Asymptotic Optimality of Base-Stock Policies for Perishable Inventory Systems

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We consider periodic-review perishable inventory systems with a fixed product lifetime. Unsatisfied demand can be either lost or backlogged. The objective is to minimize the long-run average holding, penalty, and outdating cost. The optimal policy for these systems is notoriously complex and computationally intractable due to the curse of dimensionality. Hence, various heuristic replenishment policies have been proposed in the literature, including the base-stock policy which raises the total inventory level to a constant in each review period. While various studies have shown near-optimal numerical performances of base-stock policies in the classic system with zero replenishment lead time and a first-in-first-out (FIFO) issuance policy, the results on their theoretical performances are very limited. In this paper, we first focus on this classic system and show that a simple base-stock policy is asymptotically optimal when any one of the product lifetime, demand population size, unit penalty cost, and unit outdating cost becomes large; moreover its optimality gap converges to zero exponentially fast in the first two parameters. We then study two important extensions. For a system under a last-in-first-out (LIFO) or even an arbitrary issuance policy, we prove that a simple base-stock policy is asymptotically optimal with large lifetime, large unit penalty costs, and large unit outdating costs; and for a backlogging system with positive lead times, we prove that our results continue to hold with product lifetime, demand population size, and unit outdating cost. Finally, we provide a numerical study to demonstrate the performances of base-stock policies in these systems.

Key words: perishable inventory, base-stock policy, asymptotic analysis, inventory issuance, lead times

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1. Introduction

In this paper, we consider periodic-review inventory systems for a perishable product with a fixed product lifetime over an infinite planning horizon. The demands in different review periods are independent and identically distributed (*i.i.d.*) random variables. In each review period, the firm first makes an ordering decision, and receives the order immediately or after some positive lead times. Then, random demand for this review period is realized and satisfied by the on-hand inventory as much as possible following some pre-specified issuance policy, e.g., first-in-first-out (FIFO) or last-in-first-out (LIFO). Unsatisfied demand can be either lost or backlogged. Finally, leftover inventory reaching the end of its lifetime perishes and is removed, and the rest inventories are carried to the next period. The firm's objective is to minimize the long-run average holding, penalty and outdating cost.

Perishable inventory systems with a fixed product lifetime are a fundamental class of inventory systems that have been studied extensively since the 1960's (see, e.g., Veinott 1960, Van Zyl 1964). The optimal policy for these systems is notoriously complex. This is because one needs to keep track of the inventory level of each age group due to the fixed product lifetime, which leads to a multi-dimensional system state. It has been shown that the optimal policy depends on the entire system state and does not have a simple structure (see, e.g., Nahmias 1975, Cohen and Pekelman 1978), and that computing the optimal order quantities using dynamic programming is intractable due to the *curse of dimensionality* even if the lifetime is at a moderate size (e.g., four) *and* the replenishment lead time is zero. See Karaesmen et al. (2011) or Nahmias (2011) for a comprehensive review of perishable inventory systems.

Given the complexity of the optimal policy, researchers have quickly switched their efforts to develop effective heuristic replenishment policies for these systems since the 1960's, such as base-stock policies, constant-order policies, and approximation algorithms. We refer to §1.2 for a review on different heuristics. A majority of the literature have considered the classic perishable inventory system with zero lead time and the FIFO issuance policy (i.e., the oldest inventory is used first to satisfy demand). Among these studies, the base-stock (or order-up-to/critical-number/target inventory position) policy, which raises the total inventory level to a constant in each review period, has received the most attention. Through extensive numerical studies, previous studies have shown that base-stock policies perform close to optimal in the classic system. For example, after numerically comparing base-stock policies with the optimal policy, Cooper (2001) concludes that *"the performance of the critical-number policies was nearly as good as that of an optimal policy in the classic perishable system, thereby supporting the assertion that, in the absence of significant fixed-charge order costs, critical-number policies provide a simple and effective means for managing inventories of a perishable product."*

However, there is little theory on the near-optimality of base-stock policies for the classic perishable inventory system. To our best knowledge, Zhang et al. (2020) is the only study that analyzes theoretical performances of base-stock policies for perishable inventory systems. Under a discounted profit criterion, they focus on two heuristic base-stock policies which ignore perishability and analyze their asymptotic properties when the market size is large. In this paper, we consider the classic infinite-horizon average-cost system under general *i.i.d.* demands, and establish a variety of asymptotic optimality results for a different base-stock policy by considering four asymptotic regimes of system parameters: (i) large product lifetime; (ii) large demand population sizes; (iii) large unit penalty costs; and (iv) large unit outdating costs. Needless to say, perishability is crucial and has to be included in the construction of our base-stock policy. Please refer to §1.2 for a more detailed comparison between this paper and Zhang et al. (2020).

We further extend most of these results for the classic system to a perishable system under the LIFO or general issuance policy and a backlogging perishable system with positive replenishment lead times. In brick-and-mortar retailing, customers can observe the expiration dates of perishable items and decide by themselves which items to purchase. In this context, the LIFO issuance policy (i.e., the youngest inventory is used first to satisfy demands) or a more general issuance policy (i.e., a general issuance sequence is followed) may better fit into the practice. This inspires us to consider the first extension. Besides, inventory replenishment usually takes time. However, the studies on perishable inventory systems with positive lead times are very limited, due to the additional complexity arising from pipeline inventories. This motivates us to consider the second extension.

1.1. Main Results and Contributions

We summarize the main results and contributions of this paper as follows.

First, we focus on the classic system with zero lead time and the FIFO issuance policy. We construct a simple base-stock policy for this system, and establish its asymptotic optimality in four parameters regimes. i) We prove that its optimality gap (i.e., the difference between its long-run average cost and the optimal cost) decays to zero exponentially fast in the product lifetime. ii) When the number of customers in one period follows a renewal process with arrival rate n (referred to as the *demand population size*) and different customers request *i.i.d.* units of the product, we show that its optimality gap decays to zero exponentially fast in the demand population size. iii) We consider the regime of large unit penalty costs, motivated by high service levels typically required in practice, and prove that its optimality gap converges to zero as the unit penalty cost goes to infinity. The convergence rate, however, depends on the class of demand distributions, and we characterize the convergence rate of the optimality gap for four classes of continuous demands. iv) Finally, we consider the regime of large unit outdating costs, inspired by high disposal costs for many perishable products in practice, and prove that its optimality gap also converges to zero as the unit outdating cost goes to infinity. These results provide a solid theoretical foundation for the near-optimal numerical performances of base-stock policies in the classic perishable inventory system which have been widely reported in the literature.

Second, we construct another simple base-stock policy for a perishable system under the LIFO or general issuance policy, and establish its asymptotic performances in three parameter regimes. Specifically, we prove the following results. i) Its optimality gap converges to zero in the product lifetime, at a rate at least in the order of reciprocal of lifetime. ii) When demand is unbounded, the *relative* gap between its long-run average cost and the optimal cost converges to zero when the unit penalty cost goes to infinity. iii) When the demand lower support is zero, its optimality gap converges to zero when the unit outdating cost goes to infinity. For the system under the LIFO issuance policy, we also prove that when the demand lower support is positive, the best base-stock policy is in general *not* asymptotically optimal with large unit outdating costs. These results, for the first time, establish theoretical performances of base-stock policies in the system under the LIFO or general issuance policy.

Third, we construct a simple base-stock policy for a backlogging perishable system with positive lead times, and establish its asymptotic optimality in three parameter regimes. Specifically, we prove that its optimality gap decays to zero exponentially fast in the lifetime and in the demand population size, and it converges to zero when the unit outdating cost goes to infinity. We also provide intuitive explanations on why the best base-stock policy is in general *not* asymptotically optimal with large unit backlogging costs. These results, for the first time, establish theoretical performances of base-stock policies in the backlogging system with positive lead times.

Finally, we conduct a numerical study to test the performances of base-stock policies in these systems. For the classic system, the best base-stock policy, our simple base-stock policy, and a base-stock policy proposed by Cooper (2001) all perform very well consistently under various parameter settings. By contrast, a naive base-stock policy which ignores perishability performs poorly under geometric demand. For the system under the LIFO issuance policy, base-stock policies do not perform as well as they do in the classic FIFO system, especially when the unit outdating cost is moderately large. For the backlogging system with positive lead times, for which no other base-stock policy has been proposed in the literature, both the best and our simple base-stock policies perform better as the lifetime or unit outdating cost increases, which is consistent with our theoretical results, but worse as the unit backlogging cost or the lead time increases.

Highlights of methodology. To prove the various asymptotic optimality results, we construct three lower bounds on the optimal costs, which, as a certain system parameter goes to infinity, converge to the long-run average costs of our simple base-stock policies. See Propositions 2 to 4 in §6.1. The lower bounds in Proposition 2 and Proposition 4 hold true for the classic system and the backlogging system (with positive lead times), respectively, under any inventory issuance policy. We prove both bounds by a simple sample-path approach. Specifically, we employ a simple lower bound on the cumulative outdating quantity in any duration of lifetime-long periods plus lead-time-long periods, and apply the conditional Jensen's inequality to prove these bounds. The lower bound in Proposition 3 holds true for the classic system under bounded demand. The key step in our proof is to establish a uniform lower bound on the order-up-to level under the optimal policy in each review period. To this end, we apply a vanishing discount factor approach from Schäl (1993) to translate an existing result in Nandakumar and Morton (1993) for the discounted-cost perishable inventory system to our desired result for the average-cost system.

Besides, our proofs rely on various bounds on the long-run average costs of base-stock policies in different systems. For the classic system, we borrow an upper bound from Cooper (2001) and an upper bound and a lower bound from Chazan and Gal (1977) on the long-run average outdated inventory (see Lemma 2 in §2.1). For the system under the LIFO issuance policy, we construct an upper bound on the long-run average outdated inventory based on a simple observation on the cumulative amount of outdates during any consecutive lifetime-long periods, and a lower bound from a recursion on the outdating process (see Lemma 3 in §4.1). For the system under the general issuance policy, we show that the long-run average cost of any given base-stock policy is bounded from above by that under the LIFO issuance policy, and bounded from below by that under the FIFO issuance policy (see Lemma 4 in §4.2). For the backlogging system with positive lead times, we construct the upper and lower bounds on the long-run average cost of base-stock policies by generalizing the corresponding bounds for the zero-lead-time system (see Lemma 5 in §5). All these bounds are proved by a sample-path approach.

We end this subsection by providing Table 1 as a road map for the main asymptotic-optimality results derived in this paper.

Table 1 Summary of main results for three perishable inventory systems

	FIFO zero lead time lost-sales (in §2 – §3)	LIFO and general issuance policies zero lead time lost-sales (in §4)	FIFO positive lead times backorder (in §5)
Product lifetime	**Theorem** 1	**Theorem** 5(a), **Theorem** 6	**Theorem** 7(a)
Demand population size	**Theorem** 2	N.A.	**Theorem** 7(b)
Unit penalty cost	**Theorem** 3	**Theorem** 5(b), **Theorem** 6	N.A.
Unit outdating cost	**Theorem** 4	**Theorem** 5(c), **Proposition** 1, **Theorem** 6	**Theorem** 7(c)

[1] The "N.A." means that no result is derived for that system in the corresponding parameter regime.

1.2. Literature Review

This paper is related to two streams of literature: perishable inventory systems with a fixed product lifetime, and asymptotic analysis of simple heuristic policies for complex inventory systems. In the following, we briefly review each stream of literature.

