (\tilde{u}, v)(x,0) = (\tilde{u}_0, v_0)(x) = (u_0 + 1, v_0)(x), \ \tilde{u}_0(x) \geq 1, \quad x \in [0,1], \\ \tilde{u}(0,t) = \tilde{u}(1,t) = \alpha(t) + 1 \geq 1, \ v(0,t) = v(1,t) = \beta(t). \end{cases} \tag{3.5}$$

Multiplying the first equation of (3.5) by $\ln \tilde{u}$ and integrating the result by parts over $[0,1]$, we have

$$\begin{aligned} \frac{d}{dt}\int_0^1 \eta dx + \int_0^1 \frac{(\tilde{u}_x)^2}{\tilde{u}} dx + \int_0^1 \tilde{u}_x v dx &= (\tilde{u}_x \ln \tilde{u} + uv \ln \tilde{u})\Big|_{x=0}^{x=1} + \int_0^1 v \frac{\tilde{u}_x}{\tilde{u}} dx \\ &= \tilde{u}_x \ln \tilde{u}\Big|_{x=0}^{x=1} + \int_0^1 v \frac{\tilde{u}_x}{\tilde{u}} dx, \end{aligned} \tag{3.6}$$

where $\eta = \tilde{u}\ln\tilde{u} - \tilde{u} + 1 + R$ and R is a positive constant to be determined later. Multiplying the second equation of (3.5) by v and integrating the result by parts over $[0,1]$, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 v^2 dx - \int_0^1 \tilde{u}_x v dx + \varepsilon \|v_x\|^2 = \varepsilon v_x v\Big|_{x=0}^{x=1}. \tag{3.7}$$

Adding (3.6) to (3.7) and integrating the result over $(0,t)$ yield that

$$\begin{aligned}
&\int_0^1 \eta dx + \frac{1}{2}\|v\|^2 + \int_0^t \left(\int_0^1 \frac{(\tilde{u}_x)^2}{\tilde{u}} dx + \varepsilon \|v_x\|^2\right) d\tau \\
&\le \int_0^1 \eta_0 dx + \frac{1}{2}\|v_0\|^2 + \varepsilon \int_0^t v_x v\Big|_{x=0}^{x=1} d\tau + \int_0^t \tilde{u}_x \ln\tilde{u}\Big|_{x=0}^{x=1} d\tau + \int_0^t\int_0^1 v\frac{\tilde{u}_x}{\tilde{u}} dx d\tau.
\end{aligned} \tag{3.8}$$

Using the fact $\tilde{u} \ge 1$ and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\int_0^t\int_0^1 v\frac{\tilde{u}_x}{\tilde{u}} dx d\tau \le & \frac{1}{2}\int_0^t\int_0^1 \frac{(\tilde{u}_x)^2}{\tilde{u}} dx d\tau + \frac{1}{2}\int_0^t\int_0^1 \frac{v^2}{\tilde{u}} dx d\tau \\
\le & \frac{1}{2}\int_0^t\int_0^1 \frac{(\tilde{u}_x)^2}{\tilde{u}} dx d\tau + \frac{1}{2}\int_0^t\int_0^1 v^2 dx d\tau,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \eta_0 dx =& \int_0^1 (\tilde{u}_0\ln\tilde{u}_0 - \tilde{u}_0 + 1 + R) dx \\
\le& \int_0^1 \tilde{u}_0^2 dx - \int_0^1 u_0 dx + \int_0^1 R dx \le 2\|u_0\|^2 + 2 + R.
\end{aligned}$$

On the other hand, from the boundary conditions in (2.1), for any $t > 0$ there is a constant $c_1(t)$ which may depend on t such that

$$\|(\alpha,\beta)(\cdot)\|_{C^2[0,t)} \le c_1(t). \tag{3.9}$$

Thus, using Lemma 3.2, (3.9), integration by parts and Cauchy-Schwarz inequality, we can estimate the third term on the right-hand side of (3.8) as follows:

$$\begin{aligned}
\varepsilon\int_0^t v_x v\Big|_{x=0}^{x=1} d\tau =& \varepsilon\int_0^t \beta(\tau)\left(v_x(1,\tau) - v_x(0,\tau)\right) d\tau \\
=& \int_0^t \frac{d}{d\tau}\left(\int_0^1 v(x,\tau)dx\right)\beta(\tau)d\tau \\
=& \int_0^1 v(x,t)\beta(t)dx - \int_0^1 v(x,0)\beta(0)dx - \int_0^t\left(\int_0^1 v(x,\tau)dx\right)\beta'(\tau)d\tau \\
\le& c_1(t)\int_0^1 |v| dx + c_1(t)\int_0^t\int_0^1 |v| dx d\tau + C \\
\le& \frac{1}{4}\|v\|^2 + C(t)\int_0^t \|v\|^2 d\tau + C(t).
\end{aligned} \tag{3.10}$$

Noting that $\tilde{u}_x = u_x$, for the fourth term on the right-hand side of (3.8), we use Lemma 3.2, (2.1) and the integration by parts to get

$$
\begin{aligned}
\int_0^t \tilde{u}_x \ln \tilde{u}\Big|_{x=0}^{x=1} d\tau =& \int_0^t \ln(\alpha(\tau)+1)\left(\tilde{u}_x(1,\tau)-\tilde{u}_x(0,\tau)\right)d\tau = \int_0^t \ln(\alpha(\tau)+1)\frac{d}{d\tau}\left(\int_0^1 u(x,\tau)dx\right)d\tau \\
=& \left(\int_0^1 u(x,t)dx\right)\ln(\alpha(t)+1) - \left(\int_0^1 u(x,0)dx\right)\ln(\alpha(0)+1) \\
&- \int_0^t \left(\int_0^1 \frac{u(x,\tau)\alpha'(\tau)}{\alpha(\tau)+1}dx\right)d\tau \\
\leq& \ln(\alpha(t)+1)\int_0^1 u dx + C(t)\int_0^t\int_0^1 u dx d\tau + C(t) \\
\leq& d_1 \int_0^1 u dx + C(t)\int_0^t\int_0^1 u dx d\tau + C(t),
\end{aligned}
$$

where we have used the fact that $\alpha(t) \geq 0$, (3.9) and $\ln(\alpha(t)+1) \leq \ln(c_0+1) \leq d_1$, where $d_1 \geq 1$ is a constant. If we choose $R = 2d_1 e^{2d_1+2}$, then it holds that

$$
0 \leq u \leq d_1 u \leq d_1 \tilde{u} \leq \frac{1}{2}(\tilde{u}\ln\tilde{u} - \tilde{u} + 1 + 2d_1 e^{2d_1+2}) = \frac{1}{2}\eta.
$$

Inserting the above estimates into (3.8) yields

$$
\begin{aligned}
&\int_0^1 u dx + \frac{1}{4}\|v\|^2 + \frac{1}{2}\int_0^t\int_0^1 \frac{(\tilde{u}_x)^2}{\tilde{u}} dx d\tau + \varepsilon \int_0^t \|v_x\|^2 d\tau \\
\leq& C(t) + C(t)\int_0^t\int_0^1 u dx d\tau + C(t)\int_0^t \|v\|^2 d\tau \\
\leq& C(t) + C(t)\int_0^t \left(\int_0^1 u dx d\tau + \|v\|^2\right) d\tau,
\end{aligned}
$$

which results in (3.4) by the Gronwall's inequality. □

Lemma 3.4. *Let the assumptions in Theorem* 2.1 *hold. Then for any $t > 0$, there exists a constant $C(t) > 0$ which is dependent on t but independent of ε, such that*

$$
\|u(\cdot,t)\|^2 + \|v(\cdot,t)\|^2 + \int_0^t (\|u_x\|^2 + \varepsilon\|v_x\|^2)d\tau \leq C(t). \tag{3.11}
$$

Proof. Multiplying the first equation of (1.4) by u, integrating the result by parts over $[0,1]$, and adding the resultant equality to (3.7), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u\|^2 + \|v\|^2) + \|u_x\|^2 + \varepsilon\|v_x\|^2 \\
=& -\int_0^1 u v u_x dx + \int_0^1 u_x v dx + u^2 v\Big|_{x=0}^{x=1} + u_x u\Big|_{x=0}^{x=1} - \frac{2\varepsilon}{3}v^3\Big|_{x=0}^{x=1} + \varepsilon v_x v\Big|_{x=0}^{x=1}.
\end{aligned} \tag{3.12}
$$

Integrating (3.12) with respect to t and using the boundary conditions in (1.4), we have

$$
\begin{aligned}
&\frac{1}{2}(\|u\|^2 + \|v\|^2) + \int_0^t (\|u_x\|^2 + \varepsilon\|v_x\|^2)d\tau \\
=& \frac{1}{2}(\|u_0\|^2 + \|v_0\|^2) - \int_0^t\int_0^1 u v u_x dx d\tau + \int_0^t\int_0^1 u_x v dx d\tau + \int_0^t u_x u\Big|_{x=0}^{x=1} d\tau + \varepsilon\int_0^t v_x v\Big|_{x=0}^{x=1} d\tau.
\end{aligned} \tag{3.13}
$$

For the second and third terms on the right-hand side of (3.13), by the Gagliardo-Nirenberg and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}
&\int_0^t \int_0^1 uvu_x dx d\tau + \int_0^t \int_0^1 u_x v dx d\tau \\
\le& \frac{1}{4}\int_0^t \|u_x\|^2 d\tau + \int_0^t \|u\|_{L^\infty}^2 \|v\|^2 d\tau + \int_0^t \|v\|^2 d\tau \\
\le& \frac{1}{4}\int_0^t \|u_x\|^2 d\tau + C(t)\int_0^t (\|u\|^2 + \|u\|\,\|u_x\|) d\tau + C(t) \\
\le& \frac{1}{2}\int_0^t \|u_x\|^2 d\tau + C(t)\int_0^t \|u\|^2 d\tau + C(t),
\end{aligned}$$

where in the second inequality we have used (3.4). The last term on the right-hand side of (3.13) has been well estimated in (3.10). Therefore, we get from (3.4) and (3.10) that

$$\varepsilon \int_0^t v_x v\Big|_{x=0}^{x=1} d\tau \le \frac{1}{4}\|v\|^2 + C(t)\int_0^t \|v\|^2 d\tau + C(t) \le C(t).$$

For the fourth term on the right-hand side of (3.13), by Lemma 3.2 and integration by parts, we have

$$\begin{aligned}
\int_0^t u_x u\Big|_{x=0}^{x=1} d\tau =& \int_0^t \frac{d}{d\tau}\left(\int_0^1 u(x,\tau)dx\right)\alpha(\tau) d\tau \\
=& \alpha(t)\int_0^1 u(x,t)dx - \alpha(0)\int_0^1 u(x,0)dx - \int_0^t \int_0^1 u(x,\tau)\alpha'(\tau) dx d\tau \\
\le& C(t),
\end{aligned}$$

where we have used (3.4) and (3.9). Substituting these estimates into (3.13), we obtain

$$\frac{1}{2}(\|u\|^2 + \|v\|^2) + \frac{1}{2}\int_0^t (\|u_x\|^2 + \varepsilon\|v_x\|^2) d\tau \le C(t)\int_0^t \|u\|^2 d\tau + C(t),$$

which, together with Gronwall's inequality, yields (3.11). □

Lemma 3.5. *Let the assumptions in Theorem* 2.1 *hold. Then for any* $t > 0$*, it holds that*

$$\|u_t(\cdot,t)\|^2 + \|v_t(\cdot,t)\|^2 + \|u_x(\cdot,t)\|^2 + \int_0^t \left(\|u_t\|^2 + \|u_{xt}\|^2 + \varepsilon\|v_{xt}\|^2\right) d\tau \le C(t), \tag{3.14}$$

where the constant $C(t)$ *is independent of* ε *but dependent on* t.

Proof. We first multiply the first equation of (1.4) by u_t and integrate the resulting equation over $[0,1]\times[0,t]$ to get

$$\begin{aligned}
&\frac{1}{2}\|u_x\|^2 + \int_0^t \|u_t\|^2 d\tau \\
=& \frac{1}{2}\|u_{0x}\|^2 + \int_0^t \int_0^1 (uv)_x u_t dx d\tau + \int_0^t u_x u_t\Big|_{x=0}^{x=1} d\tau \\
=& \frac{1}{2}\|u_{0x}\|^2 - \int_0^t \int_0^1 uvu_{xt} dx d\tau + \int_0^t uvu_t\Big|_{x=0}^{x=1} d\tau + \int_0^t u_x u_t\Big|_{x=0}^{x=1} d\tau \\
=& \frac{1}{2}\|u_{0x}\|^2 - \int_0^t \int_0^1 uvu_{xt} dx d\tau + \int_0^t u_x u_t\Big|_{x=0}^{x=1} d\tau,
\end{aligned} \tag{3.15}$$

where in the last equality we have used the boundary conditions in (1.4). For the second term on the right-hand side of (3.15), by the Gagliardo-Nirenberg and Cauchy-Schwarz inequalities, (3.4) and (3.11), we have

$$
\begin{aligned}
-\int_0^t\int_0^1 uvu_{xt}dxd\tau \le& \frac{1}{8}\int_0^t \|u_{xt}\|^2 d\tau + 2\int_0^t \|u\|_{L^\infty}^2 \|v\|^2 d\tau \\
\le& \frac{1}{8}\int_0^t \|u_{xt}\|^2 d\tau + C\int_0^t (\|u\|^2+\|u_x\|^2)\|v\|^2 d\tau \\
\le& C(t) + \frac{1}{8}\int_0^t \|u_{xt}\|^2 d\tau.
\end{aligned} \tag{3.16}
$$

