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Social Optima of Backward Linear-Quadratic-Gaussian Mean-Field Teams

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Abstract This paper studies a class of stochastic linear-quadratic-Gaussian (LQG) dynamic optimization problems involving a large number of weakly-coupled *heterogeneous* agents. By "heterogeneous," we mean agents are endowed with different types of parameters thus they are not statistically identical. Specifically, *discrete-type* heterogeneous agents are considered here which are more practical than *homogeneous-type* agents, and at the same time, more tractable than *continuum-type* heterogeneous agents. Unlike well-studied mean-field-game, these agents formalize a team with cooperation to minimize some social cost functional. Moreover, unlike standard social optima literature, the state here evolves by some backward stochastic differential equation (BSDE) in which the *terminal* instead *initial* condition is specified. Accordingly, the related social cost is represented by some recursive functional for which the *initial* state is considered. Applying a backward version of person-by-person optimality, we construct an auxiliary control problem for each agent based on decentralized information. The decentralized social strategy is derived by a class of new consistency condition (CC) systems, which are mean-field-type forward-backward stochastic differential equations (FBSDEs). The well-posedness of such consistency condition system is obtained via Riccati decoupling method. The related asymptotic social optimality is also verified.

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1 Introduction

The large-population system arises naturally in various fields in decision making such as economics, engineering, social science and operational research, and has been extensively studied from different perspectives. Its most striking feature is the existence of considerable negligible agents which are highly interactive through their coupled state-average or possible admissible controls. Although the effect of each individual agent on overall population scale is negligible, the effects of their statistical behaviors cannot be ignored at the population scale. The central goal of individual agent is to obtain *decentralized* strategies based on limited information of the individual agent since it is unrealistic for a given agent to synthesize all other agents' information instantaneously when the number of agents is sufficiently high. This is contrast to the classical *centralized* control which assumes the full information upon all agents is relatively tractable to be synthesized in a simultaneous manner. The mean-field game offers a powerful scheme to obtain the decentralized strategies through the limiting auxiliary control problem and the related consistency condition. For this direction, the interested readers are referred to [4], [7], [17], [19], [21]. In the basic mean field decision model, all agents are supposed to be statistically identical. However, this assumption is over simplified, because all the agents are "atomic" without diversities. Therefore, we continue to study more realistic heterogeneous model by incorporating parameter diversity.

In some real models there exists an agent with a significant influence upon other agents. Thus a modified framework is to introduce a major agent interacting with a large number of minor agents. [16] considered linear-quadratic-Gaussian (LQG) games with a major player and a large number of minor players. [26] studied large population dynamic games involving nonlinear stochastic dynamical systems with a major agent and N minor agents. [6] studied two-person zero-sum stochastic differential games, in which one player is a major one and the other player is a group of N minor agents which are collectively interactive, statistically identical, and have the same cost functional. [13] considered LQG mean-field games with a major agent and considerable heterogeneous minor agents where the individual admissible controls are constrained in closed convex subsets.

The decisions in all aforementioned works are competitive, i.e., the agents involved may have conflictive objectives and some (asymptotic) Nash equilibrium among them should be pursued. On the other hand, cooperative team optimization in dynamic multi-agent decision has also been well addressed in literature. Accordingly, it is necessary to discuss the mean-field team in the context of large population system where considerable weakly-coupled agents are cooperative to optimize some common objective functional. [18] studied a class of linear-quadratic-Gaussian control problems with N decision makers, where the basic objective is to minimize a social cost as the sum of N individual costs containing mean field coupling. [34] investigated social optima of mean field linear-quadratic-Gaussian control models with Markov jump parameters. [3] considered LQ mean field team-optimal problem by assuming mean field sharing for a given population size N, which gives an optimal control problem with special partial state information. [20] studied a linear-quadratic mean field control problem involving a major player and a large number of minor players, where the objective is to optimize a social cost as a weighted sum of the individual costs under decentralized information. For other research and applications of cooperative mean field control problems, interested readers are referred to [2], [8], [27], [31], [33], [37] and the references therein.

In this paper, we investigate a class of stochastic linear-quadratic-Gaussian (LQG) optimization involving a large number of weakly-coupled heterogeneous agents, where the dynamic is driven by some backward stochastic differential equation (BSDE). Moreover, all the heterogeneous agents are cooperative to minimize a social cost as the sum of some individual costs. Note that such optimization problem may occur for cooperative team with distributed information but recursive utility or cost functionals. In LQG setup, this is also the case where team agents aim to minimize some quadratic deviations with prescribed terminal target. The feature of *backward* state makes our setting rather different to existing works of mean-field LQG team wherein the individual states evolve by some forward stochastic differential equations (SDEs). Different to SDE, the terminal instead initial condition of BSDE should be specified as the priori. As a consequence, the BSDE will admit one adapted solution pair (y_t, z_t) where the second solution component z_t (it is also called the diffusion component) is naturally presented here due to the martingale representation and the adaptiveness requirement. The linear BSDEs were introduced in [5] and the general nonlinear BSDEs were first introduced in [28]. Based on them, the study of BSDE has undergone extensive discussions and it has been found many applications in various areas. For instance, the BSDEs has been found to be very important to characterize the nonlinear expectation in decision making, or the stochastic differential recursive utility (say, [9]). Later, [10] presented many applications of BSDEs in mathematical finance and optimal control theory.

Considering the various applications of BSDE, it is also very promising to study its associated dynamic control and game problems. Actually, there already accumulated considerable literature along this line. For example, see [36] and [11] for maximum principle of system driven by BSDE; [23] for linear-quadratic backward control with related Riccati representation, [14] and [35] for backward LQ control problems with partial information and related filtering results, [22] for backward LQ control in mean-field type. There also arise some works on mean-field game for large-population system driven by BSDE such as [15], etc.

The innovative aspects of the obtained results in this paper are as follows: Firstly, the BSDE drivers of agents depend on the state process itself and the state-average thus all agents become weakly-coupled. This bring additional difficulties when we apply the variational method to obtain the auxiliary control problem. Specifically, we need carefully introduce some adjoint processes to tackle the cross-terms in cost functional variation. Secondly, the social cost variation can be divided into K parts due to the discrete-type heterogeneous assumption. Thus the computations involved in person-by-person become more intractable and involved. Thirdly, owning to the structure of backward dynamics, the consistency system becomes mean-field backward forward stochastic differential equations (BFSDE) which are mixed at the initial condition. In addition, the consistency system becomes a highly-dimensional augmentation because of the discrete-type heterogeneous setup. Its well-posedness also becomes intractable.

Let us now briefly explain how to solve the LQG backward team optimization problem. Firstly, under some backward version of person-by-person optimality principle, we can construct some auxiliary LQG control problem by applying variational synthesization technique and solve it using stochastic maximum principle ([30, 14]). In this step, some frozen mean-field team is introduced with related adjoint process. Secondly, to determine such frozen mean field terms, we construct the consistency condition (CC) system by some fixed point analysis in backward version. Finally, by applying standard estimations of solutions of backward stochastic differential equations, we can verify that the decentralized strategy obtained from the auxiliary control problem turns to be some "good" approximation of centralized optimal control strategy (i.e., the social optima loss tend to 0 as the population number N tends to infinity).

The remaining of the paper is organized as follows: In Section 2, we give the formulation of the LQG recursive heterogeneous agents problem. In Section 3, we apply person-by-person optimality to find the auxiliary control problem of the individual agent. The consistency conditions and well-posedness of consistency systems are established in Section 4. In Section 5, we obtain the asymptotic optimality of the decentralized strategy.

2 Problem formulation

Consider a finite time horizon $[0, T]$ for fixed $T > 0$. Assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ is a complete filtered probability space satisfying the usual conditions and $\{W_i(t), 1 \leq i \leq N\}_{0\leq t\leq T}$ is a $N \times d$-dimensional Brownian motion on this space. Let $\mathcal{F}_t$ be the filtration generated by $\{W_i(s), 1 \leq i \leq N\}_{0\leq s\leq t}$ and augmented by $\mathcal{N}_{\mathbb{P}}$ (the class of all $\mathbb{P}$-null sets of $\mathcal{F}$). Let $\mathcal{F}^i_t$ be the augmentation of $\sigma\{W_i(s), 0 \leq s \leq t\}$ by $\mathcal{N}_{\mathbb{P}}$.

Let $\langle \cdot, \cdot \rangle$ denote standard Euclidean inner product. $x^\top$ denotes the transpose of a vector (or matrix) x. $M \in \mathbb{S}^n$ denotes the set of symmetric $n \times n$ matrices with real elements. $M > (\geq) 0$ denotes that $M \in \mathbb{S}^n$ which is positive (semi)definite, while $M \gg 0$ denotes that, for some $\varepsilon > 0$, $M - \varepsilon I \geq 0$. We introduce the following spaces which will be used in the paper:

- $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) := \Big\{\eta : \Omega \to \mathbb{R}^n | \eta \text{ is } \mathcal{F}_T\text{-measurable such that } \mathbb{E}|\eta|^2 < \infty\Big\}$

- $L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^n)) := \Big\{\zeta(\cdot) : [0,T]\times\Omega \to \mathbb{R}^n | \zeta(\cdot)$ is $\mathcal{F}_t$-adapted, continuous, such that $\mathbb{E}\Big[\sup_{s\in[0,T]} |\zeta(s)|^2\Big] < \infty\Big\}$
- $L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) := \Big\{\zeta(\cdot) : [0,T]\times\Omega \to \mathbb{R}^n | \zeta(\cdot)$ is $\mathcal{F}_t$-progressively measurable process such that
  $\mathbb{E}\int_0^T |\zeta(\cdot)|^2 dt < \infty\Big\}$
- $L^\infty(0,T;\mathbb{R}^{n\times n}) := \Big\{\zeta(\cdot) : [0,T] \to \mathbb{R}^{n\times n} | \zeta(\cdot)$ is uniformly bounded$\Big\}$

We consider a weakly coupled large population system with K-type *discrete* heterogeneous agents $\{\mathcal{A}_i : 1 \le i \le N\}$. The dynamics of the agents are given by a system of linear backward stochastic differential equations with mean-field coupling: that is, for $1 \le i \le N$,

$$\begin{cases} dy_i(t) = -\Big[A_{\theta_i}(t)y_i(t) + B(t)u_i(t) + C(t)y^{(N)}(t) + f(t)\Big]dt \\ \qquad\qquad + z_i(t)dW_i(t) + \displaystyle\sum_{j=1,j\neq i}^{N} z_{ij}(t)dW_j(t), \\ y_i(T) = \xi_i, \end{cases} \tag{1}$$

where $y^{(N)}(\cdot) = \frac{1}{N}\sum_{i=1}^N y_i(\cdot)$ denotes the state-average of the agents. It is remarkable that $(z_i(\cdot), z_{ij}(\cdot), 1 \le j \le N, j \neq i)$ is part of our solution in (1) which are introduced here to enable y_i to satisfy the adaptation requirement. Note that while the coefficients $(A_{\theta_i}(\cdot), B(\cdot), C(\cdot), f(\cdot))$ are dependent on the time variable t, in what follows the variable t will usually be suppressed if no confusion would occur. The number θ_i is a parameter of the agent $\mathcal{A}_i$ to model a heterogeneous population. For simplicity, we only assume that the coefficients A to be dependent on θ_i. Similar analysis can be proceeded in case that all other coefficients are also dependent on θ_i. Moreover, we assume that θ_i takes values in a finite set $\Theta := \{1, 2, \cdots, K\}$. We call $\mathcal{A}_i$ a k-type agent if $\theta_i = k \in \Theta$. In this paper, we are interested in the asymptotic behavior as N tends to infinity. For $1 \le k \le K$, introduce

$$\mathcal{I}_k = \{i|\theta_i = k, 1 \le i \le N\}, \qquad N_k = |\mathcal{I}_k|,$$

where N_k is the cardinality of index set $\mathcal{I}_k$. For $1 \le k \le K$, let $\pi_k^{(N)} = \frac{N_k}{N}$, then $\pi^{(N)} = (\pi_1^{(N)}, \cdots, \pi_K^{(N)})$ is a probability vector representing the empirical distribution of $\theta_1, \cdots, \theta_N$. We introduce the following assumption:

(A1) There exists a probability mass vector $\pi = (\pi_1, \cdots, \pi_K)$ such that $\lim_{N\to+\infty} \pi^{(N)} = \pi$, $\min_{1\le k\le K} \pi_k > 0$.

(A2) For $i = 1, \cdots, n$, $\xi_i = \xi_i(\omega; \theta_i)$ are $\mathcal{F}_T^i$-measurable random variables. If $\theta_i = \theta_j = k$, ξ_i and ξ_j are identically distributed and the common distribution is denoted by η_k.

(A3) $A_{\theta_i}(\cdot), C(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$ $(i = 1, \cdots, N)$, $B(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m})$, $f(\cdot) \in L^\infty(0,T;\mathbb{R}^n)$.

It follows that under (A1)-(A3), the state equation in (1) admits a unique solution for all $u_i \in \mathcal{U}_i$. In fact, if we denote by

$$
\begin{cases}
\widehat{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}, \widehat{Z} = \begin{pmatrix} z_1 & z_{12} & \cdots & z_{1,N-1} & z_{1N} \\ z_{21} & z_2 & \cdots & z_{2,N-1} & z_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{N1} & z_{N2} & \cdots & z_{N,N-1} & z_N \end{pmatrix}, \widehat{U} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, \widehat{W} = \begin{pmatrix} W_1 \\ \vdots \\ W_N \end{pmatrix}, \widehat{\xi} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix}, \\
\widehat{A} = \begin{pmatrix} A_{\theta_1} & & \\ & \ddots & \\ & & A_{\theta_N} \end{pmatrix}, \widehat{B} = \begin{pmatrix} B & & \\ & \ddots & \\ & & B \end{pmatrix}, \widehat{C} = \begin{pmatrix} C & & \\ & \ddots & \\ & & C \end{pmatrix}, f' = \begin{pmatrix} f \\ \vdots \\ f \end{pmatrix}, J_N = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.
\end{cases}
$$

Then (1) can be rewritten as

$$
d\widehat{Y}(t) = -\left[\Big(\widehat{A}(t)+\frac{1}{N}\widehat{C}(t)J_N\Big)\widehat{Y}(t)+\widehat{B}(t)\widehat{U}(t)+f'(t)\right]dt+\widehat{Z}(t)d\widehat{W}(t), \qquad \widehat{Y}(T) = \widehat{\xi},
$$

which is a Linear BSDE of vector value and admits a unique solution $(\widehat{Y}, \widehat{Z}) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{Nn}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{Nn\times Nd})$ for $U \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{Nm})$, (see [28]). Thus, for any $1 \le i \le N$, the state equation (1) admits a unique solution $\big(y_i, z_i, z_{ij}(j \neq i)\big) \in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n\times d}) \times \underbrace{L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n\times d}) \times \cdots \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n\times d})}_{N-1}$.

Let $u = (u_1, \cdots, u_N)$ be the set of strategies of all N agents and $u_{-i} = (u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_N)$, $1 \le i \le N$. The cost functional for $\mathcal{A}_i$, $1 \le i \le N$, is given by

$$
\begin{aligned}
&\mathcal{J}_i(u_i(\cdot), u_{-i}(\cdot)) \\
=&\frac{1}{2}\mathbb{E}\Big\{ \int_0^T \big[\big\langle S(t)(y_i(t) - G(t)y^{(N)}(t)), y_i(t) - G(t)y^{(N)}(t)\big\rangle + \big\langle R_{\theta_i}(t)u_i(t), u_i(t)\big\rangle\big] dt \\
&\qquad + \big\langle Q(y_i(0) - Hy^{(N)}(0)), y_i(0) - Hy^{(N)}(0)\big\rangle \Big\}.
\end{aligned} \tag{2}
$$

The aggregate team functional of N agents is

$$
\mathcal{J}^{(N)}_{soc}(u(\cdot)) = \sum_{i=1}^{N} \mathcal{J}_i(u_i(\cdot), u_{-i}(\cdot)). \tag{3}
$$

We impose the following assumptions on the coefficients of the cost functionals:

(A4) $S(\cdot) \in L^\infty(0,T;\mathbb{S}^n)$, $S(\cdot) \ge 0$, $G(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, $R_{\theta_i}(\cdot) \in L^\infty(0,T;\mathbb{S}^m)$, $R_{\theta_i}(\cdot) \gg 0$ $(i = 1, \cdots, N)$, $Q \in \mathbb{S}^{n\times n}$, $H \in \mathbb{R}^{n\times n}$, $Q \ge 0$.