First, perishable inventory systems with a fixed product lifetime are a fundamental class of inventory systems and have been studied extensively since the early studies by Veinott (1960) and Van Zyl (1964). We refer to Karaesmen et al. (2011) and Nahmias (2011) for detailed reviews on the earlier studies and Chao et al. (2018) for a review on recent studies. As mentioned, the optimal policy for these systems is complex and computationally intractable in general. Thus, researchers have switched to designing heuristic replenishment policies, such as base-stock policies, constant-order policies (e.g., Deniz et al. 2010, Deniz et al. 2020), heuristics based on higher-order approximations (e.g., Nahmias 1977a, Sun et al. 2016), and approximation algorithms (e.g., Chao et al. 2015, Chao et al. 2018, Zhang et al. 2021). We next provide a detailed review on the studies on base-stock policies for perishable inventory systems with a fixed product lifetime, which are more related to this paper.

Due to their simplicity and effectiveness, base-stock policies have been widely studied for the classic perishable inventory system, under both average and discounted cost criteria. We first review the studies under the average cost criterion. Van Zyl (1964) first studies the base-stock policy for a system with two-period lifetime and derives an explicit expression for the steady-state distribution of the inventory process. For the system with general lifetime, Cohen (1976) analyzes the steady-state distribution of the inventory process under a base-stock policy, Chazan and Gal (1977) prove that the long-run average outdating is convex in the base-stock level and derive its upper and lower bounds, and Cooper and Tweedie (2002) introduce a technique to estimate the steady-state distribution of the inventory process. Cooper (2001) constructs two heuristic base-stock levels by approximating the long-run average outdating by the mid-points of its best upper and lower bounds derived by himself and its upper and lower bounds derived by Chazan and Gal (1977), respectively. He also conducts an extensive numerical study showing that base-stock policies perform close to optimal under various parameter settings. We next review the studies under the discounted cost criterion. Nahmias (1976) constructs a base-stock policy based on a myopic approximation and a bound on the expected outdating cost. Nahmias (1977b) constructs two base-stock policies for systems with random lifetimes using different approximations on the expected outdating cost. Nandakumar and Morton (1993) construct a base-stock policy for an infinite-horizon system based on the bounds for the order-up-to level of the optimal policy. These studies also show numerically that base-stock policies perform close to optimal under various parameter settings. None of the studies reviewed above analyze theoretical performances of base-stock policies.

By contrast, the studies on base-stock policies for the systems under the LIFO or a general issuance policy or with positive lead times are very limited. We refer to Cohen and Pekelman (1978) and Cohen and Prastacos (1981) for early studies on base-stock policies for the systems under the LIFO issuance policy. We are not aware of any literature studying perishable inventory systems under a general issuance policy. As mentioned before, Zhang et al. (2020) analyze theoretical performances of two heuristic base-stock policies for perishable inventory systems. Different from this paper, they focus on *finite-horizon* and *discounted-profit* systems under *i.i.d.* compound Poisson demands, and both policies they propose ignore product perishability. They analyze the asymptotic properties of both policies when the market size is large. Our paper is partly inspired by theirs but is significantly different in three major aspects. First, we consider an *infinite-horizon average-cost* system under general *i.i.d.* demands, which leads to a different methodology from theirs. Second, we establish a variety of asymptotic optimality results of base-stock policies in three perishable inventory systems in four parameter regimes. Finally, our base-stock policies take product perishability into account and have better theoretical and numerical performances.

Second, this paper is related to the growing literature on asymptotic analysis of simple heuristics for complex inventory systems. We refer to Goldberg et al. (2021) for a detailed review on this topic. In particular, there have been extensive studies on lost-sales non-perishable inventory systems with lead times, including asymptotic analysis of base-stock policies with large unit penalty costs or high service-level requirements (Reiman 2004, Huh et al. 2009, Bijvank et al. 2014, Arts et al. 2015, Wei et al. 2021), and constant-order or capped base-stock policies with large lead times (Goldberg et al. 2016, Xin and Goldberg 2016, Bu et al. 2020, Xin 2021b). Other studies on nonperishable inventory systems can be classified into the following categories: i) base-stock policies for assemble-to-order systems with large lead times (Reiman and Wang 2015); ii) echelon base-stock policies for series systems with large unit penalty costs (Huh et al. 2016); iii) tailored base-surge policies for dual-sourcing systems with large lead-time differences (Xin and Goldberg 2018, Xin et al. 2018); iv) constant-order dynamic pricing policies for joint inventory and pricing systems with large lead times (Chen et al. 2019); and v) constant-order (L, U) threshold policy for hybrid manufacturing/remanufacturing systems with large manufacturing lead times (Xin 2021a). For perishable inventory systems, Zhang et al. (2020) provide the first asymptotic analysis of base-stock policies with large market sizes. Our paper contributes to this stream of literature by establishing a variety of asymptotic optimality results of base-stock policies in three perishable inventory systems in four parameter regimes.

1.3. Structure and Notation

The rest of this paper is organized as follows. In §2, we focus on the classic perishable inventory system with zero lead time and the FIFO issuance policy, and construct a simple base-stock policy. In §3, we present our asymptotic-optimality results for this policy in four parameter regimes. In §4 and §5, we consider a system under the LIFO or a general issuance policy and a backlogging system with positive lead times, respectively. In §6, we sketch the proofs of our main results. In §7, we present a numerical study. We conclude the paper in §8 with future research directions. In this paper, we denote $x^+ = \max\{0, x\}$ for real number x and denote $\triangleq$ as "equal by definition". For an increasing function $F(\cdot)$, its inverse function is defined as $F^{-1}(x) \triangleq \inf\{y \in \mathbb{R} : F(y) \geq x\}$.

2. Model Formulation and Base-Stock Policy

Consider a periodic-review inventory system of a perishable product over a planning horizon of infinitely many periods, indexed by $t = 1, 2, \ldots$. The product has a fixed lifetime of m periods, where m is a positive integer. That is, the items of the product perish (or outdate) after staying in the system for m periods and then are removed from the system. The demands in periods $1, 2, \ldots$, denoted by $D_1, D_2, \ldots$, are a sequence of *i.i.d.* non-negative random variables (r.v.'s) with the same distribution as random variable D, with $0 < \mathbb{E}[D] < \infty$ and a cumulative distribution function $F(\cdot)$. In each review period t, the firm decides on an order quantity q_t (≥ 0). There is no ordering capacity constraint. We assume zero replenishment lead time, which is an assumption widely adopted in the literature (see, e.g., Karaesmen et al. 2011). We also assume unsatisfied demand in each period is lost. Under the zero-lead-time assumption, all the results in §3 and §4 remain true when unsatisfied demand is backlogged. We will consider a backlogging model with positive lead times in §5.

The sequence of events in each review period t is described as follows. First, the firm reviews the system state $\mathbf{x}_t \triangleq (x_{t,1}, x_{t,2}, \ldots, x_{t,m-1})$, where $x_{t,i}$ denotes the inventory level of the items whose remaining lifetimes are at most i periods, $i = 1, 2, \ldots, m-1$. Second, it places an order and raises the total inventory level to $x_{t,m} \triangleq x_{t,m-1} + q_t$ immediately, due to zero lead time. Third, random demand D_t is realized and satisfied by the on-hand inventory to the maximum possible extent through the FIFO issuance policy. The FIFO issuance policy is commonly adopted in the literature (see, e.g., Karaesmen et al. 2011 and Nahmias 2011). The LIFO or a general issuance policy will be considered in §4. Unsatisfied demand is lost, incurring a unit penalty cost p (> 0). Finally, leftover inventory reaching the end of its lifetime perishes and is removed from the system, and the rest inventories are carried to the next period, with their remaining lifetimes reduced by one. Let o_t denote the amount of outdated inventory in period t, i.e., $o_t \triangleq (x_{t,1} - D_t)^+$. Then, the system dynamics from period t to period $t+1$ are given by

$$x_{t+1,i} = (x_{t,i+1} - D_t - o_t)^+, \quad i = 1, 2, \ldots, m-1. \tag{1}$$

For simplicity, we assume the system is initially empty, i.e., $x_{1,1} = x_{1,2} = \cdots = x_{1,m-1} = 0$. Following the tradition in the literature (see, e.g., Nahmias 1975 and Chao et al. 2015, 2018), all leftover inventories incur a unit holding cost h (> 0), and the outdated inventory incurs an additional unit outdating cost θ (≥ 0). The firm's objective is to minimize the long-run average holding, penalty, and outdating cost.

We next define an admissible policy π. Following definitions in Huh et al. (2011) and Bu et al. (2020), we call a policy π *admissible* if, for each review period $t \geq 1$, the order placed under policy π is given by a non-negative measurable function $\varphi_t^\pi(\mathbf{x}_t)$. Thus, an admissible policy π can be written as $\{\varphi_t^\pi : t \geq 1\}$. Let Π denote the set of all admissible policies. Given an admissible policy $\pi \in \Pi$, the total cost incurred in period t is

$$C_t^\pi \triangleq h(x_{t,m}^\pi - D_t)^+ + p(D_t - x_{t,m}^\pi)^+ + \theta o_t^\pi, \tag{2}$$

where $x_{t,m}^\pi$ and o_t^π denote the total inventory level and the amount of outdated inventory in period t under policy π, respectively. The long-run average cost of policy π is defined as $C^\pi \triangleq \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^T \mathbb{E}[C_t^\pi]$. The optimal long-run average cost (or simply, the optimal cost) over all admissible policies is defined as $\mathrm{OPT} \triangleq \inf_{\pi\in\Pi} C^\pi$.

Remark 1. We assume zero ordering cost in this paper. All of our results remain true when there is a positive unit ordering cost. This is because a perishable inventory system (regardless of the issuance policy and order lead times) with positive unit ordering cost $\hat{c}$, unit holding cost $\hat{h}$, unit penalty cost $\hat{p}$, and unit outdating cost $\hat{\theta}$ can be transformed into an equivalent system with cost parameters $c = 0$, $h = \hat{h}$, $\theta = \hat{\theta} + \hat{c}$, and $p = \hat{p} - \hat{c}$ (for the lost-sales system) or $p = \hat{p}$ (for the backlogging system), and the long-run average cost of any admissible policy in the former equals that of the same policy in the latter plus $\hat{c}\mathbb{E}[D]$ (see Chao et al. 2015, 2018).

2.1. Base-stock Policy

An admissible policy is called a base-stock policy with base-stock level S, denoted by π_S, if it places an order in each period t to raise the total inventory level to S. Under policy π_S, the order quantity $q_t^{\pi_S}$ for period t is given by $q_t^{\pi_S} = (S - x_{t,m-1}^{\pi_S})^+, \forall\, t \geq 1$. We denote the long-run average cost under policy π_S by $C(S)$. The following lemma gives an expression of the function $C(S)$.

Lemma 1. *For any $S \geq 0$, there exists a non-negative random variable $O_\infty(S)$, independent of initial system state $\mathbf{x}_1$, such that*

$$C(S) = h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] + \theta\mathbb{E}[O_\infty(S)]. \tag{3}$$

Lemma 1 generalizes a similar result established by Chazan and Gal (1977) and Cooper (2001) for discrete random demands to general random demands. We note that the Markov chain induced by policy π_S can be periodic when $\mathbb{P}(D < S/m) = 1$ and under certain initial states $\mathbf{x}_1$. Thus, it does not always have a steady-state distribution. Nevertheless, Lemma 1 shows that the long-run average cost $C(S)$ can always be expressed by the right-hand-side (RHS) of equation (3), regardless of whether the steady-state distribution exists.