From Lemma 3.2 and integration by parts, the last term on the right-hand side of (3.15) can be estimated as follows:

$$
\begin{aligned}
\int_0^t u_x u_t\Big|_{x=0}^{x=1} d\tau =& \int_0^t (u_x(1,\tau)-u_x(0,\tau))\,\alpha'(\tau)d\tau = \int_0^t \frac{d}{d\tau}\left(\int_0^1 u(x,\tau)dx\right)\alpha'(\tau)d\tau \\
=& \alpha'(t)\int_0^1 u(x,t)dx - \alpha'(0)\int_0^1 u(x,0)dx - \int_0^t\int_0^1 u(x,\tau)\alpha''(\tau)dxd\tau \\
\le& C(t),
\end{aligned} \tag{3.17}
$$

where (3.4) and (3.9) have been used. Substituting (3.16) and (3.17) into (3.15), we have

$$
\frac{1}{2}\|u_x\|^2 + \int_0^t \|u_t\|^2 \le \frac{1}{8}\int_0^t \|u_{xt}\|^2 d\tau + C(t). \tag{3.18}
$$

Next, in order to obtain the estimate of $\int_0^t \|u_{xt}\|^2 d\tau$, differentiating (1.3) with respect time t, we get

$$
\begin{cases}
u_{tt} - u_{xxt} = (uv)_{xt}, \\
v_{tt} - \varepsilon v_{xxt} = (u-\varepsilon v^2)_{xt}.
\end{cases} \tag{3.19}
$$

Multiplying the first equation of (3.19) by u_t and the second by v_t, adding the results and integrating it over $[0,1]\times[0,t]$, we have

$$
\begin{aligned}
\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|v_t\|^2 =& \frac{1}{2}\left(\|u_t(0,x)\|^2 + \|v_t(0,x)\|^2\right) + \int_0^t\int_0^1 (u_{xxt}+(uv)_{xt})\,u_t dxd\tau \\
&+ \int_0^t\int_0^1 u_{xt}v_t dxd\tau + \varepsilon\int_0^t\int_0^1 v_{xxt}v_t dxd\tau - \varepsilon\int_0^t\int_0^1 (v^2)_{xt}v_t dxd\tau \\
=& \frac{1}{2}\left(\|u_t(0,x)\|^2 + \|v_t(0,x)\|^2\right) + \sum_{i=1}^4 I_i.
\end{aligned} \tag{3.20}
$$

For I_1, integrating by parts and using the boundary conditions in (1.4), we obtain

$$
\begin{aligned}
I_1 =& -\int_0^t \|u_{xt}\|^2 d\tau - \int_0^t\int_0^1 (uv)_t u_{xt} dxd\tau + \int_0^t u_{xt}u_t\Big|_{x=0}^{x=1} d\tau + \int_0^t (uv)_t u_t\Big|_{x=0}^{x=1} d\tau \\
=& -\int_0^t \|u_{xt}\|^2 d\tau - \int_0^t\int_0^1 (uv_t+u_tv)u_{xt} dxd\tau + \int_0^t u_{xt}u_t\Big|_{x=0}^{x=1} d\tau.
\end{aligned} \tag{3.21}
$$

Using (3.11), Gagliardo-Nirenberg and Cauchy-Schwarz inequalities, we can estimate the second term on the right-hand side of (3.21) as

$$\begin{aligned}
&-\int_0^t\int_0^1 (uv_t+u_tv)u_{xt}dxd\tau\\
\le&\frac{1}{8}\int_0^t \|u_{xt}\|^2 d\tau + C\int_0^t(\|u_tv\|^2+\|uv_t\|^2)d\tau\\
\le&\frac{1}{8}\int_0^t \|u_{xt}\|^2 d\tau + C\int_0^t \|u_t\|^2_{L^\infty}\|v\|^2 d\tau + C\int_0^t \|u\|^2_{L^\infty}\|v_t\|^2 d\tau\\
\le&\frac{1}{8}\int_0^t \|u_{xt}\|^2 d\tau + C(t)\int_0^t(\|u_t\|^2+\|u_t\|\|u_{xt}\|)d\tau + C(t)\int_0^t(\|u\|^2+\|u\|\|u_x\|)\|v_t\|^2 d\tau\\
\le&\frac{1}{4}\int_0^t \|u_{xt}\|^2 d\tau + C(t)\int_0^t \|u_t\|^2 d\tau + C(t)\int_0^t(1+\|u_x\|^2)\|v_t\|^2,
\end{aligned}$$

which updates (3.21) as

$$I_1 \le -\frac{3}{4}\int_0^t \|u_{xt}\|^2 d\tau + C(t)\int_0^t\|u_t\|^2 d\tau + C(t)\int_0^t(1+\|u_x\|^2)\|v_t\|^2 + \int_0^t u_{xt}u_t\Big|_{x=0}^{x=1}d\tau.$$

By Cauchy-Schwarz inequality, we have

$$I_2 \le\frac{1}{4}\int_0^t\|u_{xt}\|^2 d\tau + \int_0^t \|v_t\|^2 d\tau.$$

Integration by parts implies

$$I_3 =\varepsilon\int_0^t\int_0^1 v_{xxt}v_t dxd\tau = -\varepsilon\int_0^t\|v_{xt}\|^2 d\tau + \varepsilon\int_0^t v_{xt}v_t\Big|_{x=0}^{x=1}d\tau. \tag{3.22}$$

In order to estimate the boundary terms in (3.21) and (3.22), we follow the same procedure as in Lemma 3.2 and get

$$u_{xt}(1,t)-u_{xt}(0,t)=\frac{d}{dt}\left(\int_0^1 u_t(x,t)dx\right) \tag{3.23}$$

and

$$v_{xt}(1,t)-\varepsilon v_{xt}(0,t)=\frac{d}{dt}\left(\int_0^1 v_t(x,t)dx\right). \tag{3.24}$$

Using (3.9) and (3.23), integration by parts and Cauchy-Schwarz inequality, we can estimate the last term on the right-hand side of (3.21) as follows:

$$\begin{aligned}
\int_0^t u_{xt}u_t\Big|_{x=0}^{x=1}d\tau =& \int_0^t(u_{xt}(1,\tau)-u_{xt}(0,\tau))\,\alpha'(\tau)d\tau = \int_0^t\frac{d}{d\tau}\left(\int_0^1 u_\tau(x,\tau)dx\right)\alpha'(\tau)d\tau\\
=&\alpha'(t)\int_0^1 u_t(x,t)dx - \alpha'(0)\int_0^1 u_t(x,0)dx - \int_0^t\int_0^1 u_t(x,\tau)\alpha''(\tau)dxd\tau\\
\le&\frac{1}{4}\|u_t\|^2 + \int_0^t\|u_t\|^2 d\tau + C(t).
\end{aligned}$$

Similar to (3.10), we can estimate the last term on the right-hand side of (3.22) as

$$\begin{aligned}
\varepsilon\int_0^t v_{xt}v_t\Big|_{x=0}^{x=1}d\tau =& \int_0^t\frac{d}{d\tau}\left(\int_0^1\int_x^1 v_t(\xi,\tau)d\xi dx\right)\beta'(\tau)d\tau\\
\le&\int_0^1|v_t||\beta'(\tau)|dx + \int_0^1|v_t(0,t)||\beta'(0)|dx + \int_0^t\int_0^1|v_t||\beta''(\tau)|dxd\tau\\
\le&\frac{1}{4}\|v_t\|^2 + \int_0^t\|v_t\|^2 d\tau + C(t),
\end{aligned}$$

where we have used (3.24). Next, we need to estimate I_4. Integrating by parts, using the boundary conditions in (1.4), Gagliardo-Nirenberg and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned}
I_4 =& -\varepsilon\int_0^t\int_0^1 (v^2)_{xt}v_t dxd\tau = \varepsilon\int_0^t\int_0^1 (v^2)_t v_{xt}dxd\tau - \varepsilon\int_0^t (v^2)_t v_t\Big|_{x=0}^{x=1}d\tau\\
\le& \frac{\varepsilon}{2}\int_0^t \|v_{xt}\|^2 d\tau + 2\varepsilon\int_0^t \|v\|_{L^\infty}^2\|v_t\|^2 d\tau\\
\le& \frac{\varepsilon}{2}\int_0^t \|v_{xt}\|^2 d\tau + C\int_0^t (\varepsilon\|v\|^2+\varepsilon\|v_x\|^2)\|v_t\|^2 d\tau\\
\le& \frac{\varepsilon}{2}\int_0^t \|v_{xt}\|^2 d\tau + C(t)\int_0^t (1+\varepsilon\|v_x\|^2)\|v_t\|^2 d\tau.
\end{aligned}$$

Substituting the estimates of $I_i\,(i=1,2,3,4)$ into (3.20), we get

$$\begin{aligned}
&\frac{1}{4}\|u_t\|^2+\frac{1}{4}\|v_t\|^2+\frac{1}{2}\int_0^t(\|u_{xt}\|^2+\varepsilon\|v_{xt}\|^2)d\tau\\
\le& C(t)+C(t)\int_0^t(1+\|u_x\|^2+\varepsilon\|v_x\|^2)(\|u_t\|^2+\|v_t\|^2)d\tau.
\end{aligned}$$

Using Gronwall's inequality and (3.11), we obtain

$$\|u_t\|^2+\|v_t\|^2+\int_0^t(\|u_{xt}\|^2+\varepsilon\|v_{xt}\|^2)d\tau\le C(t), \tag{3.25}$$

which together with (3.18) leads to $\|u_x\|^2+\int_0^t\|u_t\|^2 d\tau\le C(t)$. This, along with (3.25), leads immediately to (3.14). □

The next lemma gives the estimate of L^∞-norm of (u_x, v). It turns out it is not easy to gain them by the routine procedure like the iteration method. Motivated by the studies for the Navier-Stokes equations (cf. [17, 19, 37]), we here introduce the following so-called "effective viscous flux $G(x,t)$":

$$G = u_x + uv. \tag{3.26}$$

From the first equation of (1.3), it is easy to see that

$$G_x = u_t. \tag{3.27}$$

The quantity effective viscous flux G will play an important role deriving the L^∞-norm of (u_x, v).

Lemma 3.6. *Let the assumptions in Theorem* 2.1 *hold. Then for any $t>0$, there exists a constant $C(t)>0$ which is independent of ε, such that*

$$\|u_x(\cdot,t)\|_{L^\infty}+\|v(\cdot,t)\|_{L^\infty}\le C(t). \tag{3.28}$$

Proof. Multiplying the second equation of (1.4) by $2nv^{2n-1}$($n\ge 1$ is an integer), integrating the result by parts over $(0,1)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 v^{2n}dx+2n(2n-1)\varepsilon\int_0^1 v^{2n-2}v_x^2dx\\
=&2n\int_0^1 v^{2n-1}u_xdx+\underbrace{2n\varepsilon\left[\beta(t)^{2n-1}v_x(1,t)-\beta(t)^{2n-1}v_x(0,t)\right]}_{R_1}-\frac{4n\varepsilon}{2n+1}\int_0^1(v^{2n+1})_xdx\\
\le&2n\int_0^1 v^{2n-1}Gdx-2n\int_0^1 v^{2n}udx+R_1\\
\le&2n\int_0^1(v^{2n}+1)|G|dx+R_1\\
\le&2n\|G\|_{L^\infty}\int_0^1 v^{2n}dx+2n\|G\|_{L^\infty}+R_1,
\end{aligned} \tag{3.29}$$

where we have used the boundary conditions in (1.4) and the non-negativity of u and v^{2n}. Now, we need to control $\|G\|_{L^\infty}$. Using Gagliardo-Nirenberg inequality, (3.26)-(3.27), (3.11) and (3.14), we get

$$\|G\|^2 \leq C(\|u_x\|^2 + \|uv\|^2) \leq C(\|u_x\|^2 + \|u\|_{L^\infty}^2 \|v\|^2) \leq C(t)$$

and

$$\|G\|_{L^\infty}^2 \leq C(\|G\|^2 + \|G\| \|G_x\|) \leq C(t)(1 + \|u_t\|^2) \leq C(t). \tag{3.30}$$

Using Lemma 3.2 and integration by parts, we have

$$\begin{aligned}
\int_0^t R_1 d\tau =& 2n\varepsilon \int_0^t \beta^{2n-1}(\tau)\left(v_x(1,\tau) - v_x(0,\tau)\right) d\tau \\
=& 2n \int_0^t \beta^{2n-1}(\tau) \frac{d}{d\tau}\left(\int_0^1 v(x,\tau)dx \right) d\tau \\
\leq& 2n \int_0^1 |v||\beta^{2n-1}(t)|dx + 2n \int_0^1 |v_0||\beta^{2n-1}(0)|dx \\
&+ 2n(2n-1) \int_0^t \int_0^1 |v||\beta^{2n-2}(\tau)||\beta'(\tau)|dxd\tau \\
\leq& 2nC^{2n}(t) + 2nC^{2n}(t) \int_0^1 |v|dx + 2n(2n-1)C^{2n}(t) \int_0^t \int_0^1 |v|dxd\tau \\
\leq& Cn^2 C^{2n}(t),
\end{aligned} \tag{3.31}$$

where we have used (3.9) and (3.11). Then it follows from (3.29)-(3.31) and Gronwall's inequality that

$$\int_0^1 v^{2n} dx \leq Cn^2 C^{2n}(t) \exp\left\{ 2n \int_0^t \|G\|_{L^\infty} d\tau \right\} \leq Cn^2 C^{2n}(t) \exp\{C(t)n\}. \tag{3.32}$$

Then, raising the power $\frac{1}{2n}$ to both sides of (3.32) and letting $n \to \infty$, we obtain that

$$\|v\|_{L^\infty} \leq C(t). \tag{3.33}$$

From (3.26), (3.11), (3.14), (3.30) and (3.33), we conclude that

$$\|u_x\|_{L^\infty} \leq \|G\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{L^\infty} \leq C(t).$$

Thus, the proof of (3.28) is completed. □

The following refined estimates of (u, v) will play an important role in the study of vanishing diffusion limit.

Lemma 3.7. *Let the assumptions in Theorem* 2.1 *hold. Then for any $t > 0$, it holds that*

$$\varepsilon^{\frac{1}{2}} \|v_x(\cdot,t)\|^2 + \int_0^t \left(\varepsilon^{\frac{1}{2}} \|u_{xx}\|^2 + \varepsilon^{\frac{3}{2}} \|v_{xx}\|^2 \right) d\tau \leq C(t), \tag{3.34}$$

where the constant $C(t)$ is independent of ε but depends on t.