For $i = 1, \cdots, N$, the centralized admissible strategy set for the i^{th} agent is given by

$$
\mathcal{U}^c_i = \Big\{u_i(\cdot) | u_i(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)\Big\}.
$$

Correspondingly, the decentralized admissible strategy set for the i^{th} agent is given by

$$
\mathcal{U}^d_i = \Big\{u_i(\cdot) | u_i(\cdot) \in L^2_{\mathcal{F}^i}(0,T;\mathbb{R}^m)\Big\}.
$$

We propose the following optimal problem:

Problem 1. Find a strategy set $\bar{u}=(\bar{u}_1,\cdots,\bar{u}_N)$ where $\bar{u}_i(\cdot)\in\mathcal{U}_i^c$, $1\le i\le N$, such that

$$\mathcal{J}_{soc}^{(N)}(\bar{u}(\cdot))=\inf_{u_i\in\mathcal{U}_i^c,1\le i\le N}\mathcal{J}_{soc}^{(N)}(u_1(\cdot),\cdots,u_i(\cdot),\cdots,u_N(\cdot)). \tag{4}$$

Definition 1 A strategy $\widetilde{u}_i(\cdot)\in\mathcal{U}_i^d$, $i=1,\cdots,N$ is an ε-social decentralized optimal strategy if there exists $\varepsilon=\varepsilon(N)>0$, $\lim_{N\to\infty}\varepsilon(N)=0$ such that

$$\frac{1}{N}\Big(\mathcal{J}_{soc}^{(N)}(\widetilde{u}(\cdot))-\inf_{u_i(\cdot)\in\mathcal{U}_i^c,1\le i\le N}\mathcal{J}_{soc}^{(N)}(u(\cdot))\Big)\le\varepsilon.$$

Remark 1 In reality, the linear BSDE in (1) stands for the dynamics of some investment behaviors such as in stocks and bonds in a self-financed market, that is, there is no infusion or withdrawal of funds over $[0,T]$. In recursive or hedging problems *(*finance, optimal control, etc.*)*, the BSDE dynamics have been deeply studied in the existing literature, such as *[11]*, *[35]* and so on. The individual cost used to be applied in some terminal hedging problems with possible nonlinear expectation, taking mean variance model as an example. In particular, the initial state $y_i(0)$ in our cost (2) can be viewed as the initial hedging cost *(*or, cash surplus*)* for the i^{th} participant, which aims to reach some future payoff or obligation target ξ_i at given time T. In our social optima setting, we eager to minimize the aggregate team functional of N agents, which contains N benchmarks between the individual costs and the average costs. Besides, the constrained forward LQ control problem with state average coupling in state dynamics can also be transferred to the backward LQ control with state given by the linear BSDE, as given in (1).

Remark 2 Note that $y_i(\cdot)$ is $\mathcal{F}$-adapted due to the existence of $y^{(N)}(\cdot)$. Thus, in (1) we write $z_i(\cdot)$ as a part of solution to stand for the information of $\mathcal{A}_i$ corresponding to $W_i(\cdot)$; while $z_{ij}(\cdot)$ as a part of solution to stand for the information of $\mathcal{A}_i$ corresponding to $W_j(\cdot),1\le j\le N,j\ne i$. This is the feature of backward stochastic problems.

Remark 3 In (1) and (2), $z_i(\cdot),z_{ij}(\cdot),1\le j\le N,j\ne i$ does not enter in the drift term of the state equation and the cost functional. The reason is that there exists essential difficulties while doing the error estimations, because if $z_i(\cdot),z_{ij}(\cdot)$ enter in the drift term or the cost functional, we have to deal with the error estimation of $\sum_{i=1}^N z_i(\cdot)$, which is an impossible task under the existing BSDE theory. In the future, we may focus on this problem and try to derive some new technique to overcome this difficulty.

3 Stochastic optimal control problem for the agents $\mathcal{A}_i$

In this section, we try to solve the optimal control problem and derive the decentralized control.

3.1 Backward person-by-person optimality

In mean-field social optima scheme (mean-field team), person-by-person optimality is a critical technique, which has been used in the recent social optima literature, e.g. [34], etc. There is significant difference between mean-field team scheme and mean-field game scheme, where the auxiliary control problem is usually derived directly by fixing the state-average. This would lead to some ineffective control in social optima scheme. Thus, in this section under the person-by-person optimality principle, variation method will be applied to analyze the mean-field approximation.

Let $\{\bar{u}_i, \bar{u}_{-i} \in \mathcal{U}_i^c\}_{i=1}^N$ be centralized optimal strategy of all the agents. Now consider the perturbation that the agent $\mathcal{A}_i$ use the strategy $u_i \in \mathcal{U}_i^c$ and all the other agents still apply the strategy $\bar{u}_{-i} = (\bar{u}_1, \cdots, \bar{u}_{i-1}, \bar{u}_{i+1}, \cdots, \bar{u}_N)$. The realized states (1) corresponding to $(u_i, \bar{u}_{-i})$ and $(\bar{u}_i, \bar{u}_{-i})$ are denoted by $((y_1, z_1, z_{1j}), \cdots, (y_N, z_N, z_{Nj}))$ and $((\bar{y}_1, \bar{z}_1, \bar{z}_{1j}), \cdots, (\bar{y}_N, \bar{z}_N, \bar{z}_{Nj}))$, respectively. For $j = 1, \cdots, N$, denote the perturbation

$$\delta u_j = u_j - \bar{u}_j, \quad \delta y_j = y_j - \bar{y}_j, \quad \delta z_j = z_j - \bar{z}_j,$$
$$\delta z_{jl} = z_{jl} - \bar{z}_{jl} \ (l \neq j), \quad \delta \mathcal{J}_j = \mathcal{J}_j(u_i, \bar{u}_{-i}) - \mathcal{J}_j(\bar{u}_i, \bar{u}_{-i}).$$

Therefore, the variation of the state for $\mathcal{A}_i$ is given by

$$\begin{aligned} d\delta y_i = & - \Big[A_{\theta_i} \delta y_i + B \delta u_i + C \delta y^{(N)} \Big] dt + \delta z_i(t) dW_i(t) \\ & + \sum_{l=1, l\neq i}^{N} \delta z_{il}(t) dW_l(t), \quad \delta y_i(T) = 0, \end{aligned} \tag{5}$$

and for $\mathcal{A}_j$, $j \neq i$,

$$\begin{aligned} & d\delta y_j = - \Big[A_{\theta_j} \delta y_j + C \delta y^{(N)} \Big] dt + \delta z_j(t) dW_j(t) + \sum_{l=1, l\neq j}^{N} \delta z_{jl}(t) dW_l(t), \\ & \delta y_j(T) = 0. \end{aligned} \tag{6}$$

For $k = 1, \cdots, K$, define $\delta y_{(k)} = \sum_{j \in \mathcal{I}_k, j \neq i} \delta y_j$, thus

$$\begin{cases} d\delta y_{(k)} = - \Big[A_k \delta y_{(k)} + (N_k - I(\{i \in \mathcal{I}_k\})) C \delta y^{(N)} \Big] dt + \displaystyle\sum_{j \in \mathcal{I}_k, j\neq i} \delta z_j dW_j(t) \\ \qquad\qquad + \displaystyle\sum_{j \in \mathcal{I}_k, j\neq i} \sum_{l=1, l\neq j}^{N} \delta z_{jl}(t) dW_l(t), \\ \delta y_{(k)}(T) = 0. \end{cases}$$

By some elementary calculations, we can further obtain the variation of the cost functional of $\mathcal{A}_i$ as follows

$$\begin{aligned} \delta \mathcal{J}_i = & \mathbb{E} \Big\{ \int_0^T \big[\langle S(\bar{y}_i - G\bar{y}^{(N)}), \delta y_i - G\delta y^{(N)} \rangle + \langle R_{\theta_i} \bar{u}_i, \delta u_i \rangle \big] dt \\ & + \langle Q(\bar{y}_i(0) - H\bar{y}^{(N)}(0)), \delta y_i(0) - H\delta y^{(N)}(0) \rangle \Big\}. \end{aligned}$$

For $j \neq i$, the variation of the cost functional of $\mathcal{A}_j$ is given by

$$
\begin{aligned}
\delta \mathcal{J}_j =& \mathbb{E}\Big\{ \int_0^T \big[\big\langle S(\bar{y}_j - G\bar{y}^{(N)}), \delta y_j - G\delta y^{(N)}\big\rangle\big] dt \\
&+ \big\langle Q(\bar{y}_j(0) - H\bar{y}^{(N)}(0)), \delta y_j(0) - H\delta y^{(N)}(0)\big\rangle\Big\}.
\end{aligned}
$$

Therefore, by combining above equalities, the variation of the social cost satisfies

$$
\begin{aligned}
\delta \mathcal{J}_{soc}^{(N)} =& \mathbb{E}\Big\{ \int_0^T \Big[\big\langle S(\bar{y}_i - G\bar{y}^{(N)}), \delta y_i - G\delta y^{(N)}\big\rangle \\
&+ \sum_{j\neq i} \big\langle S(\bar{y}_j - G\bar{y}^{(N)}), \delta y_j - G\delta y^{(N)}\big\rangle + \big\langle R_{\theta_i}\bar{u}_i, \delta u_i\big\rangle\Big] dt \\
&+ \sum_{j=1}^{N} \big\langle Q(\bar{y}_j(0) - H\bar{y}^{(N)}(0)), \delta y_j(0) - H\delta y^{(N)}(0)\big\rangle\Big\}.
\end{aligned}
\tag{7}
$$

Step 1: First, replacing $\bar{y}^{(N)}$ in (7) by some mean-field term $\hat{y}$ which will be determined later,

$$
\begin{aligned}
\delta\mathcal{J}_{soc}^{(N)} =&\mathbb{E}\Big\{\int_0^T\Big[\big\langle S(\bar{y}_i-G\bar{y}^{(N)}),\delta y_i\big\rangle-\big\langle S(\bar{y}_i-G\bar{y}^{(N)}),G\delta y^{(N)}\big\rangle-\sum_{j\neq i}\big\langle SG\bar{y}^{(N)},\delta y_j\big\rangle\\
&+\sum_{j\neq i}\big\langle S\bar{y}_j,\delta y_j\big\rangle-\sum_{j\neq i}\big\langle S(\bar{y}_j-G\bar{y}^{(N)}),G\delta y^{(N)}\big\rangle+\big\langle R_{\theta_i}\bar{u}_i,\delta u_i\big\rangle\Big]dt\\
&+\sum_{j=1}^{N}\big\langle Q(\bar{y}_j(0)-H\bar{y}^{(N)}(0)),\delta y_j(0)-H\delta y^{(N)}(0)\big\rangle\Big\}\\
=&\mathbb{E}\Big\{\int_0^T\Big[\big\langle S(\bar{y}_i-G\hat{y}),\delta y_i\big\rangle-\sum_{j\neq i}\big\langle SG\hat{y},\delta y_j\big\rangle+\sum_{j\neq i}\big\langle S\bar{y}_j,\delta y_j\big\rangle\\
&-\big\langle G^\top S(\hat{y}-G\hat{y}),N\delta y^{(N)}\big\rangle+\big\langle R_{\theta_i}\bar{u}_i,\delta u_i\big\rangle\Big]dt+\big\langle Q(\bar{y}_i(0)-H\hat{y}(0)),\delta y_i(0)\big\rangle\\
&-\sum_{j\neq i}\big\langle QH\hat{y}(0),\delta y_j(0)\big\rangle+\sum_{j\neq i}\big\langle Q\bar{y}_j(0),\delta y_j(0)\big\rangle\\
&-\big\langle H^\top Q(\hat{y}(0)-H\hat{y}(0)),N\delta y^{(N)}(0)\big\rangle\Big\}+\sum_{l=1}^{2}\varepsilon_l\\
=&\mathbb{E}\Big\{\int_0^T\Big[\langle S\bar{y}_i,\delta y_i\rangle-\langle(SG+G^\top S-G^\top SG)\hat{y},\delta y_i\rangle\\
&-\sum_{k=1}^{K}\langle(SG+G^\top S-G^\top SG)\hat{y},\delta y_{(k)}\rangle+\sum_{k=1}^{K}\frac{1}{N_k}\sum_{j\in\mathcal{I}_k,j\neq i}\langle S\bar{y}_j,N_k\delta y_j\rangle\\
&+\langle R_{\theta_i}\bar{u}_i,\delta u_i\rangle\Big]dt+\langle Q\bar{y}_i(0),\delta y_i(0)\rangle-\langle(QH+H^\top Q-H^\top QH)\hat{y}(0),\delta y_i(0)\rangle\\
&-\sum_{k=1}^{K}\langle(QH+H^\top Q-H^\top QH)\hat{y}(0),\delta y_{(k)}(0)\rangle\\
&+\sum_{k=1}^{K}\frac{1}{N_k}\sum_{j\in\mathcal{I}_k,j\neq i}\langle Q\bar{y}_j(0),N_k\delta y_j(0)\rangle\Big\}+\sum_{l=1}^{2}\varepsilon_l,
\end{aligned}
$$

where

$$
\begin{cases}
\varepsilon_1=\mathbb{E}\displaystyle\int_0^T\langle(G^\top S+SG-G^\top SG)(\hat{y}-\bar{y}^{(N)}),N\delta y^{(N)}\rangle dt,\\
\varepsilon_2=\langle(H^\top Q+QH-H^\top QH)(\hat{y}(0)-\bar{y}^{(N)}(0)),N\delta y^{(N)}(0)\rangle.
\end{cases}
$$

Step 2: Next, for $k=1,\cdots,K$, introduce the limit y_k^{**} to replace $\delta y_{(k)}$, and for $j\in\mathcal{I}_k$, introduce the limit (y_j^*,z_j^*) to replace $(N_k\delta y_j, N_k\delta z_j)$, where

$$\begin{cases} dy_j^* = -\Big[A_k y_j^* + C\pi_k\delta y_i + C\pi_k\sum_{l=1}^{K} y_l^{**}\Big]dt + z_j^* dW_j(t) + \sum_{l=1,l\neq j}^{N} z_{jl}^* dW_l(t), \\ dy_k^{**} = -\Big[A_k y_k^{**} + C\pi_k\delta y_i + C\pi_k\sum_{l=1}^{K} y_l^{**}\Big]dt + \sum_{l=1}^{N} z_{kl}^{**} dW_l(t), \\ y_j^*(T)=0, \qquad y_k^{**}(T)=0. \end{cases} \tag{8}$$

Therefore,

$$\begin{aligned} \delta\mathcal{J}_{soc}^{(N)} =& \mathbb{E}\Big\{\int_0^T \Big[\langle S\bar{y}_i,\delta y_i\rangle - \langle (SG+G^\top S-G^\top SG)\hat{y},\delta y_i\rangle \\ & - \sum_{k=1}^{K}\langle (SG+G^\top S-G^\top SG)\hat{y}, y_k^{**}\rangle \\ & + \sum_{k=1}^{K}\frac{1}{N_k}\sum_{j\in\mathcal{I}_k,j\neq i}\langle S\bar{y}_j, y_j^*\rangle + \langle R_{\theta_i}\bar{u}_i,\delta u_i\rangle\Big]dt + \langle Q\bar{y}_i(0),\delta y_i(0)\rangle \\ & - \langle (QH+H^\top Q-H^\top QH)\hat{y}(0),\delta y_i(0)\rangle \\ & - \sum_{k=1}^{K}\langle (QH+H^\top Q-H^\top QH)\hat{y}(0), y_k^{**}(0)\rangle \\ & + \sum_{k=1}^{K}\frac{1}{N_k}\sum_{j\in\mathcal{I}_k,j\neq i}\langle Q\bar{y}_j(0), y_j^*(0)\rangle\Big\} + \sum_{l=1}^{6}\varepsilon_l, \end{aligned} \tag{9}$$

where

$$\begin{cases} \varepsilon_3 = \sum_{k=1}^{K}\mathbb{E}\int_0^T \langle (SG+G^\top S-G^\top SG)\hat{y}, y_k^{**}-\delta y_{(k)}\rangle dt, \\ \varepsilon_4 = \sum_{k=1}^{K}\mathbb{E}\int_0^T \frac{1}{N_k}\sum_{j\in\mathcal{I}_k,j\neq i}\langle S\bar{y}_j, N_k\delta y_j - y_j^*\rangle dt, \\ \varepsilon_5 = \sum_{k=1}^{K}\langle (QH+H^\top Q-H^\top QH)\hat{y}(0), y_k^{**}(0)-\delta y_{(k)}(0)\rangle, \\ \varepsilon_6 = \sum_{k=1}^{K}\frac{1}{N_k}\sum_{j\in\mathcal{I}_k,j\neq i}\langle Q\bar{y}_j(0), N_k\delta y_j(0) - y_j^*(0)\rangle. \end{cases}$$

Step 3: Finally, we will substitute y_j^* and y_k^{**} by dual method. It is very important to construct an auxiliary control problem for investigating decentralized control in social optimal problem (see e.g. [18,34]). We may use a duality procedure to break away $\delta\mathcal{J}_{soc}^{(N)}$ from the dependence on y_j^* and y_k^{**}.