The next lemma provides a lower bound and two upper bounds on $\mathbb{E}[O_\infty(S)]$. These bounds come from Theorem 2 of Chazan and Gal (1977) and Proposition 3 of Cooper (2001), and they will be useful in our analysis.

Lemma 2. *For any $S \geq 0$, the following inequalities hold:*

$$\frac{1}{m}\mathbb{E}\Big[\Big(S-\sum_{i=1}^{m}D_i\Big)^+\Big] \leq \mathbb{E}[O_\infty(S)] \leq \min\left\{\mathbb{E}\Big[\Big(\frac{S}{m}-D\Big)^+\Big], \mathbb{E}\Big[\Big(S-\sum_{i=1}^{m}D_i\Big)^+\Big]\right\}.$$

Define S^* as a minimizer of the function $C(S)$ over $[0,+\infty)$. Then, policy π_{S^*} is the best base-stock policy among the class of base-stock policies. Previous studies have shown that the function $C(S)$ is convex in S on $[0,+\infty)$ (see Theorem 1 of Chazan and Gal 1977 and Theorem 1 of Zhang et al. 2018). However, since the term $\mathbb{E}[O_\infty(S)]$ does not have an explicit expression and needs to be evaluated numerically via simulating the perishable inventory system, computing S^* is time-consuming (see §7 for more discussions on its computational efficiency). Consequently, several heuristic base-stock levels have been proposed in the literature based on different approximations on $\mathbb{E}[O_\infty(S)]$. However, few studies analyze the theoretical performance of base-stock policies compared with the optimal policy, leaving it as an important open problem in the literature.

In this paper, we address this open problem by considering a heuristic base-stock level $\widetilde{S}$. Here, $\widetilde{S}$ is a minimizer of the function $\widetilde{C}(S)$ over $[0,+\infty)$, where

$$\widetilde{C}(S) \triangleq h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] + \frac{\theta}{m}\mathbb{E}\Big[\Big(S-\sum_{i=1}^{m}D_i\Big)^+\Big]. \tag{4}$$

That is, the heuristic base-stock level $\widetilde{S}$ is constructed by approximating $\mathbb{E}[O_\infty(S)]$ with its lower bound in Lemma 2. Clearly, $C(S) \geq \widetilde{C}(S)$ for any $S \geq 0$. One can easily verify that $\widetilde{S}$ has the following lower and upper bounds:

$$F^{-1}\Big(\frac{p}{p+h+\theta}\Big) \leq \widetilde{S} \leq F^{-1}\Big(\frac{p}{p+h}\Big). \tag{5}$$

Since $\widetilde{C}(S)$ has an explicit form and is convex in S, the heuristic base-stock level $\widetilde{S}$ can be computed efficiently by Monte Carlo simulation and golden-section search within these bounds. For convenience, we denote $S^{NP} \triangleq F^{-1}(\frac{p}{p+h})$.

3. Asymptotic Optimality in Four Parameter Regimes

In this section, we analyze the optimality gap of base-stock policy $\pi_{\widetilde{S}}$ in the classic system, defined by $C(\widetilde{S}) - \text{OPT}$, in four parameter regimes. In §3.1, we characterize its rate of convergence to zero in product lifetime and in demand population size. In §3.2 and §3.3, we show the asymptotic optimality of policy $\pi_{\widetilde{S}}$ with large unit penalty costs and large unit outdating costs, respectively. To highlight the dependency of relevant quantities on a specific parameter, whenever necessary, we make the dependency explicit by including the parameter in their subscripts.

3.1. Exponential Decay in Lifetime and in Demand Population Size

We first consider the regime of product lifetime m. The following theorem provides an upper bound on the optimality gap of policy $\pi_{\widetilde{S}}$. The only requirement for Theorem 1 is that $\mathbb{E}[D]>0$, which we have assumed.

THEOREM 1. *For any $\lambda>0$ and $m\geq 1$, the following inequality holds:*

$$C_m(\widetilde{S}_m)-\mathrm{OPT}_m\leq \frac{(m-1)\theta}{\lambda me}e^{\lambda S^{NP}}\cdot(\mathbb{E}[e^{-\lambda D}])^m. \tag{6}$$

Since $\mathbb{E}[e^{-\lambda D}]<1$ for any $\lambda>0$, Theorem 1 shows that the optimality gap of policy $\pi_{\widetilde{S}}$ decays to zero exponentially fast in the lifetime m. We explain this result intuitively as follows. As m increases, the product can stay in the system for longer time, and so the long-run average outdated inventory under the optimal policy decreases and the perishable inventory system behaves closer to its non-perishable counterpart. For the latter system, base-stock policy $\pi_{S^{NP}}$ is known to be optimal (see, e.g., Scarf 1960, Esmaili et al. 2019). Thus, policy $\pi_{S^{NP}}$ is asymptotically optimal with large lifetime in the perishable inventory system. Further, under the FIFO issuance policy, from Lemma 2, $\mathbb{E}[O_\infty(S)]$ is bounded from above by $\mathbb{E}[(S-\sum_{i=1}^m D_i)^+]$, implying that $\mathbb{E}[O_\infty(S)]$ under any base-stock policy π_S and the optimality gap of $\pi_{S^{NP}}$ decay to zero exponentially fast in the lifetime m. The above results also hold for policy $\pi_{\widetilde{S}}$ because, by design, base-stock level $\widetilde{S}$ converges to S^{NP} when $m\to\infty$ and the term $\mathbb{E}[(S-\sum_{i=1}^m D_i)^+]/m$ for constructing $\widetilde{C}(S)$ decays to zero exponentially fast in m for any fixed S.

Next, we consider the regime of demand population size n, defined as follows. For any positive integer n, suppose that the generic one-period demand D takes the form

$$D=\sum_{j=1}^{N(n)}\hat{D}_j, \tag{7}$$

where $\{\hat{D}_j:\ j\geq 1\}$ are *i.i.d.* non-negative r.v.'s with finite mean and variance, and $N(n)\triangleq\sup\{N\in\mathbb{N}:\sum_{k=1}^N X_k/n\leq 1\}$. Here, $\{X_k:k\geq 1\}$ are *i.i.d.* non-negative r.v.'s with mean one and finite variance, and independent of $\{\hat{D}_j:\ j\geq 1\}$. Note that $N(n)$ is the random number of arrivals during one unit of time in a renewal process with inter-arrival times $\{X_k/n:k\geq 1\}$. We refer to n as the demand population size, since $N(n)\equiv n$ when $X_k\equiv 1$, and $\lim_{n\to\infty}\mathbb{E}[N(n)]/n=1$ by Elementary Renewal Theorem when X_k follows a general distribution. The demand form (7) can be interpreted as follows: there are $N(n)$ customers in each period, with customer j independently requesting $\hat{D}_j$ units of the product, so the total demand D in a period is given by (7). For technical reasons, we assume that there exists some constant $s>0$ such that $\mathbb{E}[e^{sX_1}]<\infty$. This is a mild assumption and satisfied by many distributions used in the literature.

Two special cases of demand form (7) are as follows: i) D is the sum of n *i.i.d.* non-negative r.v.'s (by letting $X_k \equiv 1$), and ii) D is a compound Poisson random variable with $\mathbb{E}[N(n)] = n$ (by letting X_k be exponentially distributed). As mentioned in §1.2, Zhang et al. (2020) consider the latter for a finite-horizon perishable inventory system under the discounted-profit criterion.

THEOREM 2. *Suppose the generic demand D takes form* (7). *There exist two positive constants K_1 and K_2 such that $C_n(\widetilde{S}_n) - \mathrm{OPT}_n \le K_1 \cdot e^{-K_2 n}$ when n is sufficiently large.*

We explain Theorem 2 intuitively as follows. Similar to Theorem 1, Theorem 2 holds because under the FIFO issuance policy, the upper bound $\mathbb{E}[(\widetilde{S}_n - \sum_{i=1}^m D_{n,i})^+]$ on the long-run average outdated inventory $\mathbb{E}[O(\widetilde{S}_n)]$ from Lemma 2 decays to zero exponentially fast in the demand population size n. This result holds under the assumption of Theorem 2, since when n is sufficiently large, base-stock level $\widetilde{S}_n$ approximately equals $n\mathbb{E}[\hat{D}_1]$ ($\approx \mathbb{E}[D]$) (since the effect of demand variation is at a lower order), while the term $\mathbb{E}[\sum_{i=1}^m D_{n,i}]$ approximately equals $mn\mathbb{E}[\hat{D}_1]$. When $m \ge 2$ and n is sufficiently large, their difference $(m-1)n\mathbb{E}[\hat{D}_1]$ increases linearly in n, which leads to the exponential decay of $\mathbb{E}[(\widetilde{S}_n - \sum_{i=1}^m D_{n,i})^+]$ to zero in n.

Finally, we shed some insights on the numerical performance of policy $\pi_{\widetilde{S}}$ with different product lifetimes and demand population sizes. To this end, we compute the following upper bound of the *relative* optimality gap of policy $\pi_{\widetilde{S}}$:

$$\frac{C(\widetilde{S}) - \mathrm{OPT}}{\mathrm{OPT}} \le \frac{(m-1)\theta}{m\widetilde{C}(\widetilde{S})}\mathbb{E}\left[\left(\widetilde{S} - \sum_{i=1}^m D_i\right)^+\right] \times 100\%, \tag{8}$$

which follows from Proposition 2 and inequality (21) to be derived in §6. Suppose $D_i = \sum_{j=1}^n \hat{D}_{i,j}$, $i = 1, \dots, m$, where $\{\hat{D}_{i,j} : 1 \le i \le m,\ 1 \le j \le n\}$ are *i.i.d.* r.v.'s. Table 2 reports the values of the RHS of (8) under different m, n and distributions of $\hat{D}_{i,j}$, with $h = 1$, $p = 6$, $\theta = 3$, $\mathbb{E}[\hat{D}_{i,j}] = 5$, and for normal demand, $\mathrm{Var}[\hat{D}_{i,j}] = 1.5^2$. It shows that the upper bound in (8) is very close to zero under Poisson and normal demands, and decays to zero drastically as m or n increases under geometric and exponential demands. It is important to note that the results in Table 2 are upper bounds of the relative optimality gap; the true optimality gaps are smaller (or at least no larger). Table 2 also shows that the marginal improvement in the upper bound decreases in both m and n. For example, for geometric demand with fixed $m = 3$, the marginal improvement is 13.36% (=17.39%-4.03%) when n increases from 1 to 3, and decreases to 3.08% (=4.03%-0.95%) when n increases from 3 to 5. A similar pattern can be observed when n is fixed and m increases. This indicates that some minimal effort in lifetime expansion for perishable goods (e.g., by better preservation of food products) or in boosting its demand (e.g., by means of advertisement or promotion) can have a significant impact on the firm's cost.