Proof. Multiplying the first equation of (1.4) by $-2\varepsilon u_{xx}$ in L^2, using Cauchy-Schwarz inequality, (3.9) and Lemmas 3.4-3.6, we have

$$\begin{aligned}
\varepsilon \frac{d}{dt} \|u_x\|^2 + 2\varepsilon \|u_{xx}\|^2 =& -2\varepsilon \int_0^1 (uv)_x u_{xx} dx + 2\varepsilon u_x u_t \Big|_{x=0}^{x=1} \\
\leq& \frac{\varepsilon}{4} \|u_{xx}\|^2 + 4\varepsilon(\|u\|_{L^\infty}^2 \|v_x\|^2 + \|u_x\|_{L^\infty}^2 \|v\|^2) + 4\varepsilon c_1(t) \|u_x\|_{L^\infty} \\
\leq& \frac{\varepsilon}{4} \|u_{xx}\|^2 + C(t)\varepsilon \|v_x\|^2 + C(t)\varepsilon.
\end{aligned} \tag{3.35}$$

Next, we differentiate the second equation of (1.3) with respect to x, and subtract the resulting equation from the first equation of (1.3), to get

$$v_{xt} - \varepsilon v_{xxx} = u_t - (uv)_x - \varepsilon(v^2)_{xx}. \tag{3.36}$$

Multiplying (3.36) by $2\varepsilon v_x$ and integrating the result over $(0,1)$ yield

$$\begin{aligned}
&\varepsilon \frac{d}{dt} \|v_x\|^2 + 2\varepsilon^2 \|v_{xx}\|^2 \\
=&2\varepsilon \int_0^1 (v_x u_t - (uv)_x v_x)\, dx - 2\varepsilon^2 \int_0^1 v_x (v^2)_{xx} dx + 2\varepsilon^2 v_x v_{xx}\Big|_{x=0}^{x=1} \\
=&I_5 + I_6 + I_7.
\end{aligned} \tag{3.37}$$

I_5 can be estimated by Cauchy-Schwarz inequality and Lemmas 3.4-3.6 as

$$I_5 \le \varepsilon \|v_x\|^2 + 2\varepsilon(\|u_t\|^2 + \|uv_x\|^2 + \|u_x v\|^2) \le C(t)\varepsilon \|v_x\|^2 + C(t)\varepsilon.$$

For I_6, we use integration by parts, Cauchy-Schwarz inequality, Sobolev embedding theorem and Lemmas 3.4-3.6, to get

$$\begin{aligned}
I_6 =& -2\varepsilon^2 \int_0^1 v_x (v^2)_{xx} dx \\
=&2\varepsilon^2 \int_0^1 v_{xx}(v^2)_x dx - 2\varepsilon^2 v_x (v^2)_x\Big|_{x=0}^{x=1} \\
\le&\frac{\varepsilon^2}{4} \|v_{xx}\|^2 + 16\varepsilon^2 \|v\|_{L^\infty}^2 \|v_x\|^2 + 4\varepsilon^2 \|v\|_{L^\infty} \|v_x\|_{L^\infty}^2 \\
\le&\frac{\varepsilon^2}{4} \|v_{xx}\|^2 + C(t)\varepsilon^2 \|v_x\|^2 + C(t)\varepsilon^2 \Big(\|v_x\|^2 + \|v_x\| \|v_{xx}\|\Big) \\
\le&\frac{\varepsilon^2}{2} \|v_{xx}\|^2 + C(t)\varepsilon \|v_x\|^2.
\end{aligned}$$

Noting that $\varepsilon v_{xx} = v_t - u_x + \varepsilon(v^2)_x$, using (3.9), (3.28), the Gagliardo-Nirenberg and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}
I_7 =&2\varepsilon^2 v_x v_{xx}\Big|_{x=0}^{x=1} = 2\varepsilon v_x \left(v_t - u_x + \varepsilon(v^2)_x\right)\Big|_{x=0}^{x=1} \\
=&2\varepsilon v_x v_t\Big|_{x=0}^{x=1} - 2\varepsilon v_x u_x\Big|_{x=0}^{x=1} + 2\varepsilon^2 v_x (v^2)_x\Big|_{x=0}^{x=1} \\
\le&2\beta'(t)\varepsilon \|v_x\|_{L^\infty} + 2\varepsilon \|u_x\|_{L^\infty} \|v_x\|_{L^\infty} + 4\varepsilon^2 \|v\|_{L^\infty} \|v_x\|_{L^\infty}^2 \\
\le&C(t)\varepsilon \|v_x\|_{L^\infty} + C(t)\varepsilon^2(\|v_x\|^2 + \|v_x\| \|v_{xx}\|) \\
\le&C(t)\varepsilon \|v_x\|^2 + \frac{1}{8}\varepsilon^2 \|v_{xx}\|^2 + C(t)\varepsilon^{\frac{1}{2}} + C(t)\varepsilon^2 \|v_x\|^2 + \frac{1}{8}\varepsilon^2 \|v_{xx}\|^2 \\
\le&C(t)\varepsilon \|v_x\|^2 + \frac{1}{4}\varepsilon^2 \|v_{xx}\|^2,
\end{aligned}$$

where we have used the following inequality derived from Gagliardo-Nirenberg inequality and Young inequality

$$\begin{aligned}
C(t)\varepsilon \|v_x\|_{L^\infty} \le&C(t)\varepsilon(\|v_x\| + \|v_x\|^{\frac{1}{2}} \|v_{xx}\|^{\frac{1}{2}}) \\
\le&C(t)\varepsilon \|v_x\|^2 + \frac{1}{8}\varepsilon^2 \|v_{xx}\|^2 + C(t)\varepsilon^{\frac{1}{2}}.
\end{aligned}$$

Substituting the estimates of $I_i\,(i = 5, 6, 7)$ into (3.37) and adding the resulting inequality to (3.35) yield

$$\varepsilon \frac{d}{dt}(\|u_x\|^2 + \|v_x\|^2) + \varepsilon \|u_{xx}\|^2 + \varepsilon^2 \|v_{xx}\|^2 \le C(t)\varepsilon \|v_x\|^2 + C(t)\varepsilon^{\frac{1}{2}}.$$

Then the Gronwall's inequality leads to

$$\varepsilon(\|u_x\|^2 + \|v_x\|^2) + \int_0^t \left(\varepsilon \|u_{xx}\|^2 + \varepsilon^2 \|v_{xx}\|^2\right) d\tau \le C(t)\varepsilon^{\frac{1}{2}},$$

which immediately gives (3.34) and completes the proof of Lemma 3.7. □

Finally, Theorem 2.1 results from Lemmas 3.1-3.7.

4. VANISHING DIFFUSION LIMIT AND BOUNDARY LAYER SOLUTIONS

This section is concerned with the vanishing diffusion limit and boundary layer solutions. We first give the global existence of solutions to the non-diffusion problem (1.5).

Lemma 4.1. *Assume that the initial and boundary data satisfy*

$$(u_0, v_0) \in H^2,\ u_0 \ge 0,\ u(0,t) = u(1,t) = \alpha(t) \ge 0,\ \alpha(t) \in C^2([0,\infty)),\ |\alpha(t)| \le c_0.$$

Then for any $0 < T < \infty$, there exists a unique strong solution (u,v) to (1.5) in $[0,1]\times[0,T]$ satisfying $(u,v) \in L^\infty\left([0,T]; H^2(0,1)\right) \cap L^2\left([0,T]; H^2(0,1)\right)$.

Proof. Noting that the energy estimates established in Theorem 2.1 still hold true for $\varepsilon = 0$, i.e., for any $t > 0$, there is a constant $C(t) > 0$, such that

$$\begin{aligned} \|u(\cdot,t)\|_{H^1}^2 + \|u_t(\cdot,t)\|^2 + \|u_x(\cdot,t)\|_{L^\infty}^2 + \|v(\cdot,t)\|^2 + \|v(\cdot,t)\|_{L^\infty}^2 + \|v_t(\cdot,t)\|^2 \\ + \int_0^t \left(\|u_t\|^2 + \|v_t\|^2 + \|u_{xt}\|^2\right) d\tau \le C(t). \end{aligned} \tag{4.1}$$

Next, we will give the estimate of $\|v_x\|$. Differentiating the second equation of (1.5) with respect to x, then subtracting the resulting equation from the first equation of (1.5), we have

$$v_{xt} = u_t - (uv)_x. \tag{4.2}$$

Multiplying (4.2) by $2v_x$, integrating by parts over $(0,1)$ and using Cauchy-Schwarz inequality and (4.1), we deduce

$$\begin{aligned} \frac{d}{dt}\|v_x\|^2 =& 2\int_0^1 v_x u_t dx - 2\int_0^1 (uv)_x v_x dx \\ \le& \|v_x\|^2 + 2\|u_t\|^2 + 4\|uv_x\|^2 + 4\|u_x v\|^2 \\ \le& C(t)\|v_x\|^2 + C(t). \end{aligned}$$

Applying Gronwall's inequality, we have

$$\|v_x\|^2 \le C(t). \tag{4.3}$$

This together with the first equation of (1.5) and (4.1) means

$$\|u_{xx}\| \le \|u_t\| + \|uv_x\| + \|u_x v\| \le C(t). \tag{4.4}$$

Next differentiating (4.2) with respect to x, we have

$$v_{xxt} = u_{tx} - (uv)_{xx}. \tag{4.5}$$

Multiplying (4.5) by $2v_{xx}$, integrating by parts over $(0,1)$, using Cauchy-Schwarz inequality, (4.1) and (4.3), we deduce

$$\begin{aligned} \frac{d}{dt}\|v_{xx}\|^2 =& 2\int_0^1 v_{xx} u_{xt} dx - 2\int_0^1 (uv)_{xx} v_{xx} dx \\ \le& \|v_{xx}\|^2 + 2\|u_{xt}\|^2 + 6\|uv_{xx}\|^2 + 24\|u_x v_x\|^2 + 6\|u_{xx} v\|^2 \\ \le& C(t)\|v_{xx}\|^2 + 2\|u_{xt}\|^2 + C(t). \end{aligned}$$

Applying Gronwall's inequality and (4.1), we have

$$\|v_{xx}\|^2 \le C(t).$$

This together with (4.1), (4.3)-(4.4) and the local existence of solutions to (1.5) (see Lemma 3.1) completes the proof of Lemma 4.1. □

4.1. **Proof of Theorem 2.2 (i).** Let $(u^\varepsilon, v^\varepsilon)$ and (u^0, v^0) be the solutions to the initial boundary value problems (1.4) and (1.5), respectively. Let us set

$$\varphi^\varepsilon = u^\varepsilon - u^0, \quad \theta^\varepsilon = v^\varepsilon - v^0.$$

Then, by a straightforward calculation, we find that $(\varphi^\varepsilon, \theta^\varepsilon)$ satisfies the following the initial boundary value problem:

$$\begin{cases} \varphi_t^\varepsilon - \left(u^\varepsilon\theta^\varepsilon + v^0\varphi^\varepsilon\right)_x = \varphi_{xx}^\varepsilon, \\ \theta_t^\varepsilon - (\varphi^\varepsilon - \varepsilon\,(v^\varepsilon)^2)_x = \varepsilon v_{xx}^\varepsilon, \end{cases} \tag{4.6}$$

with initial data

$$(\varphi^\varepsilon, \theta^\varepsilon)\,(x,0) = (0,0), \tag{4.7}$$

and boundary condition:

$$\varphi^\varepsilon(0,t) = \varphi^\varepsilon(1,t) = 0. \tag{4.8}$$

Lemma 4.2. *Assume that the assumptions listed in Theorem 2.1 and Lemma 4.1 are satisfied. Then for any $t > 0$, there exists a positive constant $C(t)$ which is independent of ε, such that*

$$\left\|(u^\varepsilon - u^0)(\cdot,t)\right\|^2 + \left\|(v^\varepsilon - v^0)(\cdot,t)\right\|^2 + \int_0^t \left(\left\|(u^\varepsilon - u^0)_x\right\|^2 + \varepsilon\left\|v_x^\varepsilon\right\|^2\right) d\tau \le C(t)\varepsilon^{\frac{1}{2}} \tag{4.9}$$

and

$$\left\|(u^\varepsilon - u^0)_x(\cdot,t)\right\|^2 + \varepsilon\left\|v_x^\varepsilon(\cdot,t)\right\|^2 + \int_0^t \left(\left\|(u^\varepsilon - u^0)_t\right\|^2 + \left\|(v^\varepsilon - v^0)_t\right\|^2\right) d\tau \le C(t)\varepsilon^{\frac{1}{2}}. \tag{4.10}$$

Proof. Multiplying the first and second equations of (4.6) by $2\varphi^\varepsilon$ and $2\theta^\varepsilon$ respectively, integrating the result by parts on $[0,1]$, using the boundary condition (4.8), we have

$$\begin{aligned} &\frac{d}{dt}(\|\varphi^\varepsilon\|^2 + \|\theta^\varepsilon\|^2) + 2\left\|\varphi_x^\varepsilon\right\|^2 \\ =&-2\int_0^1 \left(u^\varepsilon\theta^\varepsilon + v^0\varphi^\varepsilon\right)\varphi_x^\varepsilon dx + 2\int_0^1 (\varphi^\varepsilon - \varepsilon\,(v^\varepsilon)^2)_x\theta^\varepsilon dx + 2\varepsilon\int_0^1 v_{xx}^\varepsilon\theta^\varepsilon dx \\ =&J_1 + J_2 + J_3. \end{aligned} \tag{4.11}$$

By Cauchy-Schwarz inequality and Theorem 2.1, we have

$$\begin{aligned} J_1 \le&\frac{1}{2}\left\|\varphi_x^\varepsilon\right\|^2 + C\left\|u^\varepsilon\right\|_{L^\infty}^2\left\|\theta^\varepsilon\right\|^2 + C\left\|v^0\right\|_{L^\infty}^2\left\|\varphi^\varepsilon\right\|^2 \le \frac{1}{2}\left\|\varphi_x^\varepsilon\right\|^2 + C(t)(\|\theta^\varepsilon\|^2 + \|\varphi^\varepsilon\|^2), \\ J_2 \le&\frac{1}{2}\left\|\varphi_x^\varepsilon\right\|^2 + C\left\|\theta^\varepsilon\right\|^2 + C\varepsilon^2\left\|v^\varepsilon\right\|_{L^\infty}^2\left\|v_x^\varepsilon\right\|^2 \le \frac{1}{2}\left\|\varphi_x^\varepsilon\right\|^2 + C\left\|\theta^\varepsilon\right\|^2 + C(t)\varepsilon^{\frac{3}{2}}, \\ J_3 \le& \left\|\theta^\varepsilon\right\|^2 + \varepsilon^2\left\|v_{xx}^\varepsilon\right\|^2. \end{aligned}$$

Substituting the estimates of $J_i\,(i = 1,2,3)$ into (4.11), we get

$$\frac{d}{dt}(\|\varphi^\varepsilon\|^2 + \|\theta^\varepsilon\|^2) + \left\|\varphi_x^\varepsilon\right\|^2 + \varepsilon\left\|v_x^\varepsilon\right\|^2 \le C(t)(\|\varphi^\varepsilon\|^2 + \|\theta^\varepsilon\|^2) + \varepsilon^2\left\|v_{xx}^\varepsilon\right\|^2 + C(t)\varepsilon^{\frac{3}{2}},$$

which, along with Gronwall's inequality, (4.7) and (3.34), leads to

$$\|\varphi^\varepsilon\|^2 + \|\theta^\varepsilon\|^2 + \int_0^t (\left\|\varphi_x^\varepsilon\right\|^2 + \varepsilon\left\|v_x^\varepsilon\right\|^2) d\tau \le C(t)\varepsilon^2\int_0^t \left\|v_{xx}^\varepsilon\right\|^2 d\tau + C(t)\varepsilon^{\frac{3}{2}} \le C(t)\varepsilon^{\frac{1}{2}}. \tag{4.12}$$

Then (4.9) follows from (4.12).

