https://doi.org/10.1016/j.trb.2018.10.012

A CLOSED-FORM ESTIMATION OF THE TRAVEL TIME PERCENTILE FUNCTION FOR CHARACTERIZING TRAVEL TIME RELIABILITY

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ABSTRACT

Travel time reliability (TTR) has received great attention in the past decades. The majority of TTR measures rely on the travel time percentile function as a basic element for performance evaluation. There are two main approaches for deriving the travel time percentile function: simple unimodal probability distribution models and mixture/nonparametric models. Despite the tractability of the former approach, they cannot sufficiently capture the travel time distributions (TTDs) due to their heterogeneity, and also often encounters many issues such as the failure of significance tests and the indecisiveness among multiple fitted distributions. On the other hand, the latter approach possesses greater flexibility for capturing diverse TTDs, but it does not have a simple and closed-form travel time percentile function. Motivated by the above drawbacks, this paper proposes a closed-form and flexible approach for estimating the travel time percentile function of diverse TTDs based on the Cornish-Fisher expansion without the need to assume/fit a certain distribution type. To ensure a high-quality estimation, we introduce and integrate two improvements with theoretically proven foundation into the Cornish-Fisher expansion while guaranteeing a closed-form expression of the travel time percentile function. Specifically, the first improvement, logarithm transformation, increases the probability of satisfying the validity domain of the Cornish-Fisher expansion; while the second improvement, rearrangement, guarantees a monotone travel time percentile function when travel time datasets cannot satisfy the validity domain after the logarithm transformation. Realistic travel time datasets are used to examine the accuracy and robustness of the proposed method. Compared to five widely-used probability distributions, the proposed method is sufficiently adaptable to estimating percentile function of diverse TTDs with lower estimation error. More importantly, it has a closed-form expression of the travel time percentile function, which would facilitate characterizing TTR in large-scale network applications.

Keywords: travel time reliability, percentile function, travel time distribution, heterogeneity, Cornish-Fisher expansion

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1 INTRODUCTION

1.1 Motivations and Observations

(1) Percentile Function as a Basic Element in Travel Time Reliability Measures

Travelers prefer a reliable travel choice to ensure on-time arrival (van Lint *et al*., 2008), and they value travel time reliability (TTR) almost as much as they value mean travel time (e.g., Hollander, 2006; Asensio and Matas, 2008; Li *et al*., 2010). Thus, TTR plays an important role in travelers' decision making and planners' cost-benefit assessment of transportation projects. In order to better quantify TTR and provide travelers with more accurate travel time information, many travel time reliability measures have been proposed in the literature, e.g., 90th or 95th travel time (FHWA, 2009), alpha-reliable travel time (Chen and Ji, 2005; Ji *et al*., 2011) or *p*-percentile travel time (Nie, 2011), travel time index, planning time index (NCHRP, 2008), buffer index (Lomax *et al*., 2003; SHRP, 2009), skewness-width (van Lint *et al*., 2008), failure rate (Lomax *et al*., 2003), frequency of congestion, misery index (FHWA, 2009), travel time budget (Lo *et al*., 2006, Shao *et al*., 2006), mean-excess travel time (Chen and Zhou, 2010; Chen *et al*., 2011; Xu *et al*., 2013, 2017), and travel time reliability ratio (Fosgerau, 2017; Taylor, 2017). The above reliability measures and their formulas are summarized in Table 1. One can see that all these TTR measures are calculated based on either the percentile travel time (PTT) function or the distribution tail of travel time percentile function. In other words, **the travel time percentile function** (equivalent to inverse cumulative distribution function (CDF)) **is the key to calculate TTR measures for characterizing travel time variability**. As in many transportation research topics such as route choice, traffic assignment, and network optimization problems, a *closed-form* expression of travel time percentile function would facilitate the computations of these TTR measures and promote their applications in large-scale network applications.

Table 1. A summary of travel time reliability measures.

Type	Reliability measure	Formula
Based on PTT	90th or 95th travel time (90th or 95th TT)	$PTT(90\%)$ or $PTT(95\%)$
	p-percentile travel time (*p*-PTT)	$PTT(p)$
	travel time index (TTI)	$TTI = \dfrac{M}{PTT(15\%)}$
	planning time index (PTI)	$PTI = \dfrac{PTT(95\%)}{PTT(15\%)}$

	buffer index (BI)	$BI = \frac{PTT(95\%) - PTT(50\%)}{PTT(50\%)}$
	skewness-width (λ^{skew}, λ^{var})	$\lambda^{skew} = \frac{PTT(90\%) - PTT(50\%)}{PTT(50\%) - PTT(10\%)}$ $\lambda^{var} = \frac{PTT(90\%) - PTT(10\%)}{PTT(50\%)}$
Based on distribution tail of percentile function	failure rate (FR)	$FR = 100\% - P\left(TT_i < (1+p)\cdot PTT(50\%)\right)$
	frequency of congestion (FoC)	$FoC = P\left(TT_i > (1+p)\cdot PTT(50\%)\right)$
	misery index (MI)	$MI = \frac{\int_{0.8}^{1} PTT(x)\,dx - M}{M}$
	travel time budget (TTB)	$TTB(p) = \min\left\{\overline{T} \mid P\left(TT \le \overline{T}\right) \ge p\right\}$
	mean-excess travel time (METT)	$METT = \frac{1}{1-p}\int_{p}^{1} PTT(x)\,dx$
	travel time reliability ratio (TTRR)	$TTRR = \frac{\beta+\gamma}{\alpha}\int_{\frac{\gamma}{\beta+\gamma}}^{1} PTT(x)\,dx$

Note: M: mean travel time; $PTT(p)$: p-percentile travel time; p: a given probability; TT: travel time; $\bar{T}$: travel time corresponding to the user-specified confidence level p; α, β, and γ: travelers' preference parameters in schedule delay model.

(2) Challenges in Fitting Travel Time Distributions

Another important observation is the heterogeneity of travel time distributions (TTDs), which brings great challenges to fitting TTDs. Many factors, such as unpredictable traffic incidents (Cohen and Southworth, 1999), adverse weather conditions (Lam *et al*., 2008), and different traffic management and control measures (e.g., road pricing in Garnder *et al*. (2008), and speed limit control in Xu *et al*. (2018)), can significantly affect the travel demand and traffic supply, resulting in multiple or mixed traffic states. This means that when evaluating TTR, we would obviously encounter travel time observations with different statistical features, i.e., the heterogeneity of TTDs. The empirical travel time observations may render multiple statistical characteristics as shown in Figure 1. For example, van Lint and van Zuylen (2005) distinguished four traffic states (i.e., free-flow traffic, congestion onset, congested traffic, and congestion dissipation) corresponding to different shapes of TTDs based on empirical travel time observations. Crawford *et al*. (2017) also identified the predictable difference in daily traffic flow profiles due to known explanatory factors, such as the day of the week or the season. In other words, TTDs could not only be right-skewed with a fat tail (van Lint and van Zuylen, 2005; Fosgerau and Fukuda, 2012; Susilawati *et al*., 2013; Kim and Mahmassani, 2015; Delhomme *et al*., 2015) but also be left-skewed or even close to the normal distribution as shown in Figure 1.

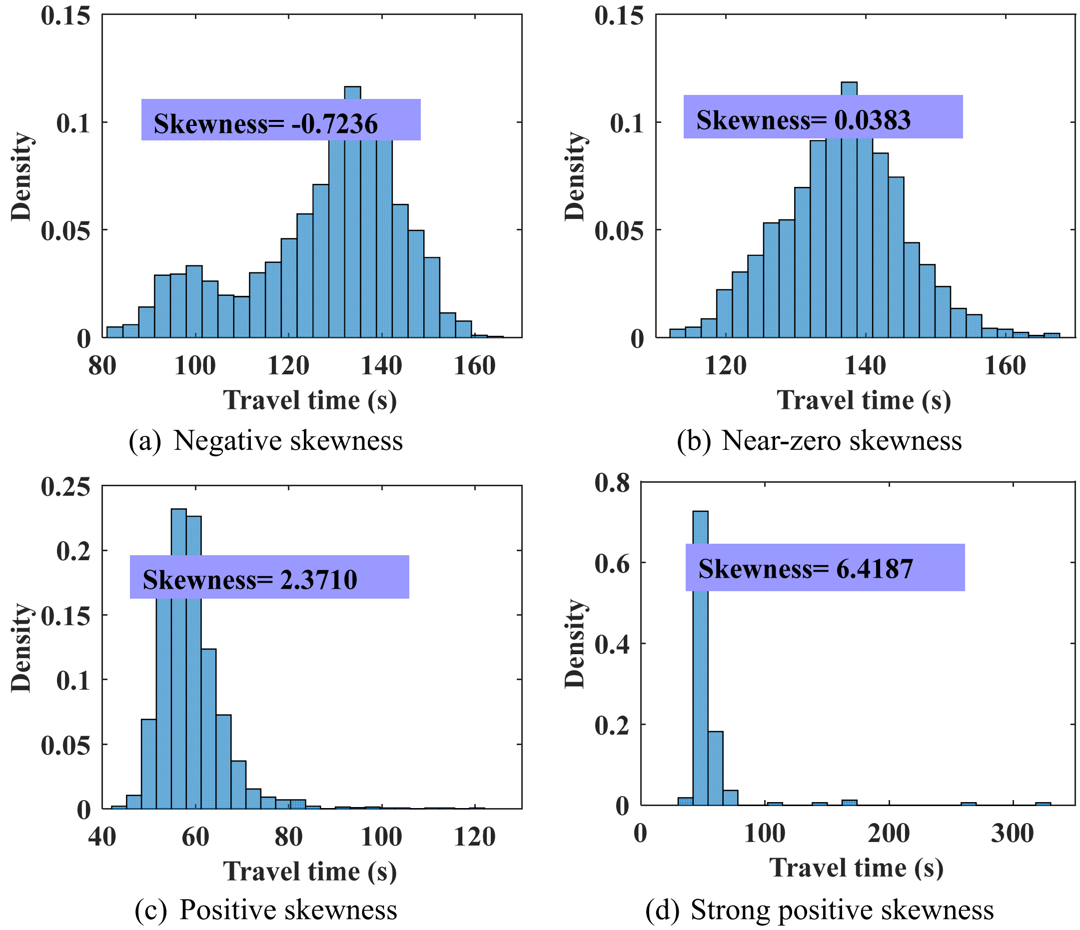


Figure 1. Histogram of empirical travel time datasets with various skewness values.

The diversity of TTDs leads to various difficulties and challenges in identifying an optimal probability distribution fitting function. Therefore, many probability distributions, such as Normal, (Shifted) Lognormal, (Compound) Gamma, Weibull, Generalized Beta, Stable, and Burr distributions, have been proposed to fit TTDs in the literature (e.g., Polus, 1979; Arroyo and Kornhauser, 2005; Al-Deek and Emam, 2006; Mazloumi et al., 2010; Rakha *et al*., 2010; Castillo *et al*., 2012; Fosgerau and Fukuda, 2012; Susilawati *et al*., 2013; Srinivasan *et al*., 2014; Taylor, 2017). Besides the diversity of distribution types, using a distribution to fit TTDs is also associated with many issues such as the failure of significance test of some distributions, and indecisiveness among multiple fitted distributions or among different goodness-of-fit (GoF) measures. Plötz *et al*. (2017) demonstrated that different GoF measures may lead to different results of the best fitted distribution types even for the same travel time dataset.