Table 2 RHS of (8) under different m, n and demand distributions

Distribution of $\hat{D}_{ij}$	$m=3$			$m=4$			$m=5$		
	$n=1$	$n=3$	$n=5$	$n=1$	$n=3$	$n=5$	$n=1$	$n=3$	$n=5$
Poisson	0.60%	0.00%	0.00%	0.02%	0.00%	0.00%	0.00%	0.00%	0.00%
Normal	0.04%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Geometric	17.39%	4.03%	0.95%	11.32%	0.57%	0.02%	4.74%	0.04%	0.00%
Exponential	13.18%	2.32%	0.32%	6.42%	0.15%	0.00%	2.27%	0.00%	0.00%

3.2. Asymptotic Optimality with Large Unit Penalty Costs

In this subsection, we consider the regime of large unit penalty cost p. In practice, the unit penalty cost p of a perishable product is often much larger than the unit holding cost h. Many firms, especially those in the manufacturing and retailing industries, often require a high service level to reduce stock-out frequencies and guarantee customer satisfaction (see, e.g., Huh et al. 2009 and Wei et al. 2021). This translates to a much larger unit penalty cost than the unit holding cost in our cost-based inventory model.

Our next theorem shows that base-stock policy $\pi_{\widetilde{S}}$ is asymptotically optimal with large unit penalty costs.

Theorem 3. $\lim_{p\to\infty}\left(C_p(\widetilde{S}_p)-\mathrm{OPT}_p\right)=0$.

We explain this result intuitively as follows. First, suppose the generic one-period demand D is bounded (i.e., $\bar{D}\triangleq\sup\{x:F(x)<1\}<\infty$). As the unit penalty cost p becomes large, both the order-up-to level under the optimal policy and base-stock level $\widetilde{S}$ should converge to $\bar{D}$ to achieve zero long-run average lost-sales penalty cost. In this case, policy $\pi_{\widetilde{S}}$ is asymptotically optimal with large p. Now suppose demand D is unbounded. Intuitively, the order-up-to level under the optimal policy has a steady-state distribution and we denote it by random variable $\mathcal{X}_m^*$. Consider base-stock policy with base-stock level $\mathbb{E}[\mathcal{X}_m^*]$. By the conditional Jensen's inequality, we have

$$h\mathbb{E}[(\mathcal{X}_m^*-D)^+]+p\mathbb{E}[(D-\mathcal{X}_m^*)^+]\geq h\mathbb{E}[(\mathbb{E}[\mathcal{X}_m^*]-D)^+]+p\mathbb{E}[(D-\mathbb{E}[\mathcal{X}_m^*])^+].$$

Since D is unbounded, when p is large, the steady-state order-up-to level $\mathcal{X}_m^*$ and its mean $\mathbb{E}[\mathcal{X}_m^*]$ are also large in order to reduce lost sales, and so, inventory outdating should occur with a high probability in each period under both the optimal policy and base-stock policy $\pi_{\mathbb{E}[\mathcal{X}_m^*]}$. Under the FIFO issuance policy, it follows from the system dynamics in (1) that $o_{t+m-1}>0$ implies $\sum_{i=t}^{t+m-1}o_i=x_{t,m}-\sum_{i=t}^{t+m-1}D_i$ for any period t. Thus, when p is large, the difference between the long-run average outdated inventory under the optimal policy (approximately $\mathbb{E}[\mathcal{X}_m^*-\sum_{i=1}^m D_i]/m$) and that under $\pi_{\mathbb{E}[\mathcal{X}_m^*]}$ (approximately $\mathbb{E}[\mathbb{E}[\mathcal{X}_m^*]-\sum_{i=1}^m D_i]/m$) is small. Altogether, the optimality gap of policy $\pi_{\mathbb{E}[\mathcal{X}_m^*]}$ is small when p is large, and it converges to zero as $p\to\infty$. Further, we can prove that the gap between $C(\widetilde{S})$ and $C(\mathbb{E}[\mathcal{X}_m^*])$ converges to zero as $p \to \infty$. As a result, policy $\pi_{\widetilde{S}}$ is asymptotically optimal with large p when D is unbounded.

Huh et al. (2009) prove that, for lost-sales non-perishable inventory systems with positive lead times, the *relative* optimality gap of certain base-stock policies converges to zero as the unit penalty cost grows. Theorem 3 for the classic perishable inventory system is stronger than their result, since it shows that the *absolute* optimality gap of policy $\pi_{\widetilde{S}}$ converges to zero as the unit penalty cost grows. On methodology, the analysis in Huh et al. (2009) is built on the similarity between the lost-sales system and the counterpart backlogging system when the unit penalty cost is large (where the stock-out probabilities in both systems are small) and that the base-stock policy is optimal for the latter. Their analysis based on this analogy does not work in perishable inventory systems because base-stock policies are sub-optimal under both lost sales and backlogging settings. Partly inspired by our above intuitive explanations for Theorem 3, we develop two lower bounds on the optimal cost (see Propositions 2 and 3 in §6.1) and prove Theorem 3 by considering bounded and unbounded demands separately.

In contrast to the results in §3.1 that the optimality gap always converges to zero exponentially fast in the product lifetime and in the demand population size, the convergence rate on the optimality gap of policy $\pi_{\widetilde{S}}$ in the unit penalty cost p differs for different demand distributions. We can characterize it for various classes of demand distributions, and present our results for four classes of continuous distributions, two for unbounded demands and two for bounded demands, in Appendix F for interested readers.

3.3. Asymptotic Optimality with Large Unit Outdating Costs

Finally, we consider the regime of large unit outdating cost θ. In practice, the unit outdating cost θ can be much larger than the unit holding cost h for two major reasons. First, many perishable products are expensive (e.g., those in blood supply chains) and their outdating results in a significant cost for firms and even for the society. For example, Slonim et al. (2014) report that the cost of the components of each unit of blood sold to hospitals in the U.S. is approximately \$570, and hospitals transfuse this blood at estimated costs of between \$522 and \$1,183 per unit in the United States and Europe. Second, it can be very costly to dispose outdated products. For example, supermarkets have to pay their employees overtime wages to dispose outdated vegetables, bakery items, and packaged meats. Also, due to safety concerns, it often requires professional processing before disposing outdated blood products, which can be very costly.

Our next theorem shows that base-stock policy $\pi_{\widetilde{S}}$ is asymptotically optimal with large unit outdating costs.

Theorem 4. $\lim_{\theta\to\infty}\left(C_\theta(\widetilde{S}_\theta) - \mathrm{OPT}_\theta\right) = 0.$

We offer an intuitive explanation on Theorem 4. As the unit outdating cost θ grows large, the optimal policy should asymptotically reduce the long-run average outdated inventory to zero. To this end, the order-up-to level under the optimal policy must be asymptotically no more than $m\underline{D}$ with probability one, where $\underline{D} \triangleq \inf\{x : F(x) > 0\}$. This is because otherwise, under the FIFO issuance policy, the optimal policy would incur a positive long-run average outdated inventory and thus an asymptotically infinite long-run average outdating cost, leading to a contradiction. Define

$$\widetilde{S}_\infty = \underset{0\le S\le m\underline{D}}{\arg\min} \big\{ h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] \big\}. \tag{9}$$

Since the order-up-to level under the optimal policy is no more than $m\underline{D}$, intuitively the optimal cost would be $h\mathbb{E}[(\widetilde{S}_\infty - D)^+] + p\mathbb{E}[(D-\widetilde{S}_\infty)^+]$ asymptotically, because the latter gives the minimum expected single-period holding and penalty cost when restricting the order-up-to level within $[0, m\underline{D}]$. On the other hand, since $\widetilde{S}_\infty \le m\underline{D}$, it follows from Lemma 2 that base-stock policy $\pi_{\widetilde{S}_\infty}$ incurs zero outdating cost and long-run average cost $h\mathbb{E}[(\widetilde{S}_\infty - D)^+] + p\mathbb{E}[(D-\widetilde{S}_\infty)^+]$. Noticing that $\widetilde{S}_\theta$ converges to $\widetilde{S}_\infty$ as $\theta \to \infty$, it follows that base-stock policy $\pi_{\widetilde{S}_\theta}$ is asymptotically optimal with large unit outdating cost θ.

Remark 2. We consider the heuristic base-stock policy $\pi_{\widetilde{S}}$, because it is asymptotically optimal in all four parameter regimes. It also performs very well numerically, when compared with the best base-stock and other heuristic base-stock policies. See §7.1 for details. In Appendix G, we construct a class of heuristic base-stock policies $\{\pi_{\widetilde{S}^{\alpha,\beta}} : \alpha \ge 0, \beta \ge 0\}$ by defining $\widetilde{S}^{\alpha,\beta}$ as a minimizer of the function $\widetilde{C}^{\alpha,\beta}(S)$ over $[0,\infty)$, where

$$\widetilde{C}^{\alpha,\beta}(S) \triangleq h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] + \frac{\alpha}{m}\mathbb{E}\Big[\Big(S - \sum_{i=1}^m D_i\Big)^+\Big] + \beta\mathbb{E}\Big[\Big(\frac{S}{m} - D\Big)^+\Big]. \tag{10}$$

Note that $\widetilde{S} = \widetilde{S}^{1,0}$ and $S^{NP} = \widetilde{S}^{0,0}$. For this class of policies, we identify conditions on parameters (α,β) under which Theorems 1 to 4 hold. For example, for policy $\pi_{S^{NP}}$, Theorems 1 and 2 hold while Theorems 3 and 4 fail (except for a few special cases). In §7.1, we will also show that policy $\pi_{S^{NP}}$ performs much worse than policy $\pi_{\widetilde{S}}$ numerically under geometric demand.

4. LIFO and General Issuance Policies

In §2 and §3, we assume that demands are satisfied by the on-hand inventory through the FIFO issuance policy. In brick-and-mortar retailing, e.g., milk selling, customers can observe the expiration dates of the perishable items, and decide by themselves the consumption sequence. In this context, the LIFO issuance policy, which assumes that the youngest inventories are consumed first, or a general issuance policy, which does not make any assumption on the consumption sequence, might better capture the real-world scenario than the FIFO issuance policy. In this section, we study the asymptotic properties of base-stock policies in the perishable inventory systems under LIFO and general issuance policies in §4.1 and §4.2, respectively.

4.1. LIFO Issuance Policy

Under the LIFO issuance policy, the model formulation remains almost the same as that in §2 except that the system dynamics in (1) need to be modified to

$$x_{t+1,i} = \min\left\{(x_{t,m} - D_t)^+, x_{t,i+1}\right\} - o_t, \quad \forall i = 1, 2, \ldots, m-1,$$

where $o_t = \min\{(x_{t,m} - D_t)^+, x_{t,1}\}$. The definitions of an admissible policy π and its long-run average cost C^π, the optimal cost OPT, a base-stock policy π_S and its long-run average cost $C(S)$, and the best base-stock level S^* all remain the same. To highlight the dependency on the LIFO issuance policy, whenever necessary, we add "LIFO" in the superscripts of relevant quantities.

Recall that Lemma 2 implies three bounds on the long-run average cost of policy π_S in the FIFO system. In the LIFO system, the following lemma provides a lower bound and an upper bound on the long-run average cost of policy π_S.