To this end, we introduce the following adjoint equations x_1^j and x_2^k of the terms y_j^* and y_k^{**}, respectively, which are shown as follows

$$\begin{cases} dx_1^j = \alpha_1^j dt, & x_1^j(0) = -Q\bar{y}_j(0), \quad j = 1, \cdots, N, \\ dx_2^k = \alpha_2^k dt, & x_2^k(0) = (QH + H^\top Q - H^\top QH)\hat{y}(0), \quad k = 1, \cdots, K. \end{cases}$$

Applying Itô's formula to $\langle x_1^j, y_j^* \rangle$, we have

$$d\langle x_1^j, y_j^* \rangle = \Big[\langle x_1^j, -(A_k y_j^* + C\pi_k \delta y_i + C\pi_k \sum_{l=1}^K y_l^{**}) \rangle + \langle \alpha_1^j, y_j^* \rangle \Big] dt + \sum_{j=1}^N (\cdots) dW_j(t).$$

For $j \in \mathcal{I}_k$, integrating from 0 to T and taking expectation, we obtain

$$\begin{aligned} &\mathbb{E}\langle Q\bar{y}_j(0), y_j^*(0) \rangle \\ =&\mathbb{E}\langle x_1^j(T), y_j^*(T) \rangle - \mathbb{E}\langle x_1^j(0), y_j^*(0) \rangle \\ =&\mathbb{E} \int_0^T \Big[\langle x_1^j, -(A_k y_j^* + C\pi_k \delta y_i + C\pi_k \sum_{l=1}^K y_l^{**}) \rangle + \langle \alpha_1^j, y_j^* \rangle \Big] dt \\ =&\mathbb{E} \int_0^T \Big[\langle \alpha_1^j - A_k^\top x_1^j, y_j^* \rangle - \sum_{l=1}^K \langle \pi_k C^\top x_1^j, y_l^{**} \rangle - \langle \pi_k C^\top x_1^j, \delta y_i \rangle \Big] dt. \end{aligned} \tag{10}$$

Similarly, we have

$$\begin{aligned} &-\mathbb{E}\langle (QH + H^\top Q - H^\top QH)\hat{y}(0), y_k^{**}(0) \rangle \\ =&\mathbb{E}\langle x_2^k(T), y_k^{**}(T) \rangle - \mathbb{E}\langle x_2^k(0), y_k^{**}(0) \rangle \\ =&\mathbb{E} \int_0^T \Big[\langle \alpha_2^k - A_k^\top x_2^k, y_k^{**} \rangle - \sum_{l=1}^K \langle \pi_k C^\top x_2^k, y_l^{**} \rangle - \langle \pi_k C^\top x_2^k, \delta y_i \rangle \Big] dt. \end{aligned} \tag{11}$$

Letting

$$\begin{cases} \alpha_1^j = A_k^\top x_1^j - S\bar{y}_j, \\ \alpha_2^k = A_k^\top x_2^k + (SG + G^\top S - G^\top SG)\hat{y} + \sum_{l=1}^K \pi_l C^\top \mathbb{E} x_1^l + \sum_{l=1}^K \pi_l C^\top x_2^l, \end{cases}$$

substituting (10) and (11) into (9), we have

$$\begin{aligned} \delta \mathcal{J}_{soc}^{(N)} =&\mathbb{E}\Big\{ \int_0^T \Big[\langle S\bar{y}_i, \delta y_i \rangle - \langle (SG + G^\top S - G^\top SG)\hat{y}, \delta y_i \rangle - \sum_{k=1}^K \langle \pi_k C^\top x_2^k, \delta y_i \rangle \\ &- \sum_{k=1}^K \langle \pi_k C^\top \mathbb{E} \mathbf{x}_k, \delta y_i \rangle + \langle R_{\theta_i} \bar{u}_i, \delta u_i \rangle \Big] dt \\ &+ \langle Q\bar{y}_i(0), \delta y_i(0) \rangle - \langle (QH + H^\top Q - H^\top QH)\hat{y}(0), \delta y_i(0) \rangle \Big\} + \sum_{l=1}^8 \varepsilon_l, \end{aligned}$$

where

$$\begin{cases} dx_1^j = [A_k^\top x_1^j - S\bar{y}_j]dt, \qquad x_1^j(0) = -Q\bar{y}_j(0), \quad j = 1, \cdots, N, \\ dx_2^k = [A_k^\top x_2^k + (SG + G^\top S - G^\top SG)\hat{y} + \sum_{l=1}^{K} \pi_l C^\top \mathbb{E}\mathbf{x}_1^l + \sum_{l=1}^{K} \pi_l C^\top x_2^l]dt, \\ x_2^k(0) = (QH + H^\top Q - H^\top QH)\hat{y}(0), \quad k = 1, \cdots, K, \end{cases} \tag{12}$$

and

$$\begin{aligned} \varepsilon_7 &= \sum_{k=1}^{K} \mathbb{E} \int_0^T \Big\langle \sum_{l=1}^{K} \pi_l C^\top \mathbb{E}\mathbf{x}_l - \sum_{l=1}^{K} \frac{\pi_l}{N_l} \sum_{j\in\mathcal{I}_l, j\neq i} C^\top x_1^j, y_k^{**} \Big\rangle dt, \\ \varepsilon_8 &= \sum_{k=1}^{K} \mathbb{E} \int_0^T \langle \pi_k C^\top \mathbb{E}\mathbf{x}_k - \frac{1}{N_k} \sum_{j\in\mathcal{I}_k, j\neq i} \pi_k C^\top x_1^j, \delta y_i \rangle dt. \end{aligned}$$

Note that the states $x_1^j, j \in \mathcal{I}_k, j \neq i$ are exchangeable. When we consider the expectations, we will use $\mathbf{x}_k$ denote the process x_1^j defined in (12) of the representative agent of type k. Moreover, in Section 5 we still use this kind of notations, i.e., use $\mathbf{x}_k, \mathbf{y}_k, \mathbf{p}_k$ to denote the involved processes of the representative agent of type k $\varepsilon_1 - \varepsilon_8$ are actually $o(1)$ order and the rigorous proof will be shown in Section 5. Therefore, we introduce the decentralized auxiliary cost functional J_i with perturbation as follows:

$$\begin{aligned} \delta J_i =& \mathbb{E}\Big\{ \int_0^T \Big[\langle S\bar{y}_i, \delta y_i\rangle - \langle (SG + G^\top S - G^\top SG)\hat{y}, \delta y_i\rangle - \sum_{k=1}^{K} \langle \pi_k C^\top x_2^k, \delta y_i\rangle \\ &- \sum_{k=1}^{K} \langle \pi_k C^\top \hat{x}_k, \delta y_i\rangle + \langle R_{\theta_i}\bar{u}_i, \delta u_i\rangle\Big] dt + \langle Q\bar{y}_i(0), \delta y_i(0)\rangle \\ &- \langle (QH + H^\top Q - H^\top QH)\hat{y}(0), \delta y_i(0)\rangle \Big\}. \end{aligned} \tag{13}$$

Remark 4 It is remarkable that due to the perturbation of control $\delta u_i \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$, δy_i is an $\mathcal{F}_t$-adapted stochastic process in (5) and $\delta z_i, \delta z_{il}, 1 \leq l \leq N, l \neq i$ cannot vanish, though the terminal value $\delta y_i(T)$ is zero. So does δy_j in (6), because $\delta y^{(N)}$ is $\mathcal{F}_t$-adapted. In the same way, we suppose y_j^*, y_k^{**} are $\mathcal{F}_t$-adapted stochastic processes satisfying the BSDE in (8). This is rather different from the forward case, where one usually does not need to consider these items.

Actually x_1^j is an $\mathcal{F}_t$-adapted stochastic process, satisfying the SDE in (12). x_2^k depends on $\hat{y}$ and $\mathbb{E}\mathbf{x}_l$, and the initial value $x_2^k(0)$ is deterministic. By Theorem 1 (see below), we know $\hat{y}$ is deterministic. Thus, in (12) x_2^k is the solution of ODE indeed. That is, the system (8) is a coupled BSDE system, while the adjoint system is composed by a SDE and an ODE.

Remark 5 In above analysis, we introduce $N+K$ adjoint processes to break away $\delta\mathcal{J}_{soc}^{(N)}$ from the dependence on y_j^* and y_k^{**}. This difficulty is brought by the existence of $y^{(N)}$ in the drift item of state equation, that is $C(\cdot)\neq 0$. By contrast, if $C(\cdot)\equiv 0$, then $y_j^*(\cdot)\equiv 0$ and $y_k^{**}(\cdot)\equiv 0$. There's no additional adjoint processes are needed to derive the auxiliary problem.

3.2 Decentralized strategy

Motivated by (13), introduce the following auxiliary backward LQG control problem:

Problem 2. Minimize $J_i(u_i)$ over $u_i\in\mathcal{U}_i^d$ subject to

$$\begin{cases} dy_i(t)=-\Big[A_{\theta_i}(t)y_i(t)+B(t)u_i(t)+C(t)\hat{y}(t)+f(t)\Big]dt+z_i(t)dW_i(t),\\ y_i(T)=\xi_i, \end{cases}\tag{14}$$

where

$$\begin{aligned} J_i(u_i)=&\frac{1}{2}\Big\{\mathbb{E}\int_0^T\Big[\langle Sy_i,y_i\rangle-2\langle\Theta_1,y_i\rangle+\langle R_{\theta_i}u_i,u_i\rangle\Big]dt\\ &\qquad\qquad+\langle Qy_i(0),y_i(0)\rangle-2\langle\Theta_2,y_i(0)\rangle\Big\}, \end{aligned}\tag{15}$$

$$\begin{aligned} &\Theta_1=(SG+G^\top S-G^\top SG)\hat{y}-\sum_{k=1}^K\pi_kC^\top x_2^k-\sum_{k=1}^K\pi_kC^\top\hat{x}_k,\\ &\Theta_2=(QH+H^\top Q-H^\top QH)\hat{y}(0), \end{aligned}$$

and $\hat{y},x_2^k,\hat{x}_k$ will be determined by the consistency condition in the following section.

Similar to [30] and [14], we will apply stochastic maximum principle to study **Problem 2**. First introduce the following first order adjoint equation:

$$dp_i(t)=\Big[A_{\theta_i}^\top p_i+Sy_i-\Theta_1\Big]dt,\quad p_i(0)=Qy_i(0)-\Theta_2,$$

and the Hamiltonian function

$$H_i(t,y,u,p)=\Big\langle p,A_{\theta_i}y+Bu+C\hat{y}+f\Big\rangle+\frac{1}{2}\Big[\langle Sy,y\rangle+\langle R_{\theta_i}u,u\rangle-2\langle\Theta_1,y\rangle\Big].$$

The global stochastic maximum principle takes the following form:

$$\frac{\partial H_i}{\partial u}(t,y_i,\bar{u}_i,p_i)=B^\top p_i+R_{\theta_i}^\top\bar{u}_i=0,\ a.e.,\ t\in[0,T],\ \mathbb{P}-a.s..\tag{16}$$

Therefore, the optimal control is given by

$$\bar{u}_i(t)=-R_{\theta_i}^{-1}(t)B^\top(t)p_i(t),\ \mathbb{P}-a.s..$$

The related Hamiltonian system becomes

$$\begin{cases} dy_i(t) = -\Big[A_{\theta_i}(t)y_i(t) - B(t)R_{\theta_i}(t)^{-1}B^\top(t)p_i(t) + C(t)\hat{y}(t) + f(t)\Big]dt \\ \qquad\qquad + z_i(t)dW_i(t), \\ dp_i(t) = \Big[A_{\theta_i}^\top p_i + Sy_i - \Theta_1\Big]dt, \\ y_i(T) = \xi_i, \quad p_i(0) = Qy_i(0) - \Theta_2. \end{cases}$$

4 Consistency condition

In this section, we focus on the solution of **Problem 2** by constructing the consistency condition system and decoupling it.

Theorem 1 *Let (A1)-(A4) hold. The parameters in* ***Problem 2*** *can be determined by*

$$\hat{y}, x_2^k, \hat{x}_k = (\sum_{l=1}^{K} \pi_l \mathbb{E}\alpha_l, \check{x}_2^k, \mathbb{E}\check{x}_1^k),$$

where $(\alpha_k, \beta_k, \gamma_k, \check{x}_1^k, \check{x}_2^k)$ *is the solution of the following mean-field FBSDEs, which is so-called consistency condition (CC) system: for* $k = 1, \cdots, K$,

$$\begin{cases} d\alpha_k(t) = -\Big[A_k(t)\alpha_k(t) - B(t)R_k(t)^{-1}B^\top(t)\gamma_k(t) + C(t)\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l(t) \\ \qquad\qquad + f(t)\Big]dt + \beta_k(t)dW_k(t), \\ d\gamma_k(t) = \Big[A_k^\top\gamma_k + S\alpha_k - (SG + G^\top S - G^\top SG)\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l(t) \\ \qquad\qquad - \sum_{l=1}^{K}\pi_l C^\top \check{x}_2^l - \sum_{l=1}^{K}\pi_l C^\top \check{x}_1^l\Big]dt, \\ d\check{x}_1^k(t) = [A_k^\top \check{x}_1^k - S\alpha_k]dt, \\ d\check{x}_2^k(t) = [A_k^\top \check{x}_2^k + (SG + G^\top S - G^\top SG)\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l + \sum_{l=1}^{K}\pi_l C^\top \mathbb{E}\check{x}_1^l \\ \qquad\qquad + \sum_{l=1}^{K}\pi_l C^\top \check{x}_2^l]dt, \\ \alpha_k(T) = \xi_k, \quad \gamma_k(0) = Q\alpha_k(0) - (QH + H^\top Q - H^\top QH)\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l(0), \\ \check{x}_1^k(0) = -Q\alpha_k(0), \quad \check{x}_2^k(0) = (QH + H^\top Q - H^\top QH)\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l(0). \end{cases} \tag{17}$$

Remark 6 (i) It is remarkable that if (17) is solved, by the estimates of BSDE, we can easily obtain $\hat{y} = \sum_{l=1}^{K} \pi_l \mathbb{E}\alpha_l$, so does $x_2^k = \check{x}_2^k$, $\hat{x}_k = \check{x}_1^k$. Actually, the *(*CC*)* system (17) is a coupled FBSDE composed by not only three forward SDEs and a BSDE, but also the mean-field terms. If taking the expectation to (17), we can also obtain $\hat{y} = \sum_{l=1}^{K} \pi_l \mathbb{E}\alpha_l$. However, in the consideration of the generalization, we focus the coupled FBSDE with mean-field terms here.

(ii) In (17), for $k = 1, \cdots, K$, the subscript k *(*e.g., $\alpha_k, \beta_k, \cdots$*)* stands for a representative agent in the k-type.