To deal with the heterogeneity of TTDs, mixture or nonparametric models have been proposed to fit TTDs due to their greater flexibility, e.g., Normal mixture model (Guo *et al*., 2010), Lognormal mixture model (Kazagli and Koutsopoulos, 2012), Gamma mixture model (Yang and Wu, 2016), finite mixture of regression model (Chen *et al*., 2014), kernel density estimation such as Hasofer–Lind–Rackwitz–Fiessler algorithm (Yang *et al*., 2014), and the fast Fourier transform (Ng and Waller, 2010). For fitting TTDs with multi-modality, the non-parametric estimation / mixture models can provide a better fitting than unimodal probability distribution models as these models can account for the multi-modality (Guo *et al*., 2010; Rahmani *et al*., 2015). However, it is difficult, if not impossible, to derive a closed-form travel time *percentile function* from these complex models, which is quite important for guaranteeing the applicability of TTRs in large-scale networks. Based on the above issues, a natural question arises: *Can we develop a closed-form with high-quality estimation of the travel time percentile function that circumvents the issues associated with distribution fitting method while being sufficiently adaptable to capturing the heterogeneity of TTDs*?

1.2 Main Contributions of This Paper

Motivated by the above challenges and difficulties, this paper proposes a closed-form and flexible approach for estimating the travel time percentile function based on the Cornish-Fisher expansion without the need to assume/fit a certain distribution type. Besides the widely used mean and variance, skewness and kurtosis are also used in the Cornish-Fisher expansion to characterize the asymmetry and flatness of TTDs, respectively. Therefore, the proposed method is more adaptable to accurately capturing the empirical characteristics of TTDs.

It should be noted that the Cornish-Fisher expansion (Cornish and Fisher, 1937) has received very limited attention in transportation research field to our best knowledge. It was used to derive theoretical route choice and network assessment models, such as travel time risk model (Lu *et al*., 2005, 2006; Di *et al*., 2008), perceived mean-excess travel time model (Chen *et al*., 2011; Xu *et al*., 2013), and mean-excess total travel time model (Xu *et al*., 2014). Although Zang *et al.* (2018) has used it for calculating travel time reliability ratio, the Cornish-Fisher expansion is only an approximation of the travel time percentile, and thus its estimation accuracy and the monotonicity of the estimated percentile function may not be necessarily guaranteed. High-quality estimation and monotonicity of the travel time percentile function are critical for characterizing TTR. Besides, this paper focuses on a more fundamental subject– the *whole* percentile function of TTDs; while Zang *et al.* (2018) calculates a particular metric –TTRR, which only involves the distribution tail of the inverse CDF. Therefore, Zang *et al.* (2018) emphasizes the accuracy of estimating TTRR, while this paper emphasizes the monotonicity required to be a percentile function in mathematical sense and the adaptability/flexibility for capturing heterogeneous TTDs.

In this paper, instead of a simple and straightforward adoption, we propose and integrate two important methodological improvements with theoretically proven foundation into the Cornish-Fisher expansion for estimating the travel time percentile function of diverse TTDs. These two methodological improvements, i.e., logarithm transformation and rearrangement, can greatly improve the estimation accuracy of the Cornish-Fisher expansion while guaranteeing a closed-form expression of travel time percentile function. (1) A logarithm transformation is first introduced by making use of the fact that the Cornish-Fisher expansion performs better when the unknown distribution is closer to the normal distribution. The validity domain of the Cornish-Fisher expansion with logarithm transformation (i.e., the applicable condition that guarantees the estimated travel time percentile function to be monotone) is also rigorously derived. Compared to the original Cornish-Fisher expansion, the probability of travel time datasets being out of the validity domain is reduced after logarithm transformation. (2) When travel time datasets cannot satisfy the validity domain after logarithm transformation, rearrangement is further integrated to make the estimated percentile function to be monotone. We rigorously prove that the rearrangement can strictly reduce the estimation error of the Cornish-Fisher expansion. In addition, since Pichler and Selitsch (2000) recommended the sixth-order rather than the typical fourth-order Cornish-Fisher expansion in value-at-risk model for portfolio optimization, the effect of higher-order (up to the sixth-order) moments in the Cornish-Fisher expansion is examined. Realistic datasets extracted from the License Plate Recognition (LPR) system in Shenzhen, China, are used to test the accuracy and robustness of the proposed method compared to five widely-used probability distributions (i.e., Lognormal, Weibull, Gamma, Normal, and Burr).

In summary, *the main contribution of this paper is the development of a closed-form,* *flexible, and high-quality estimation of the travel time percentile function for characterizing TTR while being adaptable to capturing the heterogeneity of TTDs. Since no assumption is needed for the probability distribution type, the proposed method circumvents the issues and challenges in fitting unimodal travel time distributions, while inheriting the advantages of unimodal distributions due to a simple expression of percentile function. The closed-form estimation of travel time percentile function would facilitate the computation of the TTR measures and promote their applications in large-scale network applications. The proposed method has three main features: (1) the distribution-fitting-free characteristic; (2) theoretically proven foundation based on the two improvements; and (3) a closed-form expression of the travel time percentile function.*

The remainder of this paper is organized as follows. Travel time percentile function is estimated by the proposed approach in Section 2. Section 3 describes the datasets extracted from the LPR system and performance assessment. Section 4 explores the accuracy and features of the proposed method, particularly the effects of logarithm transformation, rearrangement, and higher-order moments in the Cornish-Fisher expansion. Conclusions are summarized in Section 5.

2 Modeling Travel Time Percentile Function

In this section, we propose a closed-form estimation approach for the travel time percentile function based on the Cornish-Fisher expansion with two improvements. The applications of the proposed method are also briefly presented.

2.1 Approximating Percentile Function Based on Cornish-Fisher Expansion

The Cornish-Fisher expansion proposed by Cornish and Fisher (1937) can be used to estimate any percentile by using high-order moments of a random variable. Zang *et al.* (2018) adopted the Cornish-Fisher expansion by using the first four moments (i.e., mean, variance, skewness, and (excess) kurtosis) to estimate the PTT as shown below.

$$PTT(p) \approx k_1(TT) + \varphi(p)\sqrt{k_2(TT)} \tag{1}$$

where $k_1(TT)$ and $\sqrt{k_2(TT)}$ are the mean and standard deviation of travel time (TT); p is a given probability; and $\varphi(p)$ is related to the skewness (denoted by S) and (excess) kurtosis (denoted by K) of TT:

$$\varphi(p) = U_p + \frac{S}{6}\left(U_p^2 - 1\right) + \frac{K}{24}\left(U_p^3 - 3U_p\right) - \frac{S^2}{36}\left(2U_p^3 - 5U_p\right) \tag{2}$$

where U_p is p-quantile of the inverse standard normal CDF.

One can see that the Cornish-Fisher expansion is only a polynomial approximation of the travel time percentile. The accuracy and monotonicity of estimating percentile function may not be necessarily guaranteed. Below we introduce two important improvements with theoretically proven foundation to improve its estimation accuracy and to guarantee its monotonicity in estimating PTTs.

(1) Improvement 1: Logarithm Transformation

Jaschke (2001) and Zang *et al*. (2018) pointed out that the accuracy of the Cornish-Fisher expansion in estimating PTTs is better when the unknown distribution is closer to the normal distribution. Motivated with this observation, we introduce the logarithm transformation to transfer the original travel time dataset $TT\,(tt_1, tt_2, \dots, tt_n)$ into a new dataset $TT_{\ln}$ $(tt_1', tt_2', \dots, tt_n')$ as shown in Eq. (3), which is closer to the normal distribution.

$$tt_i' = \ln\left(tt_i\right) \tag{3}$$

Then we use the first four moments of the new dataset to estimate PTTs of the original travel time observations, and rewrite Eqs. (1) and (2) as follows:

$$PTT(p) \approx \exp\left(k_1(TT_{\ln}) + \varphi_{\ln}(p)\sqrt{k_2(TT_{\ln})}\right) \tag{4}$$

$$\varphi_{\ln}(p) = U_p + \frac{S_{\ln}}{6}\left(U_p^2 - 1\right) + \frac{K_{\ln}}{24}\left(U_p^3 - 3U_p\right) - \frac{S_{\ln}^2}{36}\left(2U_p^3 - 5U_p\right) \tag{5}$$

where $k_1(TT_{\ln})$, $\sqrt{k_2(TT_{\ln})}$, $S_{\ln}$, and $K_{\ln}$ are the mean, standard deviation, skewness, and kurtosis of the new travel time dataset with the logarithm transformation (i.e., $TT_{\ln}$).

We can estimate the PTT at any given probability through Eq. (4). In other words, we can derive a simple closed-form *travel time percentile function* via Eq. (4) without the need to fit a predefined distribution type of TTDs. From this perspective, this approach can also be viewed as a semi-parametric modeling approach in the sense that it inherits the following advantages of fitting probability distributions: (1) it is more statistically representative than empirical travel time datasets, as a fitted distribution can circumvent missing empirical values in the collected sample datasets which may indeed exist in the real traffic situation; and (2) it is a powerful means to understand the TTDs, which can contribute to the development of improved TTR measures (Susilawati et al., 2013) and the investigation of analytical relationship among many existing TTR measures (Pu, 2011). Although the non-parametric estimation / mixture models can provide a better fitting for TTDs with multi-modality than the proposed method as these models can account for the multi-modality (Guo *et al*., 2010; Rahmani *et al*., 2015), the widely-used techniques to estimate their parameters (e.g., the Expectation Maximization algorithm and the Bayesian approach based on Markov Chain Monte Carlo) are computationally demanding (Redner and Walker, 1984; Chen *et al*., 2014). Therefore, the proposed method is a better choice for the large-scale network applications of calculating TTR measures that require many computational efforts due to its distribution-fitting-free nature and closed-form expression of the travel time percentile function. Besides, it is a flexible approach in capturing diverse TTDs, which will be discussed in detail later.

Maillard (2012) and Zang *et al*. (2018) derived the validity domain of the Cornish-Fisher expansion to ensure a monotone estimated percentile function. The validity domain of the Cornish-Fisher expansion *with the logarithm transformation* is further rigorously derived below.

Proposition 1. The validity domain of the Cornish-Fisher expansion remains intact with and without the logarithm transformation.