LEMMA 3. *For any $S \geq 0$, we have $\hat{C}_L(S) \leq C^{LIFO}(S) \leq \hat{C}_U(S)$, where*

$$\hat{C}_L(S) \triangleq h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] + \frac{\theta}{m}\mathbb{E}[(S - \max\{D_1, D_2, \ldots, D_m\})^+]; \tag{11}$$

$$\hat{C}_U(S) \triangleq \left(h + \frac{\theta}{m}\right)\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+]. \tag{12}$$

The lower bound $\hat{C}_L(S)$ is derived from a recursion on the outdating process under the LIFO issuance policy. The upper bound $\hat{C}_U(S)$ is derived from the following simple observation under any issuance policy: $\sum_{j=t}^{t+m-1} o_j^{\pi_S} \leq (S - D_t)^+$ for any period t and demand sample path. Thus, it is also an upper bound on the long-run average cost of policy π_S in the system under any issuance policy. Define $\hat{S}$ as a minimizer of the function $(\hat{C}_L(S) + \hat{C}_U(S))/2$ over $[0, \infty)$. The following theorem presents our asymptotic-optimality results for policy $\pi_{\hat{S}}$ in the LIFO system.

THEOREM 5. *(a) $C^{LIFO}(\hat{S}) - \mathrm{OPT}^{LIFO} \leq 3\theta\mathbb{E}[(S^{NP} - D)^+]/(2m)$;*

(b) If demand D is unbounded, then $\lim_{p\to\infty}(C_p^{LIFO}(\hat{S}_p) - \mathrm{OPT}_p^{LIFO})/\mathrm{OPT}_p^{LIFO} = 0$;

(c) If the demand lower support $\underline{D} = 0$, then $\lim_{\theta\to\infty}(C_\theta^{LIFO}(\hat{S}_\theta) - \mathrm{OPT}_\theta^{LIFO}) = 0$.

Part (a) of Theorem 5 shows that as the lifetime m increases, the optimality gap of policy $\pi_{\hat{S}}$ converges to zero at a rate at least in the order of $1/m$. This result is weaker than the exponential-rate result established in Theorem 1 for the FIFO system. This is because from Lemma 3, the long-run average outdated inventory under policy π_S is bounded from above by $\mathbb{E}[(S-D)^+]/m$ in the LIFO system, which converges to zero in the order of $1/m$. By contrast, from Lemma 2, it is bounded from above by $\mathbb{E}[(S - \sum_{i=1}^m D_i)^+]$ in the FIFO system, which converges to zero exponentially fast in m. It remains unknown whether the optimality gap of policy $\pi_{\hat{S}}$ in the LIFO system converges to zero exponentially fast in m.

Part (b) shows that when demand D is unbounded, policy $\pi_{\hat{S}}$ is asymptotically optimal (in the *relative* sense) in the LIFO system with large unit penalty cost p. This result is weaker than Theorem 3 for the FIFO system (which is in the *absolute* sense). Intuitively, the best base-stock policy may also be asymptotically optimal with large p in the *absolute* sense in the LIFO system. This is because in the LIFO system, $o_{t+m} > 0$ implies $\sum_{i=t}^{t+m-1} o_i = (x_{t,m} - D_t)^+$ for any period t, and thus when p is large, the long-run average outdated inventory under the optimal policy is approximately $\mathbb{E}[(\mathcal{X}_m^* - D)^+]/m$, and that under base-stock policy $\pi_{\mathbb{E}[\mathcal{X}_m^*]}$ is approximately $\mathbb{E}[(\mathbb{E}[\mathcal{X}_m^*] - D)^+]/m$. Thus, the optimality gap of policy $\pi_{\mathbb{E}[\mathcal{X}_m^*]}$ should be small when p is large, and converge to zero as $p \to \infty$. When demand D is bounded, it remains unknown whether policy $\pi_{\hat{S}}$ or the best base-stock policy is asymptotically optimal with large p in the LIFO system. By contrast, Theorem 3 shows the asymptotic optimality of policy $\pi_{\widetilde{S}}$ in the FIFO system. Intuitively, policy $\pi_{\hat{S}}$ or the best base-stock policy may also be asymptotically optimal with large values of p in the LIFO system, since both the order-up-to level under the optimal policy in the LIFO system and base-stock level $\hat{S}$ should converge to $\bar{D}$ to achieve zero long-run average lost-sales penalty as p grows. We leave both unresolved issues for future research.

Finally, part (c) shows that policy $\pi_{\hat{S}}$ is asymptotically optimal with large unit outdating cost θ when $\underline{D} = 0$. Intuitively, as θ goes to infinity, both the order-up-to level under the optimal policy and base-stock level $\hat{S}$ would converge to zero to achieve zero long-run average outdating cost. However, this result is only true when $\underline{D} = 0$ and two other trivial cases when $m = 1$ or $S^{NP} \leq \underline{D}$ where $\pi_{\hat{S}}$ is the optimal policy. Otherwise, the following result shows that the best base-stock policy is *not* asymptotically optimal with large values of θ in the LIFO system.

Proposition 1. *If $m \geq 2$ and $S^{NP} > \underline{D} > 0$, then $\lim_{\theta\to\infty}(C_\theta^{LIFO}(S_\theta^{LIFO,*}) - \mathrm{OPT}_\theta^{LIFO}) > 0$.*

We describe the main ideas for proving Proposition 1 and leave the detailed proof in Appendix C.3. First, we prove that $\lim_{\theta\to\infty} C_\theta^{LIFO}(S_\theta^{LIFO,*}) = p(\mathbb{E}[D] - \underline{D})$. This is intuitive, since to achieve zero outdating under the LIFO issuance policy, one would expect $S_\theta^{LIFO,*}$ to converge to $\underline{D}$ as $\theta \to \infty$. Second, we construct the following admissible policy: order $\min\{S^{NP}, m\underline{D}\}$ units in periods 1, $m+1$, $2m+1$, $\ldots$, and use the base-stock policy with level $\underline{D}$ in all other periods. Under this policy, we verify that there is no inventory outdating, and that the expected holding and penalty cost is strictly less than $p(\mathbb{E}[D] - \underline{D})$ in each of the periods 1, $m+1$, $2m+1$, $\ldots$ when $m \geq 2$ and $S^{NP} > \underline{D} > 0$, and it is at most $p(\mathbb{E}[D] - \underline{D})$ in each of all other periods. As a result, the long-run average cost under this policy is strictly less than $p(\mathbb{E}[D] - \underline{D})$ and independent of θ. Combining the above two results and the definition of OPT^{LIFO} leads to the result in Proposition 1.

Similar to the FIFO system, we can construct a class of base-stock policies $\{\pi_{\hat{S}^{\alpha,\beta}} : \alpha \geq 0, \beta \geq 0\}$ by approximating $C^{LIFO}(S)$ through a non-negative linear combination of its upper and lower bounds in Lemma 3. Then, we can partly extend the results in Theorem 5 to this class of policies as follows: part (a) (after a minor modification) holds for any $\alpha \geq 0$ and $\beta \geq 0$; part (b) holds for any $\alpha + \beta = 1$; and part (c) holds for any $\beta > 0$. We leave the details to interested readers.

Finally, we comment on the performance of base-stock policies with large demand population sizes in the LIFO system. First, our proof of Theorem 2 fails in the LIFO system, because it is built on the inequality $\mathbb{E}[O_\infty(S)] \leq \mathbb{E}[(S - \sum_{i=1}^{m} D_i)^+]$ from Lemma 2 for the FIFO system (also see our discussions after Theorem 2), which cannot be implied from the inequality $C^{LIFO}(S) \leq \hat{C}_U(S)$ in Lemma 3. Second, our numerical results in §7.2 show that for most instances tested under the LIFO system, a larger demand population size worsens the performances of the best base-stock policy and policy $\pi_{\hat{S}}$. So, in the LIFO system, the best base-stock policy may not be asymptotically optimal with large values of n.

4.2. General Issuance Policy

In this subsection, we consider the perishable system under a general issuance policy. In this case, the model formulation again remains the same as that of the FIFO system in §2, except that the state dynamics become more complicated and need to be specified by the given issuance policy. We omit the details here due to page limit. The definitions and notations of the relevant quantities are similar to those in §2, and we impose "GI" in their superscripts to highlight the dependency on a general issuance policy.

The following lemma shows that the long-run average cost of base-stock policy π_S under a general issuance policy is bounded from above by that under the LIFO policy and from below by that under the FIFO policy.

Lemma 4. *For any $S \geq 0$, $C^{FIFO}(S) \leq C^{GI}(S) \leq C^{LIFO}(S)$.*

This result can be explained as follows. First, since policy π_S always maintains the total inventory level at S in each period, its long-run average holding and penalty cost equals $h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+]$, regardless of the inventory issuance policy. Second, for any demand sample path $D_1, \ldots, D_t$, one can verify that the total amount of outdated inventories in periods $1, \ldots, t$ of policy π_S under a general issuance policy is at least that under the FIFO issuance policy and at most that under the LIFO issuance policy. Then, the long-run average outdating cost of policy π_S under a general issuance policy is at least that under the FIFO issuance policy and at most that under the LIFO issuance policy. Thus, the results in Lemma 4 hold.

Lemma 4 implies that policy π_S incurs less long-run average cost under a general issuance policy than under the LIFO issuance policy. As will be seen from Proposition 2 in §6.1, the lower bound $\widetilde{C}(\widetilde{S})$ on OPT^{LIFO} which we establish for proving Theorem 5 is also a lower bound on OPT^{GI}. Consequently, all the results in Theorem 5 established for policy $\pi_{\hat{S}}$ under the LIFO issuance policy also hold under a general issuance policy. We formalize this claim in the following theorem.

Theorem 6. *All the results established for base-stock policy $\pi_{\hat{S}}$ under the LIFO issuance policy in Theorem 5 hold under a general issuance policy.*

5. Backlogging System with Positive Lead Times

So far, we have considered the systems with zero replenishment lead time, where unsatisfied demand can be either lost or backlogged. In this section, we extend most of the results in the earlier sections to a backlogging system with positive lead times of L periods, where L is an arbitrary positive integer.

For the backlogging system with lead times L, the sequence of events in each review period t is described as follows. First, the order placed in period $t-L$ arrives and is used to satisfy backorders, if any, as much as possible. Second, the firm reviews the system state $\mathbf{x}_t \triangleq (x_{t,1}, \ldots, x_{t,m+L-1})$. For $1 \le i \le m$, $x_{t,i}$ denotes the amount of on-hand inventories whose remaining lifetimes are at most i periods *minus* the amount of backorders if the firm orders nothing after period $t+i-m-L$; and for $m+1 \le i \le m+L-1$, $x_{t,i}$ equals $x_{t,i-1}$ plus the amounts of pipeline inventory that will arrive in period $t+i-m$. Third, the firm makes the order decision q_t (≥ 0) and raises the *inventory position* to $x_{t,m+L} \triangleq x_{t,m+L-1} + q_t$. Fourth, random demand D_t is realized and satisfied from the on-hand inventory as much as possible following the FIFO issuance policy. Unsatisfied demand is backlogged, incurring backlogging cost b per unit per period. Same as before, all leftover inventory incurs a unit holding cost h and the outdated inventory incurs an additional unit outdating cost θ. We assume $\theta \ge hL$, which can be mostly satisfied in practice since θ is typically much larger than h and the lead times L for perishable products are typically short. Finally, the system proceeds to the next period, with the system dynamics given by $x_{t+1,i} = x_{t,i+1} - D_t - o_t$, $\forall i = 1, 2, \ldots, m+L-1$, where, same as before, o_t denotes the outdated quantity in period t.