Next, we derive the estimates for $(\varphi_x^\varepsilon, \theta_x^\varepsilon)$. To this end, multiplying the first and second equations of (4.6) by $2\varphi_t^\varepsilon$ and $2\theta_t^\varepsilon$, respectively, and then integrating the results over $[0,1]$, we have

$$\begin{aligned}&\frac{d}{dt}\|\varphi_x^\varepsilon\|^2+2\|\varphi_t^\varepsilon\|^2+2\|\theta_t^\varepsilon\|^2\\=&2\int_0^1\left(u^\varepsilon\theta^\varepsilon+v^0\varphi^\varepsilon\right)_x\varphi_t^\varepsilon dx+2\int_0^1(\varphi^\varepsilon-\varepsilon(v^\varepsilon)^2)_x\theta_t^\varepsilon dx+2\varepsilon\int_0^1 v_{xx}^\varepsilon\theta_t^\varepsilon dx\\=&J_4+J_5+J_6.\end{aligned}\tag{4.13}$$

Next, we estimate $J_i\,(i=4,5,6)$. First, we write J_4 as follows:

$$J_4=2\int_0^1\left(u_x^\varepsilon\theta^\varepsilon\varphi_t^\varepsilon+v_x^0\varphi^\varepsilon\varphi_t^\varepsilon+v^0\varphi_x^\varepsilon\varphi_t^\varepsilon\right)dx+2\int_0^1 u^\varepsilon\theta_x^\varepsilon\varphi_t^\varepsilon dx=H_1+H_2.$$

It follows from Cauchy-Schwarz inequality, Lemma 4.1, Theorem 2.1 and (4.9) that

$$\begin{aligned}H_1\le&\frac{1}{2}\|\varphi_t^\varepsilon\|^2+C\|u_x^\varepsilon\|_{L^\infty}^2\|\theta^\varepsilon\|^2+C\left\|v_x^0\right\|_{L^\infty}^2\|\varphi^\varepsilon\|^2+C\left\|v^0\right\|_{L^\infty}^2\|\varphi_x^\varepsilon\|^2\\\le&\frac{1}{2}\|\varphi_t^\varepsilon\|^2+C(t)\|\varphi_x^\varepsilon\|^2+C(t)\varepsilon^{\frac{1}{2}}.\end{aligned}$$

For H_2, integrating by parts and using Cauchy-Schwarz inequality, Gagliardo-Nirenberg inequality, Theorem 2.1 and (4.9), we have

$$\begin{aligned}H_2=&-2\int_0^1 u^\varepsilon\theta^\varepsilon\varphi_{xt}^\varepsilon dx-2\int_0^1 u_x^\varepsilon\theta^\varepsilon\varphi_t^\varepsilon dx\\=&-2\frac{d}{dt}\int_0^1 u^\varepsilon\theta^\varepsilon\varphi_x^\varepsilon dx+2\int_0^1 u_t^\varepsilon\theta^\varepsilon\varphi_x^\varepsilon dx+2\int_0^1 u^\varepsilon\theta_t^\varepsilon\varphi_x^\varepsilon dx-2\int_0^1 u_x^\varepsilon\theta^\varepsilon\varphi_t^\varepsilon dx\\\le&-2\frac{d}{dt}\int_0^1 u^\varepsilon\theta^\varepsilon\varphi_x^\varepsilon dx+\|\theta^\varepsilon\|^2+C\|u_t^\varepsilon\|_{L^\infty}^2\|\varphi_x^\varepsilon\|^2+\frac{1}{2}\|\theta_t^\varepsilon\|^2+C\|u^\varepsilon\|_{L^\infty}^2\|\varphi_x^\varepsilon\|^2\\&+\frac{1}{2}\|\varphi_t^\varepsilon\|^2+C\|u_x^\varepsilon\|_{L^\infty}^2\|\theta^\varepsilon\|^2\\\le&-2\frac{d}{dt}\int_0^1 u^\varepsilon\theta^\varepsilon\varphi_x^\varepsilon dx+\frac{1}{2}(\|\theta_t^\varepsilon\|^2+\|\varphi_t^\varepsilon\|^2)+C(t)(1+\|u_t^\varepsilon\|^2+\|u_{xt}^\varepsilon\|^2)\|\varphi_x^\varepsilon\|^2+C(t)\varepsilon^{\frac{1}{2}}\\\le&-2\frac{d}{dt}\int_0^1 u^\varepsilon\theta^\varepsilon\varphi_x^\varepsilon dx+\frac{1}{2}(\|\theta_t^\varepsilon\|^2+\|\varphi_t^\varepsilon\|^2)+C(1+\|u_{xt}^\varepsilon\|^2)\|\varphi_x^\varepsilon\|^2+C(t)\varepsilon^{\frac{1}{2}}.\end{aligned}$$

Next, using Cauchy-Schwarz inequality and Theorem 2.1, we obtain

$$\begin{aligned}J_5=&2\int_0^1\varphi_x^\varepsilon\theta_t^\varepsilon dx-4\varepsilon\int_0^1 v^\varepsilon v_x^\varepsilon\theta_t^\varepsilon dx\\\le&\frac{1}{4}\|\theta_t^\varepsilon\|^2+C\|\varphi_x^\varepsilon\|^2+C\varepsilon^2\|v^\varepsilon\|_{L^\infty}^2\|v_x^\varepsilon\|^2\\\le&\frac{1}{4}\|\theta_t^\varepsilon\|^2+C\|\varphi_x^\varepsilon\|^2+C(t)\varepsilon^{\frac{3}{2}}\end{aligned}$$

and

$$J_6\le\frac{1}{4}\|\theta_t^\varepsilon\|^2+4\varepsilon^2\|v_{xx}^\varepsilon\|^2.$$

Substituting above estimates of $J_i\,(i=4,5,6)$ into (4.13), integrating the resulting inequality over $[0,t]$ and using Theorem 2.1 and (4.9), we get

$$\begin{aligned}
&\|\varphi_x^\varepsilon\|^2+\varepsilon\|v_x^\varepsilon\|^2+\int_0^t(\|\varphi_t^\varepsilon\|^2+\|\theta_t^\varepsilon\|^2)d\tau\\
\le&-2\int_0^1 u^\varepsilon\theta^\varepsilon\varphi_x^\varepsilon dx+C(t)\int_0^t(1+\|u_{xt}^\varepsilon\|^2)\|\varphi_x^\varepsilon\|^2d\tau+C\varepsilon^2\int_0^t\|v_{xx}^\varepsilon\|^2d\tau+C(t)\varepsilon^{\frac12}\\
\le&\frac12\|\varphi_x^\varepsilon\|^2+2\|u^\varepsilon\|_{L^\infty}^2\|\theta^\varepsilon\|^2+C(t)\int_0^t(1+\|u_{xt}^\varepsilon\|^2)\|\varphi_x^\varepsilon\|^2d\tau+C(t)\varepsilon^{\frac12}\\
\le&\frac12\|\varphi_x^\varepsilon\|^2+C(t)\int_0^t(1+\|u_{xt}^\varepsilon\|^2)\|\varphi_x^\varepsilon\|^2d\tau+C(t)\varepsilon^{\frac12}.
\end{aligned}$$

It follows from Gronwall's inequality and Theorem 2.1 that

$$\|\varphi_x^\varepsilon\|^2+\varepsilon\|v_x^\varepsilon\|^2+\int_0^t(\|\varphi_t^\varepsilon\|^2+\|\theta_t^\varepsilon\|^2)d\tau\le C(t)\varepsilon^{\frac12},$$

which gives (4.10) and the proof of Lemma 4.2 is completed. □

Finally, Theorem 2.2 is a consequence of Lemma 4.2.

4.2. **Proof of Theorem 2.2 (ii).** Inspired by a recent work [19], we first establish the following lemma by the weighted L^2-method dedicating to the boundary layer solutions.

Lemma 4.3. *Assume that the assumptions listed in Theorem 2.1 and Lemma 4.1 are satisfied. Then for any $t>0$, there exists a positive constant $C(t)$ which is independent of ε, such that*

$$\int_0^1\xi(x)|\theta_x^\varepsilon|^2dx\le C(t)\varepsilon^{\frac12},\tag{4.14}$$

where the weight function $\xi(x)$ is defined as $\xi(x)=x^2(1-x)^2,\ x\in[0,1]$.

Proof. We differentiate the second equation of (4.6) with respect to x to get

$$\theta_{xt}^\varepsilon-(\varphi^\varepsilon-\varepsilon(v^\varepsilon)^2)_{xx}=\varepsilon\theta_{xxx}^\varepsilon+\varepsilon v_{xxx}^0.\tag{4.15}$$

Multiplying (4.15) by $\xi(x)\theta_x^\varepsilon$ and integrating by parts over $[0,1]\times[0,T)$, one gets

$$\begin{aligned}
&\frac12\int_0^1\xi(x)|\theta_x^\varepsilon|^2dx+\varepsilon\int_0^t\int_0^1\xi(x)|\theta_{xx}^\varepsilon|^2dxd\tau\\
=&\frac{\varepsilon}{2}\int_0^t\int_0^1\xi''(x)|\theta_x^\varepsilon|^2dxd\tau+\int_0^t\int_0^1\varphi_{xx}^\varepsilon\xi(x)\theta_x^\varepsilon dxd\tau\\
&+\varepsilon\int_0^t\int_0^1\left((v^\varepsilon)^2+v_x^0\right)_{xx}\xi(x)\theta_x^\varepsilon dxd\tau\\
=&K_1+K_2+K_3.
\end{aligned}\tag{4.16}$$

First, by Lemma 4.1 and Theorem 2.1, we have

$$K_1\le C\varepsilon\int_0^t\|\theta_x^\varepsilon\|^2d\tau\le C\varepsilon\int_0^t\left(\|v_x^\varepsilon\|^2+\|v_x^0\|^2\right)d\tau\le C(t)\varepsilon^{\frac12}.\tag{4.17}$$

For K_2, using the second equation of (4.6), Cauchy-Schwarz inequality, Lemma 4.1, Theorem 2.1 and (4.9)-(4.10), we obtain

$$\begin{aligned}
K_2 =& \int_0^t \int_0^1 \xi(x)\theta_x^\varepsilon \left(\varphi_t^\varepsilon - \left(u^\varepsilon\theta^\varepsilon + v^0\varphi^\varepsilon\right)_x\right) dxd\tau \\
=& \int_0^t \int_0^1 \xi(x)\theta_x^\varepsilon \left(\varphi_t^\varepsilon - u_x^\varepsilon\theta^\varepsilon - u^\varepsilon\theta_x^\varepsilon - v^0\varphi_x^\varepsilon - v_x^0\varphi^\varepsilon\right) dxd\tau \\
\le& C\int_0^t \Big(1 + \|u_x^\varepsilon\|_{L^\infty}^2 + \|u^\varepsilon\|_{L^\infty}^2 + \left\|v^0\right\|_{L^\infty}^2 + \left\|v_x^0\right\|_{L^\infty}^2\Big) \int_0^1 \xi(x)|\theta_x^\varepsilon|^2 dxd\tau \\
&+ \int_0^t \Big(\|\varphi_t^\varepsilon\|^2 + \|\theta^\varepsilon\|^2 + \|\varphi_x^\varepsilon\|^2 + \|\varphi^\varepsilon\|^2\Big) d\tau \\
\le& C(t)\int_0^t\int_0^1 \xi(x)|\theta_x^\varepsilon|^2 dxd\tau + C(t)\varepsilon^{\frac{1}{2}}.
\end{aligned}$$

For K_3, integrating by parts, using Cauchy-Schwarz inequality, Lemma 4.1, Theorem 2.1 and (4.17), we obtain

$$\begin{aligned}
K_3 =& -\varepsilon \int_0^t\int_0^1 \Big((v^\varepsilon)^2 + v_x^0\Big)_x \xi(x)\theta_{xx}^\varepsilon dxd\tau - \varepsilon\int_0^t\int_0^1 \Big((v^\varepsilon)^2 + v_x^0\Big)_x \xi'(x)\theta_x^\varepsilon dxd\tau \\
\le& \frac{\varepsilon}{2}\int_0^t\int_0^1 \xi(x)|\theta_{xx}^\varepsilon|^2 dxd\tau + C\varepsilon\int_0^t \Big(\|v^\varepsilon\|_{L^\infty}^2 \|v_x^\varepsilon\|^2 + \left\|v_{xx}^0\right\|^2\Big) d\tau + C\varepsilon\int_0^t \|\theta_x^\varepsilon\|^2 d\tau \\
\le& \frac{\varepsilon}{2}\int_0^t\int_0^1 \xi(x)|\theta_{xx}^\varepsilon|^2 dxd\tau + C(t)\varepsilon^{\frac{1}{2}}.
\end{aligned}$$