Proof. Let $f_{\ln}(p)$ and $f(p)$ respectively denote the travel time percentile functions derived from the Cornish-Fisher expansion with and without the logarithm transformation.

$$f(p) \approx k_1(TT) + \varphi(p)\sqrt{k_2(TT)} \tag{6}$$

$$f_{\ln}(p) \approx \exp\left(k_1(TT_{\ln}) + \varphi_{\ln}(p)\sqrt{k_2(TT_{\ln})}\right) \tag{7}$$

where $\varphi(p)$ and $\varphi_{\ln}(p)$ are referred to Eqs. (2) and (5), respectively. In order to be a monotone function, $df(p)/dp$ and $df_{\ln}(p)/dp$ should be nonnegative:

$$\frac{df(p)}{dp} = \sqrt{k_2(TT)} \cdot \frac{d\varphi(p)}{dp} \geq 0 \tag{8}$$

$$\frac{df_{\ln}(p)}{dp} = \exp\left(k_1(TT_{\ln}) + \varphi_{\ln}(p)\sqrt{k_2(TT_{\ln})}\right) \cdot \sqrt{k_2(TT_{\ln})} \cdot \frac{d\varphi_{\ln}(p)}{dp} \geq 0 \tag{9}$$

Namely,

$$\frac{d\varphi(p)}{dp} \geq 0 \tag{10}$$

$$\frac{d\varphi_{\ln}(p)}{dp} \geq 0 \tag{11}$$

The right-hand side of both Eqs. (2) and (5) can be viewed as a quadratic function of the skewness (i.e., S and $S_{\ln}$). Also, since the coefficients of the polynomial function in terms of skewness are the same for the right-hand side of Eqs. (2) and (5), $d\varphi_{\ln}(p)/dp \geq 0$ equals $d\varphi(p)/dp \geq 0$. Therefore, the Cornish-Fisher expansion with and without the logarithm transformation have the same validity domain. Zang *et al.* (2018) derived the validity domain of the Cornish-Fisher expansion without the logarithm transformation. We can further rewrite it in a simpler form as shown in Eq. (12) and visualized in Figure 2. For a more detailed derivation of Eq. (12), please refer to Appendix A.

$$\begin{aligned} &S \in \left[-6\left(\sqrt{2}-1\right), 6\left(\sqrt{2}-1\right)\right] \\ &K \in \left[4+\frac{11}{9}S^2-\sqrt{\frac{1}{81}S^4-\frac{8}{3}S^2+16},\ 4+\frac{11}{9}S^2+\sqrt{\frac{1}{81}S^4-\frac{8}{3}S^2+16}\right] \end{aligned} \tag{12}$$

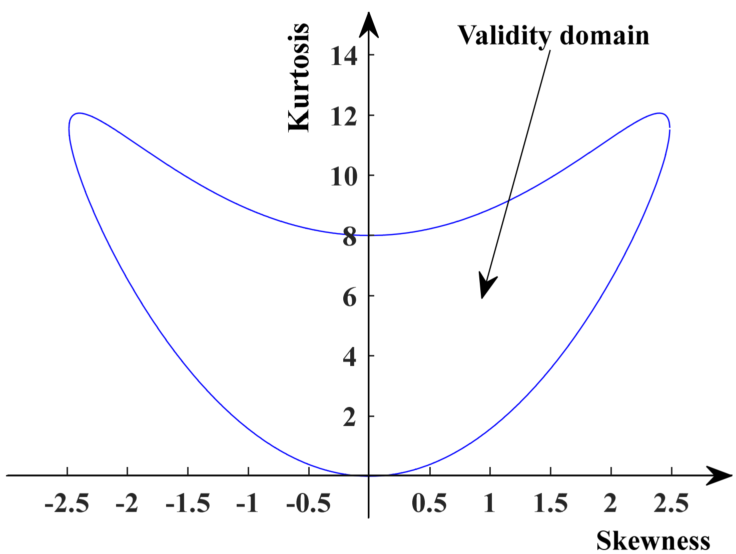


Figure 2. The validiy domain of the Cornish-Fisher expansion with and without the logarithm transformation.

This completes the proof. □

Remark 1. The logarithm transformation can make highly skewed datasets to be less skewed (McDonald, 2009). Although the validity domain of the Cornish-Fisher expansion with the logarithm transformation remains intact, the probability of travel time datasets being out of the validity domain is reduced after the logarithm transformation. In other words, the proposed logarithm transformation expands the applicable domain of the analytical estimation for travel time datasets with larger skewness and kurtosis. The empirical results in Section 4 will demonstrate the promising improvement of the Cornish-Fisher expansion in estimating PTTs contributed by the logarithm transformation.

(2) Improvement 2: Rearrangement

Although the logarithm transformation can increase the probability of satisfying the validity domain, extremely skewed travel time datasets may still be out of the validity domain. This phenomenon will not only result in a large estimation error, but will also render the estimated percentile function invalid. To ensure a monotone estimated percentile function when travel time datasets cannot satisfy the validity domain, we integrate increasing rearrangement proposed by Chernozhukov *et al*. (2010) (i.e., sort the estimated values in increasing order) into the Cornish-Fisher expansion. The rearrangement can reduce the estimation error of the Cornish-Fisher expansion as shown in Proposition 2.

Lemma 1. If there exist two regions $X_1 \subset X$ $(X = [0,1])$ and $X_2 \subset X$, such that for all $x_1 \in X_1$ and $x_2 \in X_2$, we have that (i) $x_1 < x_2$, (ii) $\hat{f}(x_1) > \hat{f}(x_2)$, and (iii) $f_0(x_1) < f_0(x_2)$. Then the rearrangement for any $p \in [1, \infty]$ strictly reduces the estimation error:

$$\left[\int_{x\in X} \left|\hat{f}^*(x) - f_0(x)\right|^p dx\right]^{1/p} < \left[\int_{x\in X} \left|\hat{f}(x) - f_0(x)\right|^p dx\right]^{1/p} \quad (13)$$

where f_0 is the true percentile function of the travel time datasets, i.e., the target monotone function to be estimated; $\hat{f}$ is the predicted percentile function, i.e., an initial estimation to the target function f_0; and $\hat{f}^*$ is the rearrangement of $\hat{f}$.

Proof. See Chernozhukov *et al*. (2010).

Proposition 2. When travel time datasets exceed the validity domain, the rearrangement can strictly *reduce* the estimation error of the Cornish-Fisher expansion in estimating PTT.

Proof. We use *ptt* and *Sptt* to stand for the **p**ercentile **t**ravel **t**ime (before rearrangement) and the **s**orted **p**ercentile **t**ravel **t**ime (after rearrangement). Let $ptt_r(i)$ and $ptt_f(i)$ denote the *i*th empirical and estimated percentile value of the observed dataset at the cumulative probability $P_i = \Pr(tt < TT_i)$, respectively. When travel time datasets cannot satisfy the validity domain, some estimated PTTs would not monotonically increase as the cumulative probability increases. So, there must exist two regions X_1 and X_2 for the estimated travel time percentile function ptt_f and the real travel tine percentile function ptt_r satisfying: (i) $i < j$, (ii) $ptt_f(i) > ptt_f(j)$, and (iii) $ptt_r(i) < ptt_r(j)$, $i \in X_1$ and $j \in X_2$. Therefore, according to Lemma 1, the rearrangement can strictly *reduce* the estimation error of the Cornish-Fisher expansion in estimating PTT. This completes the proof. □

Corollary 1. When travel time datasets cannot satisfy the validity domain, the rearrangement can strictly *improve* the estimation accuracy of the Cornish-Fisher expansion in estimating PTT in terms of RMSE (root mean squared error) and R^2 (coefficient of determination).

Proof. We use *Sptt* to stand for the **s**orted **p**ercentile **t**ravel **t**ime (after rearrangement). Let $Sptt_f(i)$ denote the *i*th value of the estimated percentile in increasing order; let $\overline{ptt}$ denote the mean of the empirical percentile values; $(TT_1, TT_2, \ldots, TT_n)$ is the increasing order of the observed dataset $(tt_1, tt_2, \ldots, tt_n)$; and *n* is the sample size. Note that the estimation accuracy is better when R^2 is higher and when RMSE is lower. According to Lemma 1, we have Eq. (14) for any *i* and *j* ($i \in X_1$ and $j \in X_2$):

$$\left(\left|Sptt_f(i) - ptt_r(i)\right|^2 + \left|Sptt_f(j) - ptt_r(j)\right|^2\right)^{\frac{1}{2}} < \left(\left|ptt_f(i) - ptt_r(i)\right|^2 + \left|ptt_f(j) - ptt_r(j)\right|^2\right)^{\frac{1}{2}} \quad (14)$$

Recall that $\sum_{i=1}^{n}\left(ptt_r(i) - \overline{ptt}\right)^2 > 0$, and $f(x) = x^{1/2}$ is a monotonically increasing function of *x*. Thus, we obtain Eqs. (15)-(16) from Eq. (14).

$$\frac{\left|Sptt_f(i) - ptt_r(i)\right|^2 + \left|Sptt_f(j) - ptt_r(j)\right|^2}{n} < \frac{\left|ptt_f(i) - ptt_r(i)\right|^2 + \left|ptt_f(j) - ptt_r(j)\right|^2}{n} \quad (15)$$

$$\frac{\left|Sptt_f(i) - ptt_r(i)\right|^2}{\sum_{i=1}^{n}\left(ptt_r(i) - \overline{ptt}\right)^2} + \frac{\left|Sptt_f(j) - ptt_r(j)\right|^2}{\sum_{i=1}^{n}\left(ptt_r(i) - \overline{ptt}\right)^2} < \frac{\left|ptt_f(i) - ptt_r(i)\right|^2}{\sum_{i=1}^{n}\left(ptt_r(i) - \overline{ptt}\right)^2} + \frac{\left|ptt_f(j) - ptt_r(j)\right|^2}{\sum_{i=1}^{n}\left(ptt_r(i) - \overline{ptt}\right)^2} \quad (16)$$

Therefore,

$$RMSE: \sqrt{\frac{\sum_{i=1}^{n}\left(Sptt_f(i) - ptt_r(i)\right)^2}{n}} < \sqrt{\frac{\sum_{i=1}^{n}\left(ptt_f(i) - ptt_r(i)\right)^2}{n}} \quad (17)$$

$$R^2: 1 - \frac{\sum_{i=1}^{n}\left(Sptt_f(i) - ptt_r(i)\right)^2}{\sum_{i=1}^{n}\left(ptt_r(i) - \overline{ptt}\right)^2} > 1 - \frac{\sum_{i=1}^{n}\left(ptt_f(i) - ptt_r(i)\right)^2}{\sum_{i=1}^{n}\left(ptt_r(i) - \overline{ptt}\right)^2} \quad (18)$$

Namely, the RMSE and R^2 of the Cornish-Fisher expansion are strictly improved due to the rearrangement. This completes the proof. □

Corollary 2. When travel time datasets cannot satisfy the validity domain, if we have

$\left|Sptt_f(i)-ptt_r(i)\right|<\left|ptt_f(i)-ptt_r(i)\right|$ and $\left|Sptt_f(j)-ptt_r(j)\right|<\left|ptt_f(j)-ptt_r(j)\right|$ for any i and j ($i \in X_1$ and $j \in X_2$), the rearrangement can strictly *improve* the estimation accuracy of the Cornish-Fisher expansion in estimating PTT in terms of MAPE (mean absolute percentage error).