A base-stock policy π_S for this system is defined as the policy that raises the inventory position to a constant S in each review period. That is, $q_t^{\pi_S} = (S - x_{t,m+L-1})^+$ for each period $t \ge 1$. The notations and definitions of different policies and long-run average costs are similar to those in §2, and thus omitted. To highlight the dependency on the lead times L, whenever necessary, we make the dependency explicit by adding "L" in the superscript.

The following lemma provides an upper bound and a lower bound on the long-run average cost of base-stock policies. Note that this lemma also holds when $L = 0$ (i.e., for the backlogging system with zero lead time).

Lemma 5. *For any $L \ge 0$ and $S \ge 0$, the following inequalities hold:*

$$C^L(S) \le h\mathbb{E}\Big[\Big(S - \sum_{i=1}^{L+1} D_i\Big)^+\Big] + b\mathbb{E}\Big[\Big(\sum_{i=1}^{L+1} D_i - S\Big)^+\Big] + (\theta + bL)\mathbb{E}\Big[\Big(S - \sum_{i=1}^{m+L} D_i\Big)^+\Big], \tag{13}$$

$$C^L(S) \ge \widetilde{C}^L(S) \triangleq h\mathbb{E}\Big[\Big(S - \sum_{i=1}^{L+1} D_i\Big)^+\Big] + b\mathbb{E}\Big[\Big(\sum_{i=1}^{L+1} D_i - S\Big)^+\Big] + \frac{\theta - hL}{m+L}\mathbb{E}\Big[\Big(S - \sum_{i=1}^{m+L} D_i\Big)^+\Big]. \tag{14}$$

Since $\theta \geq hL$, the function $\widetilde{C}^L(S)$ is convex in S on $[0,\infty)$. Define $\widetilde{S}^L$ as a minimizer of $\widetilde{C}^L(S)$ over $[0,\infty)$. The following theorem presents our asymptotic-optimality results for policy $\pi_{\widetilde{S}^L}$.

Theorem 7. *For the backlogging system with positive lead times L and under the FIFO issuance policy, the following results hold:*

(a) The optimality gap of policy $\pi_{\widetilde{S}^L}$ converges to zero exponentially fast in the lifetime m;

(b) Under the assumption in Theorem 2, the optimality gap of policy $\pi_{\widetilde{S}^L}$ converges to zero exponentially fast in the demand population size n;

(c) The optimality gap of policy $\pi_{\widetilde{S}^L}$ converges to zero as the unit outdating cost θ goes to infinity.

Parts (a) and (b) of Theorem 7 can be explained similarly to Theorems 1 and 2 for the system with zero lead time, because the best base-stock policy is optimal for the backlogging non-perishable inventory system with positive lead times (see, e.g., Karlin and Scarf 1958). Part (c) can also be explained similarly to that of Theorem 4, except that the quantity $\widetilde{S}_\infty$ defined there needs to be generalized to

$$\widetilde{S}_\infty^L \triangleq \underset{0\leq S\leq (m+L)\underline{D}}{\arg\min} \left\{ h\mathbb{E}\Big[\Big(S-\sum_{i=1}^{L+1} D_i\Big)^+\Big] + b\mathbb{E}\Big[\Big(\sum_{i=1}^{L+1} D_i - S\Big)^+\Big]\right\}.$$

Theorem 7 does not address the asymptotic performance of policy $\pi_{\widetilde{S}^L}$ or the best base-stock policy with large unit backlogging cost b. When the lead time is positive, our intuitions for Theorem 3 become invalid. In this case, a base-stock policy maintains a constant inventory position, which is different from maintaining a constant total on-hand inventory level. When the unit backlogging cost b is large, this difference is amplified because the best base-stock policy maintains a high inventory position to reduce backorders and results in large outdated inventories. Thus, the best base-stock policy is likely *not* asymptotically optimal with large b for the backlogging system with positive lead times. In §7.3, we will show numerically that both policy $\pi_{\widetilde{S}^L}$ and the best base-stock policy perform worse when the unit backlogging cost b or the lead times L increases.

Similar to the system with zero lead time, we can extend most of the results in Theorem 7 to a class of base-stock policies by approximating $C^L(S)$ through a non-negative linear combination of its upper and lower bounds in Lemma 5. Besides, we can extend the results in Lemma 3 and parts (a) and (c) of Theorem 5 to a backlogging system with positive lead times under the LIFO or a general issuance policy. We leave the details to interested readers.

6. Sketched Proofs of Main Results

In this section, we provide sketched proofs of Theorems 1 to 7 in §3 to §5. The common strategy for proving each of these results is to construct a tight upper bound on the long-run average cost of a certain base-stock policy and a tight lower bound on the optimal cost, and then prove that their absolute or relative gap approaches zero as a certain system parameter goes to infinity. In Lemmas 2 to 5, we have constructed various upper bounds on the long-run average costs of base-stock policies in different systems. In this section, we first construct three important lower bounds on the optimal costs in §6.1 and then sketch the proofs of Theorems 1 to 7 in §6.2.

6.1. Three Lower Bounds on the Optimal Cost

First, we present a lower bound on the optimal cost for the lost-sales system with zero lead time. This bound will be used extensively in the proofs of Theorems 1 to 6.

PROPOSITION 2. *Under any issuance policy, the optimal cost for the lost-sales system with zero lead time is bounded from below by $\widetilde{C}(\widetilde{S})$. That is,* $\mathrm{OPT} \geq \widetilde{C}(\widetilde{S})$ *under any issuance policy.*

We prove Proposition 2 below by a simple sample-path approach. For any admissible policy $\pi \in \Pi$ and under any issuance policy, the following inequality

$$\sum_{i=t}^{t+m-1} o_i^{\pi} \geq \Big(x_{t,m}^{\pi} - \sum_{i=t}^{t+m-1} D_i \Big)^+ \tag{15}$$

holds for any period $t \geq 1$ under any demand sample path. To see this, note that the total inventory $x_{t,m}^{\pi}$ in period t, regardless of the issuance policy, either satisfies demands or outdates in periods t to $t+m-1$. Since it can satisfy at most $\min\{x_{t,m}^{\pi}, \sum_{i=t}^{t+m-1} D_i\}$ units of demands in those periods, the cumulative outdated quantity in periods t to $t+m-1$ is at least $(x_{t,m}^{\pi} - \sum_{i=t}^{t+m-1} D_i)^+$. For any $T \geq 1$, it follows from inequality (15) that

$$\sum_{t=1}^{T+m-1} o_t^{\pi} \geq \frac{1}{m} \sum_{t=1}^{T} \sum_{i=t}^{t+m-1} o_i^{\pi} \geq \frac{1}{m} \sum_{t=1}^{T} (x_{t,m}^{\pi} - \sum_{i=t}^{t+m-1} D_i)^+. \tag{16}$$

From the definition of C_t^{π} in (2), we have

$$\begin{aligned} \sum_{t=1}^{T+m-1} \mathbb{E}[C_t^{\pi}] &\geq \sum_{t=1}^{T} \mathbb{E}\big[h(\mathbb{E}[x_{t,m}^{\pi}] - D_t)^+ + p(D_t - \mathbb{E}[x_{t,m}^{\pi}])^+ + \frac{\theta}{m}(\mathbb{E}[x_{t,m}^{\pi}] - \sum_{i=t}^{t+m-1} D_i)^+\big] \\ &\geq T\widetilde{C}(\widetilde{S}), \end{aligned} \tag{17}$$

where the first inequality follows from (16), the independence between $x_{t,m}^{\pi}$ and $(D_t, \ldots, D_{t+m-1})$ for each period t and the conditional Jensen's inequality, and the second inequality follows from the definition of $\widetilde{C}(\cdot)$ and $\widetilde{S}$. Then, it follows from the definitions of C^{π} and inequality (17) that

$$C^{\pi} = \limsup_{T \to \infty} \frac{1}{T+m-1} \sum_{t=1}^{T+m-1} \mathbb{E}[C_t^{\pi}] \geq \widetilde{C}(\widetilde{S}).$$

Since the above inequality holds for any admissible policy $\pi \in \Pi$ and under any issuance policy, the proof of Proposition 2 is complete.

Next, we present a lower bound on the optimal cost for the lost-sales system with zero lead time under bounded demand and FIFO issuance policy. This bound is crucial to the proof of Theorem 3 for the case of bounded demand.

PROPOSITION 3. *Suppose that demand D is bounded. Then,*

$$\mathrm{OPT}^{FIFO} \geq h\mathbb{E}[(\underline{S}-D)^+] + \theta\mathbb{E}[O_\infty(\underline{S})], \tag{18}$$

where $\underline{S} \triangleq \arg\min_{S\geq 0}\{h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] + \theta S\}$.

The proof of Proposition 3 is based on the following lemma, which characterizes a uniform lower bound on the order-up-to level under the optimal policy.

LEMMA 6. *Suppose that demand D is bounded. Then, there exists an optimal policy for the lost-sales system with zero lead time and under the FIFO issuance policy, denoted by $\pi^{FIFO,*}$, such that $x_{t,m}^{\pi^{FIFO,*}} \geq \underline{S}$ for any period $t \geq 1$ and under any demand sample path.*

We describe the main ideas of proving Lemma 6 below. Proposition 3 of Nandakumar and Morton (1993) shows that the order-up-to level under the optimal policy for the counterpart system under the discounted-cost criterion with discount factor $\alpha \in (0,1)$ is uniformly bounded from below by

$$\underline{S}^\alpha \triangleq \arg\min_{S\geq 0}\{h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] + \alpha^{m-1}\theta S\}.$$

Since $\underline{S}^\alpha \geq \underline{S}$ for all $\alpha \in (0,1)$, Lemma 6 holds under the discounted-cost criterion. To prove that it also holds under the average-cost criterion, we apply a *vanishing discount factor* approach from Schäl (1993). Specifically, by verifying all the conditions in Theorem 3.8 of Schäl (1993), we prove that when demand D is bounded, there exist an optimal policy for the average-cost system and a sequence of discount factors approaching one such that the discounted optimal policy converges to the average optimal policy when the discount factor in that sequence approaches one. Through this, we complete the proof of Lemma 6.

We now prove Proposition 3. From the system dynamics in (1), we have the following recursion for any admissible policy π under any demand sample path:

$$o_{t+m-1}^\pi = \left(x_{t,m}^\pi - \sum_{i=t}^{t+m-1} D_i - \sum_{j=t}^{t+m-2} o_i^\pi\right)^+, \quad \forall t \geq 1. \tag{19}$$

By applying (19) and Lemma 6, one can easily prove by induction on T the following inequality under any demand sample path:

$$\sum_{t=1}^T o_t^{\pi^{FIFO,*}} \geq \sum_{t=1}^T o_t^{\pi_{\underline{S}}}, \quad \forall\ T \geq 1. \tag{20}$$

Then, it follows from Lemma 1, Lemma 6, and (20) that

$$\mathrm{OPT}^{FIFO} \geq \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^T (h\mathbb{E}[(x_{t,m}^{\pi^{FIFO,*}} - D)^+] + \theta\mathbb{E}[o_t^{\pi^{FIFO,*}}]) \geq h\mathbb{E}[(\underline{S}-D)^+] + \theta\mathbb{E}[O_\infty(\underline{S})].$$

This completes the proof of Proposition 3.

Finally, we present a lower bound on the optimal cost for the backlogging system with positive lead times. This bound is crucial to the proof of Theorem 7.