Substituting above estimates for $K_i\,(i = 1, 2, 3)$ into (4.16), we get

$$\int_0^1 \xi(x)|\theta_x^\varepsilon|^2 dx + \varepsilon\int_0^t\int_0^1 \xi(x)|\theta_{xx}^\varepsilon|^2 dxd\tau \le C(t)\int_0^t\int_0^1 \xi(x)|\theta_x^\varepsilon|^2 dxd\tau + C(t)\varepsilon^{\frac{1}{2}},$$

which, together with Gronwall's inequality, leads to (4.14) and completes the proof of Lemma 4.3. ☐

Next, we show Theorem 2.2 (ii). For any $\delta \in (0, \frac{1}{2})$, by (4.14), we have

$$\begin{aligned}
\delta^2\int_\delta^{1-\delta} |\theta_x^\varepsilon|^2 dx =& \delta^2\int_\delta^{\frac{1}{2}} |\theta_x^\varepsilon|^2 dx + \delta^2\int_{\frac{1}{2}}^{1-\delta} |\theta_x^\varepsilon|^2 dx \\
\le& \int_\delta^{\frac{1}{2}} x^2|\theta_x^\varepsilon|^2 dx + \int_{\frac{1}{2}}^{1-\delta} (1-x)^2|\theta_x^\varepsilon|^2 dx \\
\le& 4\int_\delta^{\frac{1}{2}} x^2(1-x)^2|\theta_x^\varepsilon|^2 dx + 4\int_{\frac{1}{2}}^{1-\delta} x^2(1-x)^2|\theta_x^\varepsilon|^2 dx \\
\le& 4\int_\delta^{1-\delta} x^2(1-x)^2|\theta_x^\varepsilon|^2 dx \le C(t)\varepsilon^{\frac{1}{2}}.
\end{aligned}$$

This gives for any $\delta \in (0, \frac{1}{2})$ that

$$\|(v^\varepsilon - v^0)_x\|_{L^2[\delta,1-\delta]} \le C(t)\delta^{-1}\varepsilon^{\frac{1}{4}}. \tag{4.18}$$

Then, using the Morrey and Gagliardo-Nirenberg inequalities, (4.9) and (4.18), we end up with

$$\begin{aligned}
\|v^\varepsilon - v^0\|_{C[\delta,1-\delta]}^2 \le& C\|v^\varepsilon - v^0\|_{L^2[\delta,1-\delta]}^2 + C\|v^\varepsilon - v^0\|_{L^2[\delta,1-\delta]}\|(v^\varepsilon - v^0)_x\|_{L^2[\delta,1-\delta]} \\
\le& C\|v^\varepsilon - v^0\|_{L^2[0,1]}^2 + C\|v^\varepsilon - v^0\|_{L^2[0,1]}\|(v^\varepsilon - v^0)_x\|_{L^2[\delta,1-\delta]} \\
\le& C(t)\delta^{-1}\varepsilon^{\frac{1}{2}} \to 0, \quad \text{as} \quad \varepsilon \to 0,
\end{aligned}$$

for any function $\delta = \delta(\varepsilon)$ satisfying (2.2). Thus (2.3) is proved. We proceed to prove (2.4). To this end, integrating the second equation of (1.5) over $[0, t]$ and then setting $x = 0$, we have

$$v^0(0,t) = \int_0^t u_x^0(0,t)ds + v^0(0,0).$$

Thus, if we choose the appropriate boundary value $v^\varepsilon(0,t)$ such that

$$v^\varepsilon(0,t) \neq v^0(0,t), \text{ namely } \beta(t) \neq \int_0^t u_x^0(0,s)ds + v_0(0),$$

then we arrive at (2.4). Thus we complete the proof of Theorem 2.2 (ii). □

5. Long-time behavior

In this section, we prove Theorem 2.3. For the reader's convenience, we restate the initial-boundary value problem, which reads as

$$\begin{cases} u_t - (uv)_x = u_{xx}, & x \in (0,1), \quad t > 0, \\ v_t - u_x = \varepsilon v_{xx} - \varepsilon (v^2)_x, & \\ (u,v)(x,0) = (u_0, v_0)(x), & x \in [0,1], \\ u|_{x=0,x=1} = \alpha(t), \quad v|_{x=0,x=1} = \beta(t), & t \geq 0. \end{cases} \tag{5.1}$$

The proof of Theorem 2.3 is divided into four steps contained in a series of subsections. First of all, we note that, due to the conditions of Theorem 2.3 and maximum principle, it holds that $u(x,t) \geq 0$, provided that the solution exists. We depart with a basic estimate involving the logarithmic expansion of u.

5.1. **Entropy estimates.**

Lemma 5.1. *Let the assumptions in Theorem* 2.3 *hold. Then there exists a constant* $C > 0$ *which is independent on* t *and* ε*, such that*

$$E(u(\cdot,t),\alpha(t)) + \|v(\cdot,t) - \beta(t)\|^2 + \int_0^t \int_0^1 \frac{(u_x)^2}{u} dx d\tau + \varepsilon \int_0^t \|v_x\|^2 d\tau \leq C,$$

where

$$E(u,\alpha) \equiv \int_0^1 \Big\{ (u \ln u - u) - (\alpha \ln \alpha - \alpha) - (u - \alpha) \ln \alpha \Big\} dx \geq 0$$

denotes the entropy expansion.

Proof. We divide the proof into three steps.

Step 1. By a direct calculation, we can show that

$$\begin{aligned} &(u \ln u - u)_t - (\alpha \ln \alpha - \alpha)_t - [(u-\alpha)\ln \alpha]_t \\ =& u_t \ln u \ - \alpha_t \ln \alpha - (u-\alpha)_t \ln \alpha - (u-\alpha)\frac{\alpha_t}{\alpha} \\ =& (\ln u - \ln \alpha) u_t - (u - \alpha)\frac{\alpha_t}{\alpha}. \end{aligned} \tag{5.2}$$

By using the first equation of (5.1) and noting α depends only on t, we deduce that

$$\begin{aligned} (\ln u - \ln \alpha) u_t &= (\ln u - \ln \alpha)[(uv)_x + u_{xx}] \\ &= [(\ln u - \ln \alpha) uv]_x + [(\ln u - \ln \alpha) u_x]_x - v\, u_x - \frac{(u_x)^2}{u}. \end{aligned} \tag{5.3}$$

Then plugging (5.3) into (5.2), we find

$$\begin{aligned} &(u \ln u - u)_t - (\alpha \ln \alpha - \alpha)_t - [(u-\alpha)\ln \alpha]_t \\ =& [(\ln u - \ln \alpha) uv]_x + [(\ln u - \ln \alpha) u_x]_x - v\, u_x - \frac{(u_x)^2}{u} - (u-\alpha)\frac{\alpha_t}{\alpha}. \end{aligned} \tag{5.4}$$

After integrating (5.4) over $[0,1]$, and using the boundary conditions we have

$$\begin{aligned}&\frac{d}{dt}\left(\int_0^1 [(u\ln u-u)-(\alpha\ln\alpha-\alpha)-(u-\alpha)\ln\alpha]\,dx\right)+\int_0^1\frac{(u_x)^2}{u}dx\\ =&-\int_0^1 v\,u_x\,dx-\int_0^1(u-\alpha)\frac{\alpha_t}{\alpha}dx.\end{aligned}\tag{5.5}$$

Since β is independent of x, we derive from the second equation of (5.1) that

$$\begin{aligned}(v-\beta)_t-u_x&=\varepsilon(v-\beta)_{xx}-2\varepsilon\,v\,(v-\beta)_x-\beta_t\\ &=\varepsilon(v-\beta)_{xx}-2\varepsilon\,(v-\beta)\,(v-\beta)_x-2\varepsilon\,\beta\,(v-\beta)_x-\beta_t.\end{aligned}\tag{5.6}$$

Taking the L^2 inner product of (5.6) with $v-\beta$, we have

$$\frac{1}{2}\frac{d}{dt}\|v-\beta\|^2+\varepsilon\|v_x\|^2=\int_0^1(v-\beta)\,u_x\,dx-\int_0^1(v-\beta)\,\beta_t\,dx.\tag{5.7}$$

Note that

$$\begin{aligned}\int_0^1(v-\beta)\,u_x\,dx&=\int_0^1 v\,u_x\,dx-\int_0^1\beta\,u_x\,dx\\ &=\int_0^1 v\,u_x\,dx-\beta(\alpha-\alpha)\\ &=\int_0^1 v\,u_x\,dx.\end{aligned}$$

So we update (5.7) as

$$\frac{1}{2}\frac{d}{dt}\|v-\beta\|^2+\varepsilon\|v_x\|^2=\int_0^1 v\,u_x\,dx-\int_0^1(v-\beta)\,\beta_t\,dx.\tag{5.8}$$

By adding (5.8) to (5.5), we get that

$$\begin{aligned}&\frac{d}{dt}\left(E(u,\alpha)+\frac{1}{2}\|v-\beta\|^2\right)+\int_0^1\frac{(u_x)^2}{u}dx+\varepsilon\|v_x\|^2\\ =&-\int_0^1(u-\alpha)\frac{\alpha_t}{\alpha}dx-\int_0^1(v-\beta)\,\beta_t\,dx\\ \le&\frac{|\alpha_t|}{\alpha}\int_0^1|u-\alpha|dx+|\beta_t|\int_0^1|v-\beta|dx,\end{aligned}\tag{5.9}$$

where

$$E(u,\alpha)\equiv\int_0^1[(u\ln u-u)-(\alpha\ln\alpha-\alpha)-(u-\alpha)\ln\alpha]\,dx\ge 0.\tag{5.10}$$

We remark that in [30] the two terms on the right hand side of (5.9) vanish, due to the constant boundary conditions. The treatment of these non-constant terms is one of the major differences between this paper and [30].

Step 2. In this step, we derive an energy bound for the L^1 norm of u in terms of the entropy expansion defined by (5.10). We remark that under the Dirichlet type boundary conditions, the L^1 norm of u is not a conserved quantity. Hence, the energy method established in [32] for the mixed Neumann-Dirichlet boundary value problem can not be utilized for the Dirichlet boundary conditions. Luckily, such an issue was previously resolved in [30] for constant Dirichlet boundary data by developing a new approach through higher order nonlinear cancellation. Though such a technique also works for the time-dependent Dirichlet boundary conditions and can produce a uniform-in-time energy estimate for the low frequency part of the solution, the proof is lengthy and one needs more constraints on the boundary data to close the energy estimate. In this paper, we develop a very new approach (which has never appeared in any related work) to settle down the energy estimate for the low frequency part of the solution. The idea is to fully explore the convexity of the entropy expansion $E(u,\alpha)$ and compare it with a linear function. For this purpose, we set

$$F_\alpha(u) \equiv (u\ln u - u) - (\alpha\ln\alpha - \alpha) - (u-\alpha)\ln\alpha + (e-1)\alpha - u.$$

Then it can be readily checked that

$$\begin{aligned}
&F_\alpha(0) = e\alpha > 0,\\
&F_\alpha'(u) = \ln u - \ln\alpha - 1,\\
&F_\alpha''(u) = \frac{1}{u} \geq 0,\\
&F_\alpha'(e\alpha) = 0,\\
&F_\alpha(e\alpha) = 0,
\end{aligned}$$

which imply that $F_\alpha(u) \geq 0$ for any $u \geq 0$. This leads to

$$0 \leq u \leq (u\ln u - u) - (\alpha\ln\alpha - \alpha) - (u-\alpha)\ln\alpha + (e-1)\alpha,$$

and therefore,

$$0 \leq \int_0^1 u(x,t)\,dx \leq E(u,\alpha) + (e-1)\alpha. \tag{5.11}$$

Step 3. By plugging (5.11) into (5.9), we see that

$$\begin{aligned}
&\frac{d}{dt}\left(E(u,\alpha) + \frac{1}{2}\|v-\beta\|^2\right) + \int_0^1 \frac{(u_x)^2}{u}dx + \varepsilon\|v_x\|^2\\
&\leq \frac{|\alpha_t|}{\underline{\alpha}}E(u,\alpha) + e|\alpha_t| + \frac{|\beta_t|}{2} + \frac{|\beta_t|}{2}\|v-\beta\|^2,
\end{aligned} \tag{5.12}$$

where we used the first assumption of Theorem 2.3 and the Cauchy-Schwarz inequality. By applying the Gronwall's inequality to (5.12), we have

$$\begin{aligned}
&E(u(\cdot,t),\alpha(t)) + \frac{1}{2}\|v(\cdot,t) - \beta(t)\|^2\\
&\leq \exp\left\{\int_0^t \left(\frac{|\alpha_\tau|}{\underline{\alpha}} + |\beta_\tau|\right)d\tau\right\} \times \left[\int_0^t \left(e|\alpha_\tau| + \frac{|\beta_\tau|}{2}\right)d\tau\right.\\
&\qquad\qquad\left. + E(u_0,\alpha_0) + \frac{1}{2}\|v_0 - \beta_0\|^2\right].
\end{aligned} \tag{5.13}$$

By using the second assumption of Theorem 2.3, we deduce from (5.13) that

$$E(u(\cdot,t),\alpha(t)) + \frac{1}{2}\|v(\cdot,t) - \beta(t)\|^2 \leq C, \quad \forall\, t > 0, \quad \forall\, \varepsilon \geq 0, \tag{5.14}$$

where the constant C is independent of time and ε. By plugging (5.14) into (5.12), then integrating the resulting inequality with respect to time, we have in particular,

$$\int_0^t \int_0^1 \frac{(u_x)^2}{u}dxd\tau + \varepsilon\int_0^t \|v_x\|^2 d\tau \leq C, \quad \forall\, t > 0, \quad \forall\, \varepsilon \geq 0, \tag{5.15}$$

where the constant C is independent of time and ε. This together with (5.14) completes the entropy estimate and hence the proof of Lemma 5.1. □