In the following, we give two propositions to obtain the well-posedness of (CC) system (17). Before solving (17), let us make some transformations and introduce some notations. Define $Y = (\alpha_1^\top, \cdots, \alpha_K^\top)^\top$, $X = (\gamma_1^\top, \cdots, \gamma_K^\top, (\check{x}_1^1)^\top, \cdots, (\check{x}_1^K)^\top, (\check{x}_2^1)^\top, \cdots, (\check{x}_2^K)^\top)^\top$, $W = (W_1^\top, \cdots, W_K^\top)^\top$ and

$$Z = \begin{pmatrix} \beta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_K \end{pmatrix}_{(Kn \times Kd)},$$

the mean-field FBSDEs (17) then take the following form:

$$\begin{cases} dY = -\Big[\mathbb{A}_1 Y + \bar{\mathbb{A}}_1 \mathbb{E}[Y] + \mathbb{B}_1 X + \hat{f}\Big] dt + Z dW(t), \\ dX = \Big[\mathbb{A}_2 X + \bar{\mathbb{A}}_2 \mathbb{E}[X] + \mathbb{B}_2 Y + \bar{\mathbb{B}}_2 \mathbb{E}[Y]\Big] dt, \\ X(0) = \mathbb{H}_1 Y(0) + \mathbb{H}_2 \mathbb{E} Y(0), \qquad Y(T) = (\xi_1^\top, \cdots, \xi_K^\top)^\top, \end{cases} \tag{18}$$

where

$$\mathbb{A}_1=\begin{pmatrix} A_1^\top & 0 & \cdots & 0\\ 0 & A_2^\top & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & A_K^\top \end{pmatrix}_{(Kn\times Kn)},\bar{\mathbb{A}}_1=\begin{pmatrix} C\pi_1 & C\pi_2 & \cdots & C\pi_K\\ C\pi_1 & C\pi_2 & \cdots & C\pi_K\\ \vdots & \vdots & \ddots & \vdots\\ C\pi_1 & C\pi_2 & \cdots & C\pi_K \end{pmatrix}_{(Kn\times Kn)},\hat{f}=\begin{pmatrix} f\\ f\\ \vdots\\ f \end{pmatrix}_{(Kn\times 1)},$$

$$\mathbb{B}_1=\begin{pmatrix} -BR_1^{-1}B^\top & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & -BR_2^{-1}B^\top & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -BR_K^{-1}B^\top & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{(Kn\times 3Kn)},\mathbb{H}_1=\begin{pmatrix} Q & 0 & \cdots & 0\\ 0 & Q & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & Q\\ -Q & 0 & \cdots & 0\\ 0 & -Q & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -Q\\ 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 \end{pmatrix}_{(3Kn\times Kn)},$$

$$\mathbb{A}_2=\begin{pmatrix} A_1^\top & 0 & \cdots & 0 & -\pi_1C^\top & -\pi_2C^\top & \cdots & -\pi_KC^\top & -\pi_1C^\top & -\pi_2C^\top & \cdots & -\pi_KC^\top\\ 0 & A_2^\top & \cdots & 0 & -\pi_1C^\top & -\pi_2C^\top & \cdots & -\pi_KC^\top & -\pi_1C^\top & -\pi_2C^\top & \cdots & -\pi_KC^\top\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & A_K^\top & -\pi_1C^\top & -\pi_2C^\top & \cdots & -\pi_KC^\top & -\pi_1C^\top & -\pi_2C^\top & \cdots & -\pi_KC^\top\\ 0 & 0 & \cdots & 0 & A_1^\top & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & A_2^\top & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & A_K^\top & 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & A_1^\top+\pi_1C^\top & \pi_2C^\top & \cdots & \pi_KC^\top\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \pi_1C^\top & A_2^\top+\pi_2C^\top & \cdots & \pi_KC^\top\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \pi_1C^\top & \pi_2C^\top & \cdots & A_K^\top+\pi_KC^\top \end{pmatrix}_{(3Kn\times 3Kn)},$$

$$
\bar{\mathbb{A}}_2 = \begin{pmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \pi_1 C^\top & \cdots & \pi_K C^\top & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \pi_1 C^\top & \cdots & \pi_K C^\top & 0 & \cdots & 0
\end{pmatrix}_{(3Kn\times 3Kn)}, \mathbb{B}_2 = \begin{pmatrix}
S & 0 & \cdots & 0 \\
0 & S & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & S \\
-S & 0 & \cdots & 0 \\
0 & -S & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -S \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}_{(3Kn\times Kn)},
$$

$$
\bar{\mathbb{B}}_2 = \begin{pmatrix}
-(SG+G^\top S-G^\top SG)\pi_1 & -(SG+G^\top S-G^\top SG)\pi_2 & \cdots & -(SG+G^\top S-G^\top SG)\pi_K \\
-(SG+G^\top S-G^\top SG)\pi_1 & -(SG+G^\top S-G^\top SG)\pi_2 & \cdots & -(SG+G^\top S-G^\top SG)\pi_K \\
\vdots & \vdots & \ddots & \vdots \\
-(SG+G^\top S-G^\top SG)\pi_1 & -(SG+G^\top S-G^\top SG)\pi_2 & \cdots & -(SG+G^\top S-G^\top SG)\pi_K \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
(SG+G^\top S-G^\top SG)\pi_1 & (SG+G^\top S-G^\top SG)\pi_2 & \cdots & (SG+G^\top S-G^\top SG)\pi_K \\
(SG+G^\top S-G^\top SG)\pi_1 & (SG+G^\top S-G^\top SG)\pi_2 & \cdots & (SG+G^\top S-G^\top SG)\pi_K \\
\vdots & \vdots & \ddots & \vdots \\
(SG+G^\top S-G^\top SG)\pi_1 & (SG+G^\top S-G^\top SG)\pi_2 & \cdots & (SG+G^\top S-G^\top SG)\pi_K
\end{pmatrix}_{(3Kn\times Kn)},
$$

$$
\mathbb{H}_2 = \begin{pmatrix}
-(QH+H^\top Q-H^\top QH)\pi_1 & -(QH+H^\top Q-H^\top QH)\pi_2 & \cdots & -(QH+H^\top Q-H^\top QH)\pi_K \\
-(QH+H^\top Q-H^\top QH)\pi_1 & -(QH+H^\top Q-H^\top QH)\pi_2 & \cdots & -(QH+H^\top Q-H^\top QH)\pi_K \\
\vdots & \vdots & \ddots & \vdots \\
-(QH+H^\top Q-H^\top QH)\pi_1 & -(QH+H^\top Q-H^\top QH)\pi_2 & \cdots & -(QH+H^\top Q-H^\top QH)\pi_K \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
(QH+H^\top Q-H^\top QH)\pi_1 & (QH+H^\top Q-H^\top QH)\pi_2 & \cdots & (QH+H^\top Q-H^\top QH)\pi_K \\
(QH+H^\top Q-H^\top QH)\pi_1 & (QH+H^\top Q-H^\top QH)\pi_2 & \cdots & (QH+H^\top Q-H^\top QH)\pi_K \\
\vdots & \vdots & \ddots & \vdots \\
(QH+H^\top Q-H^\top QH)\pi_1 & (QH+H^\top Q-H^\top QH)\pi_2 & \cdots & (QH+H^\top Q-H^\top QH)\pi_K
\end{pmatrix}_{(3Kn\times Kn)}.
$$

The above system is highly-augmented due to the coupling in discrete-type heterogenous agents. In the following, we will use the Riccati equation theory to discuss the well-posedness of system (18), which is shown as a coupled backward-forward stochastic differential equation.

Proposition 1 *Let (A1)-(A4) hold. Suppose the Riccati equation*

$$
\begin{cases}
\dot{\Phi} + \Phi(\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1) + (\mathcal{A}_1 + \mathcal{B}_1\mathcal{H})\Phi + \Phi\big[\mathcal{H}\mathcal{A}_1 + (\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1)\mathcal{H} + \mathcal{B}_2\big]\Phi + \mathcal{B}_1 = 0, \\
\Phi(T) = 0,
\end{cases} \tag{19}
$$

where

$$\mathcal{A}_1=\begin{pmatrix}\mathbb{A}_1+\bar{\mathbb{A}}_1 & 0\\ 0 & \mathbb{A}_1\end{pmatrix},\mathcal{B}_1=\begin{pmatrix}\mathbb{B}_1 & 0\\ 0 & \mathbb{B}_1\end{pmatrix},\mathcal{F}=\begin{pmatrix}\hat{f}\\ 0\end{pmatrix},\mathcal{Z}=\begin{pmatrix}0\\ Z\end{pmatrix},$$
$$\mathcal{A}_2=\begin{pmatrix}\mathbb{A}_2+\bar{\mathbb{A}}_2 & 0\\ 0 & \mathbb{A}_2\end{pmatrix},\mathcal{B}_2=\begin{pmatrix}\mathbb{B}_2+\bar{\mathbb{B}}_2 & 0\\ 0 & \mathbb{B}_2\end{pmatrix},\mathcal{H}=\begin{pmatrix}\mathbb{H}_1+\mathbb{H}_2 & 0\\ 0 & \mathbb{H}_1\end{pmatrix},$$
$$\Xi=(\mathbb{E}\xi_1^\top,\cdots,\mathbb{E}\xi_K^\top,\xi_1^\top-\mathbb{E}\xi_1^\top,\cdots,\xi_K^\top-\mathbb{E}\xi_K^\top)^\top,$$

admits a unique solution $\Phi(\cdot)$ over $[0,T]$ such that $I+\Phi\mathcal{H}$ is invertible, then the well-posedness of (CC) *system* (17) (*which is equivalent to* (18)) *is obtained.*

Proof Taking the expectation of (18), we can get

$$\begin{cases} d\mathbb{E}[Y]=-\Big[(\mathbb{A}_1+\bar{\mathbb{A}}_1)\mathbb{E}[Y]+\mathbb{B}_1\mathbb{E}[X]+\hat{f}\Big]dt,\\ d\mathbb{E}[X]=\Big[(\mathbb{A}_2+\bar{\mathbb{A}}_2)\mathbb{E}[X]+(\mathbb{B}_2+\bar{\mathbb{B}}_2)\mathbb{E}[Y]\Big]dt,\\ \mathbb{E}[X](0)=(\mathbb{H}_1+\mathbb{H}_2)\mathbb{E}[Y](0),\qquad \mathbb{E}[Y](T)=(\mathbb{E}\xi_1^\top,\cdots,\mathbb{E}\xi_K^\top)^\top.\end{cases}\tag{20}$$

From (18) and (20), it follows that

$$\begin{cases} d\Big(Y-\mathbb{E}[Y]\Big)=-\Big[\mathbb{A}_1\Big(Y-\mathbb{E}[Y]\Big)+\mathbb{B}_1\Big(X-\mathbb{E}[X]\Big)\Big]dt+ZdW,\\ d\Big(X-\mathbb{E}[X]\Big)=\Big[\mathbb{A}_2\Big(X-\mathbb{E}[X]\Big)+\mathbb{B}_2\Big(Y-\mathbb{E}[Y]\Big)\Big]dt,\\ X(0)-\mathbb{E}[X](0)=\mathbb{H}_1(Y(0)-\mathbb{E}[Y](0)),\\ Y(T)-\mathbb{E}[Y](T)=(\xi_1^\top-\mathbb{E}\xi_1^\top,\cdots,\xi_K^\top-\mathbb{E}\xi_K^\top)^\top.\end{cases}\tag{21}$$

Denote

$$\mathcal{Y}=\begin{pmatrix}\mathbb{E}[Y]\\ Y-\mathbb{E}[Y]\end{pmatrix},\qquad \mathcal{X}=\begin{pmatrix}\mathbb{E}[X]\\ X-\mathbb{E}[X]\end{pmatrix}.$$

Then the well-posedness of the mean-field FBSDE (18) is equivalent to the following FBSDE

$$\begin{cases} d\mathcal{Y}=-\Big[\mathcal{A}_1\mathcal{Y}+\mathcal{B}_1\mathcal{X}+\mathcal{F}\Big]dt+\mathcal{Z}dW,\\ d\mathcal{X}=\Big[\mathcal{A}_2\mathcal{X}+\mathcal{B}_2\mathcal{Y}\Big]dt,\\ \mathcal{X}(0)=\mathcal{H}\mathcal{Y}(0),\qquad \mathcal{Y}(T)=\Xi.\end{cases}\tag{22}$$

Define

$$\tilde{\mathcal{X}}(t)=\mathcal{X}(t)-\mathcal{H}\mathcal{Y}(t),\qquad t\in[0,T].$$

$\mathcal{X}(0)=\mathcal{H}\mathcal{Y}(0)$ implies $\tilde{\mathcal{X}}(0)=0$. By (22) and

$$d\tilde{\mathcal{X}}=d\mathcal{X}-\mathcal{H}d\mathcal{Y},$$

we have

$$\begin{cases} d\tilde{\mathcal{X}} = \Big[(\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1)\tilde{\mathcal{X}} + \Big(\mathcal{H}\mathcal{A}_1 + \big(\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1\big)\mathcal{H} + \mathcal{B}_2\Big)\mathcal{Y} + \mathcal{H}\digamma\Big]dt \\ \qquad\quad - \mathcal{H}\mathcal{Z}dW(t), \\ d\mathcal{Y} = -\Big[(\mathcal{A}_1 + \mathcal{B}_1\mathcal{H})\mathcal{Y} + \mathcal{B}_1\tilde{\mathcal{X}} + \digamma\Big]dt + \mathcal{Z}dW(t), \\ \tilde{\mathcal{X}}(0) = 0, \qquad \mathcal{Y}(T) = \Xi, \end{cases} \tag{23}$$

which is a *common* fully-coupled FBSDE.

We assume that $\tilde{\mathcal{X}}$ and $\mathcal{Y}$ are related by

$$\mathcal{Y}(t) = \Phi(t)\tilde{\mathcal{X}}(t) + \Psi(t), \qquad t \in [0,T], \quad a.s.$$

where $\Phi : [0,T] \to \mathbb{R}^{2Kn\times 6Kn}$ is a deterministic matrix-valued function and $\Psi : [0,T]\times\Omega \to \mathbb{R}^{2Kn}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process. We are going to derive the equation for $\Phi(\cdot)$ and $\Psi(\cdot)$. It follows from the initial value of $\tilde{\mathcal{X}}$ and $\mathcal{Y}$ that

$$\Phi(T) = 0, \qquad \Psi(T) = \Xi.$$

Since $\Xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^{2Kn})$ and $\Psi(\cdot)$ is required to be $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, we should assume that $\Psi(\cdot)$ satisfies a BSDE:

$$\begin{cases} d\Psi(t) = a(t)dt + b(t)dW(t), \\ \Psi(T) = \Xi, \end{cases} \tag{24}$$

where $a(\cdot), b(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{2Kn})$ being undetermine. Applying Itô's formula, we have

$$\begin{aligned} d\mathcal{Y} &= \dot{\Phi}\tilde{\mathcal{X}}dt + \Phi d\tilde{\mathcal{X}} + adt + bdW(t) \\ &= \Big\{\Big[\dot{\Phi} + \Phi\big(\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1\big) + \Phi\big[\mathcal{H}\mathcal{A}_1 + \big(\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1\big)\mathcal{H} + \mathcal{B}_2\big]\Phi\Big]\tilde{\mathcal{X}} \\ &\quad + \Phi\big[\mathcal{H}\mathcal{A}_1 + \big(\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1\big)\mathcal{H} + \mathcal{B}_2\big]\Psi + \Phi\mathcal{H}\digamma + a\Big\}dt + (b - \Phi\mathcal{H}\mathcal{Z})dW(t). \end{aligned}$$

Comparing with the second equation in (23), we obtain that

$$\begin{aligned} &\Big[\dot{\Phi} + \Phi\big(\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1\big) + \Phi\big[\mathcal{H}\mathcal{A}_1 + \big(\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1\big)\mathcal{H} + \mathcal{B}_2\big]\Phi + \big(\mathcal{A}_1 + \mathcal{B}_1\mathcal{H}\big)\Phi + \mathcal{B}_1\Big]\tilde{\mathcal{X}} \\ &\quad + \Phi\big[\mathcal{H}\mathcal{A}_1 + \big(\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1\big)\mathcal{H} + \mathcal{B}_2\big]\Psi + \Phi\mathcal{H}\digamma + a + \big(\mathcal{A}_1 + \mathcal{B}_1\mathcal{H}\big)\Psi + \digamma = 0 \end{aligned}$$

and

$$b - \Phi\mathcal{H}\mathcal{Z} = \mathcal{Z}.$$

Since $I + \Phi\mathcal{H}$ is invertible, it follows that

$$\mathcal{Z} = (I + \Phi\mathcal{H})^{-1}b.$$

Noting (19), we have

$$a = -\Big[\Phi\big[\mathcal{H}\mathcal{A}_1 + \big(\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1\big)\mathcal{H} + \mathcal{B}_2\big]\Psi + \Phi\mathcal{H}\digamma + \big(\mathcal{A}_1 + \mathcal{B}_1\mathcal{H}\big)\Psi + \digamma\Big].$$

Then the equation (24) has the following form

$$\begin{cases} d\Psi = -\Big\{\Big[\mathcal{A}_1 + \mathcal{B}_1\mathcal{H} + \Phi\big(\mathcal{H}\mathcal{A}_1 + \big(\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1\big)\mathcal{H} + \mathcal{B}_2\big)\Big]\Psi \\ \qquad\qquad + \Phi\mathcal{H}\digamma + \digamma\Big\}dt + bdW(t), \\ \Psi(T) = \Xi. \end{cases} \tag{25}$$

When (19) admits a solution $\Phi(\cdot)$ such that $I + \Phi\mathcal{H}$ is invertible, then BSDE (25) admits a unique adapted solution $(\Psi(\cdot), b(\cdot))$. Then the equation of $\tilde{\mathcal{X}}$

$$\begin{cases} d\tilde{\mathcal{X}} = \Big\{\Big[\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1 + \big(\mathcal{H}\mathcal{A}_1 + \big(\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1\big)\mathcal{H} + \mathcal{B}_2\big)\Phi\Big]\tilde{\mathcal{X}} \\ \qquad\quad + \big(\mathcal{H}\mathcal{A}_1 + \big(\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1\big)\mathcal{H} + \mathcal{B}_2\big)\Psi \\ \qquad\quad + \mathcal{H}\digamma\Big\}dt - \mathcal{H}(I + \Phi\mathcal{H})^{-1}bdW(t), \\ \tilde{\mathcal{X}}(0) = 0, \end{cases} \tag{26}$$

admits a unique solution $\tilde{\mathcal{X}}(\cdot)$. Furthermore, the second equation in (23) (BS-DE) admits a unique solution $(\mathcal{Y}(\cdot), \mathcal{Z}(\cdot))$. Then the well-posedness of $\mathcal{X}(\cdot)$ is obtained. The proof is complete.

In the rest of this section, we will give another thinking about the well-posedness of (17), where we decouple (22) directly.