Proof. For any i and j ($i \in X_1$ and $j \in X_2$), if $\left|Sptt_f(i)-ptt_r(i)\right|<\left|ptt_f(i)-ptt_r(i)\right|$ and $\left|Sptt_f(j)-ptt_r(j)\right|<\left|ptt_f(j)-ptt_r(j)\right|$, then we have:

$$\frac{\left|Sptt_f(i)-ptt_r(i)\right|}{ptt_r(i)}+\frac{\left|Sptt_f(j)-ptt_r(j)\right|}{ptt_r(j)}<\frac{\left|ptt_f(i)-ptt_r(i)\right|}{ptt_r(i)}+\frac{\left|ptt_f(j)-ptt_r(j)\right|}{ptt_r(j)} \tag{19}$$

Therefore,

$$MAPE:\ 100\%\cdot\frac{1}{n}\sum_{i=1}^{n}\frac{\left|Sptt_f(i)-ptt_r(i)\right|}{ptt_r(i)}<100\%\cdot\frac{1}{n}\sum_{i=1}^{n}\frac{\left|ptt_f(i)-ptt_r(i)\right|}{ptt_r(i)} \tag{20}$$

Note that the estimation accuracy is better when MAPE is lower. Namely, the MAPE of the Cornish-Fisher expansion is strictly improved due to the rearrangement. This completes the proof. □

Corollary 3. When travel time datasets cannot satisfy the validity domain, if for any $i \in X_1$, there exists one $j \in X_2$ satisfying $Sptt_f(i)=ptt_f(j)$ and $Sptt_f(j)=ptt_f(i)$, the rearrangement can also strictly *improve* the estimation accuracy of the Cornish-Fisher expansion in estimating PTT in terms of χ^2.

Proof. See Appendix B.

In summary, when travel time datasets cannot satisfy the validity domain, the logarithm transformation can be used to increase the probability of satisfying the validity domain as presented in Proposition 1 and Remark 1. If travel time datasets still exceed the validity domain after the logarithm transformation, the rearrangement can not only guarantee the monotonicity of the estimated travel time percentile function, but also improve the estimation accuracy of the Cornish-Fisher expansion as proved in Proposition 2 and its corollaries. Therefore, our proposed method integrates both the logarithm transformation and rearrangement into the Cornish-Fisher expansion for more accurately estimating the travel time percentile function.

2.2 Effect of Higher-Order Moments in Cornish-Fisher Expansion

Pichler and Selitsch (2000) recommended up to the sixth-order Cornish-Fisher expansion after comparing with Johnson transformation, Delta-Normal, and the fourth-order Cornish-Fisher expansion. However, previous studies (Lu *et al*., 2005, 2006; Di *et al*., 2008; Chen *et al*., 2011; Xu *et al*., 2013, 2014; Zang *et al*., 2018) in transportation domain only used up to the fourth-order Cornish-Fisher expansion (i.e., the first four moments). The difference between the fourth-order, fifth-order and sixth-order Cornish-Fisher expansion lies in the term of $\varphi_{\ln}(p)$ in Eq. (4). Eq. (21) provides the formulas for $\varphi_{\ln}(p)$ when using the first four, five, and six moments (see Section 12.5 of Johnson and Kotz (1970) for more details).

$$\begin{aligned}
\varphi_{\ln}(p) = & U_p + \frac{1}{6}\left(U_p^2 - 1\right)k_3 \\
& + \frac{1}{24}\left(U_p^3 - 3U_p\right)k_4 - \frac{1}{36}\left(2U_p^3 - 5U_p\right)k_3^2 \quad \text{up to the fourth-order} \\
& + \frac{1}{120}\left(U_p^4 - 6U_p^2 + 3\right)k_5 - \frac{1}{24}\left(U_p^4 - 5U_p^2 + 2\right)k_3 k_4 \\
& + \frac{1}{324}\left(12U_p^4 - 53U_p^2 + 17\right)k_3^3 \quad \text{up to the fifth-order} \\
& + \frac{1}{720}\left(U_p^5 - 10U_p^3 + 15U_p\right)k_6 \\
& - \frac{1}{180}\left(2U_p^5 - 17U_p^3 + 21U_p\right)k_3 k_5 \\
& - \frac{1}{384}\left(3U_p^5 - 24U_p^3 + 29U_p\right)k_4^2 \\
& + \frac{1}{288}\left(14U_p^5 - 103U_p^3 + 107U_p\right)k_3^2 k_4 \\
& - \frac{1}{7776}\left(252U_p^5 - 1688U_p^3 + 1511U_p\right)k_3^4 \quad \text{up to the sixth-order}
\end{aligned} \tag{21}$$

where k_i is the ith cumulant of travel time, which can be derived from the standardized central moment as shown below:

$$\begin{aligned}
k_3 &= \mu_3 \\
k_4 &= \mu_4 - 3 \\
k_5 &= \mu_5 - 10\mu_3 \\
k_6 &= \mu_6 - 15\mu_4 - 10\mu_3^2 + 30
\end{aligned} \tag{22}$$

where μ_i is the ith standardized central moment of travel time. In Section 4.3, we will examine the effect of higher-order moments in estimating the travel time percentile function. More specifically, we will examine the benefit of considering up to the fifth or sixth order in the Cornish-Fisher expansion for estimating travel time percentiles.

2.3 Application of The Proposed Method

As shown in Table 1, most TTR measures are based on the travel time percentile function as a basic element. The percentile (p) is usually fixed for a particular reliability measure, such as 95% and 50% percentiles in the buffer index. Recall that PTTs estimated by our proposed method in Eq. (4) are based on the percentile of standard normal distribution (U_p). A fixed PTT ($PTT(p)$) corresponds to a fixed percentile of standard normal distribution (U_p) in Eq. (5). The most widely-used fixed percentiles of the standard normal distribution and their corresponding PTT expressions can be readily computed for calculating TTR measures as shown in Table 2 and Table 3.

Table 2. Values of the typical percentiles of the standard normal distribution

Percentile	$U_{10\%}$	$U_{15\%}$	$U_{50\%}$	$U_{80\%}$	$U_{90\%}$	$U_{95\%}$
Value	-1.2816	-1.0364	0	0.8416	1.2816	1.6449

Table 3. Closed-form expressions of the most widely-used PTTs in reliability measures.

Percentile	Closed-form expression
PTT(10%)	$\exp\left(\mu_{\ln}+\sigma_{\ln}\left(-1.2816+0.1071S_{\ln}+0.0725K_{\ln}-0.0611{S_{\ln}}^{2}\right)\right)$
PTT(15%)	$\exp\left(\mu_{\ln}+\sigma_{\ln}\left(-1.0364+0.0124S_{\ln}+0.0831K_{\ln}-0.0821{S_{\ln}}^{2}\right)\right)$
PTT(50%)	$\exp\left(\mu_{\ln}-0.1667\sigma_{\ln}S_{\ln}\right)$
PTT(80%)	$\exp\left(\mu_{\ln}+\sigma_{\ln}\left(0.8416-0.0486S_{\ln}-0.0804K_{\ln}+0.0838{S_{\ln}}^{2}\right)\right)$
PTT(90%)	$\exp\left(\mu_{\ln}+\sigma_{\ln}\left(1.2816+0.1071S_{\ln}-0.0725K_{\ln}+0.0611{S_{\ln}}^{2}\right)\right)$
PTT(95%)	$\exp\left(\mu_{\ln}+\sigma_{\ln}\left(1.6449+0.2843S_{\ln}-0.0202K_{\ln}-0.0188{S_{\ln}}^{2}\right)\right)$

Note: $\mu_{\ln}$ and $\sigma_{\ln}$ are the mean and standard deviation of the travel time datasets after the logarithm transformation, i.e., $k_1(TT_{\ln})$ and $\sqrt{k_2(TT_{\ln})}$ in Eq. (4).

With Table 3, we can obtain a closed-form expression for any PTT-based reliability measures (e.g., 90th or 95th TT, p-PTT, PTI, TTI, BI, λ^{skew} and λ^{var} in Table 1) without the need to assume/fit TTDs. From this perspective, the proposed method is an easy and parsimonious way to calculate reliability measures for large-scale applications, especially when working with a large number of links/paths and time periods. For the TTR measures based on the distribution tail or the integral of percentile function (e.g., FR, FoC, MI, TTB, METT, and TTRR in Table 1), numerical integration (e.g., using Simpson's rule) is straightforward to calculate the integral of percentile function based on the closed-form travel time percentile function. Therefore, the proposed method is much easier to apply than probability distributions without closed-form inverse CDF. This is also the reason why Taylor (2017) highly recommended the Burr distribution to calculate TTRR owing to the existence of an analytical expression for the inverse CDF, but one cannot obtain a closed-form expression of TTRR for the Burr distribution.

3 Datasets and Performance Assessment

In this section, we describe the empirical travel time datasets and the GoF measures used in the accuracy analysis.

3.1 Empirical Travel Time Datasets

The empirical travel time datasets were extracted from the License Plate Recognition (LPR) system in Shenzhen, China. As shown in Figure 3, the studied route comprises of five consecutive links numbered from 1 to 5. Their travel time observations were collected during 6:00 AM to 10:00 AM (including peak and off-peak hours) from December 3 (Tuesday) to December 5 (Thursday) and December 7 (Saturday), 2013. Table 4 shows the detailed information about these five links, including the link length, mean travel time and standard deviation of the collected travel time datasets.

TTDs constructed over a 30-minute or 15-minute interval are usually used to validate the performance of TTD models (e.g., Ramezani and Geroliminis, 2012; Chen *et al*., 2017; Zheng *et al*., 2017). In this paper, for each studied link, the datasets of each day (from 6:00 AM to 10:00 AM) are divided into 8 groups with a time interval of 30 minutes per group unless specified otherwise. It should be noted that with 30 minutes as the time interval of TTDs, we can have more groups of TTDs with diverse skewness, including the heavily right-skewed TTDs (i.e., with long tails), to validate the flexibility of the proposed method. In total, there are 152 groups of travel time datasets after deleting those with less than 50 samples. The minimum, mean, median, and maximum sample sizes of these 152 groups are 122, 1195, 1063, and 2910, respectively. All the travel time observations are normalized by the link length to exclude travel time variation resulted from the varied link length, which allows us to focus on the travel time variability due to speed variation (Daganzo, 1997; Mahmassani *et al*., 2013; Saberi *et al*., 2014). For simplicity, travel time is referred to as normalized travel time hereinafter.

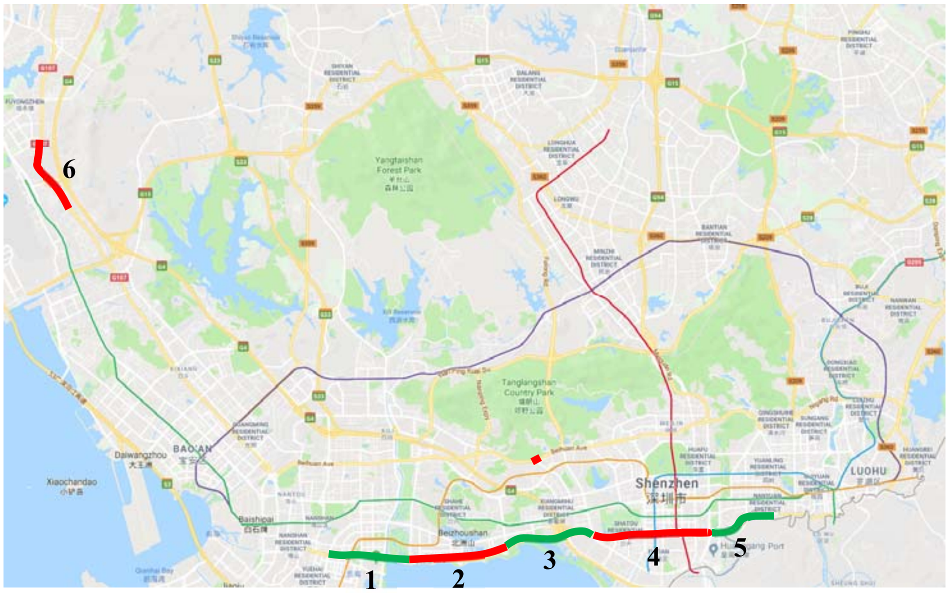


Figure 3. Location of the five consecutive studied links in Shenzhen, China.