Proposition 4. *Under any issuance policy, the optimal cost for the backlogging system with positive lead times L is bounded from below by $\widetilde{C}^L(\widetilde{S}^L)$. That is,* $\mathrm{OPT}^L \geq \widetilde{C}^L(\widetilde{S}^L)$ *for any* $L \geq 1$ *under any issuance policy.*

Similar to Proposition 2, we prove Proposition 4 by a simple sample-path approach. Generally speaking, we can establish a similar inequality to (16) through a similar inequality to (15) after incorporating the effect of lead times L, and then apply similar arguments to establish a similar inequality to (17). For brevity, we leave the detailed proof to Appendix E.2.

6.2. Sketched Proofs of Theorems 1 to 7

In this subsection, we sketch the proofs of Theorems 1 to 7. For brevity, we will focus on the key upper bounds on the optimality gaps of our simple base-stock policies, which are obtained by applying the lower bounds on the optimal costs established in §6.1 and various upper bounds on the long-run average costs of base-stock policies established in §2.1, §4 and §5. The missing details can be found in the appendices.

Sketched Proofs of Theorems 1 to 4. First, the proofs of Theorems 1, 2 and 4 are all based on the following upper bound on the optimality gap of policy $\pi_{\widetilde{S}}$:

$$C(\widetilde{S}) - \mathrm{OPT} \leq \frac{(m-1)\theta}{m} \mathbb{E}\Big[\Big(\widetilde{S} - \sum_{i=1}^{m} D_i\Big)^+\Big], \tag{21}$$

which follows from Lemmas 1 and 2, and Proposition 2.

Since $x^+ \leq e^{x-1}$ for any $x \in \mathbb{R}$ (see, e.g., Lemma 1 in Kingman 1962), $\{D_i : 1 \leq i \leq m\}$ are *i.i.d.* r.v.'s, and $\widetilde{S} \leq S^{NP}$ from inequality (5), it follows from (21) that for any $\lambda > 0$,

$$C(\widetilde{S}) - \mathrm{OPT} \leq \frac{(m-1)\theta}{\lambda m e} \cdot \mathbb{E}\big[e^{\lambda(\widetilde{S} - \sum_{i=1}^{m} D_i)}\big] \leq \frac{(m-1)\theta}{\lambda m e} \cdot e^{\lambda S^{NP}} (\mathbb{E}[e^{-\lambda D}])^m. \tag{22}$$

This completes the proof of Theorem 1.

To prove Theorem 2, after plugging $D = \sum_{j=1}^{N(n)} \hat{D}_j$ into the RHS of (22), we obtain

$$C(\widetilde{S}) - \mathrm{OPT} \leq \frac{(m-1)\theta}{\lambda m e} \big(e^{\frac{1}{m}\lambda S_n^{NP}} \mathbb{E}[e^{-\lambda \sum_{j=1}^{N(n)} \hat{D}_j}]\big)^m, \quad \forall \lambda > 0. \tag{23}$$

To proceed, we construct an upper bound on S_n^{NP} by applying a classic result from Scarf (1958) for a robust newsvendor problem, and an upper bound on the $\mathbb{E}[e^{-\lambda \sum_{j=1}^{N(n)} \hat{D}_j}]$ by employing Bernstein's inequality (see, e.g., Theorem 1.13 in Rigollet and Hütter 2015). We leave the rest of the proof to Appendix B.1.

Based on inequality (21), Theorem 4 is implied by the following identity:

$$\lim_{\theta \to \infty} \theta \mathbb{E}\Big[\Big(\widetilde{S}_\theta - \sum_{i=1}^{m} D_i\Big)^+\Big] = 0. \tag{24}$$

Recall that $\widetilde{S}_\theta$ is a minimizer of the function $\widetilde{C}_\theta(S)$ over $[0,\infty)$, defined in (4). We leave the proof of identity (24) to Appendix B.3.

Second, the proof of Theorem 3 is based on the following two upper bounds on the optimality gap of policy $\pi_{\widetilde{S}}$:

$$C(\widetilde{S}) - \text{OPT} \le \theta\Big(\mathbb{E}\Big[\Big(D - \frac{\widetilde{S}}{m}\Big)^+\Big] - \mathbb{E}\Big[\Big(\frac{1}{m}\sum_{i=1}^m D_i - \frac{\widetilde{S}}{m}\Big)^+\Big]\Big), \tag{25}$$

and when $\bar{D} = \sup\{x : F(x) < 1\} < \infty$,

$$C(\widetilde{S}) - \text{OPT} \le (h+\theta)(\bar{D} - \underline{S}). \tag{26}$$

Inequality (25) follows from Lemma 2 and Proposition 2, and inequality (26) follows from Proposition 3. Theorem 3 follows directly from these bounds by considering the cases $\bar{D} = \infty$ and $\bar{D} < \infty$ separately. We leave the proofs of inequalities (25) and (26) to Appendix B.2.

Sketched Proofs of Theorems 5 and 6. The proof of Theorem 5 is built on inequalities $C^{LIFO}(S) \le \hat{C}_U(S)$ and $\text{OPT}^{LIFO} \ge \widetilde{C}(\widetilde{S})$ from Lemma 3 and Proposition 2. First, they imply the following upper bound on the optimality gap of policy $\pi_{\hat{S}}$:

$$\begin{aligned} C^{LIFO}(\hat{S}) - \text{OPT}^{LIFO} \le & \frac{\theta}{2m}\Big(\mathbb{E}[(\hat{S}-D)^+] + \mathbb{E}[(\widetilde{S}-D)^+] - \mathbb{E}[(\hat{S} - \max\{D_1,\dots,D_m\})^+] \\ & + \mathbb{E}[(\widetilde{S} - \max\{D_1,\dots,D_m\})^+] - 2\mathbb{E}[(\widetilde{S} - \sum_{i=1}^m D_i)^+]\Big), \end{aligned} \tag{27}$$

which follows from $\hat{C}_L(\hat{S}) + \hat{C}_U(\hat{S}) \le \hat{C}_L(\widetilde{S}) + \hat{C}_U(\widetilde{S})$ by the definition of $\hat{S}$, and the definitions of $\hat{C}_L(\cdot)$, $\hat{C}_U(\cdot)$ and $\widetilde{C}(\cdot)$. Parts (a) and (b) follow from inequality (27). Second, they, together with a similar proof to Theorem 4, imply part (c). We leave the detailed proof to Appendix C.2. From Lemmas 3 and 4 and Proposition 2, inequalities $C^{GI}(S) \le \hat{C}_U(S)$ and $\text{OPT}^{GI} \ge \widetilde{C}(\widetilde{S})$ hold under a general issuance policy. Therefore, Theorem 6 holds.

Sketched Proof of Theorem 7. The proofs of parts (a) to (c) in Theorem 7 are based on the following upper bound on the optimality gap of policy $\pi_{\widetilde{S}^L}$:

$$C^L(\widetilde{S}^L) - \text{OPT}^L \le \frac{(m-1)\theta + (b(m+L)+h+\theta)L}{m+L} \cdot \mathbb{E}\Big[\Big(\widetilde{S}^L - \sum_{i=1}^{m+L} D_i\Big)^+\Big], \tag{28}$$

which follows from Lemma 5 and Proposition 4. The rest of the proofs of parts (a) to (c) follow similar arguments to those for Theorems 1, 2 and 4, respectively. We omit the details for brevity.

7. Numerical Study

In this section, we conduct a numerical study to test the performances of our base-stock policies under three perishable inventory systems. In §7.1 and §7.2, we consider lost-sales systems with zero lead time under the FIFO and LIFO issuance policies, respectively. In §7.3, we consider the backlogging system with positive lead times. The performance of base-stock policy π_S is measured by its *relative* optimality gap, defined as $\Delta(S) \triangleq (C(S) - \text{OPT})/\text{OPT} \times 100\%$.

In our study, the optimal policy is computed by a relative value iteration method (see, e.g., Bertsekas et al. 2000), and the best base-stock policy is computed by simulating the inventory system under different integer base-stock levels. Our simple base-stock levels proposed in §2 to §5, and other heuristic base-stock levels tested are solved by minimizing some convex functions with closed-form expressions, and their long-run average costs are evaluated by simulating the inventory systems. Over all the instances tested in this section, the average computation times for the long-run average costs of the optimal policy, the best base-stock policy, and heuristic base-stock policies are around 2.7 hours, 1.6 hours, and 4.2 minutes, respectively. All computations were done using Matlab R2020a on a laptop with an Intel Core i7-8705G, 3.10GHz CPU.

7.1. Lost-sales System under FIFO Issuance Policy

In this subsection, we investigate the performances of four base-stock policies in the classic FIFO system: π_{S^*}, $\pi_{\widetilde{S}}$, $\pi_{S^{CCG}}$ and $\pi_{S^{NP}}$. The superscript "*CCG*" in $\pi_{S^{CCG}}$ stands for Cooper-Chazan-Gal, as base-stock level S^{CCG} was proposed in Cooper (2001) by approximating $\mathbb{E}[O_\infty(S)]$ with the mid-point of its upper and lower bounds provided in Chazan and Gal (1977) (see Lemma 2). We employ $\pi_{S^{CCG}}$ as a benchmark, since previous studies show that it performs very well in the classic perishable inventory system (see, e.g., Nahmias 2011). To examine the effect of unit ordering cost, we also incorporate it into our system and denote the unit ordering, holding, penalty, and outdating costs by $\hat{c}$, $\hat{h}$, $\hat{p}$, and $\hat{\theta}$, respectively. For fixed $m = 3$ and $\hat{h} = 1$, Table 3 reports the optimal cost and the relative optimality gaps of the four base-stock policies under different $(\hat{c}, \hat{p}, \hat{\theta})$ combinations, and Poisson and geometric distributions with mean 5.

We make the following observations from Table 3. First, policies π_{S^*}, $\pi_{\widetilde{S}}$ and $\pi_{S^{CCG}}$ perform consistently close to optimal under Poisson demand, and slightly worse but still very well under geometric demand. By contrast, policy $\pi_{S^{NP}}$ performs well only under Poisson demand, but much worse than the other three policies under geometric demand. This is expected since $\pi_{S^{NP}}$ ignores product perishability and its consequence becomes more serious under more variable geometric demand. Second, policy $\pi_{S^{CCG}}$ performs consistently very close to the best base-stock policy π_{S^*}, and $\pi_{\widetilde{S}}$ performs slightly worse than $\pi_{S^{CCG}}$ in several instances. This indicates that approximating $\mathbb{E}[O_\infty(S)]$ by the mid-point of its upper and lower bounds is numerically more effective than by its lower bound. Third, as $\hat{\theta}$ increases, π_{S^*}, $\pi_{\widetilde{S}}$ and $\pi_{S^{CCG}}$ perform worse when $\hat{\theta}$ is moderate while they perform better when $\hat{\theta}$ is very small or very large. This is expected, because all the three policies are optimal when $\hat{\theta} = 0$ and asymptotically optimal when $\hat{\theta}$ is large (see part (d) of Proposition EC.2).