Proposition 2 *Let (A1)-(A4) hold. Suppose the Riccati equation*

$$\begin{cases} \dot{\phi} - \phi\mathcal{A}_1 - \mathcal{A}_2\phi - \phi\mathcal{B}_1\phi - \mathcal{B}_2 = 0, \\ \phi(0) = \mathcal{H}, \end{cases} \tag{27}$$

admits a unique solution $\phi(\cdot)$ over $[0,T]$, and $I + P\phi$ is invertible, then the well-posedness of (CC) *system* (17) (*which is equivalent to* (18) *and* (22)) *is obtained.*

Proof Define

$$\mathcal{X}(t) = \phi(t)\mathcal{Y}(t) + \psi(t), \qquad t \in [0,T].$$

$\mathcal{X}(0) = \mathcal{H}\mathcal{Y}(0)$ implies $\phi(0) = \mathcal{H}$, $\psi(0) = 0$. Applying Itô's formula, we have

$$\begin{aligned} d\mathcal{X} &= \dot{\phi}\mathcal{Y} + \phi d\mathcal{Y} + d\psi \\ &= \big(\dot{\phi} - \phi\mathcal{A}_1 - \phi\mathcal{B}_1\phi\big)\mathcal{Y}dt - \big(\phi\mathcal{B}_1\psi + \phi\digamma\big)dt + d\psi + \phi\mathcal{Z}dW(t) \\ &= \big(\mathcal{A}_2\phi + \mathcal{B}_2\big)\mathcal{Y}dt + \mathcal{A}_2\psi dt. \end{aligned}$$

Comparing the coefficients, we obtain (27) and

$$\begin{cases} d\psi = \Big[\big(\mathcal{A}_2 + \phi\mathcal{B}_1\big)\psi + \phi\digamma\Big]dt - \phi\mathcal{Z}dW(t), \\ \psi(0) = 0. \end{cases} \tag{28}$$

Under the assumption, (27) admits a unique solution $\phi(\cdot)$ over $[0,T]$. Actually, (25) is a SDE depending on ϕ and $\mathcal{Z}$. By noting the first equation of (22), we find $\mathcal{Y}$ and ψ are coupled together:

$$\begin{cases} d\psi = \Big[\big(\mathcal{A}_2 + \phi\mathcal{B}_1\big)\psi + \phi\digamma\Big]dt - \phi\mathcal{Z}dW(t), \\ d\mathcal{Y} = -\Big[\big(\mathcal{A}_1 + \mathcal{B}_1\phi\big)\mathcal{Y} + \mathcal{B}_1\psi + \digamma\Big]dt + \mathcal{Z}dW(t), \\ \psi(0) = 0, \quad \mathcal{Y}(T) = \varXi. \end{cases} \tag{29}$$

Here, in (29) there's no couple structure in the initial or terminal value, which seems similar to (23). Applying the same method as Proposition 1, we define

$$\mathcal{Y}(t) = P(t)\psi(t) + \hat{P}(t), \qquad t \in [0,T], \quad a.s.$$

By Itô's formula and comparing the coefficients, we obtain

$$\begin{cases} \dot{P} + P\big(\mathcal{A}_2 + \phi\mathcal{B}_1\big) + \big(\mathcal{A}_1 + \mathcal{B}_1\phi\big)P + \mathcal{B}_1 = 0, \\ P(T) = 0, \end{cases} \tag{30}$$

$$\begin{cases} d\hat{P} = -\Big[\big(\mathcal{A}_1 + \mathcal{B}_1\phi\big)\hat{P} + P\phi\digamma + \digamma\Big]dt + qdW(t), \\ \hat{P}(T) = \varXi, \end{cases} \tag{31}$$

and

$$\mathcal{Z} = (I + P\phi)^{-1}q.$$

Actually we can derive the explicit solution of (30) (see Proposition 3). Then we obtain that BSDE (31) admits a unique solution $(\hat{P}(\cdot), q(\cdot))$. Then the equation of ψ

$$\begin{cases} d\psi = \Big[\big(\mathcal{A}_2 + \phi\mathcal{B}_1\big)\psi + \phi\digamma\Big]dt - \phi(I + P\phi)^{-1}qdW(t), \\ \psi(0) = 0, \end{cases} \tag{32}$$

admits a unique solution $\psi(\cdot)$. Furthermore, the second equation (BSDE) in (29) admits a unique solution $(\mathcal{Y}(\cdot), \mathcal{Z}(\cdot))$. Then the well-posedness of $\mathcal{X}(\cdot)$ is obtained.

Remark 7 For the fully-coupled FBSDE (22), from the initial value relationship $\mathcal{X}(0) = \mathcal{H}\mathcal{Y}(0)$, we conjecture that $\mathcal{X}$ has some representation of $\mathcal{Y}$, like $\mathcal{X}(\cdot) = \phi(\cdot)\mathcal{Y}(\cdot) + \psi(\cdot)$. However, ψ in (28) depends on $\mathcal{Z}$, which cannot be determined off-line. Actually, ψ and $\mathcal{Y}$ are still coupled, where ψ enters in the generator of $\mathcal{Y}$, while ψ does not depends on $\mathcal{Y}$ explicitly. Until now, the decouple process has not completed.

Remark 8 In both of two decoupling processes, we assume the Riccati equations (19) and (27) admit unique solutions and some invertible conditions hold. In Proposition 1, we use Itô's formula and compare the coefficients for one time, but the coefficients of (19) seem to be complicated. In Proposition 2, the decoupling process is more complex, while the coefficients of (27) is relatively simple. In the following, we will give some explicit representations and numerical solutions of (19) and (27).

Remark 9 (22) is a fully-coupled FBSDE, and the initial value of SDE $\mathcal{X}$ depends on that of BSDE $\mathcal{Y}$, which has some difference as the common sense. This type of FBSDE has attracted some attentions, see e.g., [22,23], etc. However, there's essential difference between the coupled FBSDE in this paper and those in the existing works. Note that $\mathcal{A}_2 + \mathcal{H}\mathcal{B}_1 \neq (\mathcal{A}_1 + \mathcal{B}_1\mathcal{H})^\top$, $\mathcal{A}_2 \neq \mathcal{A}_1^\top$, which means Φ and ϕ are both asymmetric. From the decoupling process, we can see the reason why the Riccati equations are asymmetric is that Y and X have different dimension, which is caused by the characteristics of the system itself. To obtain the proof of Riccati equations of the solvability of (19) and (27) is challenging. In the future, we may focus on this problem and try to get some meaningful results on it.

First of all, we introduce a lemma as follows.

Lemma 1 (*Existence and Uniqueness of Solutions,* [1]) *Let $I_0 \in \mathbb{R}$ be an open interval with $t_0 \in I_0$, $A \in L^\infty(I_0, \mathbb{C}^{n\times n})$, $B \in L^\infty(I_0, \mathbb{C}^{m\times m})$, $C \in L^\infty(I_0, \mathbb{C}^{n\times m})$ and $D \in \mathbb{C}^{n\times m}$. The differential Sylvester equation*

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + C(t), \quad X(t_0) = D,$$

has the unique solution

$$X(t) = \Pi_A(t,t_0)D\Big(\Pi_{B^\top}(t,t_0)\Big)^\top + \int_{t_0}^{t} \Pi_A(t,s)C(s)\Big(\Pi_{B^\top}(t,s)\Big)^\top ds.$$

$\Pi_A(t,t_0)$ and $\Pi_{B^\top}(t,t_0)$ are the unique state-transition matrices with respect to $t_0 \in I_0$ defined by

$$\begin{cases} \dot{\Pi}_A(t,t_0) := \dfrac{\partial}{\partial t}\Pi_A(t,t_0) = A(t)\Pi_A(t,t_0), \\ \Pi_A(t_0,t_0) = I_{n\times n}, \end{cases}$$

and

$$\begin{cases} \dot{\Pi}_{B^\top}(t,t_0) := \dfrac{\partial}{\partial t}\Pi_{B^\top}(t,t_0) = \Big(B(t)\Big)^\top \Pi_{B^\top}(t,t_0), \\ \Pi_{B^\top}(t_0,t_0) = I_{m\times m}. \end{cases}$$

Similarly, we can also expand the existence and uniqueness of solution of the Sylvester equation to the case that the terminal value is given.

Lemma 2 (*Existence and Uniqueness of Solutions*) *Let $T > 0$, $A \in L^\infty(0,T;\mathbb{C}^{n\times n})$, $B \in L^\infty(0,T;\mathbb{C}^{m\times m})$, $C \in L^\infty(0,T;\mathbb{C}^{n\times m})$ and $D \in \mathbb{C}^{n\times m}$. The (backward) differential Sylvester equation*

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + C(t), \quad X(T) = D, \tag{33}$$

has the unique solution

$$X(t) = \Pi_A(t,T)D\Big(\Pi_{B^\top}(t,T)\Big)^\top - \int_t^T \Pi_A(t,s)C(s)\Big(\Pi_{B^\top}(t,s)\Big)^\top ds. \tag{34}$$

$\mathit{\Pi}_A(t,T)$ and $\mathit{\Pi}_{B^\top}(t,T)$ are the unique state-transition matrices with respect to T defined by

$$\begin{cases} \dot{\mathit{\Pi}}_A(t,T) := \dfrac{\partial}{\partial t}\mathit{\Pi}_A(t,T) = A(t)\mathit{\Pi}_A(t,T), \\ \mathit{\Pi}_A(T,T) = I_{n\times n}, \end{cases}$$

and

$$\begin{cases} \dot{\mathit{\Pi}}_{B^\top}(t,T) := \dfrac{\partial}{\partial t}\mathit{\Pi}_{B^\top}(t,T) = \Big(B(t)\Big)^\top \mathit{\Pi}_{B^\top}(t,T), \\ \mathit{\Pi}_{B^\top}(T,T) = I_{m\times m}. \end{cases}$$

The proof is trivial. Actually, it is not hard to verify that $X(\cdot)$ defined by (34) satisfies the Sylvester equation (33).

Proposition 3 *Let (A1)-(A4) hold. For any $s \in [0,T]$, let $\Psi_1(\cdot,s)$ and $\Psi_2(\cdot,s)$ be the solutions of the following ODEs:*

$$\begin{cases} \dfrac{d}{dt}\Psi_1(t,s) = \widehat{\mathbf{A}}_1(t)\Psi_1(t,s), \quad t\in[s,T], \\ \Psi_1(s,s) = I, \end{cases} \tag{35}$$

and

$$\begin{cases} \dfrac{d}{dt}\Psi_2(t,s) = \widehat{\mathbf{A}}_2(t)\Psi_2(t,s), \quad t\in[s,T], \\ \Psi_2(s,s) = I, \end{cases} \tag{36}$$

respectively, where

$$\begin{cases} \widehat{\mathbf{A}}_1(\cdot) = \begin{pmatrix} \mathcal{A}_2(\cdot)+\mathcal{H}\mathcal{B}_1(\cdot) & \mathcal{H}\mathcal{A}_1(\cdot)+[\mathcal{A}_2(\cdot)+\mathcal{H}\mathcal{B}_1(\cdot)]\mathcal{H}+\mathcal{B}_2(\cdot) \\ -\mathcal{B}_1(\cdot) & -[\mathcal{A}_1(\cdot)+\mathcal{B}_1(\cdot)\mathcal{H}] \end{pmatrix}, \\ \widehat{\mathbf{A}}_2(\cdot) = \begin{pmatrix} \mathcal{A}_1(T-\cdot)+\mathcal{B}_1(T-\cdot)\mathcal{H} & \mathcal{B}_1(T-\cdot) \\ -[\mathcal{H}\mathcal{A}_1(T-\cdot)+\mathcal{A}_2(T-\cdot)\mathcal{H}+\mathcal{H}\mathcal{B}_1(T-\cdot)\mathcal{H}+\mathcal{B}_2(T-\cdot)] & -[\mathcal{A}_2(T-\cdot)+\mathcal{H}\mathcal{B}_1(T-\cdot)] \end{pmatrix}. \end{cases}$$

Suppose

$$\left[\begin{pmatrix} 0 & I \end{pmatrix}\Psi_1(T,t)\begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1} \in L^1(0,T;\mathbb{R}^{2Kn\times 6Kn}),$$

$$\left[\begin{pmatrix} 0 & I \end{pmatrix}\Psi_2(T,t)\begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1} \in L^1(0,T;\mathbb{R}^{6Kn\times 2Kn}).$$

Then Riccati equation (19) and (27) admit unique solutions $\Phi(\cdot)$ and $\phi(\cdot)$, which are given by the followings:

$$\Phi(t) = -\left[\begin{pmatrix} 0 & I \end{pmatrix}\Psi_1(T,t)\begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1}\begin{pmatrix} 0 & I \end{pmatrix}\Psi_1(T,t)\begin{pmatrix} I \\ 0 \end{pmatrix}, \qquad t\in[0,T], \tag{37}$$

and

$$\phi(t) = \mathcal{H} - \left[\begin{pmatrix} 0 & I \end{pmatrix}\Psi_2(T,T-t)\begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1}\begin{pmatrix} 0 & I \end{pmatrix}\Psi_2(T,T-t)\begin{pmatrix} I \\ 0 \end{pmatrix}, \tag{38}$$

respectively. The solution of (backward) Sylvester equation (30) *is given as follows:*

$$P(t)=\int_t^T e^{\int_t^s \left(\mathcal{A}_1(r)+\mathcal{B}_1(r)\phi(r)\right)dr}\mathcal{B}_1(s)e^{\int_t^s \left(\mathcal{A}_2(r)+\phi(r)\mathcal{B}_1(r)\right)dr}ds. \tag{39}$$

Proof We can refer to [38, Theorem 5.3] for (37). Define

$$\Gamma(t)=\phi(T-t)-\mathcal{H}, \qquad t\in[0,T].$$

$\phi(0)=\mathcal{H}$ implies $\Gamma(T)=0$. By $\dot{\Gamma}(t)=-\dot{\phi}(T-t)$, we obtain

$$\begin{cases}\dot{\Gamma}+\Gamma\Big(\mathcal{A}_1(T-t)+\mathcal{B}_1(T-t)\mathcal{H}\Big)+\Big(\mathcal{A}_2(T-t)+\mathcal{H}\mathcal{B}_1(T-t)\Big)\Gamma+\Gamma\mathcal{B}_1(T-t)\Gamma\\ \qquad +\mathcal{H}\mathcal{A}_1(T-t)+\mathcal{A}_2(T-t)\mathcal{H}+\mathcal{H}\mathcal{B}_1(T-t)\mathcal{H}+\mathcal{B}_2(T-t)=0,\\ \Gamma(T)=0.\end{cases} \tag{40}$$

Then we have

$$\Gamma(t)=-\left[\begin{pmatrix}0 & I\end{pmatrix}\Psi_2(T,t)\begin{pmatrix}0\\ I\end{pmatrix}\right]^{-1}\begin{pmatrix}0 & I\end{pmatrix}\Psi_2(T,t)\begin{pmatrix}I\\ 0\end{pmatrix}, \qquad t\in[0,T]. \tag{41}$$

Furthermore, we get (38). With the help of Lemma 2, we can also derive (39).

In the following, we further discuss the explicit solutions of Riccati equations. We give the following proposition.

Proposition 4 *Let* $\widehat{\mathbf{A}}_1(\cdot)$, $\widehat{\mathbf{A}}_2(\cdot)$ *be constant-valued matrices and denote by* $\widehat{\mathbf{A}}_1(t)\equiv\Lambda$, $\widehat{\mathbf{A}}_2(t)\equiv\Delta$. *Suppose*

$$\det\left\{\begin{pmatrix}0 & I\end{pmatrix}e^{\Lambda t}\begin{pmatrix}0\\ I\end{pmatrix}\right\}>0,\ \det\left\{\begin{pmatrix}0 & I\end{pmatrix}e^{\Delta t}\begin{pmatrix}0\\ I\end{pmatrix}\right\}>0, \quad \forall\ t\in[0,T] \tag{42}$$

holds, then (37) *and* (38) *admit unique solutions* $\Phi(\cdot)$ *and* $\phi(\cdot)$*, which have the following representations:*

$$\Phi(t)=-\left[\begin{pmatrix}0 & I\end{pmatrix}e^{\Lambda(T-t)}\begin{pmatrix}0\\ I\end{pmatrix}\right]^{-1}\begin{pmatrix}0 & I\end{pmatrix}e^{\Lambda(T-t)}\begin{pmatrix}I\\ 0\end{pmatrix}, \qquad t\in[0,T], \tag{43}$$

$$\phi(t)=\mathcal{H}-\left[\begin{pmatrix}0 & I\end{pmatrix}e^{\Delta t}\begin{pmatrix}0\\ I\end{pmatrix}\right]^{-1}\begin{pmatrix}0 & I\end{pmatrix}e^{\Delta t}\begin{pmatrix}I\\ 0\end{pmatrix}, \qquad t\in[0,T]. \tag{44}$$

Proof (43) is the direct corollary of [25]. Define $\Pi(t)=\phi(T-t)-\mathcal{H},\ t\in[0,T]$, by [25] we easily have

$$\Pi(t)=-\left[\begin{pmatrix}0 & I\end{pmatrix}e^{\Delta(T-t)}\begin{pmatrix}0\\ I\end{pmatrix}\right]^{-1}\begin{pmatrix}0 & I\end{pmatrix}e^{\Delta(T-t)}\begin{pmatrix}I\\ 0\end{pmatrix}, \tag{45}$$

which implies (44).