Table 4. Characteristics of the five consecutive studied links.

Link no.	Length (km)	Date collected	Mean travel time (s)	Standard deviation (s)
1	4.1	Dec 3, 2013	243.49	33.40
		Dec 4, 2013	241.68	22.19
		Dec 5, 2013	240.66	24.52
		Dec 7, 2013	221.00	18.25
2	2.6	Dec 3, 2013	182.88	25.27
		Dec 4, 2013	184.65	28.30
		Dec 5, 2013	183.68	33.28
		Dec 7, 2013	162.26	18.30
3	3.0	Dec 3, 2013	167.77	12.72
		Dec 4, 2013	168.72	13.74
		Dec 5, 2013	173.47	20.82
		Dec 7, 2013	160.69	11.17
4	3.2	Dec 3, 2013	244.98	97.32
		Dec 4, 2013	229.75	86.32
		Dec 5, 2013	281.23	109.13
		Dec 7, 2013	168.53	14.10
5	3.5	Dec 3, 2013	191.57	22.00
		Dec 4, 2013	200.61	36.27
		Dec 5, 2013	191.13	20.48
		Dec 7, 2013	176.72	7.24

The outlier filtering algorithm used in this paper is the same as Oliveira-Neto *et al*.

(2012), which was proposed by Clark *et al*. (2002) specifically for removing travel time outliers of LPR datasets. In this algorithm, any travel time observation lying outside the interval determined by Eq. (23) is classified as an outlier.

$$M_e \pm 3 \times \frac{\sum_{i=1}^{m} \left| tt_i - M_e \right|}{m} \tag{23}$$

where tt_i denotes the travel time of vehicle i; M_e is the median for each 5-min block of travel time; and m is the number of observations within each 5-min block.

3.2 Performance Assessment

To assess the performance of the proposed method for travel time datasets with various statistical characteristics, five widely-used travel time probability distributions (i.e., Lognormal, Weibull, Gamma, Normal, and Burr) are used to estimate PTTs for accuracy comparison. The algebraic forms of probability density function (PDF) and inverse CDF (i.e., the percentile function) of the above five distributions are summarized in Table 5.

Table 5. PDF and inverse CDF formulas of five widely-used travel time distributions.

Distribution	PDF	Inverse CDF/percentile function
Lognormal	$f(x) = \frac{1}{x}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)$	$F^{-1}(p) = \exp\left(\sigma \cdot \Phi^{-1}(p) + \mu\right)$
Weibull	$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\exp\left(-\left(\frac{x}{\lambda}\right)^{k}\right)$	$F^{-1}(p) = \lambda\left(-\ln(1-p)\right)^{\frac{1}{k}}$
Gamma	$f(x) = \frac{x^{k-1}\exp\left(-\frac{x}{\theta}\right)}{\theta^k\Gamma(k)}$	Not available
Normal	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$F^{-1}(p) = \mu + \sigma \cdot \Phi^{-1}(p)$
Burr	$f(x) = \frac{ck}{\theta}\left(\frac{x}{\theta}\right)^{c-1}\left(1+\left(\frac{x}{\theta}\right)^{c}\right)^{-(k+1)}$	$F^{-1}(p) = \theta\left((1-p)^{-1/k} - 1\right)^{\frac{1}{c}}$
Remark: $\Phi^{-1}(\cdot)$ is the quantile function of standard normal distribution		

To set up a fair comparison, all the accuracy analyses are based on the empirical PTTs, i.e., the empirical PTTs are used as the true values of PTTs in calculating the GoF measures: MAPE, RMSE, χ^2, and R^2. The use of four GoF measures is motivated by the observation that different GoF measures may lead to different results of the best fitted distribution types (Plötz *et al*., 2017). Both MAPE (i.e., percentage error) and χ^2 (i.e., significant difference) focus on the relative difference; RMSE focuses on the absolute difference between estimated values and observed values; and R^2 describes the correlation between estimated values and observed values. Note that the value of PTT is positive infinity when the cumulative probability is equal to 1. Our calculation of the four GoF measures will not involve the PTT at the cumulative probability of 1.

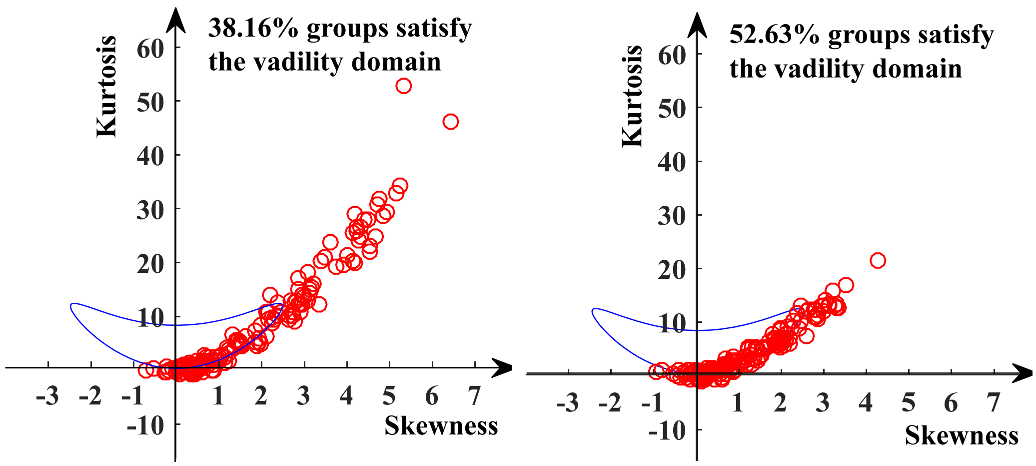
4 RESULTS AND ANALYSES

In this section, we first assess the performance of the logarithm transformation and rearrangement in estimating the travel time percentile function. Then, the estimation accuracy of the proposed method is compared with five widely-used travel time probability distributions. Finally, we examine the effect of higher-order travel time moments in the Cornish-Fisher expansion.

4.1 Effects of Logarithm Transformation and Rearrangement

In the following analysis, Analytical4, Analytical4_log, and Analytical4_log_RE are used to denote the original Cornish-Fisher expansion, the Cornish-Fisher expansion with the logarithm transformation, and the Cornish-Fisher expansion with both the logarithm transformation and rearrangement, respectively. The number 4 represents the use of the first four moments of travel time in the estimation.

The validity domain and the scatter plot of skewness and kurtosis of travel time datasets without and with the logarithm transformation are shown in Figure 4. One can see that the logarithm transformation not only increases the probability of satisfying the validity domain, but also makes the travel time datasets closer to the validity domain compared with the original datasets. Among the 152 groups, 58 groups (38.16%) satisfy the validity domain without the logarithm transformation, and 80 groups (52.63%) satisfy the validity domain after the logarithm transformation.

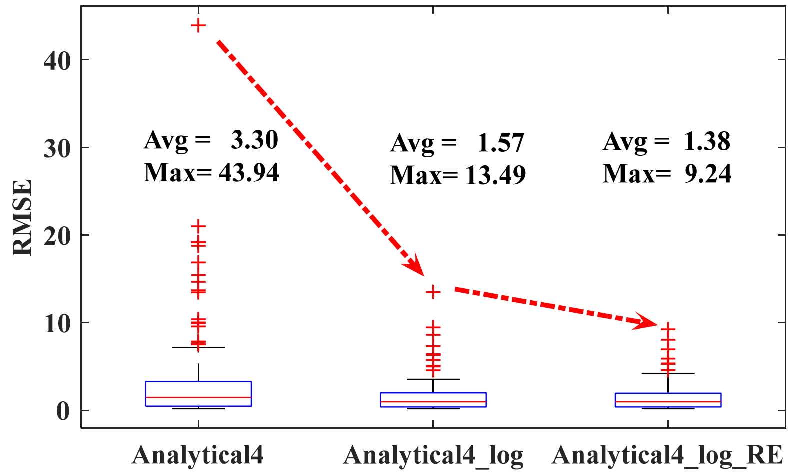


(a) Skewness and kurtosis scatter of the original travel time datasets

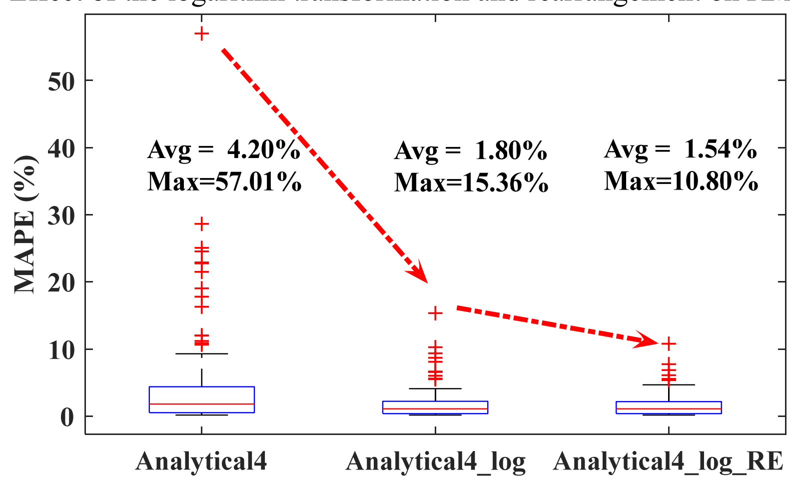
(b) Skewness and kurtosis scatter of the new travel time datasets after the logarithm transformation

Figure 4. The validity domain and the skewness-kurtosis scatter diagram.

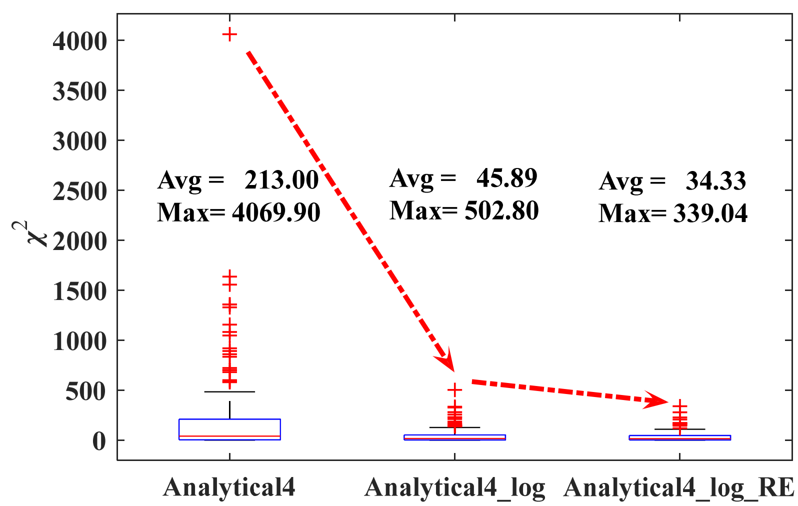
Figure 5 shows the average and the worst GoF performance derived from Analytical4, Analytical4_log, and Analytical4_log_RE for all 152 groups. We can see that the logarithm transformation greatly improves the performance of Cornish-Fisher expansion in estimating PTT, while the rearrangement further reduces the estimation error. For example, the average R^2 value has been improved from 0.71 to 0.97, which indicates that the estimated PTTs derived from Analytical4_log_RE have a much stronger positive correlation (i.e., 0.97) with the empirical PTTs than the estimated PTTs from Analytical4. More importantly, Analytical4_log_RE has a much lower worst estimation error than Analytical4. Specifically, the maximum RMSE, MAPE, and χ^2 have been reduced from 43.94 to 9.24, from 57.01% to 10.80%, from 4069.90 to 339.04, respectively; and the minimum R^2 has been increased from -9.37 to 0.54. Note that values of R^2 outside the range of 0 to 1 can occur when the model fits the data worse than a horizontal hyperplane (Kvålseth, 1985). The worst case in terms of R^2 for Analytical4 (i.e., -9.37) means that a wrong model may be used to estimate the PTTs.