5.2. **L^2-estimates.** To perform further energy estimates, we let

$$\tilde{u} \equiv u - \alpha, \quad \tilde{v} \equiv v - \beta,$$

where (u,v) satisfies (1.4). Then $(\tilde{u},\tilde{v})$ satisfies

$$\begin{cases}
\tilde{u}_t - (\tilde{u}\tilde{v})_x - \alpha\tilde{v}_x - \beta\tilde{u}_x = \tilde{u}_{xx} - \alpha_t,\\
\tilde{v}_t - \tilde{u}_x = \varepsilon\tilde{v}_{xx} - 2\varepsilon\tilde{v}\tilde{v}_x - 2\varepsilon\beta\tilde{v}_x - \beta_t,\\
(\tilde{u},\tilde{v})(x,0) = (u_0 - \alpha, v_0 - \beta)(x),\\
\tilde{u}|_{x=0,x=1} = 0, \quad \tilde{v}|_{x=0,x=1} = 0.
\end{cases} \tag{5.16}$$

Lemma 5.2. *Let the assumptions in Theorem* 2.3 *hold. Then there exists a constant* $C > 0$ *which is independent on* t *and* ε*, such that*

$$\|\tilde{u}(\cdot,t)\|^2 + \alpha(t)\|\tilde{v}(\cdot,t)\|^2 + \int_0^t \|\tilde{u}_x\|^2 d\tau \le C.$$

Proof. Taking the L^2 inner product of the first equation of (5.16) with $\tilde{u}$, we have

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}\|^2 + \|\tilde{u}_x\|^2 = -\int_0^1 \tilde{u}\,\tilde{v}\,\tilde{u}_x\,dx + \alpha\int_0^1 \tilde{u}\,\tilde{v}_x\,dx - \alpha_t\int_0^1 \tilde{u}\,dx. \tag{5.17}$$

Taking the L^2 inner product of the second equation of (5.16) with $\tilde{v}$ yields

$$\frac{1}{2}\frac{d}{dt}\|\tilde{v}\|^2 + \varepsilon\|\tilde{v}_x\|^2 = \int_0^1 \tilde{v}\,\tilde{u}_x\,dx - \beta_t\int_0^1 \tilde{v}\,dx. \tag{5.18}$$

Multiplying (5.18) by α, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\alpha\|\tilde{v}\|^2\right) + \varepsilon\alpha\|\tilde{v}_x\|^2 &= \alpha\int_0^1 \tilde{v}\,\tilde{u}_x\,dx - \alpha\,\beta_t\int_0^1 \tilde{v}\,dx + \frac{\alpha_t}{2}\|\tilde{v}\|^2\\ &= -\alpha\int_0^1 \tilde{u}\,\tilde{v}_x\,dx - \alpha\,\beta_t\int_0^1 \tilde{v}\,dx + \frac{\alpha_t}{2}\|\tilde{v}\|^2,\end{aligned} \tag{5.19}$$

where we have applied integration by parts to the first term on the right hand side of (5.19). Adding (5.19) to (5.17), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}\|^2 + \alpha\|\tilde{v}\|^2\right) + \|\tilde{u}_x\|^2 + \varepsilon\alpha\|\tilde{v}_x\|^2\\ &= -\int_0^1 \tilde{u}\,\tilde{v}\,\tilde{u}_x\,dx - \alpha_t\int_0^1 \tilde{u}\,dx - \alpha\,\beta_t\int_0^1 \tilde{v}\,dx + \frac{\alpha_t}{2}\|\tilde{v}\|^2.\end{aligned} \tag{5.20}$$

Now, we estimate the first term on the right hand side of (5.20) by using the L^1 estimate obtained from the previous subsection. To this end, we observe that

$$\left|-\int_0^1 \tilde{u}\,\tilde{v}\,\tilde{u}_x\,dx\right| \le \frac{1}{2}\|\tilde{u}\|_{L^\infty}^2\|\tilde{v}\|^2 + \frac{1}{2}\|\tilde{u}_x\|^2, \tag{5.21}$$

where $\|\tilde{u}\|_{L^\infty}^2$ can be estimated through the following procedure:

Step 1. Note that for any $x \in [0,1]$ and $t > 0$,

$$\tilde{u}(x,t) = \int_0^x \tilde{u}_y\,dy,$$

which implies

$$\|\tilde{u}\|_{L^\infty}^2 \le \left(\int_0^1 |\tilde{u}_x|\,dx\right)^2.$$

Step 2. Since $\tilde{u} = u - \alpha$ and α is independent of x, it holds that $\tilde{u}_x = u_x$. Then by Hölder's inequality and the positivity of u, we have

$$\|\tilde{u}\|_{L^\infty}^2 \le \left(\int_0^1 u\,dx\right)\left(\int_0^1 \frac{(u_x)^2}{u}\,dx\right). \tag{5.22}$$

Step 3. By applying (5.14) to (5.11) and using the first assumption of Theorem 2.3, we obtain

$$\int_0^1 u(x,t)\,dx \le C, \qquad \forall\ t > 0. \tag{5.23}$$

Step 4. By applying (5.23) to the first term on the right hand side of (5.22), we obtain

$$\|\tilde{u}\|_{L^\infty}^2 \le C\int_0^1 \frac{(u_x)^2}{u}\,dx. \tag{5.24}$$

By plugging the preceding estimate into (5.21), we find

$$\left|-\int_0^1 \tilde{u}\,\tilde{v}\,\tilde{u}_x\,dx\right| \le \frac{C}{2}\left(\int_0^1 \frac{(u_x)^2}{u}\,dx\right)\|\tilde{v}\|^2 + \frac{1}{2}\|\tilde{u}_x\|^2,$$

which updates (5.20) as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}\|^2+\alpha\|\tilde{v}\|^2\right)+\frac{1}{2}\|\tilde{u}_x\|^2+\varepsilon\alpha\|\tilde{v}_x\|^2\\ \le&\frac{1}{2}\left(C\int_0^1\frac{(u_x)^2}{u}\,dx+|\alpha_t|\right)\|\tilde{v}\|^2+|\alpha_t|\int_0^1|\tilde{u}|\,dx+\alpha\,|\beta_t|\int_0^1|\tilde{v}|\,dx.\end{aligned}\tag{5.25}$$

Note that

$$|\alpha_t|\int_0^1 |\tilde{u}|\,dx \le \frac{|\alpha_t|}{2}+\frac{|\alpha_t|}{2}\|\tilde{u}\|^2,$$

and

$$\alpha|\beta_t|\int_0^1|\tilde{v}|dx \le \frac{\alpha|\beta_t|}{2}+\frac{\alpha|\beta_t|}{2}\|\tilde{v}\|^2.$$

So we update (5.25) as

$$\begin{aligned}&\frac{d}{dt}\left(\|\tilde{u}\|^2+\alpha\|\tilde{v}\|^2\right)+\|\tilde{u}_x\|^2+2\varepsilon\alpha\|\tilde{v}_x\|^2\\ \le&\left(\frac{C}{\underline{\alpha}}\int_0^1\frac{(u_x)^2}{u}\,dx+\frac{|\alpha_t|}{\underline{\alpha}}+|\alpha_t|+|\beta_t|\right)\left(\|\tilde{u}\|^2+\alpha\|\tilde{v}\|^2\right)+|\alpha_t|+\overline{\alpha}\,|\beta_t|,\end{aligned}\tag{5.26}$$

where we have used the first assumption of Theorem 2.3. Applying the Gronwall's inequality to (5.26) and using (5.15) and the second assumption of Theorem 2.3, we find that

$$\|\tilde{u}(\cdot,t)\|^2+\alpha\|\tilde{v}(\cdot,t)\|^2 \le C, \quad \forall\ t>0,\tag{5.27}$$

for some constant C which is independent of t and ε. Plugging (5.27) back into (5.26), then integrating the resulting inequality with respect to time, we conclude that

$$\int_0^t \|\tilde{u}_x\|^2 d\tau \le C, \quad \forall\ t>0,\tag{5.28}$$

where the constant C is independent of t and ε. This completes the energy estimate for the low frequency part of the solution. □

Next, we shall move on to the estimation of the first order derivatives of the solution.

5.3. H^1-estimates.

Lemma 5.3. *Let the assumptions in Theorem* 2.3 *hold. Then it follows that*

$$\|\tilde{u}_x(\cdot,t)\|^2+\|\tilde{v}_x(\cdot,t)\|^2+\int_0^t\left(\|\tilde{u}_{xx}\|^2+\varepsilon\|\tilde{v}_{xx}\|^2\right)d\tau \le C,$$

where the constant C is independent of t, but is inversely proportional to ε.

Proof. Taking the L^2 inner products of the first equation of (5.16) with $-\tilde{u}_{xx}$, and the second with $-\tilde{v}_{xx}$, respectively, then adding the results, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}_x\|^2+\|\tilde{v}_x\|^2\right)+\|\tilde{u}_{xx}\|^2+\varepsilon\|\tilde{v}_{xx}\|^2\\ =&-\int_0^1\left(\tilde{v}\tilde{u}_x+\tilde{u}\tilde{v}_x+\alpha\tilde{v}_x+\beta\tilde{u}_x\right)\tilde{u}_{xx}\,dx+\alpha_t\int_0^1\tilde{u}_{xx}\,dx\\ &+2\varepsilon\int_0^1\tilde{v}\tilde{v}_x\tilde{v}_{xx}\,dx+2\varepsilon\beta\int_0^1\tilde{v}_x\tilde{v}_{xx}\,dx-\int_0^1\tilde{u}_x\tilde{v}_{xx}\,dx+\beta_t\int_0^1\tilde{v}_{xx}\,dx\\ =&\sum_{i=1}^6\mathcal{I}_i.\end{aligned}\tag{5.29}$$

For the right hand side of (5.29), we first apply the basic Cauchy-Schwarz inequality to deduce

$$
\begin{aligned}
\mathcal{I}_1 &\le \frac{1}{4}\|\tilde{u}_{xx}\|^2 + 4\left(\|\tilde{v}\|^2_{L^\infty}\|\tilde{u}_x\|^2 + \|\tilde{u}\|^2_{L^\infty}\|\tilde{v}_x\|^2 + \alpha^2\|\tilde{v}_x\|^2 + \beta^2\|\tilde{u}_x\|^2\right);\\
\mathcal{I}_2 &\le \frac{1}{4}\|\tilde{u}_{xx}\|^2 + |\alpha_t|^2;\\
\mathcal{I}_3 &\le \frac{\varepsilon}{8}\|\tilde{v}_{xx}\|^2 + 8\varepsilon\,\|\tilde{v}\|^2_{L^\infty}\|\tilde{v}_x\|^2;\\
\mathcal{I}_4 &\le \frac{\varepsilon}{8}\|\tilde{v}_{xx}\|^2 + 8\varepsilon\,\beta^2\|\tilde{v}_x\|^2;\\
\mathcal{I}_5 &\le \frac{\varepsilon}{8}\|\tilde{v}_{xx}\|^2 + \frac{2}{\varepsilon}\,\|\tilde{u}_x\|^2;\\
\mathcal{I}_6 &\le \frac{\varepsilon}{8}\|\tilde{v}_{xx}\|^2 + \frac{2}{\varepsilon}\,|\beta_t|^2.
\end{aligned}
$$

For the L^∞ norms appearing in the above estimates, we note that since both the functions $\tilde{u}$ and $\tilde{v}$ equal zero on the boundary, it holds that

$$
\tilde{u}(x,t) = \int_0^x \tilde{u}_y\,dy \quad\Longrightarrow\quad \|\tilde{u}\|^2_{L^\infty} \le \left(\int_0^1 |\tilde{u}_x|\,dx\right)^2 \le \|\tilde{u}_x\|^2, \tag{5.30}
$$

and the same is true for $\tilde{v}$. Hence, we can update $\mathcal{I}_1$ and $\mathcal{I}_3$ as

$$
\begin{aligned}
\mathcal{I}_1 &\le \frac{1}{4}\|\tilde{u}_{xx}\|^2 + 8\|\tilde{v}_x\|^2\|\tilde{u}_x\|^2 + 4\alpha^2\|\tilde{v}_x\|^2 + 4\beta^2\|\tilde{u}_x\|^2;\\
\mathcal{I}_3 &\le \frac{\varepsilon}{8}\|\tilde{v}_{xx}\|^2 + 8\varepsilon\,\|\tilde{v}_x\|^2\|\tilde{v}_x\|^2.
\end{aligned}
$$

Plugging these estimates and preceding estimates for $\mathcal{I}_2$, $\mathcal{I}_4$, $\mathcal{I}_5$ and $\mathcal{I}_6$ into (5.29), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}_x\|^2 + \|\tilde{v}_x\|^2\right) + \frac{1}{2}\|\tilde{u}_{xx}\|^2 + \frac{\varepsilon}{2}\|\tilde{v}_{xx}\|^2\\
\le\ & 8\|\tilde{v}_x\|^2\|\tilde{u}_x\|^2 + 4\overline{\alpha}^2\|\tilde{v}_x\|^2 + 4\overline{\beta}^2\|\tilde{u}_x\|^2 + |\alpha_t|^2\\
&+ 8\varepsilon\,\|\tilde{v}_x\|^2\|\tilde{v}_x\|^2 + 8\varepsilon\,\overline{\beta}^2\|\tilde{v}_x\|^2 + \frac{2}{\varepsilon}\,\|\tilde{u}_x\|^2 + \frac{2}{\varepsilon}\,|\beta_t|^2\\
\le\ & 8\left(\|\tilde{u}_x\|^2 + \varepsilon\|\tilde{v}_x\|^2\right)\left(\|\tilde{u}_x\|^2 + \|\tilde{v}_x\|^2\right)\\
&+ \left(\frac{4\overline{\alpha}^2}{\varepsilon} + 8\overline{\beta}^2\right)\varepsilon\|\tilde{v}_x\|^2 + \left(4\overline{\beta}^2 + \frac{2}{\varepsilon}\right)\|\tilde{u}_x\|^2 + |\alpha_t|^2 + \frac{2}{\varepsilon}\,|\beta_t|^2,
\end{aligned} \tag{5.31}
$$

where we have used the first assumption of Theorem 2.3. Applying the Gronwall's inequality to (5.31), we have

$$
\begin{aligned}
&\|\tilde{u}_x(\cdot,t)\|^2 + \|\tilde{v}_x(\cdot,t)\|^2\\
\le\ & \exp\left\{16\int_0^t \left(\|\tilde{u}_x\|^2 + \varepsilon\|\tilde{v}_x\|^2\right)d\tau\right\}\times\\
&\left\{\left(\frac{8\overline{\alpha}^2}{\varepsilon} + 16\overline{\beta}^2\right)\varepsilon\int_0^t \|\tilde{v}_x\|^2 d\tau + \left(8\overline{\beta}^2 + \frac{4}{\varepsilon}\right)\int_0^t \|\tilde{u}_x\|^2 d\tau\right.\\
&\left.+ 2\int_0^t |\alpha_t|^2 d\tau + \frac{4}{\varepsilon}\int_0^t |\beta_t|^2 d\tau + \|\tilde{u}_{0x}\|^2 + \|\tilde{v}_{0x}\|^2\right\}.
\end{aligned}
$$