In the rest of this section, we set some numerical values to verify the assumption (42) and invertible conditions in Proposition 1 and 2. Based on it, we obtain the numerical solutions of Φ, Π and ϕ.

Example 1 Let $n = m = d = 1$, $K = 1$,

$$[A, B, C, R, Q, H, S, G] = \left[5, 3, -\frac{1}{5}, 5, 2, -2, 3, -3\right],$$

and

$$[t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9] = [1, 2, 3, 4, 5, 6, 7, 8, 9],\ T = 10.$$

Thus Λ is given as $\Lambda = \begin{pmatrix} -27.4 & 0.2 & 0.2 & 0 & 0 & 0 & -362.4 & 0 \\ 3.6 & 5 & 0 & 0 & 0 & 0 & 42.2 & 0 \\ 28.8 & -0.2 & 4.8 & 0 & 0 & 0 & 320.2 & 0 \\ 0 & 0 & 0 & 1.4 & 0.2 & 0.2 & 0 & 15.4 \\ 0 & 0 & 0 & 3.6 & 5 & 0 & 0 & -15.8 \\ 0 & 0 & 0 & 0 & 0 & 4.8 & 0 & 0 \\ 1.8 & 0 & 0 & 0 & 0 & 0 & 27.6 & 0 \\ 0 & 0 & 0 & 1.8 & 0 & 0 & 0 & -1.4 \end{pmatrix}$, which is an 8×8 matrix. Then we have

t	t_1	t_2	t_3	t_4	t_5
$\det\left\{(0\ I)e^{\Lambda t}(0\ I)^\top\right\}$	4.88×10^6	3.32×10^{13}	2.19×10^{20}	1.39×10^{27}	8.47×10^{33}

and

t	t_6	t_7	t_8	t_9
$\det\left\{(0\ I)e^{\Lambda t}(0\ I)^\top\right\}$	4.91×10^{40}	2.66×10^{47}	1.29×10^{54}	5.06×10^{60}

From above tables, we can see that

$$\det\left\{ \begin{pmatrix} 0 & I \end{pmatrix} e^{\Lambda t_i} \begin{pmatrix} 0 \\ I \end{pmatrix} \right\} = \det\left\{ \begin{pmatrix} 0 & I \end{pmatrix} e^{\Lambda t_i} \begin{pmatrix} 0 & I \end{pmatrix}^\top \right\} > 0, \quad i = 1, \dots, 9.$$

Besides that, we can derive

t	t_1	t_2	t_3	t_4	t_5
$\det\left\{I + \Phi(t)\mathcal{H}\right\}$	0.0202	0.0196	0.0194	0.0193	0.0192

and

t	t_6	t_7	t_8	t_9
$\det\left\{I + \Phi(t)\mathcal{H}\right\}$	0.0191	0.0190	0.0190	0.0189

Thus $I + \Phi(t_i)\mathcal{H}$ is invertible, i=1,. . . ,9.

Example 2 Let $n = m = d = 1$, $K = 2$,

$$[A_1, A_2, B, C, \pi_1, \pi_2, R_1, R_2, Q, H, S, G] = \left[1, 2, 2, -3, \frac{1}{3}, \frac{2}{3}, 1, 2, 1, 1, 2, 3\right],$$

and

$$[t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9] = [1, 2, 3, 4, 5, 6, 7, 8, 9], \ T = 10.$$

Then we have

t	t_1	t_2	t_3	t_4	t_5
$\det\left\{(0\ I)e^{\Delta t}(0\ I)^\top\right\}$	4.25×10^{-2}	0.56×10^{-2}	7.34×10^{-4}	9.64×10^{-5}	1.27×10^{-5}

and

t	t_6	t_7	t_8	t_9
$\det\left\{(0\ I)e^{\Delta t}(0\ I)^\top\right\}$	1.66×10^{-6}	2.17×10^{-7}	3.96×10^{-8}	8.74×10^{-7}

From above tables, we can see that

$$\det\left\{ \begin{pmatrix} 0 & I \end{pmatrix} e^{\Delta t_i} \begin{pmatrix} 0 \\ I \end{pmatrix} \right\} = \det\left\{ \begin{pmatrix} 0 & I \end{pmatrix} e^{\Delta t_i} \begin{pmatrix} 0 & I \end{pmatrix}^\top \right\} > 0, \qquad i = 1, \dots, 9.$$

Furthermore, we obtain $\Pi(t_i)$ and $\phi(t_i)$, which are all 12×4 matrix, for $i = 1, \dots, 9$. Here we list $\Pi(1), \Pi(8)$ and $\phi(1), \phi(8)$ as an example.

$$\Pi(1) = \begin{pmatrix} 0.4674 & 1.1003 & 0 & 0 \\ 0.3063 & 2.3709 & 0 & 0 \\ 0.1186 & 1.5221 & 0 & 0 \\ 0.5199 & 3.6292 & 0 & 0 \\ -0.9357 & -0.9921 & 0 & 0 \\ -0.6085 & 1.7372 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Pi(8) = \begin{pmatrix} 0.5081 & 1.1239 & 0 & 0 \\ 0.3570 & 2.3024 & 0 & 0 \\ 0.2595 & 0.7019 & 0 & 0 \\ 0.4498 & -0.1523 & 0 & 0 \\ -0.5854 & -0.9060 & 0 & 0 \\ -0.6992 & -0.8458 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\phi(1) = \begin{pmatrix} 1.1707 & 0.4649 & 0 & 0 \\ 0.0215 & 2.6398 & 0 & 0 \\ -0.7511 & 0.7143 & 0 & 0 \\ 0.4558 & -1.1861 & 0 & 0 \\ -0.2646 & -0.2090 & 0 & 0 \\ -0.3544 & -0.1780 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \phi(8) = \begin{pmatrix} 1.1737 & 0.4544 & 0 & 0 \\ 0.0225 & 2.6328 & 0 & 0 \\ -0.7470 & 0.7066 & 0 & 0 \\ -0.1508 & -5.8453 & 0 & 0 \\ -0.2607 & -0.3452 & 0 & 0 \\ 0.4547 & 6.2165 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is worthy pointing out that since $P(\cdot)$ depends on $\phi(\cdot)$ in (30) (or (39)), the numerical solution of $P(\cdot)$ is very complicated. Thus we just list the solution of $\phi(\cdot)$ and do not verify the invertible condition in Proposition 2.

5 Asymptotic ε-optimality

Let $\widetilde{u} = (\widetilde{u}_1, \cdots, \widetilde{u}_N)$ be the decentralized strategy given by

$$\widetilde{u}_i(t) = -R_{\theta_i}(t)^{-1}B^\top(t)p_i(t),\ i = 1, \cdots, N,$$

where

$$\begin{cases} dy_i(t) = -\Big[A_{\theta_i}(t)y_i(t) - B(t)R_{\theta_i}(t)^{-1}B^\top(t)p_i(t) + C(t)\sum_{k=1}^{K}\pi_k\mathbb{E}\alpha_k(t) \\ \qquad\qquad + f(t)\Big]dt + z_i(t)dW_i(t), \\ dp_i(t) = \Big[A_{\theta_i}^\top p_i + Sy_i - (SG + G^\top S - G^\top SG)\sum_{k=1}^{K}\pi_k\mathbb{E}\alpha_k \\ \qquad\qquad + \sum_{k=1}^{K}\pi_k C^\top x_2^k + \sum_{k=1}^{K}\pi_k C^\top \hat{x}_k\Big]dt, \\ y_i(T) = \xi_i, \quad p_i(0) = Q^\top y_i(0) - (QH + H^\top Q - H^\top QH)\sum_{k=1}^{K}\alpha_k(0). \end{cases}$$

Correspondingly, the realized decentralized state $(\widetilde{y}_1, \cdots, \widetilde{y}_N)$ satisfy

$$\begin{cases} d\widetilde{y}_i(t) = -\Big[A_{\theta_i}(t)\widetilde{y}_i(t) - B(t)R_{\theta_i}^{-1}B^\top(t)p_i(t) + C(t)\widetilde{y}^{(N)}(t) + f(t)\Big]dt \\ \qquad\qquad + \widetilde{z}_i(t)dW_i(t) + \sum_{j=1,j\neq i}^{N}\widetilde{z}_{ij}(t)dW_j(t), \\ \widetilde{y}_i(T) = \xi_i, \end{cases} \tag{46}$$

and $\widetilde{y}^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N}\widetilde{y}_i(t)$.

5.1 Representation of social cost

Rewrite the large-population system (1) as follows:

$$\begin{cases} d\mathbb{Y} = -(\mathbf{A}\mathbb{Y} + \mathbf{B}u + \mathbf{f})dt + \mathbf{z}dW, \\ \mathbf{y}(T) = \varXi, \end{cases} \tag{47}$$

where

$$\mathbb{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}_{(Nn\times 1)}, \mathbf{A} = \begin{pmatrix} A_{\theta_1} + \frac{C}{N} & \frac{C}{N} & \cdots & \frac{C}{N} \\ \frac{C}{N} & A_{\theta_2} + \frac{C}{N} & \cdots & \frac{C}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{C}{N} & \frac{C}{N} & \cdots & A_{\theta_N} + \frac{C}{N} \end{pmatrix}_{(Nn\times Nn)},$$

$$\mathbf{B}=\begin{pmatrix} B & 0 & \cdots & 0\\ 0 & B & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & B\end{pmatrix}_{(Nn\times Nm)},\mathbf{f}=\begin{pmatrix} f\\ \vdots\\ f\end{pmatrix}_{(Nn\times 1)},u=\begin{pmatrix} u_1\\ \vdots\\ u_N\end{pmatrix}_{(Nm\times 1)},$$

$$\mathbf{z}=\begin{matrix}1\\ \vdots\\ i\\ \vdots\\ N\end{matrix}\begin{pmatrix} z_1 & z_{12} & \cdots & z_{1i} & \cdots & z_{1N}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ z_{i1} & z_{i2} & \cdots & z_i & \cdots & z_{iN}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ z_{N1} & z_{N2} & \cdots & z_{Ni} & \cdots & z_N\end{pmatrix}_{(Nn\times Nn)},\Xi=\begin{pmatrix}\xi_1\\ \vdots\\ \xi_N\end{pmatrix}_{(Nn\times 1)},W=\begin{pmatrix}W_1\\ \vdots\\ W_N\end{pmatrix}_{(Nd\times 1)}.$$

Similarly, the social cost takes the following form:

$$\begin{aligned}\mathcal{J}_{soc}^{(N)}(u)&=\sum_{i=1}^{N}\frac{1}{2}\mathbb{E}\Big\{\int_0^T\big[\big\langle S(t)(y_i(t)-G(t)y^{(N)}(t)),y_i(t)-G(t)y^{(N)}(t)\big\rangle\\&\quad+\big\langle R_{\theta_i}(t)u_i(t),u_i(t)\big\rangle\big]dt+\big\langle Q(y_i(0)-Hy^{(N)}(0)),y_i(0)-Hy^{(N)}(0)\big\rangle\Big\}\\&=\frac{1}{2}\mathbb{E}\int_0^T\Big[\langle\mathbf{S}\mathbb{Y},\mathbb{Y}\rangle+\langle\mathbf{R}u,u\rangle\Big]dt+\langle\mathbf{Q}\mathbb{Y}(0),\mathbb{Y}(0)\rangle,\end{aligned}\tag{48}$$

where

$$\mathbf{S}=\begin{pmatrix} S+\frac{1}{N}(G^\top SG-SG-G^\top S) & \frac{1}{N}(G^\top SG-SG-G^\top S) & \cdots & \frac{1}{N}(G^\top SG-SG-G^\top S)\\ \frac{1}{N}(G^\top SG-SG-G^\top S) & S+\frac{1}{N}(G^\top SG-SG-G^\top S) & \cdots & \frac{1}{N}(G^\top SG-SG-G^\top S)\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{N}(G^\top SG-SG-G^\top S) & \frac{1}{N}(G^\top SG-SG-G^\top S) & \cdots & S+\frac{1}{N}(G^\top SG-SG-G^\top S)\end{pmatrix}_{(Nn\times Nn)},$$

$$\mathbf{R}=\begin{pmatrix} R_{\theta_1} & 0 & \cdots & 0\\ 0 & R_{\theta_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & R_{\theta_N}\end{pmatrix}_{(Nm\times Nm)},$$

$$\mathbf{Q}=\begin{pmatrix} Q+\frac{1}{N}(H^\top QH-QH-H^\top Q) & \frac{1}{N}(H^\top QH-QH-H^\top Q) & \cdots & \frac{1}{N}(H^\top QH-QH-H^\top Q)\\ \frac{1}{N}(H^\top QH-QH-H^\top Q) & Q+\frac{1}{N}(H^\top QH-QH-H^\top Q) & \cdots & \frac{1}{N}(H^\top QH-QH-H^\top Q)\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{N}(H^\top QH-QH-H^\top Q) & \frac{1}{N}(H^\top QH-QH-H^\top Q) & \cdots & Q+\frac{1}{N}(H^\top QH-QH-H^\top Q)\end{pmatrix}_{(Nn\times Nn)}.$$

Therefore, there exist a bounded self-adjoint linear operators $M_2:\mathcal{U}_1^c\times\cdots\mathcal{U}_N^c\to\mathcal{U}_1^c\times\cdots\mathcal{U}_N^c$, a bounded operator $M_1:L^\infty(0,T;\mathbb{R}^{Nn})\times L^2_{\mathcal{F}_T}(\mathbb{R}^{Nn})\to L^2(0,T;\mathbb{R}^{Nm})$ and some $M_0\in\mathbb{R}$ depending on $(\mathbf{f},\Xi)$ such that

$$\mathcal{J}_{soc}^{(N)}(u)=\frac{1}{2}\Big\{\langle M_2(u)(\cdot),u(\cdot)\rangle+2\langle M_1,u(\cdot)\rangle+M_0\Big\},$$

where

$$\langle M_2(u)(\cdot)=\mathbf{R}u+\mathbf{B}^\top\varGamma_1,\quad M_1=\mathbf{B}^\top\varGamma_2,\quad M_0=\mathbb{E}\int_0^T\langle\varGamma_2,\mathbf{f}\rangle dt+\mathbb{E}\langle\varGamma_2(T),\Xi\rangle,\tag{49}$$

with

$$\begin{cases} d\mathbb{Y}_1=-(\mathbf{A}\mathbb{Y}_1+\mathbf{B}u)dt+\mathbb{Z}_1dW,\\ d\varGamma_1(s)=(\mathbf{S}\mathbb{Y}_1+\mathbf{A}^\top\varGamma_1)ds,\\ \mathbb{Y}_1(T)=0,\qquad \varGamma_1(0)=\mathbf{Q}\mathbb{Y}_1(0).\end{cases}$$

and

$$\begin{cases} d\mathbb{Y}_2 = -(\mathbf{A}\mathbb{Y}_2 + \mathbf{f})dt + \mathbb{Z}_2 dW, \\ d\Gamma_2(s) = (\mathbf{S}\mathbb{Y}_2 + \mathbf{A}^\top \Gamma_2)ds, \\ \mathbb{Y}_2(T) = \Xi, \qquad \Gamma_2(0) = \mathbf{Q}\mathbb{Y}_2(0). \end{cases}$$

5.2 Agent $\mathcal{A}_i$ perturbation

Let us consider the case that the agent $\mathcal{A}_i$ uses an alternative strategy u_i while the other agents $\mathcal{A}_j, j \neq i$ use the strategy $\widetilde{u}_{-i}$. The realized state with the i-th agent's perturbation is

$$\begin{cases} d\breve{y}_i = -\Big[A_{\theta_i}\breve{y}_i + Bu_i + C\breve{y}^{(N)} + f\Big]dt + \breve{z}_i dW_i(t) + \displaystyle\sum_{l=1, l\neq i}^{N} \breve{z}_{il} dW_l(t), \\ d\breve{y}_j = -\Big[A_{\theta_j}\breve{y}_j - BR_{\theta_j}^{-1}B^\top p_j + C\breve{y}^{(N)} + f\Big]dt + \breve{z}_j dW_j(t) + \displaystyle\sum_{l=1, l\neq j}^{N} \breve{z}_{jl} dW_l(t), \\ \breve{y}_i(T) = \xi_i, \quad \breve{y}_j(T) = \xi_j, \quad 1 \leq j \leq N, \quad j \neq i, \end{cases}$$

where $\breve{y}^{(N)} = \frac{1}{N}\sum_{i=1}^{N} \breve{y}_i$. For $j = 1, \cdots, N$, denote the perturbation

$$\delta u_j = u_j - \widetilde{u}_j, \quad \delta y_j = \breve{y}_j - \widetilde{y}_j, \quad \delta\mathcal{J}_j = \mathcal{J}_j(u_i, \widetilde{u}_{-i}) - \mathcal{J}_j(\widetilde{u}_i, \widetilde{u}_{-i}).$$