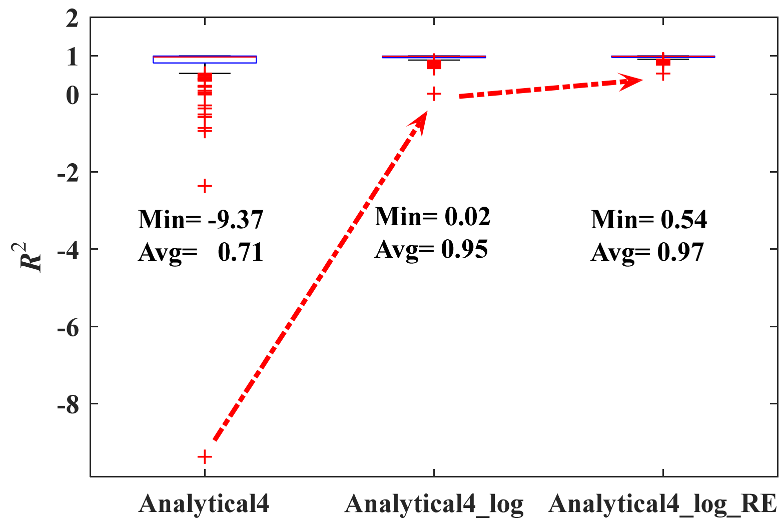
(a) Effect of the logarithm transformation and rearrangement on RMSE



(b) Effect of the logarithm transformation and rearrangement on MAPE



(c) Effect of the logarithm transformation and rearrangement on χ^2



(d) Effect of the logarithm transformation and rearrangement on R^2

Figure 5. Effect of logarithm transformation and rearrangement on the GoF measures.

The above significant improvements of estimation quality are due to three major effects of the logarithm transformation as proved in Proposition 1 and the rearrangement as proved in Proposition 2 and its corollaries: (1) the logarithm transformation increases the probability of satisfying the validity domain; (2) the logarithm transformation makes the travel time datasets closer to the validity domain even if they still cannot satisfy the validity domain; and (3) the rearrangement can guarantee a monotone estimated percentile function and simultaneously reduce the estimation error of the Cornish-Fisher expansion. Therefore, the proposed method, i.e., estimating PTTs via the Cornish-Fisher expansion with the logarithm transformation and rearrangement, is a more robust approach with a lower estimation error than estimating PTTs just via the Cornish-Fisher expansion.

4.2 Accuracy Comparison Between the Proposed Method and Five Widely-Used Probability Distributions

Among the 152 groups, the Burr distribution fails to fit 4 groups of travel time datasets due to the nonexistence of a maximum likelihood estimator. These 4 failure groups are either left-skewed or close to the normal distribution (i.e., with skewness and kurtosis of (-0.72, -0.22), (-0.25, -0.60), (-0.22, -0.87), and (0.09, -1.15), respectively). Wingo (1993) and Ghitany and Al-Awadhi (2002) pointed out that certain requirements are needed to ensure the existence and uniqueness of the maximum likelihood estimation for the Burr distribution. Since previous studies (Susilawati *et al*., 2013; Taylor, 2017) only demonstrated its promising performance in fitting right-skewed TTDs, it may not be suitable to select the Burr distribution as the candidate probability distribution for fitting left-skewed travel time datasets.

Table 6 shows the accuracy comparison of Analytical4_Log_RE and five widely-used probability distributions. One can see that Analytical4_Log_RE has the best GoF for these datasets. Even when we ignore the four groups that failed to be fitted by Burr distribution, Analytical4_Log_RE is still the best fitted model in terms of average RMSE, χ^2 and R^2. Despite that the Burr distribution is the best model in terms of average MAPE for those successfully fitted datasets, the proposed Analytical4_Log_RE has the closest accuracy to the Burr distribution (1.54% vs. 1.10%) compared with Lognormal, Weibull, Gamma, and Normal distributions. Overall, the above results demonstrate the promising performance of the proposed method in estimating PTTs compared with the five widely-used probability distributions of travel times.

Table 6. Accuracy comparison of Analytical4_Log_RE and five widely-used probability distributions.

Methods	Analytical4_Log_RE	Lognormal	Weibull	Gamma	Normal	Burr
RMSE	**1.38**	2.57	5.40	2.71	3.32	NA (1.92)
MAPE	**1.54%**	2.65%	7.37%	3.07%	4.30%	NA (**1.10%**)
χ^2	**34.33**	120.99	796.62	138.72	434.75	NA (102.54)
R^2	**0.97**	0.89	0.47	0.88	0.81	NA (0.94)

Remark: NA means that the Burr distribution fails to fit all 152 groups of datasets; the numbers in the parenthesis correspond to the average results of the successfully fitted groups (148 groups).

In the literature, the right-skewed characteristic of TTDs has received great attention (e.g., van Lint and van Zuylen, 2005; Fosgerau and Fukuda, 2012; Susilawati *et al*., 2013; Kim and Mahmassani, 2015; Delhomme *et al*., 2015). However, as discussed in Section 1 and shown in our empirical datasets, we would obviously encounter TTDs those are close to the normal distribution or left-skewed due to the heterogeneity of traffic states. Hence, the approaches of estimating the travel time percentile function should have great robustness and flexibility for fitting diverse TTDs. To test the robustness and flexibility of the proposed method and the five widely-used distributions, the 152 groups are categorized into four classes according to their skewness. The four classes are: (1) left-skewed datasets ($-2 \leq S < 0$); (2) right-skewed datasets ($0 \leq S < 2$); (3) highly right-skewed datasets ($2 \leq S < 4$); and (4) extremely right-skewed datasets ($4 \leq S$). Table 7 shows the average and the worst estimation error of Analytical4_Log_RE, Lognormal, Weibull, Gamma, Normal, and Burr distributions in terms of RMSE, χ^2, and R^2; and Table 8 shows the MAPE.

Table 7. Accuracy comparison of Analytical4_Log_RE, Lognormal, Weibull, Gamma, Normal, and Burr distributions for four classes in terms of RMSE, χ^2 and R^2.

Error metric	Method	Left skewed 7 groups (4.61%)		Right skewed 81 groups (53.29%)		Highly right skewed 41 groups (26.97%)		Extremely right skewed 23 groups (15.13%)	
		Avg	Max	Avg	Max	Avg	Max	Avg	Max
RMSE	Analytical4_log_RE	**1.13**	**2.68**	**0.69**	**3.54**	**1.50**	**5.35**	**3.68**	**9.24**
	Lognormal	2.68	5.59	1.26	6.14	3.02	9.32	6.36	11.31
	Weibull	1.96	2.73	3.12	10.13	6.72	13.70	12.10	21.18
	Gamma	2.29	4.93	1.32	6.74	3.20	9.74	6.88	12.79
	Normal	1.72	3.96	1.58	8.55	3.76	11.61	9.17	23.07
	Burr	NA	NA	NA	NA	1.56	6.68	4.46	9.60
χ^2	Analytical4_log_RE	**29.09**	**105.66**	**21.62**	**279.79**	**35.59**	**339.04**	**78.46**	**206.30**
	Lognormal	137.38	409.16	65.87	751.63	135.88	1101.84	283.56	621.70
	Weibull	87.27	130.25	397.59	2038.91	1041.52	4730.63	1981.26	4147.68
	Gamma	104.59	330.34	73.36	732.73	157.15	1286.43	346.45	744.30
	Normal	65.10	226.56	147.52	1824.15	252.83	2529.24	1883.09	20009.26
	Burr	NA	NA	NA	NA	36.76	474.26	115.96	326.39
		Avg	Min	Avg	Min	Avg	Min	Avg	Min
R^2	Analytical4_log_RE	**0.99**	**0.97**	**0.99**	**0.97**	**0.96**	**0.89**	**0.88**	**0.77**
	Lognormal	0.96	0.89	0.97	0.89	0.85	0.68	0.68	0.54
	Weibull	0.97	0.95	0.76	0.32	0.20	-0.21	-0.23	-0.71
	Gamma	0.97	0.91	0.97	0.87	0.83	0.65	0.63	0.48
	Normal	0.98	0.94	0.95	0.80	0.76	0.50	0.33	0.00
	Burr	NA	NA	NA	NA	0.96	0.83	0.84	0.67

Remark: Avg, Max and Min: the average, maximum and minimum results.

Table 8. Accuracy comparison of Analytical4_Log_RE, Lognormal, Weibull, Gamma, Normal, and Burr distributions for four classes in terms of MAPE.

Method	Left skewed		Right skewed		Highly right skewed		Extremely right skewed	
	Avg	Max	Avg	Max	Avg	Max	Avg	Max
Analytical4_log_RE	**0.77%**	1.94%	**0.77%**	**3.09%**	1.80%	5.39%	4.06%	10.80%
Lognormal	1.62%	3.54%	1.38%	4.90%	3.14%	8.99%	6.52%	13.03%
Weibull	1.15%	**1.64%**	3.79%	11.41%	9.54%	18.17%	18.00%	33.66%
Gamma	1.43%	3.23%	1.52%	5.42%	3.61%	10.38%	8.08%	18.32%
Normal	1.16%	2.68%	1.91%	8.33%	4.79%	14.19%	12.80%	36.78%
Burr	NA	NA	NA	NA	**0.95%**	**3.25%**	**1.95%**	**3.78%**

From Table 7 and Table 8, we can draw the following conclusions:

- The proposed Analytical4_Log_RE is the most *robust and flexible* approach that can work for all four classes of datasets. It has the smallest average and worst estimation errors in terms of RMSE, χ^2 and R^2 as shown in Table 7.
- One can see that the proposed Analytical4_Log_RE performs much better than the normal distribution. Improvement 1 in Section 2.1 is proposed based on the observation that the Cornish-Fisher expansion performs better when unknown distribution is closer to the normal distribution. However, the above observation does not mean that the Cornish-Fisher expansion performs just as the normal distribution would do.
- As for MAPE, the proposed Analytical4_Log_RE is still the best model for the first two classes and has the closest accuracy to the Burr distribution (i.e., the best model) for the last two classes. However, the Burr distribution may encounter the fitting failure problem for heterogeneous TTDs, e.g., the first two classes in Table 7 and Table 8. This further verifies the *flexibility* of the proposed Analytical4_Log_RE for various heterogeneous TTDs.
- The best fitted distribution can be different in terms of different GoF measures as pointed out by Plötz *et al*. (2017). This conclusion is also verified in our results. The reason is that different GoF measures focus on different aspects of estimation error as discussed in Section 3.2. However, Analytical4_Log_RE is the best model for all classes in terms of RMSE, χ^2, and R^2, and it is very competitive with the Burr distribution which works best for highly or extremely right-skewed datasets in terms of MAPE. From Eqs. (17), (18), (20), and (B-2), one can see that an inaccurate estimation of a small value (i.e., large relative error) may have a larger influence on MAPE compared to RMSE, $\boldsymbol{\chi^2}$ and R^2, while an inaccurate estimation of a large value (i.e., small relative error) may have a smaller influence on MAPE compared to RMSE, $\boldsymbol{\chi^2}$ and R^2. Therefore, the inconsistency between the results of MAPE and the three other GoF measures may suggest that the proposed method has different estimation performances between large and small PTTs.