By using (5.15), (5.28) and the second assumption of Theorem 2.3, we obtain

$$
\|\tilde{u}_x(\cdot,t)\|^2 + \|\tilde{v}_x(\cdot,t)\|^2 \le C, \quad \forall\, t > 0, \tag{5.32}
$$

where the constant C is independent of t, but depends reciprocally on ε. Further applying (5.32) to (5.31), then integrating the result with respect to time, we conclude

$$\int_0^t \left(\|\tilde{u}_{xx}\|^2 + \varepsilon\|\tilde{v}_{xx}\|^2\right) d\tau \le C, \quad \forall\, t > 0,$$

for some constant C which is independent of t, but depends reciprocally on ε. This completes the estimate of the first order spatial derivatives of the solution, and therefore the desired energy estimates stated in Theorem 2.3. □

Next, we prove the decay property recorded in Theorem 2.3.

5.4. **Decay estimate.** First, we would like to remark that a function of t, belonging to $W^{1,1}(0,\infty)$, converges to zero as time goes to infinity. In what follows, we use such a fact, together with the energy estimates obtained in the previous subsections, to establish the decay estimate stated in Theorem 2.3.

Recalling (5.15) and (5.28), we see that

$$\|\tilde{u}_x(\cdot,t)\|^2 + \varepsilon\|\tilde{v}_x(\cdot,t)\|^2 \in L^1(0,\infty).$$

Hence, for any fixed value of ε, due to the Poincaré's inequality and the first assumption of Theorem 2.3, it holds that

$$\|\tilde{u}(\cdot,t)\|^2 + \alpha\|\tilde{v}(\cdot,t)\|^2 \in L^1(0,\infty). \tag{5.33}$$

Next, we note that (5.20) can be written as

$$\begin{aligned} \frac{d}{dt}\left(\|\tilde{u}\|^2 + \alpha\|\tilde{v}\|^2\right) = &- 2\|\tilde{u}_x\|^2 - 2\varepsilon\alpha\|\tilde{v}_x\|^2 - 2\int_0^1 \tilde{u}\,\tilde{v}\,\tilde{u}_x\,dx \\ &- 2\alpha_t\int_0^1 \tilde{u}\,dx - 2\alpha\,\beta_t\int_0^1 \tilde{v}\,dx + \alpha_t\|\tilde{v}\|^2, \end{aligned} \tag{5.34}$$

from which we can deduce

$$\begin{aligned} \left|\frac{d}{dt}\left(\|\tilde{u}\|^2 + \alpha\|\tilde{v}\|^2\right)\right| \le\; & 2\|\tilde{u}_x\|^2 + 2\varepsilon\overline{\alpha}\|\tilde{v}_x\|^2 + \|\tilde{u}\|_{L^\infty}\left(\|\tilde{v}\|^2 + \|\tilde{u}_x\|^2\right) \\ &+ |\alpha_t|^2 + \|\tilde{u}\|^2 + \overline{\alpha}\left(|\beta_t|^2 + \|\tilde{v}\|^2\right) + |\alpha_t|\|\tilde{v}\|^2. \end{aligned} \tag{5.35}$$

According to (5.30), we have

$$\|\tilde{u}\|_{L^\infty} \le \|\tilde{u}_x\|, \quad \|\tilde{u}\|^2 \le \|\tilde{u}_x\|^2, \quad \|\tilde{v}\|^2 \le \|\tilde{v}_x\|^2.$$

Hence, we can update (5.35) as

$$\left|\frac{d}{dt}\left(\|\tilde{u}\|^2 + \alpha\|\tilde{v}\|^2\right)\right| \le C\left(\|\tilde{u}_x\|^2 + \|\tilde{v}_x\|^2 + |\alpha_t|^2 + |\beta_t|^2 + |\alpha_t|\right), \tag{5.36}$$

where the constant C is independent of t, and we have applied (5.32) for the uniform estimate of $\|\tilde{u}_x\|$ and (5.14) to the last term on the right hand side of (5.35). From (5.15), (5.28) and the third assumption of Theorem 2.3 we see that the right hand side of (5.36) is uniformly integrable with respect to time. Therefore,

$$\frac{d}{dt}\left(\|\tilde{u}(\cdot,t)\|^2 + \alpha\|\tilde{v}(\cdot,t)\|^2\right) \in L^1(0,\infty). \tag{5.37}$$

The combination of (5.33) and (5.37) implies that

$$\|\tilde{u}(\cdot,t)\|^2 + \alpha\|\tilde{v}(\cdot,t)\|^2 \in W^{1,1}(0,\infty).$$

Thus,

$$\lim_{t\to\infty}\left(\|\tilde{u}(\cdot,t)\|^2 + \alpha\|\tilde{v}(\cdot,t)\|^2\right) = 0.$$

Since $\alpha(t) \ge \underline{\alpha} > 0$, we conclude that

$$\lim_{t\to\infty}\left(\|\tilde{u}(\cdot,t)\|^2 + \|\tilde{v}(\cdot,t)\|^2\right) = 0.$$

In a completely similar fashion by using the estimates in Section 5.4, we can show that

$$\lim_{t\to\infty}\left(\|\tilde{u}_x(\cdot,t)\|^2+\|\tilde{v}_x(\cdot,t)\|^2\right)=0.$$

This completes the proof of the decay estimate, and thus of Theorem 2.3. □

6. Proof of Theorem 2.4

In this section, we pass the results of the transformed chemotaxis model (1.3) to the original chemotaxis system (1.1) with $m=1$. Noticing that the transformed and pre-transformed systems have the same quantity u, we are left to prove the results for w only. We start with the proof of (2.6). Let $x_0\in[0,1])$ be such that $w_0(x_0)>0$. Using $(\ln w_0(x))_x\in H^2[0,1]$ and Sobolev embedding theorem, we get $(\ln w_0(x))_x\in C^1[0,1]$. Thus,

$$\ln w_0(x)-\ln w_0(x_0)=\int_{x_0}^{x}(\ln w_0(y))_y dy,\quad x\in[0,1],$$

which leads to

$$w_0(x)=w_0(x_0)\exp\left\{\int_{x_0}^{x}(\ln w_0(y))_y dy\right\},\quad x\in[0,1].$$

This along with $(\ln w_0(x))_x\in C^1[0,1]$ yields $w_0(x)\in C^2[0,1]$. Hence there exist two positive constants $\underline{w}$ and $\overline{w}$ such that $0<\underline{w}\le w_0(x)\le\overline{w}<\infty$.

From the second equation of (1.1) with $m=1$ and the Cole-Hopf transformation (1.2), we have

$$(\ln w)_t=-u-\varepsilon v_x+\varepsilon(v)^2.$$

Integrating the above equality with respect to t to get

$$w(x,t)=w_0(x)\exp\Big\{\int_0^t[-u-\varepsilon v_x+\varepsilon(v)^2]d\tau\Big\}.\tag{6.1}$$

Using Gagliardo-Nirenberg inequality and Theorem 2.1, we have

$$\int_0^t\left(\|u\|_{L^\infty}+\varepsilon\|v_x\|_{L^\infty}+\varepsilon\|v\|_{L^\infty}^2\right)d\tau\le C(t),$$

which implies

$$e^{-C(t)}\le\exp\Big\{\int_0^t[-u-\varepsilon v_x+\varepsilon(v)^2]d\tau\Big\}\le e^{C(t)}.$$

This along with (6.1) and $0<\underline{w}\le w_0(x)\le\overline{w}<\infty$ gives

$$c_2(t)\le w(x,t)\le c_3(t),\tag{6.2}$$

where $c_2(t)=\underline{w}e^{-C(t)}$ and $c_3(t)=\overline{w}e^{C(t)}$. Noting that

$$\begin{cases}w_x=w(\ln w)_x,\\ w_{xx}=w_x(\ln w)_x+w(\ln w)_{xx},\\ w_{xxx}=w_{xx}(\ln w)_x+2w_x(\ln w)_{xx}+w(\ln w)_{xxx}.\end{cases}\tag{6.3}$$

By using the Cole-Hopf transformation (1.2), Theorem 2.1, (6.2) and (6.3), we complete the proof of (2.6).

Next, we prove (2.7). Let $(u^\varepsilon,w^\varepsilon)$ and (u^0,w^0) be the solutions to (2.5) with $\varepsilon>0$ and $\varepsilon=0$, respectively. From the second equation of (1.1) with $m=1$ and the Cole-Hopf transformation (1.2), we have

$$(\ln w^\varepsilon)_t=-u^\varepsilon-\varepsilon v_x^\varepsilon+\varepsilon(v^\varepsilon)^2$$

and

$$(\ln w^0)_t=-u^0.\tag{6.4}$$

Then, the difference of the above two equations yields

$$(\ln w^\varepsilon - \ln w^0)_t = (u^0 - u^\varepsilon) - \varepsilon v_x^\varepsilon + \varepsilon (v^\varepsilon)^2. \tag{6.5}$$

Integrating (6.5) with respect to t, we get

$$\frac{w^\varepsilon(x,t)}{w^0(x,t)} = \exp\left\{\int_0^t [(u^0-u^\varepsilon) - \varepsilon v_x^\varepsilon + \varepsilon(v^\varepsilon)^2] d\tau\right\},$$

where we have used $w^\varepsilon(x,0) = w^0(x,0)$. Subtracting 1 from both sides of above equation, we obtain

$$|w^\varepsilon(x,t) - w^0(x,t)| \le |w^0(x,t)| \cdot \left|\exp\left\{\int_0^t [(u^0-u^\varepsilon) - \varepsilon v_x^\varepsilon + \varepsilon(v^\varepsilon)^2] d\tau\right\} - 1\right|. \tag{6.6}$$

Note that Gagliardo-Nirenberg inequality and Young inequality, Theorem 2.1 and Lemma 4.2 give us that

$$\begin{aligned}
&\int_0^t [(u^0-u^\varepsilon) - \varepsilon v_x^\varepsilon + \varepsilon(v^\varepsilon)^2] d\tau \\
\le& C\int_0^t \left(\|u^\varepsilon - u^0\|_{L^\infty} + \varepsilon \|v_x^\varepsilon\|_{L^\infty} + \varepsilon \|v^\varepsilon\|_{L^\infty}^2\right) d\tau \\
\le& C\int_0^t \left[\left\|u^\varepsilon - u^0\right\|_{H^1} + C(t)(\varepsilon \left\|v_x\right\|^2 + \varepsilon^2 \left\|v_{xx}\right\|^2 + \varepsilon^{\frac12}) + C(t)\varepsilon\right] d\tau \\
\le& C(t)\varepsilon^{\frac14}.
\end{aligned} \tag{6.7}$$

On the other hand, we need to estimate $|w^0(x,t)|$. Integrating (6.4) with respect to t and using Lemma 4.1, we get

$$w^0(x,t) = w_0(x)\exp\left\{-\int_0^t u^0 d\tau\right\} \le w_0(x) e^{t\|u^0\|_{L^\infty}} \le C(t),$$

which, along with (6.6) and (6.7), gives

$$\left\|w^\varepsilon(\cdot,t) - w^0(\cdot,t)\right\|_{C[0,1]} \le C(t)|e^\kappa - 1| \le C(t)(|\kappa| + o(|\kappa|)) \le C(t)\varepsilon^{\frac14}, \tag{6.8}$$

where the Taylor expansion has been used and κ denotes the argument of the exponential function in (6.6). This together with Lemma 4.2 completes the proof of (2.7).

Next, we proceed to prove (2.8) and (2.9). Note first that

$$\begin{aligned}
w_x^\varepsilon - w_x^0 &= w^\varepsilon\left(\frac{w_x^\varepsilon}{w^\varepsilon} - \frac{w_x^0}{w^0}\right) + \frac{w_x^0(w^\varepsilon - w^0)}{w^0} \\
&= w^\varepsilon\left((\ln w^\varepsilon)_x - (\ln w^0)_x\right) + (\ln w^0)_x (w^\varepsilon - w^0),
\end{aligned} \tag{6.9}$$

which subject to (1.2), (4.1) and (6.2), yields

$$\begin{aligned}
\left\|(w_x^\varepsilon - w_x^0)(\cdot,t)\right\|_{C[\delta,1-\delta]} &\le \|w^\varepsilon\|_{C[\delta,1-\delta]} \left\|v^\varepsilon - v^0\right\|_{C[\delta,1-\delta]} + \left\|v_x^0\right\|_{C[\delta,1-\delta]} \left\|w^\varepsilon - w^0\right\|_{C[\delta,1-\delta]} \\
&\le C(t)\left\|v^\varepsilon - v^0\right\|_{C[\delta,1-\delta]} + C(t)\varepsilon^{\frac14}.
\end{aligned}$$

This, combined with Theorem 2.2, leads to (2.8).