Similar as the computations in Section 3.1, we have

$$\begin{aligned} \delta\mathcal{J}_{soc}^{(N)} =& \mathbb{E}\Big\{ \int_0^T \Big[\langle S\widetilde{y}_i, \delta y_i\rangle - \langle (SG + G^\top S - G^\top SG)\sum_{l=1}^{K} \pi_l \mathbb{E}\alpha_l, \delta y_i\rangle \\ & - \sum_{k=1}^{K} \langle \pi_k C^\top x_2^k, \delta y_i\rangle - \sum_{k=1}^{K} \langle \pi_k C^\top \mathbb{E}x_k, \delta y_i\rangle + \langle R_{\theta_i}\widetilde{u}_i, \delta u_i\rangle \Big] dt \\ & + \langle Q\widetilde{y}_i(0), \delta y_i(0)\rangle \\ & - \langle (QH + H^\top Q - H^\top QH)\sum_{l=1}^{K} \pi_l \mathbb{E}\alpha_l(0), \delta y_i(0)\rangle \Big\} + \sum_{l=1}^{8} \varepsilon_l, \end{aligned} \tag{50}$$

where

$$
\begin{cases}
\varepsilon_1 = \mathbb{E}\int_0^T \langle (G^\top S + SG - G^\top SG)(\sum_{l=1}^K \pi_l \mathbb{E}\alpha_l - \widetilde{y}^{(N)}), N\delta y^{(N)} \rangle dt, \\
\varepsilon_2 = \langle (H^\top Q + QH - H^\top QH)(\sum_{l=1}^K \pi_l \mathbb{E}\alpha_l(0) - \widetilde{y}^{(N)}(0)), N\delta y^{(N)}(0) \rangle, \\
\varepsilon_3 = \sum_{k=1}^K \mathbb{E}\int_0^T \langle (SG + G^\top S - G^\top SG)\sum_{l=1}^K \pi_l \mathbb{E}\alpha_l, y_k^{**} - \delta y_{(k)} \rangle dt, \\
\varepsilon_4 = \sum_{k=1}^K \mathbb{E}\int_0^T \frac{1}{N_k}\sum_{j\in\mathcal{I}_k, j\neq i} \langle S\bar{y}_j, N_k \delta y_j - y_j^* \rangle dt, \\
\varepsilon_5 = \sum_{k=1}^K \langle (QH + H^\top Q - H^\top QH)\sum_{l=1}^K \pi_l \mathbb{E}\alpha_l(0), y_k^{**}(0) - \delta y_{(k)}(0) \rangle, \\
\varepsilon_6 = \sum_{k=1}^K \frac{1}{N_k}\sum_{j\in\mathcal{I}_k, j\neq i} \langle Q\widetilde{y}_j(0), N_k \delta y_j(0) - y_j^*(0) \rangle, \\
\varepsilon_7 = \sum_{k=1}^K \mathbb{E}\int_0^T \Big\langle \sum_{l=1}^K \pi_l C^\top \mathbb{E}x_1^l - \sum_{l=1}^K \frac{\pi_l}{N_l}\sum_{j\in\mathcal{I}_l, j\neq i} C^\top x_1^j, y_k^{**} \Big\rangle dt, \\
\varepsilon_8 = \sum_{k=1}^K \mathbb{E}\int_0^T \langle \pi_k C^\top \mathbb{E}x_k - \frac{1}{N_k}\sum_{j\in\mathcal{I}_k, j\neq i} \pi_k C^\top x_1^j, \delta y_i \rangle dt.
\end{cases}
\tag{51}
$$

First, we need some estimations. In the proofs, L will denote a constant whose value may change from line to line.

Lemma 3 [13, Lemma 5.1] *Let (A1)-(A4) hold. Then there exists a constant L independent of N such that*

$$
\begin{aligned}
&\sum_{l=1}^K \mathbb{E}\sup_{0\le t\le T}\Big[|\alpha_l(t)|^2 + |\gamma_l(t)|^2 + |\check{x}_1^l(t)|^2 + |\check{x}_2^l(t)|^2\Big] + \sup_{1\le i\le N}\mathbb{E}\sup_{0\le t\le T}|\widetilde{y}_i(t)|^2 \\
&+ \sum_{l=1}^K \mathbb{E}\int_0^T |\beta_l(t)|^2 dt \le L.
\end{aligned}
\tag{52}
$$

Similar to Lemma 3, we have

$$
\sup_{1\le i\le N}\mathbb{E}\sup_{0\le t\le T}|\check{y}_i(t)|^2 \le L. \tag{53}
$$

where L is a constant independent of N.

Lemma 4 *Let (A1)-(A4) hold. Then there exists a constant L independent of N such that*

$$\mathbb{E}\sup_{0\le t\le T}\Big|\widetilde{y}^{(N)}(t)-\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l(t)\Big|^2\le\frac{L}{N}+L\epsilon_N^2,\tag{54}$$

where $\epsilon_N=\sup_{1\le l\le K}|\pi_l^{(N)}-\pi_l|$.

Proof For $1\le k\le K$, denote the k-type agent state average by

$$\widetilde{y}^{(k)}:=\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}\widetilde{y}_j,\tag{55}$$

thus

$$\begin{cases}d\widetilde{y}^{(k)}=-\Big[A_k\widetilde{y}^{(k)}-\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}BR_k^{-1}B^\top p_j+C\widetilde{y}^{(N)}+f\Big]dt\\ \qquad+\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}\widetilde{z}_jdW_j(t)+\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}\sum_{l=1,l\neq j}^{N}\widetilde{z}_{jl}dW_l(t),\\ \widetilde{y}^{(k)}(T)=\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}\xi_j.\end{cases}$$

Note that

$$d\mathbb{E}\alpha_k=-\Big[A_k\mathbb{E}\alpha_k-\mathbb{E}\big(BR_k^{-1}B^\top p_k\big)+C\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l+f\Big]dt,\quad \mathbb{E}\alpha_k(T)=\mathbb{E}\xi_k,$$

we have

$$\begin{cases}d\Big(\widetilde{y}^{(k)}-\mathbb{E}\alpha_k\Big)=-\Big[A_k\Big(\widetilde{y}^{(k)}-\mathbb{E}\alpha_k\Big)\\ -\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}\Big(BR_k^{-1}B^\top p_j-\mathbb{E}\big(BR_k^{-1}B^\top p_k\big)\Big)+C\Big(\widetilde{y}^{(N)}-\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l\Big)\Big]dt\\ +\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}\widetilde{z}_jdW_j(t)+\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}\sum_{l=1,l\neq j}^{N}\widetilde{z}_{jl}dW_l(t),\\ \Big(\widetilde{y}^{(k)}-\mathbb{E}\alpha_k\Big)(T)=\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}\xi_j-\mathbb{E}\xi_k.\end{cases}$$

By Cauchy-Schwartz inequality, Burkholder-Davis-Gundy inequality and the estimates of BSDE, we have

$$
\begin{aligned}
&\mathbb{E}\sup_{t\le s\le T}\Big|\widetilde{y}^{(k)}-\mathbb{E}\alpha_k\Big|^2+\mathbb{E}\int_t^T\Big(\frac{1}{N_k^2}\sum_{j\in\mathcal{I}_k}\big|\widetilde{z}_j\big|^2+\frac{1}{N_k^2}\sum_{j\in\mathcal{I}_k}\sum_{l=1,l\neq j}^{N}\big|\widetilde{z}_{jl}\big|^2\Big)ds\\
\le&\mathbb{E}\Big|\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}(\xi_j-\mathbb{E}\xi_k)\Big|^2+L\mathbb{E}\int_t^T\Big[\Big|\widetilde{y}^{(k)}-\mathbb{E}\alpha_k\Big|^2+\Big|\widetilde{y}^{(N)}-\sum_{l=1}^K\pi_l\mathbb{E}\alpha_l\Big|^2\Big]ds\\
&+L\mathbb{E}\int_t^T\Big|\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}\Big(BR_k^{-1}B^\top p_j-\mathbb{E}\big(BR_k^{-1}B^\top\mathbf{p}_k\big)\Big)\Big|^2ds.
\end{aligned}
$$

By (A2), for $1\le k\le K$, $\{\xi_j,j\in\mathcal{I}_k\}$ are independent identically distributed. Note that $p_j(\cdot)\in\mathcal{F}_t^j$, thus $\{p_j,j\in\mathcal{I}_k\}$ are independent identically distributed. Then we have

$$
\mathbb{E}\Big|\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}(\xi_j-\mathbb{E}\xi_k)\Big|^2=\frac{1}{N_k}\mathbb{E}\big|\xi_j-\mathbb{E}\xi_k\big|^2\le\frac{L}{N_k}
$$

and

$$
\begin{aligned}
&\mathbb{E}\int_t^T\Big|\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}\Big(BR_k^{-1}B^\top p_j-\mathbb{E}\big(BR_k^{-1}B^\top\mathbf{p}_k\big)\Big)\Big|^2ds\\
=&\frac{1}{N_k}\mathbb{E}\int_t^T\Big|BR_k^{-1}B^\top p_j-\mathbb{E}\big(BR_k^{-1}B^\top\mathbf{p}_k\big)\Big|^2ds\le\frac{L}{N_k}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\sup_{t\le s\le T}\Big|\widetilde{y}^{(k)}-\mathbb{E}\alpha_k\Big|^2\\
\le&L\mathbb{E}\int_t^T\Big|\widetilde{y}^{(k)}-\mathbb{E}\alpha_k\Big|^2ds+L\mathbb{E}\int_t^T\Big|\widetilde{y}^{(N)}-\sum_{l=1}^K\pi_l\mathbb{E}\alpha_l\Big|^2ds+\frac{L}{N_k}.
\end{aligned}
$$

By Gronwall inequality, we have

$$
\mathbb{E}\sup_{t\le s\le T}\Big|\widetilde{y}^{(k)}-\mathbb{E}\alpha_k\Big|^2\le L\mathbb{E}\int_t^T\Big|\widetilde{y}^{(N)}-\sum_{l=1}^K\pi_l\mathbb{E}\alpha_l\Big|^2ds+\frac{L}{N_k}.
$$

Since

$$
\begin{aligned}
&\widetilde{y}^{(N)}-\sum_{l=1}^K\pi_l\mathbb{E}\alpha_l\\
=&\sum_{l=1}^K(\pi_l^{(N)}\widetilde{y}^{(l)}-\pi_l\mathbb{E}\alpha_l)=\sum_{l=1}^K\pi_l^{(N)}(\widetilde{y}^{(l)}-\mathbb{E}\alpha_l)+\sum_{l=1}^K(\pi_l^{(N)}-\pi_l)\mathbb{E}\alpha_l,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\mathbb{E}\sup_{t\le s\le T}\left|\widetilde{y}^{(N)}-\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l\right|^2\\
\le& L\sum_{l=1}^{K}\mathbb{E}\sup_{t\le s\le T}\left|\widetilde{y}^{(l)}-\mathbb{E}\alpha_l\right|^2+L\epsilon_N^2\\
\le& L\mathbb{E}\int_t^T\left|\widetilde{y}^{(N)}-\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l\right|^2ds+\frac{L}{N}+L\epsilon_N^2.
\end{aligned}
$$

Therefore, the result follows from Gronwall inequality.

Lemma 5 *Let (A1)-(A4) hold. Then there exists a constant L independent of N such that*

$$
\begin{aligned}
&\sup_{1\le j\le N, j\ne i}\left[\mathbb{E}\sup_{0\le t\le T}|\delta y_j(t)|^2+\mathbb{E}\int_0^T\left|\delta z_j(t)\right|^2dt+\mathbb{E}\int_0^T\sum_{l=1,l\ne j}^{N}\left|\delta z_{jl}(t)\right|^2dt\right]\\
\le&\frac{L}{N^2}(1+\mathbb{E}\int_0^T|\delta u_i|^2ds).
\end{aligned}
\tag{56}
$$

Proof Recall that

$$
\begin{cases}
d\delta y_i=-\left[A_{\theta_i}\delta y_i+B\delta u_i+C\delta y^{(N)}\right]dt+\delta z_i dW_i(t)+\displaystyle\sum_{l=1,l\ne i}^{N}\delta z_{il}dW_l(t),\\
\delta y_i(T)=0,
\end{cases}
$$

for $j\ne i$,

$$
\begin{cases}
d\delta y_j=-\left[A_{\theta_j}\delta y_j+C\delta y^{(N)}\right]dt+\delta z_j dW_j(t)+\displaystyle\sum_{l=1,l\ne j}^{N}\delta z_{jl}dW_l(t),\\
\delta y_j(T)=0,
\end{cases}
$$

and for $k=1,\cdots,K$,

$$
\begin{cases}
d\delta y_{(k)}=-\left[A_k\delta y_{(k)}+(N_k-I(\{i\in\mathcal{I}_k\}))C\delta y^{(N)}\right]dt+\displaystyle\sum_{j\in\mathcal{I}_k,j\ne i}\delta z_j dW_j(t)\\
\qquad\qquad+\displaystyle\sum_{j\in\mathcal{I}_k,j\ne i}\sum_{l=1,l\ne j}^{N}\delta z_{jl}dW_l(t),\\
\delta y_{(k)}(T)=0.
\end{cases}
$$

Therefore,

$$\begin{aligned}&\mathbb{E}\sup_{t\le s\le T}|\delta y_i|^2+\mathbb{E}\int_t^T\left|\delta z_i\right|^2ds+\mathbb{E}\int_t^T\sum_{l=1,l\neq i}^N\left|\delta z_{il}\right|^2ds\\&\quad\le L(1+\mathbb{E}\int_0^T|\delta u_i|^2ds)+L\mathbb{E}\int_t^T|\delta y_i|^2ds+L\mathbb{E}\int_t^T|\delta y^{(N)}|^2ds,\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\sup_{t\le s\le T}|\delta y_j|^2+\mathbb{E}\int_t^T\left|\delta z_j\right|^2ds+\mathbb{E}\int_t^T\sum_{l=1,l\neq j}^N\left|\delta z_{jl}\right|^2ds\\&\quad\le L\mathbb{E}\int_t^T|\delta y_j|^2ds+L\mathbb{E}\int_t^T|\delta y^{(N)}|^2ds,\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\sup_{t\le s\le T}|\delta y_{(k)}|^2+\mathbb{E}\int_t^T\sum_{j\in\mathcal{I}_k,j\neq i}\left|\delta z_j\right|^2ds+\mathbb{E}\int_t^T\sum_{j\in\mathcal{I}_k,j\neq i}\sum_{l=1,l\neq j}^N\left|\delta z_{jl}\right|^2ds\\&\le L\mathbb{E}\int_t^T|\delta y_{(k)}|^2ds+LN^2\mathbb{E}\int_t^T|\delta y^{(N)}|^2ds.\end{aligned}$$

Note that

$$\delta y^{(N)}=\frac{1}{N}\delta y_i+\frac{1}{N}\sum_{l=1}^K\delta y_{(l)},$$

we have

$$\begin{aligned}&\mathbb{E}\sup_{t\le s\le T}|\delta y_i|^2+\mathbb{E}\int_t^T\left|\delta z_i\right|^2ds+\mathbb{E}\int_t^T\sum_{l=1,l\neq i}^N\left|\delta z_{il}\right|^2ds\\&\quad\le L(1+\mathbb{E}\int_0^T|\delta u_i|^2ds)+L\mathbb{E}\int_0^T|\delta y_i|^2ds+\frac{L}{N^2}\sum_{l=1}^K\mathbb{E}\int_t^T|\delta y_{(l)}|^2ds,\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\sup_{t\le s\le T}|\delta y_{(k)}|^2+\mathbb{E}\int_t^T\sum_{j\in\mathcal{I}_k,j\neq i}\left|\delta z_j\right|^2ds+\mathbb{E}\int_t^T\sum_{j\in\mathcal{I}_k,j\neq i}\sum_{l=1,l\neq j}^N\left|\delta z_{jl}\right|^2ds\\&\le L\mathbb{E}\int_0^T|\delta y_{(k)}|^2ds+L\mathbb{E}\int_t^T|\delta y_i|^2ds+L\sum_{l=1}^K\mathbb{E}\int_t^T|\delta y_{(l)}|^2ds.\end{aligned}$$

Therefore, it follows from Gronwall inequality that

$$\mathbb{E}\sup_{t\le s\le T}|\delta y_i|^2+\sum_{l=1}^K\mathbb{E}\sup_{t\le s\le T}|\delta y_{(l)}|^2\le L(1+\mathbb{E}\int_0^T|\delta u_i|^2ds).$$

Thus,

$$\mathbb{E}\sup_{t\le s\le T}|\delta y^{(N)}|^2\le\frac{L}{N^2}(1+\mathbb{E}\int_0^T|\delta u_i|^2ds).$$

By Gronwall inequality again, we have

$$\sup_{1\le j\le N, j\neq i}\left[\mathbb{E}\sup_{t\le s\le T}|\delta y_j|^2+\mathbb{E}\int_t^T\left|\delta z_j\right|^2ds+\mathbb{E}\int_t^T\sum_{l=1,l\neq j}^N\left|\delta z_{jl}\right|^2ds\right]\le\frac{L}{N^2}(1+\mathbb{E}\int_0^T|\delta u_i|^2ds).$$

Remark 10 Note that in (56), the upper bound depends on $\mathbb{E}\int_0^T|\delta u_i|^2ds$. However, when studying the asymptotic optimality, we only need to consider the perturbations satisfying (**??**). Hence, in Section 5 when applying Lemma 5, similar estimation still holds while the upper bound is $\frac{L}{N^2}$ and L is general constant.