To further validate the above conclusions, we also use the travel time observation datasets of these five consecutive links to construct 20 TTDs with a longer time interval (i.e., 240 minutes). In addition, we use travel time observation datasets over two weeks from 6:00 AM-10:00 AM on an urban expressway (i.e., the red line numbered by 6 in Figure 3) to construct another TTD. Results are shown in Table 9 and Table 10. We can draw the same conclusions as those concluded from Table 7 and Table 8.

Table 9. Accuracy comparison of Analytical4_Log_RE and five widely-used

probability distributions for TTDs constructed with a *240-minute time interval*.

Method	RMSE		MAPE		χ^2		R^2	
	Avg	Max	Avg	Max	Avg	Max	Avg	Min
Analytical4_log_RE	**3.29**	**10.04**	2.90	**7.87**	**1818.53**	**9417.73**	**0.95**	**0.78**
Lognormal	5.43	18.38	4.47	10.55	4324.02	21846.56	0.88	0.63
Weibull	7.40	12.90	9.52	19.12	12344.38	32017.19	0.62	-0.19
Gamma	5.14	13.44	4.94	10.80	4193.38	15974.99	0.88	0.61
Normal	5.75	12.36	6.62	13.95	12066.17	103938.87	0.83	0.43
Burr	5.00	17.51	**2.07**	11.13	3622.95	19631.89	0.85	0.28

Table 10. Accuracy comparison of Analytical4_Log_RE and five widely-used probability distributions for a *two-week travel time dataset.*

Methods	Analytical4_Log_RE	Lognormal	Weibull	Gamma	Normal	Burr
RMSE	**2.84**	9.53	13.12	10.07	12.40	3.79
MAPE	2.75	8.25	17.87	9.92	15.10	**2.00**
χ^2	**8265.83**	113849.35	556168.75	141210.74	329832.55	11739.40
R^2	**0.98**	0.77	0.57	0.75	0.62	0.96

In summary, compared with the widely-used mean and variance, skewness and kurtosis are used in the proposed Analytical4_Log_RE method to capture the asymmetry and flatness of TTDs, respectively. Therefore, Analytical4_Log_RE can more accurately capture the TTDs directly based on the statistical characteristics, making it more robust and flexible in estimating percentile function for heterogeneous travel time datasets.

4.3 Effect of Higher-Order Moments in the Cornish-Fisher Expansion

For estimating the travel time percentile function, higher-order moments (i.e., up to the fifth-order or sixth-order) are used in the Cornish-Fisher expansion as shown in Eq. (21) to examine their effects on the estimation accuracy. Table 11 shows the average and worst estimation errors of the fourth, fifth, and sixth-order Cornish-Fisher expansions in estimating the travel time percentile function (Analytical4_log_RE, Analytical5_log_RE, and Analytical6_log_RE for short).

Table 11. Estimation errors of using higher-order moments.

Metric	Analytical4_log_RE		Analytical5_log_RE		Analytical6_log_RE	
	Avg	Max	Avg	Max	Avg	Max
RMSE	**1.78**	**9.24**	8.71	267.20	3.07×10^{9}	3.16×10^{11}
MAPE	**1.98%**	**10.80%**	9.00%	231.23%	3.13×10^{8}%	3.23×10^{10}%
χ^2	**46.04**	**339.04**	2108.70	71456.99	4.26×10^{21}	4.38×10^{23}
R^2	Avg	Min	Avg	Min	Avg	Min
	0.96	**0.54**	-3.54	-382.68	-5.20×10^{18}	-5.36×10^{20}

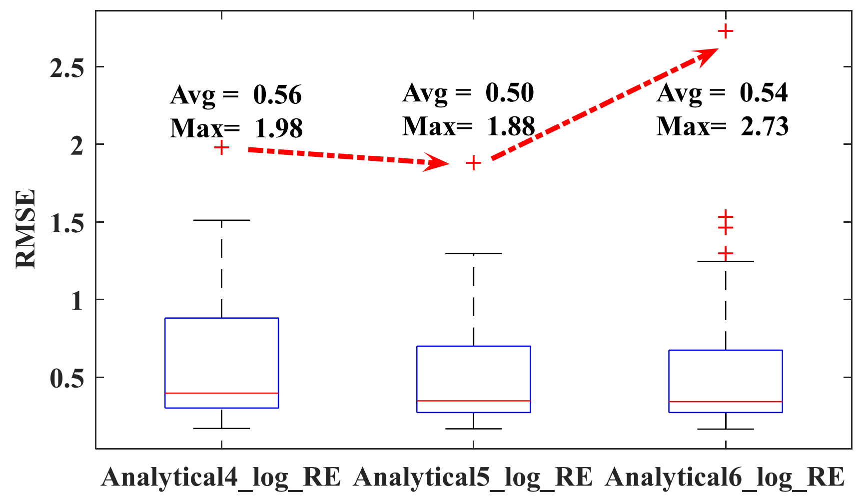
From Table 11, the fourth-order method has the lowest estimation error for all four GoF measures. Besides, both the average and the worst estimation errors have a sharp increase for all four GoF measures when taking higher-order moments into account. In other words, the consideration of the fifth and sixth moments in the Cornish-Fisher expansion would have significantly worse estimation accuracy and extremely unreasonable estimated PTTs.

To further explore why the higher-order Cornish-Fisher expansion unexpectedly performs worse in estimating PTTs, the 152 groups are divided into two classes according to the change of estimation error after introducing higher-order moments. In Class 1, higher-order moments reduce the estimation error; In Class 2, higher-order moments increase the estimation error. The more detailed information about these two classes is provided in Table 12. One can see that the average and maximum skewness and kurtosis in Class 2 are all greater than Class 1. In other words, the travel time datasets in Class 1 are closer to the normal distribution, while the travel time datasets in Class 2 are stronger right-skewed with a fatter tail.

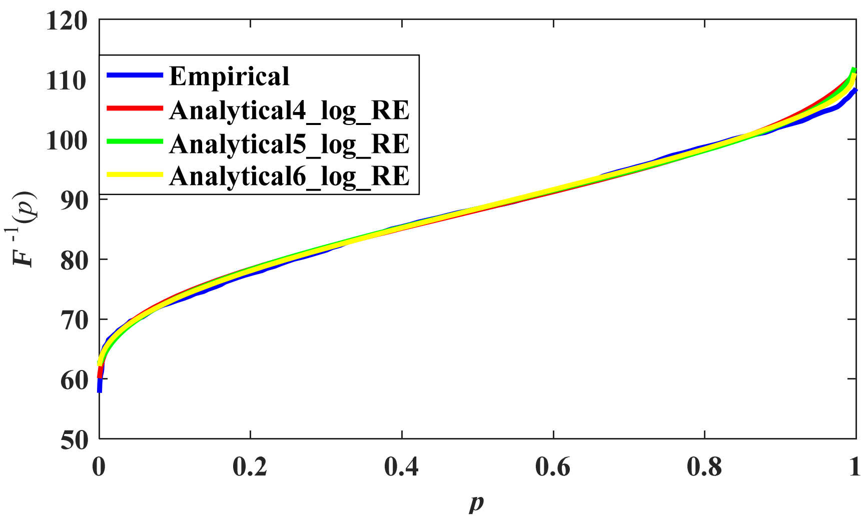
Table 12. The detailed information of Class 1 and Class 2.

Class	Group size	Percentage	Average skewness	Maximum skewness	Average kurtosis	Maximum kurtosis
Class 1	49	32.24%	0.35	1.20	0.67	4.78
Class 2	103	67.76%	1.53	4.24	5.46	21.09

Figure 6 (a) shows the boxplot of RMSE for Class 1. Without loss of generality, Figure 6 (b) shows the estimated percentile functions for Group 36 (skewness: -0.26; kurtosis: -0.56) in Class 1 derived from empirical CDF, Analytical4_log_RE, Analytical5_log_RE, and Analytical6_log_RE. Analytical5_log_RE and Analytical6_log_RE can perform better than Analytical4_log_RE when travel time datasets are close to the normal distribution or left skewed. The above result is consistent with the conclusion that higher-order Cornish-Fisher expansion leads to more accurate results in the portfolio optimization research (Pichler and Selitsch, 2000). However, the accuracy difference between the fourth-order and fifth/sixth-order Cornish-Fisher expansions is trivial. Such a small improvement is negligible in the real-world applications. However, when travel time datasets are right-skewed as in Class 2, the estimated PTTs derived from higher-order Cornish-Fisher expansion can get worse. Without loss of generality, Figure 7 shows the percentile functions estimated by the fourth, fifth, and sixth-order Cornish-Fisher expansions for Group 41 (skewness: 4.56; kurtosis: 25.11) in Class 2. One can see that the higher-order Cornish-Fisher expansion may produce extremely unreasonable estimation of PTTs when travel time datasets are extremely right-skewed.



(a) RMSE boxplot of Analytical4_log_RE, Analytical5_log_RE and Analytical6_log_RE for Class 1.



(b) Percentile function curves for Group 36 in Class 1 derived from empirical CDF, Analytical4_log_RE, Analytical5_log_RE, and Analytical6_log_RE

Figure 6. The boxplot of RMSE for Class 1 and percentile function curves for Group 36 in Class 1.

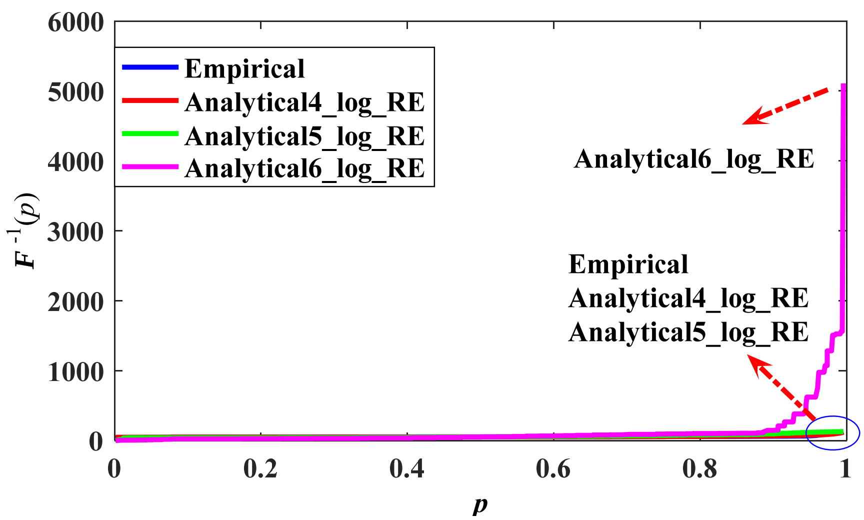


Figure 7. The percentile function curves for Group 41 in Class 2 derived from Analytical4_log_RE, Analytical5_log_RE, and Analytical6_log_RE.