Now, we turn to prove (2.9). We argue by contradiction. Suppose that

$$\liminf_{\varepsilon\to 0} \|w_x^\varepsilon - w_x^0\|_{L^\infty([0,T);\, C[0,1])} = 0. \tag{6.10}$$

It follows from (1.2) and (6.9) that

$$v^0 - v^\varepsilon = \frac{(w_x^\varepsilon - w_x^0) + (w^\varepsilon - w^0)v^0}{w^\varepsilon},$$

which, together with (6.2), implies that

$$\|(v^0 - v^\varepsilon)(\cdot,t)\|_{C[0,1]} \le \frac{1}{c_2(t)}\left(\|w_x^\varepsilon - w_x^0\|_{C[0,1]} + \|w^\varepsilon - w^0\|_{C[0,1]}\|v^0\|_{C[0,1]}\right).$$

By using (4.1) and (6.8), we can show that

$$\|(v^0 - v^\varepsilon)(\cdot,t)\|_{C[0,1]} \le C(t)\left(\|w_x^\varepsilon - w_x^0\|_{C[0,1]} + C(t)\varepsilon^{\frac{1}{4}}\right),$$

which, along with (6.10), leads to

$$\liminf_{\varepsilon\to 0} \|v^0 - v^\varepsilon\|_{L^\infty([0,T);\,C[0,1])} = 0.$$

Apparently, the above result contradicts (2.4). Therefore, Theorem 2.4 (i) is proved.

Finally, we prove Theorem 2.4 (ii). Let (u, w) be the solution to (5.1). We rewrite (6.1) as

$$\begin{aligned} w(x,t) = w_0(x)\exp&\left\{-\int_0^t(\alpha-\varepsilon\beta^2)d\tau\right\} \\ &\times\exp\left\{\int_0^t[-(u-\alpha)-\varepsilon v_x+\varepsilon(v-\beta)^2+2\varepsilon\beta(v-\beta)]d\tau\right\}. \end{aligned} \tag{6.11}$$

By the first assumption of Theorem 2.3, we have

$$\exp\left\{-\int_0^t(\alpha-\varepsilon\beta^2)d\tau\right\} \le e^{-(\underline{\alpha}-\varepsilon\overline{\beta}^2)t}.$$

Using Cauchy-Schwarz inequality, (5.24) and Lemma 5.1 yields

$$\begin{aligned} \int_0^t \|u-\alpha\|_{L^\infty} d\tau \le& \frac{\zeta_0 t}{3} + C\int_0^t \|u-\alpha\|_{L^\infty}^2 d\tau \\ \le& \frac{\zeta_0 t}{3} + C\int_0^t\int_0^1 \frac{(u_x)^2}{u} d\tau \\ \le& \frac{\zeta_0 t}{3} + C, \end{aligned}$$

where ζ_0 is a positive constant to be determined later. From Theorem 2.3, Gagliardo-Nirenberg and Cauchy-Schwarz inequalities, we get

$$\begin{aligned} \varepsilon\int_0^t \|v_x\|_{L^\infty} d\tau \le& \frac{\zeta_0 t}{3} + C\varepsilon^2\int_0^t \|v_x\|_{L^\infty}^2 d\tau \\ \le& \frac{\zeta_0 t}{3} + C\varepsilon^2\int_0^t (\|v_x\|^2 + \|v_{xx}\|^2) d\tau \\ \le& \frac{\zeta_0 t}{3} + C. \end{aligned}$$

Using Gagliardo-Nirenberg inequality, Poincaré's inequality and Theorem 2.3, we have

$$\varepsilon\int_0^t \|v-\beta\|_{L^\infty}^2 d\tau \le C\varepsilon\int_0^t(\|v-\beta\|^2 + \|v_x\|^2)d\tau \le C\varepsilon\int_0^t \|v_x\|^2 d\tau \le C.$$

In a similar way, we may readily derive that

$$\begin{aligned} 2\varepsilon\int_0^t \beta\,\|v-\beta\|_{L^\infty} d\tau \le& \frac{\zeta_0 t}{3} + C\varepsilon^2\overline{\beta}^2\int_0^t \|v-\beta\|_{L^\infty}^2 d\tau \\ \le& \frac{\zeta_0 t}{3} + C. \end{aligned}$$

Substituting the above estimates into (6.11) and choosing $\zeta_0 = \frac{\underline{\alpha}-\varepsilon\overline{\beta}^2}{2}$ yield

$$\|w(x,t)\|_{L^\infty} \le Ce^{-\frac{\underline{\alpha}-\varepsilon\overline{\beta}^2}{2}t}.$$

This completes the proof of the second part of Theorem 2.4, and thus of Theorem 2.4. □

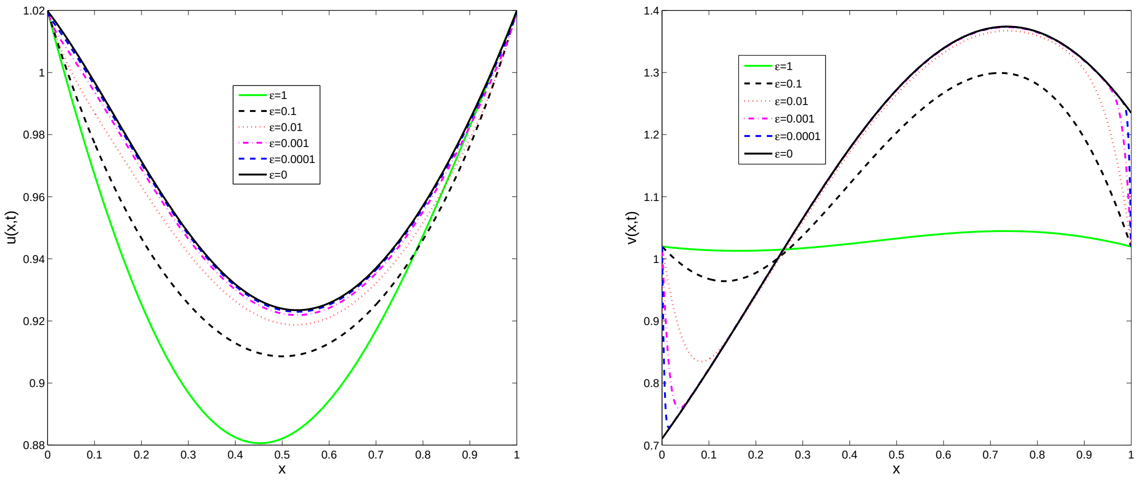

FIGURE 1. Numerical simulation of the evolution of solution profiles of the system (1.4) as $\varepsilon \to 0$ in the interval $[0,1]$, where $u|_{x=0,1} = 1 + 0.1\sin(t), v|_{x=0,1} = 1+0.1\sin(t), u_0(x) = 1-\sin(\pi x), v_0(x) = 1+x(1-x)$. The solution $(u(x,t), v(x,t)$ is plotted at time $t = 0.2$.

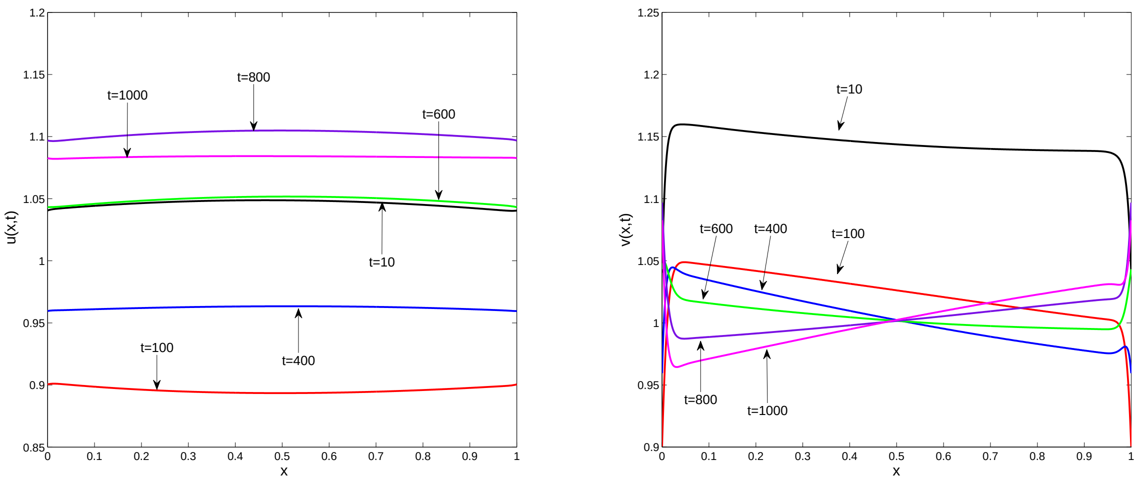

FIGURE 2. Numerical simulation of the time evolution of boundary layer solutions of (1.4) with $\varepsilon = 0.0001$ in the interval $[0,1]$, where the initial and boundary date are same as those chosen in Fig. 1.

7. Simulations and implications

In this section, we numerically solve system (1.4) to illustrate the boundary layer profile $(u^\varepsilon, v^\varepsilon)$, verify our analytical results and discuss boundary effects. The model is solved in the interval $[0,1]$ with MATLAB based on the finite difference scheme with mesh size $\Delta x = 0.001, \Delta t = 0.01$.

We first choose the initial and boundary data satisfying the requirements in (2.1) and implement numerical computations to the system (1.4). The solution profile $(u,v)(x,t)$ at time $t = 0.2$ as $\varepsilon \to 0$ is plotted in Fig.1. For the sake of comparison, we also numerically solve the non-diffusive problem (1.5) in the absence of boundary conditions for v and plot the numerical solution at $t = 0.2$ in Fig.1. We find from the simulations that the solution profile $u(x,t)$ is convergent with respect to ε in $[0,1]$, whereas the solution profile $v(x,t)$ becomes increasingly sharp near the boundary as $\varepsilon \to 0$ and boundary layers arise. Outside the boundary layer (i.e. in the interior of $[0,1]$) the solution profiles $v(x,t)$ for small $\varepsilon > 0$ and $\varepsilon = 0$ match well.

In Fig.2, we proceed to plot the time evolution of the same solution solved in Fig.1 to observe the asymptotic profiles, where we find that large-time profiles of the solution is elusive. This is because the boundary data chosen in Fig.1 vary (oscillate) in time. But the simulations show that the boundary layer profiles (sharp transition near boundaries) persist in time given small $\varepsilon > 0$. However if we impose some decay properties to the boundary data, the results of Theorem 2.3 show that the asymptotic behavior of solutions may become tractable and converge to some constant states, where the decay profiles of solutions are determined by the boundary data. Here we numerically explore this analytical finding. For this, we choose the initial and boundary data (see the caption of Fig. 3) such that the decay of boundary data for u is exponential and for v is algebraic, as well as the initial data satisfying the compatibility conditions at the end points $x = 0, 1$, as required by Theorem 2.3. We plot the numerical solution profiles in Fig.3 at different times showing that the solution (u, v) will approach constant states as time evolves. In particular, we find that the convergence of u is much faster than that of v. This complies with our analytical results in Theorem 2.3 that the decay rates of u and v are same as the boundary data $\alpha(t)$ and $\beta(t)$, respectively, where the former (exponential decay) is much fast than the latter (algebraic decay).

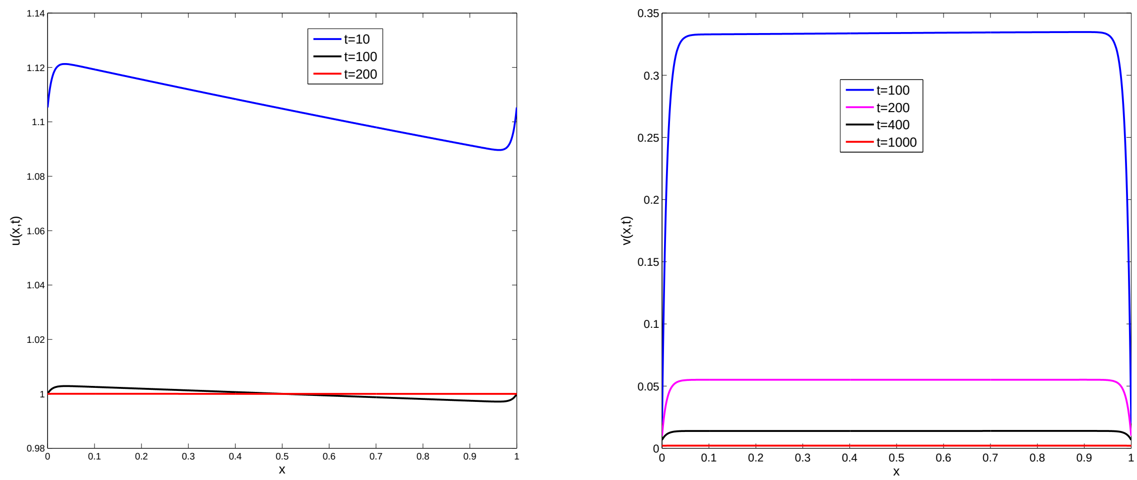

FIGURE 3. Numerical simulation of the time evolution of solutions to (1.4) in the interval $[0,1]$ with decay boundary data, where $u|_{x=0,1} = 1 + \exp(-t), v|_{x=0,1} = 1/(1+t), u_0(x) = 2 + x(1-x), v_0(x) = 1 + x(1-x)$, and $\chi = D = 1, \varepsilon = 0.0001$.

Finally we shall discuss some biological insights gained from our analytical and numerical results. In view of model (1.1) with $m = 1$ and the transformation (1.2), we see that the quantity v represents the velocity of chemotactic flux crossing the boundary. Therefore the results in Theorem 2.4 imply that if the chemical diffusion is small, although both cell density and chemical concentration have no boundary layers, the chemotactic flux (i.e. the term $u(\ln w)_x = uv$) may change drastically near the boundary since v has boundary layers. If the boundary data have oscillating properties, this phenomenon will persist in time. However if the boundary data have some decay properties, the boundary layer may vanish as time evolves. Therefore the nature of boundary date play an essential role in determining the solution behaviour near the boundary and large-time dynamics.

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