Lemma 6 *Let (A1)-(A4) hold. Then there exist constants L independent of N such that*

$$\sum_{l=1}^K\mathbb{E}\sup_{0\le t\le T}|y_l^{**}(t)-\delta y_{(l)}(t)|^2\le\frac{L}{N^2}+L\epsilon_N^2,\tag{57}$$

and for $j\in\mathcal{I}_k$, $1\le k\le K$,

$$\mathbb{E}\sup_{0\le t\le T}|N_k\delta y_j(t)-y_j^*(t)|^2\le\frac{L}{N^2}+L\epsilon_N^2.\tag{58}$$

Proof First,

$$\begin{cases}d(y_k^{**}-\delta y_{(k)})=-\Big[A_k(y_k^{**}-\delta y_{(k)})+C\Big(\pi_k-\dfrac{N_k-I(\{i\in\mathcal{I}_k\})}{N}\Big)\delta y_i\\ \quad+C\pi_k\displaystyle\sum_{l=1}^K(y_l^{**}-\delta y_{(l)})+C\Big(\pi_k-\dfrac{N_k-I(\{i\in\mathcal{I}_k\})}{N}\Big)\sum_{l=1}^K\delta y_{(l)}\Big]dt\\ \quad+\displaystyle\sum_{l=1}^N z_{kl}^{**}dW_l(t)-\sum_{j\in\mathcal{I}_k,j\neq i}\delta z_jdW_j(t)-\sum_{j\in\mathcal{I}_k,j\neq i}\sum_{l=1,l\neq j}^N\delta z_{jl}dW_l(t),\\ (y_k^{**}-\delta y_{(k)})(0)=0,\end{cases}$$

and for $j\in\mathcal{I}_k$,

$$\begin{cases}d(y_j^*-N_k\delta y_j)=-\Big[A_k(y_j^*-N_k\delta y_j)+C(\pi_k-\pi_k^{(N)})\delta y_i\\ \quad+C(\pi_k-\pi_k^{(N)})\displaystyle\sum_{l=1}^K y_l^{**}+C\pi_k^{(N)}\sum_{l=1}^K(y_l^{**}-\delta y_{(l)})\Big]dt\\ \quad+(z_j^*-N_k\delta z_j)dW_j(t)+\displaystyle\sum_{l=1,l\neq j}^N(z_{jl}^*-N_k\delta z_{jl})dW_l(t),\\ (y_j^*-N_k\delta y_j)(T)=0.\end{cases}$$

Therefore, it follows from Burkholder-Davis-Gundy inequality that

$$\begin{aligned}&\mathbb{E}\sup_{t\le s\le T}|y_k^{**}-\delta y_{(k)}|^2\\ \le& L\mathbb{E}\int_t^T|y_k^{**}-\delta y_{(k)}|^2ds+L\sum_{l=1}^K\mathbb{E}\int_t^T|y_l^{**}-\delta y_{(l)}|^2ds+\frac{L}{N^2}+L\epsilon_N^2.\end{aligned}$$

Thus,

$$\sum_{l=1}^K\mathbb{E}\sup_{t\le s\le T}|y_l^{**}-\delta y_{(l)}|^2\le L\sum_{l=1}^K\mathbb{E}\int_t^T|y_l^{**}-\delta y_{(l)}|^2+\frac{L}{N^2}+L\epsilon_N^2.$$

It then follows from Gronwall inequality that

$$\sum_{l=1}^K\mathbb{E}\sup_{0\le t\le T}|y_l^{**}(t)-\delta y_{(l)}(t)|^2\le\frac{L}{N^2}+L\epsilon_N^2.$$

Similarly, we have (58).

Lemma 7 *Let (A1)-(A4) hold. Then there exist constants L independent of N such that*

$$\sum_{k=1}^K\mathbb{E}\sup_{0\le t\le T}|\mathbb{E}\widetilde{\boldsymbol{y}}_k-\widetilde{y}^{(k)}(t)|^2\le\frac{L}{N^2}+L\epsilon_N^2,\tag{59}$$

$$\sum_{k=1}^K\mathbb{E}\sup_{0\le t\le T}|\mathbb{E}\boldsymbol{x}_k-\frac{1}{N_k}\sum_{j\in\mathcal{I}_k,j\neq i}x_1^j(t)|^2\le\frac{L}{N^2}+L\epsilon_N^2,\tag{60}$$

where $\widetilde{\boldsymbol{y}}_k$ is a representative agent of type k with the state defined in (46) and $\widetilde{y}^{(k)}$ is defined in (55).

Proof Let $y^{(k)}:=\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}y_j$, $p^{(k)}:=\frac{1}{N_k}\sum_{j\in\mathcal{I}_k}p_j$

$$\begin{cases}d(\mathbb{E}\boldsymbol{y}_k-y^{(k)})=-\Big[A_k(\mathbb{E}\boldsymbol{y}_k-y^{(k)})-BR_k^{-1}B^\top(\mathbb{E}\boldsymbol{p}_k-p^{(k)})+C\sum_{k=1}^K\pi_k\mathbb{E}\alpha_kI\{i\in\mathcal{I}_k\}+fI\{i\in\mathcal{I}_k\}\Big]dt\\ -\frac{1}{N_k}\sum_{j\in\mathcal{I}_k,j\neq i}z_jdW_j,\\ \mathbb{E}\boldsymbol{y}_k(T)-y^{(k)}(T)=\mathbb{E}\eta_k-\frac{1}{N_k}\sum_{j\in\mathcal{I}_k,j\neq i}\xi_j,\end{cases}\tag{61}$$

$$\begin{cases}d(\mathbb{E}\boldsymbol{p}_k-p^{(k)})=\Big[A_k(\mathbb{E}\boldsymbol{p}_k-p^{(k)})+S(\mathbb{E}\boldsymbol{y}_k-y^{(k)})+\Big(-(SG+G^\top S-G^\top SG)\sum_{k=1}^K\pi_k\mathbb{E}\alpha_k\\ \quad+\sum_{k=1}^K\pi_kC^\top x_2^k+\sum_{k=1}^K\pi_kC^\top\hat{x}_k\Big)I\{i\in\mathcal{I}_k\}\Big]dt,\\ \mathbb{E}\boldsymbol{p}_k(0)-p^{(k)}(0)=Q^\top(\mathbb{E}\boldsymbol{y}_k(0)-y^{(k)}(0))-(QH+H^\top Q-H^\top QH)\sum_{k=1}^K\alpha(0)I\{i\in\mathcal{I}_k\},\end{cases}\tag{62}$$

$$\begin{cases} d(\mathbb{E}\widetilde{\boldsymbol{y}}_k - \widetilde{y}^{(k)}) = -\Big[A_k(\mathbb{E}\widetilde{\boldsymbol{y}}_k - \widetilde{y}^{(k)}) - BR_k^{-1}B^\top(\mathbb{E}\boldsymbol{p}_k - p^{(k)}) + C(\mathbb{E}\widetilde{y}^{(N)} - \widetilde{y}^{(N)})\Big]dt \\ -\frac{1}{N_k}\sum_{j\in\mathcal{I}_k, j\neq i}\widetilde{z}_j dW_j - \frac{1}{N_k}\sum_{j\in\mathcal{I}_k, j\neq i}\sum_{l=1,l\neq i}^{N}\widetilde{z}_{jl}dW_l, \\ \mathbb{E}\widetilde{\boldsymbol{y}}_k(T) - \widetilde{y}^{(k)}(T) = \mathbb{E}\eta_k - \frac{1}{N_k}\sum_{j\in\mathcal{I}_k, j\neq i}\xi_j, \end{cases} \tag{63}$$

$$\begin{cases} d(\mathbb{E}\widetilde{y}^{(k)} - \widetilde{y}^{(k)}) = -\Big[A_k(\mathbb{E}\widetilde{y}^{(k)} - \widetilde{y}^{(k)}) - BR_k^{-1}B^\top(\mathbb{E}p^{(k)} - p^{(k)}) + C(\mathbb{E}\widetilde{y}^{(N)} - \widetilde{y}^{(N)})\Big]dt \\ -\frac{1}{N_k}\sum_{j\in\mathcal{I}_k, j\neq i}\widetilde{z}_j dW_j - \frac{1}{N_k}\sum_{j\in\mathcal{I}_k, j\neq i}\sum_{l=1,l\neq i}^{N}\widetilde{z}_{jl}dW_l, \\ \mathbb{E}\widetilde{y}^{(k)}(T) - \widetilde{y}^{(k)}(T) = \mathbb{E}\eta_k - \frac{1}{N_k}\sum_{j\in\mathcal{I}_k, j\neq i}\xi_j, \end{cases} \tag{64}$$

Note that

$$\widetilde{y}^{(N)} = \frac{1}{N}\widetilde{y}_i + \sum_{k=1}^{K}\pi_k^{(N)}\widetilde{y}^{(k)},$$

and

$$\mathbb{E}\Big(\mathbb{E}\eta_k - \frac{1}{N_k}\sum_{j\in\mathcal{I}_k, j\neq i}\xi_j\Big)^2 = O(\frac{1}{N_k}).$$

By the standard estimations of BSDE and SDE, we have (59)*. By* (59)*,* (12) *and the standard estimations of SDE, we have* (60)*.*

5.3 Asymptotic optimality

In order to prove asymptotic optimality, it suffices to consider the perturbations $u_i \in \mathcal{U}_i^c$ such that $\mathcal{J}_{soc}^{(N)}(u_1, \cdots, u_N) \leq \mathcal{J}_{soc}^{(N)}(\widetilde{u}_1, \cdots, \widetilde{u}_N)$. It is easy to check that

$$\mathcal{J}_{soc}^{(N)}(\widetilde{u}_1, \cdots, \widetilde{u}_N) \leq LN,$$

where L is a constant independent of N. Therefore, in the following we only consider the perturbations $u_i \in \mathcal{U}_i^c$ satisfying

$$\sum_{i=1}^{N}\mathbb{E}\int_0^T |u_i|^2 dt \leq LN.$$

Let $\delta u_i = u_i - \widetilde{u}_i$, and consider a perturbation $u = \widetilde{u} + (\delta u_1, \cdots, \delta u_N) := \widetilde{u} + \delta u$. Therefore, recalling Lemma 5, there exists a constant L independent of N such that

$$\sup_{1\leq j\leq N, j\neq i}\left[\mathbb{E}\sup_{0\leq t\leq T}|\delta y_j(t)|^2 + \mathbb{E}\int_0^T \big|\delta z_j(t)\big|^2 dt + \mathbb{E}\int_0^T \sum_{l=1,l\neq j}^{N}\big|\delta z_{jl}(t)\big|^2 dt\right] \leq \frac{L}{N}.$$

Furthermore, by Section 5.1, we have

$$
\begin{aligned}
&2\mathcal{J}_{soc}^{(N)}(\widetilde{u}+\delta u)\\
=&\langle M_2(\widetilde{u}+\delta u),\widetilde{u}+\delta u\rangle+2\langle M_1,\widetilde{u}+\delta u\rangle+M_0\\
=&\langle M_2\widetilde{u},\widetilde{u}\rangle+\langle M_2\delta u,\delta u\rangle+2\langle M_2\widetilde{u},\delta u\rangle+2\langle M_1,\widetilde{u}\rangle+2\langle M_1,\delta u\rangle+M_0\\
=&2\mathcal{J}_{soc}^{(N)}(\widetilde{u})+\langle M_2\delta u,\delta u\rangle+2\langle M_2\widetilde{u},\delta u\rangle+2\langle M_1,\delta u\rangle\\
=&2\mathcal{J}_{soc}^{(N)}(\widetilde{u})+2\langle M_2\widetilde{u}+M_1,\delta u\rangle+\langle M_2\delta u,\delta u\rangle\\
=&2\mathcal{J}_{soc}^{(N)}(\widetilde{u})+2\sum_{i=1}^{N}\langle M_2\widetilde{u}+M_1,\delta u_i\rangle+\langle M_2\delta u,\delta u\rangle,
\end{aligned}
\tag{65}
$$

where $\langle M_2\widetilde{u}+M_1,\cdot\rangle$ is the Fréchet differential of $\mathcal{J}_{soc}^{(N)}$ with $\widetilde{u}$.

Theorem 2 *Let (A1)-(A4) hold. Then $\widetilde{u}=(\widetilde{u}_1,\cdots,\widetilde{u}_N)$ is a $\left(\frac{1}{\sqrt{N}}+\epsilon_N\right)$-optimal strategy for the agents.*

Proof It follows from Cauchy-Schwarz inequality that

$$
\begin{aligned}
&\mathcal{J}_{soc}^{(N)}(\widetilde{u}+\delta u)-\mathcal{J}_{soc}^{(N)}(\widetilde{u})\\
\geq&-\sqrt{\sum_{i=1}^{N}|M_2\widetilde{u}+M_1|^2\sum_{i=1}^{N}|\delta u_i|^2}+\frac{1}{2}\langle M_2\delta u,\delta u\rangle\geq-|M_2\widetilde{u}+M_1|O(N).
\end{aligned}
$$

Therefore, in order to prove asymptotic optimality, we only need to show that

$$|M_2\widetilde{u}+M_1|=o(1).$$

From Section 5.2, we have

$$
\begin{aligned}
&\langle M_2\widetilde{u}+M_1,\delta u_i\rangle\\
=&\mathbb{E}\Big\{\int_0^T\Big[\langle S\widetilde{y}_i,\delta y_i\rangle-\langle(SG+G^\top S-G^\top SG)\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l,\delta y_i\rangle-\sum_{k=1}^{K}\langle\pi_kC^\top x_2^k,\delta y_i\rangle\\
&-\sum_{k=1}^{K}\langle\pi_kC^\top\mathbb{E}x_k,\delta y_i\rangle+\langle R_{\theta_i}\widetilde{u}_i,\delta u_i\rangle\Big]dt+\langle Q\widetilde{y}_i(0),\delta y_i(0)\rangle\\
&-\langle(QH+H^\top Q-H^\top QH)\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l(0),\delta y_i(0)\rangle\Big\}+\sum_{l=1}^{8}\varepsilon_l,
\end{aligned}
$$

It follows from the optimality of $\widetilde{u}$ that

$$
\begin{aligned}
&\mathbb{E}\Big\{\int_0^T\Big[\langle S\widetilde{y}_i,\delta y_i\rangle-\langle(SG+G^\top S-G^\top SG)\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l,\delta y_i\rangle-\sum_{k=1}^{K}\langle\pi_kC^\top x_2^k,\delta y_i\rangle\\
&-\sum_{k=1}^{K}\langle\pi_kC^\top\mathbb{E}x_k,\delta y_i\rangle+\langle R_{\theta_i}\widetilde{u}_i,\delta u_i\rangle\Big]dt+\langle Q\widetilde{y}_i(0),\delta y_i(0)\rangle\\
&-\langle(QH+H^\top Q-H^\top QH)\sum_{l=1}^{K}\pi_l\mathbb{E}\alpha_l(0),\delta y_i(0)\rangle\Big\}=0.
\end{aligned}
$$

Moreover, by Lemma 4-7, we have

$$\sum_{l=1}^{8} \varepsilon_l = O\Big(\frac{1}{\sqrt{N}} + \epsilon_N\Big).$$

Therefore,

$$|M_2\widetilde{u} + M_1| = O\Big(\frac{1}{\sqrt{N}} + \epsilon_N\Big).$$

6 Conclusion

In this paper, we mainly study a class of stochastic LQG dynamic optimization problems involving a large number of weakly-coupled heterogeneous agents. Different to the well-studied mean-field-game, these agents formalize a team with cooperation to minimize a social cost functional, while the state is driven by BSDE. With the help of a backward version of person-by-person optimality, we formulate an auxiliary control problem and derive the decentralized social strategy based on the consistency condition (CC) system. Applying the Riccati decoupling equation method, we develop two Riccati equations and a (backward) Sylvester equation to decouple this mean-field-type FBSDE. The explicit solutions and numerical solutions of Riccati equations are also investigated. Finally, we verify the related asymptotic social optimality.

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