Compared to the profit and loss distribution in portfolio optimization (i.e., the main application context of the Cornish-fisher expansion), TTDs have diverse statistical properties, i.e., TTDs can be left-skewed, close to normal, and right-skewed. However, higher-order Cornish-Fisher expansion only performs better for capturing the left tail of unknown distribution (Pichler and Selitsch, 2000), which means that higher-order Cornish-Fisher expansion is not suitable for estimating the travel time percentile function of diverse TTDs. What we see here is Ockham's razor in action: a higher-order model may be more powerful, however it is also much more sensitive to the dataset and it would perform worse than the lower-order model once the dataset is outside its input domain.

According to Section 4.2 and the above discussion, the first four moments are already able to provide high-quality estimation for capturing both left-skewed and right-skewed characteristics of TTDs. Hence, from the perspective of robustness and flexibility, we suggest using only the first four moments in estimating the travel time percentile function to make the proposed method simple and less sensitive to the datasets. Only when the accuracy requirement is extremely high or the majority of sample datasets are left-skewed, the fifth-order or the sixth-order moment could be considered in the Cornish-Fisher expansion for estimating PTTs.

5 Conclusions

In this paper, we proposed a closed-form estimation of the travel time percentile function based on the Cornish-Fisher expansion while being adaptable to accurately capturing the heterogeneity of TTDs in characterizing the TTR. For guaranteeing the high-quality estimation of the travel time percentile function, we proposed and integrated two improvements into the Cornish-Fisher expansion: (1) logarithm transformation and (2) rearrangement. The logarithm transformation was first introduced into the Cornish-Fisher expansion for 'transforming' the highly skewed datasets to be less skewed, by making use of the fact that the Cornish-Fisher expansion performs better when the unknown distribution is closer to the normal distribution. Compared with the original Cornish-Fisher expansion, the validity domain of Cornish-Fisher expansion with the logarithm transformation remains unchanged. This means that the probability of travel time datasets satisfying the validity domain increases after performing the logarithm transformation. When travel time datasets cannot satisfy the validity domain after the logarithm transformation, the rearrangement was then adopted to ensure a monotone estimated travel time percentile function. We rigorously proved that the rearrangement can strictly reduce the estimation error in terms of four GoF measures: RMSE, MAPE, χ^2, and R^2 when travel time datasets exceed the validity domain.

Realistic travel time datasets that cover five links across four days were used to examine the accuracy and robustness of the proposed method. Results demonstrated that the two improvements in the proposed method (i.e., logarithm transformation and rearrangement) can greatly reduce the estimation error while ensuring a monotone estimated travel time percentile function. Compared to the five widely-used probability distributions, the proposed method is a more robust and flexible approach to accurately estimating the travel time percentile function of TTDs with diverse characteristics. Besides, the first four moments are already able to capture the heterogeneity of TTDs compared with higher-order moments, and thus we suggest using only the first four moments in the Cornish-Fisher expansion for estimating travel time percentile function.

In summary, the advantages of the proposed method are three folds: *(1) its distribution-fitting-free nature circumvents the issues associated with fitting a predefined distribution type to TTDs; (2) it has a promising estimation quality with theoretically proven foundation based on the two improvements; and (3) it has a closed-form expression of the travel time percentile function, which could facilitate the computation of TTR measures and promote their applications in large-scale network applications*.
For future research, several directions are worthy of further investigations. As the proposed method is an easy and parsimonious way to derive a closed-form travel time

percentile function, a direct application is the online calculation of reliability measures, which is the basis of real-time reliability information dissipation or monitoring platform. Besides, Pu (2011) investigated the analytical relationship among many existing TTR measures via the Lognormal distribution assumption. Compared to the Lognormal distribution, the proposed method has a closed-form expression of percentile function directly based on the first four moments of travel time. We plan to investigate the analytical relationship between many existing TTR measures and statistical indicators of travel time via the proposed method.

APPENDIX A

From Zang *et al*. (2018), the validity domain of the Cornish-Fisher expansion without the logarithm transformation is as follows:

$$S = K = 0, \text{ or } \begin{cases} 4S^2 < 3K \\ \frac{5}{54}S^4 - \left(\frac{11K}{72} - \frac{7}{9}\right)S^2 + \frac{1}{16}K^2 - \frac{1}{2}K \le 0 \end{cases} \tag{A-1}$$

This can be rewritten as:

$$S = K = 0, \text{ or } \begin{cases} 4S^2 < 3K \\ \frac{1}{16}K^2 - \left(\frac{11S^2}{72} + \frac{1}{2}\right)K + \frac{5}{54}S^4 + \frac{7}{9}S^2 \le 0 \end{cases} \tag{A-2}$$

Let $f(K) = \frac{1}{16}K^2 - \left(\frac{11S^2}{72} + \frac{1}{2}\right)K + \frac{5}{54}S^4 + \frac{7}{9}S^2$. Obviously, $f(K)$ is a quadratic function of K. Considering that the coefficient of the quadratic term (i.e., 1/16) is greater than 0, $f(K) \le 0$ implies that: (1) $f(K)=0$ has two roots; and (2) K that satisfies $f(K) \le 0$ sits between these two roots. Therefore, the discriminant of $f(K)$ must be non-negative:

$$\begin{aligned} &\Delta = \left(\frac{11S^2}{72} + \frac{1}{2}\right)^2 - 4\left(\frac{1}{16}\right)\left(\frac{5}{54}S^4 + \frac{7}{9}S^2\right) \ge 0 \\ &\Rightarrow \left(S^2 - 108\right)^2 \ge 10368 = 2 \cdot 72^2 \end{aligned} \tag{A-3}$$

Then, we have

$$\begin{aligned} &S^2 \le 108 - 72\sqrt{2} \quad \text{or} \quad S^2 \ge 108 + 72\sqrt{2} \\ &\Rightarrow |S_1| \le 6\left(\sqrt{2} - 1\right) \quad \text{or} \quad |S_2| \ge 6\left(\sqrt{2} + 1\right) \end{aligned} \tag{A-4}$$

The first area S_1 is useful in real applications of the Cornish-Fisher expansion, while the second area S_2 is too large and abandoned given with the actual skewness of travel time datasets. Then, the two roots of $f(K) = 0$ are:

$$K_1 = 4 + \frac{11}{9}S^2 - \sqrt{\frac{1}{81}S^4 - \frac{8}{3}S^2 + 16}, \quad K_2 = 4 + \frac{11}{9}S^2 + \sqrt{\frac{1}{81}S^4 - \frac{8}{3}S^2 + 16} \qquad \text{(A-5)}$$

Namely,

$$\begin{aligned} &S \in \left[-6\left(\sqrt{2}-1\right), 6\left(\sqrt{2}-1\right)\right] \\ &K \in \left[4 + \frac{11}{9}S^2 - \sqrt{\frac{1}{81}S^4 - \frac{8}{3}S^2 + 16},\ 4 + \frac{11}{9}S^2 + \sqrt{\frac{1}{81}S^4 - \frac{8}{3}S^2 + 16}\right] \end{aligned} \qquad \text{(A-6)}$$

Obviously, the range of S and K defined by Eq. (A-1) is equal to the range of S and K defined by Eq. (A-6). Therefore, the validity domain of the Cornish-Fisher expansion can be finally written as Eq. (A-6).

APPENDIX B

We continue to consider the definition of i and j in Lemma 1 (i.e., $ptt_r(i) < ptt_r(j)$ and $ptt_f(i) > ptt_f(j)$). Note that the logarithm transformation (i.e., the exponential function in Eq. (4)) makes $ptt_f(i) > ptt_f(j) > 0$. Then, we have

$$\begin{aligned} &\frac{ptt_r(i)+ptt_r(j)}{ptt_f(i)} < \frac{ptt_r(i)+ptt_r(j)}{ptt_f(j)} \\ &\Rightarrow \frac{2ptt_f(i) - ptt_r(i) - ptt_r(j)}{ptt_f(i)} > \frac{2ptt_f(j) - ptt_r(i) - ptt_r(j)}{ptt_f(j)} \\ &\Rightarrow \frac{\left(ptt_r(i) - ptt_r(j)\right)\left(2ptt_f(i) - ptt_r(i) - ptt_r(j)\right)}{ptt_f(i)} \\ &\quad < \frac{\left(ptt_r(i) - ptt_r(j)\right)\left(2ptt_f(j) - ptt_r(i) - ptt_r(j)\right)}{ptt_f(j)} \\ &\Rightarrow \frac{\left(ptt_f(i) - ptt_r(j)\right)^2 - \left(ptt_f(i) - ptt_r(i)\right)^2}{ptt_f(i)} \\ &\quad < \frac{\left(ptt_f(j) - ptt_r(j)\right)^2 - \left(ptt_f(j) - ptt_r(i)\right)^2}{ptt_f(j)} \end{aligned} \qquad \text{(B-1)}$$

$$\Rightarrow \frac{\left(ptt_f(i)-ptt_r(j)\right)^2}{ptt_f(i)}+\frac{\left(ptt_f(j)-ptt_r(i)\right)^2}{ptt_f(j)}$$

$$<\frac{\left(ptt_f(i)-ptt_r(i)\right)^2}{ptt_f(i)}+\frac{\left(ptt_f(j)-ptt_r(j)\right)^2}{ptt_f(j)}$$

If for any $i \in X_1$, there exists one $j \in X_2$ satisfying $Sptt_f(i)=ptt_f(j)$ and $Sptt_f(j)=ptt_f(i)$, then we have

$$\frac{\left(Sptt_f(i)-ptt_r(i)\right)^2}{Sptt_f(i)}+\frac{\left(Sptt_f(j)-ptt_r(j)\right)^2}{Sptt_f(j)}$$

$$=\frac{\left(ptt_f(j)-ptt_r(i)\right)^2}{ptt_f(j)}+\frac{\left(ptt_f(i)-ptt_r(j)\right)^2}{ptt_f(i)}<\frac{\left(ptt_f(i)-ptt_r(i)\right)^2}{ptt_f(i)}+\frac{\left(ptt_f(j)-ptt_r(j)\right)^2}{ptt_f(j)} \quad \text{(B-2)}$$

$$\Rightarrow \chi^2:\ \sum_{i=1}^{n}\frac{\left(Sptt_f(i)-ptt_r(i)\right)^2}{Sptt_f(i)}<\sum_{i=1}^{n}\frac{\left(ptt_f(i)-ptt_r(i)\right)^2}{ptt_f(i)}$$

Namely, χ^2 of the Cornish-Fisher expansion in estimating the PTTs is improved due to the rearrangement. This completes the proof. □

ACKNOWLEDGEMENTS

The authors are grateful to the two anonymous referees for their constructive comments and insightful suggestions to improve the quality and clarity of the paper. The work described in this paper was jointly supported by research grants from the National Natural Science Foundation of China (U1764261), the Chenguang Program sponsored by Shanghai Education Development Foundation and Shanghai Municipal Education Commission, the Research Grants Council of the Hong Kong Special Administrative Region (Project No. 15267116), and the Kwong Wah Education Foundation of the Research Institute for Sustainable Urban Development at the Hong Kong Polytechnic University. These supports are gratefully acknowledged.

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