Table 3 Performances of base-stock policies in the lost-sales system under FIFO issuance policy

	$\hat{c}=0$					$\hat{c}=5$				
$(\hat{p},\hat{\theta})$	OPT	$\Delta(S^{*})$	$\Delta(\widetilde{S})$	$\Delta(S^{CCG})$	$\Delta(S^{NP})$	OPT	$\Delta(S^{*})$	$\Delta(\widetilde{S})$	$\Delta(S^{CCG})$	$\Delta(S^{NP})$
					Poisson Demand					
$(8,3)$	4.16	0.08%	0.08%	0.08%	0.08%	28.01	0.00%	0.00%	0.00%	0.00%
$(8,6)$	4.23	0.06%	0.06%	0.06%	0.06%	28.02	0.00%	0.00%	0.00%	0.00%
$(8,8)$	4.28	0.03%	0.03%	0.03%	0.03%	28.03	0.00%	0.00%	0.00%	0.00%
$(20,8)$	5.50	0.28%	0.28%	0.28%	0.28%	30.26	0.00%	0.00%	0.00%	0.75%
$(40,8)$	6.56	0.48%	0.97%	0.48%	0.48%	31.57	0.01%	0.01%	0.01%	1.26%
					Geometric Demand					
$(8,3)$	14.24	0.75%	0.75%	0.75%	6.00%	34.14	0.05%	0.05%	0.05%	2.62%
$(8,6)$	15.81	1.09%	1.21%	1.21%	14.84%	34.48	0.20%	0.28%	0.20%	4.30%
$(8,8)$	16.75	0.40%	2.11%	0.40%	21.08%	34.66	0.14%	0.54%	0.14%	5.53%
$(20,8)$	26.58	1.37%	1.47%	1.47%	17.82%	51.34	0.55%	1.11%	0.55%	19.25%
$(40,8)$	35.55	1.88%	1.92%	1.92%	19.35%	64.87	0.97%	1.05%	1.05%	19.79%

Finally, as $\hat{c}$ increases, the four base-stock policies in general perform better except for $\pi_{S^{NP}}$ under several instances. This is because any admissible policy incurs the same long-run average ordering cost $\hat{c}\mathbb{E}[D]$, and as $\hat{c}$ increases, this policy-independent cost constitutes a larger proportion in the overall long-run average costs of the optimal policy and base-stock policies, leading to improved performances.

Table 3 also shows that as $\hat{p}$ increases from 8 to 40, all three policies π_{S^*}, $\pi_{\widetilde{S}}$ and $\pi_{S^{CCG}}$ perform worse in most of the instances we tested. Considering that they are asymptotically optimal with large $\hat{p}$ (see Theorem 3 and part (c) of Proposition EC.2), we further investigate how their performances depend on a wider range of values of $\hat{p}$. In Figure EC.1 of Appendix H, for fixed $m=3$, $\hat{h}=1$, $\hat{c}=0$ and $\hat{\theta}=8$, we plot the relative optimality gaps of π_{S^*}, $\pi_{\widetilde{S}}$ and $\pi_{S^{CCG}}$, where $\hat{p}$ ranges from 8 to 150 (or the service level $\hat{p}/(\hat{p}+\hat{h})\times 100\%$ ranging from 88.89% to 99.34%), with mean demand 5. From that figure, the relative optimality gaps are non-monotone in $\hat{p}$, but they tend to increase in $\hat{p}$ when $\hat{p}$ is small while decrease in $\hat{p}$ when $\hat{p}$ is large. This is expected, because all three base-stock policies are optimal when $\hat{p}=0$ and asymptotically optimal when $\hat{p}$ is sufficiently large. That figure also shows that when $\hat{p}\geq 120$ (or when the service level exceeds 99.17%), all the three policies are almost optimal under Poisson demand, and close to optimal with the relative optimality gap less than 1% under geometric demand.

7.2. Lost-sales System under LIFO Issuance Policy

In this subsection, we investigate the performances of three base-stock policies in the LIFO system: π_{S^*}, $\pi_{\hat{S}}$, and $\pi_{S^{CCG}}$. To examine the effect of the demand population size n, we consider the generic one-period demand $D=\sum_{j=1}^{n}\hat{D}_j$, where each $\hat{D}_j$ follows *i.i.d.* Poisson or geometric distribution with mean 5. Table 4 reports the results for $m=3$ and $h=1$ under different (p,θ,n) combinations and demand distributions.

Table 4 Performances of base-stock policies in the lost-sales system under LIFO issuance policy

	$n=1$				$n=2$				$n=3$			
(p,θ)	OPT	$\Delta(S^*)$	$\Delta(\hat{S})$	$\Delta(S^{CCG})$	OPT	$\Delta(S^*)$	$\Delta(\hat{S})$	$\Delta(S^{CCG})$	OPT	$\Delta(S^*)$	$\Delta(\hat{S})$	$\Delta(S^{CCG})$
					Poisson Component Demand							
(8,3)	5.44	1.51%	1.51%	13.08%	7.58	2.06%	2.06%	10.07%	9.20	2.22%	3.45%	11.35%
(8,6)	6.24	6.35%	8.00%	31.49%	8.68	4.52%	4.52%	26.68%	10.56	5.51%	5.51%	28.50%
(8,8)	6.59	6.69%	6.69%	45.27%	9.21	6.83%	6.83%	38.65%	11.24	6.31%	6.31%	40.47%
(20,8)	10.08	5.43%	9.46%	30.65%	13.95	5.07%	7.64%	20.09%	16.89	4.74%	4.74%	33.91%
(40,8)	12.98	3.84%	3.84%	9.75%	17.84	5.08%	5.44%	29.77%	21.60	4.52%	4.93%	22.90%
					Geometric Component Demand							
(8,3)	15.58	1.75%	1.75%	3.65%	21.15	2.32%	2.32%	5.28%	25.30	2.50%	2.55%	7.80%
(8,6)	17.67	2.92%	2.92%	7.36%	23.94	4.82%	4.82%	12.24%	28.64	5.37%	5.37%	13.26%
(8,8)	18.68	3.85%	3.85%	6.83%	25.30	6.11%	6.11%	13.84%	30.23	7.04%	7.04%	21.07%
(20,8)	31.19	3.58%	3.58%	7.43%	40.94	5.55%	5.55%	12.05%	48.20	6.15%	6.15%	18.04%
(40,8)	42.84	3.34%	3.34%	5.80%	55.08	4.91%	4.91%	10.03%	64.25	5.23%	5.54%	12.96%

The first observation is that, compared with the results in Table 3, the best base-stock policy performs worse in the LIFO system than in the FIFO system. This is likely because the best base-stock policy in the LIFO system results in more outdated units than those in the FIFO system. Second, $\pi_{\hat{S}}$ performs very close to π_{S^*} in most cases, demonstrating its effectiveness as a simple heuristic. By contrast, $\pi_{S^{CCG}}$ performs much worse than π_{S^*} and $\pi_{\hat{S}}$, albeit its near-optimal performances in the FIFO system. This indicates that the LIFO system is different from the FIFO system, and an effective heuristic policy in the FIFO system can perform very poorly in the LIFO system. Third, when p increases, π_{S^*} and $\pi_{\hat{S}}$ perform better in most of the instances we have tested. This observation is consistent with our theoretical result in Theorem 5(b). Fourth, as demand population size n increases, π_{S^*} and $\pi_{\hat{S}}$ perform worse in most of the instances, especially under geometric demand. This is in sharp contrast with our findings for the FIFO system (see Theorem 2 and Table 2). This shows that the effect of the demand population size on the performances of base-stock policies in the LIFO system is complex and we leave its investigation for future research.

Table 4 also reveals that policies π_{S^*} and $\pi_{\hat{S}}$ perform very well when θ is small (e.g., $\theta=3$), but perform worse when θ increases. Considering that both policies are asymptotically optimal with large values of θ under Poisson and geometric demands (see Theorem 5(c)), we further explore how their performances depend on a wider range of values of θ. In Figure EC.2 of Appendix H, for fixed $m=3$, $n=1$, $h=1$ and $p=8$, we plot the relative optimality gaps of π_{S^*} and $\pi_{\hat{S}}$, where θ ranges from 3 to 400, with mean demand 5. From that figure, we observe that a very large value of θ is required to ensure near-optimal performances of policies π_{S^*} and $\pi_{\hat{S}}$ under both Poisson and geometric distributions. In particular, under Poisson demand, the relative optimality gaps for both policies are over 50% when θ is as large as 400. This, together with Proposition 1, indicates that base-stock policies in general do not perform very well when θ is reasonably large.

7.3. Backlogging System with Positive Lead Times

In this subsection, we investigate the performances of $\pi_{SL,*}$ and $\pi_{\widetilde{S}L}$ in the backlogging system with positive lead times L. For fixed $h=1$, Table 5 reports the results under different (b,θ,m,L) combinations, and Poisson and geometric demands with mean 5. Due to the computational intractability of the optimal cost, we only test the scenarios where $m+L\le 3$.

Table 5 Performances of base-stock policies in the backlogging system with positive lead times

	$(m,L)=(1,1)$			$(m,L)=(2,1)$			$(m,L)=(1,2)$		
(b,θ)	OPT	$\Delta(S^{L,*})$	$\Delta(\widetilde{S}^L)$	OPT	$\Delta(S^{L,*})$	$\Delta(\widetilde{S}^L)$	OPT	$\Delta(S^{L,*})$	$\Delta(\widetilde{S}^L)$
				Poisson Demand					
$(8,3)$	11.19	6.51%	6.51%	6.75	2.12%	2.12%	11.56	11.58%	14.57%
$(8,6)$	15.47	6.70%	7.04%	7.85	1.69%	1.69%	16.26	11.02%	11.02%
$(8,8)$	17.81	5.01%	5.01%	8.46	2.98%	2.98%	18.73	10.75%	10.75%
$(20,8)$	26.19	6.95%	6.95%	11.69	3.01%	3.01%	26.58	12.89%	18.35%
$(40,8)$	32.89	7.32%	11.01%	14.25	4.26%	4.26%	33.30	12.77%	21.55%
				Geometric Demand					
$(8,3)$	29.50	7.15%	7.20%	22.19	4.88%	4.88%	31.49	12.52%	13.05%
$(8,6)$	38.33	5.58%	5.58%	27.24	4.73%	4.92%	42.15	10.34%	10.34%
$(8,8)$	42.49	4.73%	5.20%	29.89	4.51%	4.87%	47.48	9.15%	9.32%
$(20,8)$	69.93	7.35%	7.35%	46.14	6.24%	6.24%	74.04	13.09%	13.83%
$(40,8)$	94.57	8.96%	10.33%	60.09	7.06%	7.81%	97.41	15.33%	21.43%

First, for a fixed lead time $L=1$, both $\pi_{SL,*}$ and $\pi_{\widetilde{S}L}$ perform significantly better as the lifetime m increases from one to two, which is consistent with our theoretical result in Theorem 7(a). Second, for fixed lifetime $m=1$, both policies perform significantly worse as the lead time L increases from one to two, and the relative optimality gaps when $L=2$ are over 9% in all problem instances. This observation is expected, since it is harder for base-stock policies (which maintain a constant inventory position) to control the total on-hand inventory level under a larger L. Third, when θ increases, both policies perform better in general, which is in alignment with our theoretical result in Theorem 7(c). Finally, both $\pi_{SL,*}$ and $\pi_{\widetilde{S}L}$ perform significantly worse as the unit backlogging cost b increases. This observation matches our intuition that base-stock policies are not asymptotically optimal with large unit backlogging costs as explained in §5.

8. Conclusion

In this paper, we conduct an in-depth study on the theoretical performances of base-stock policies for perishable inventory systems over an infinite planning horizon. For the classic system with zero lead time and the FIFO issuance policy, we construct a simple base-stock policy and show that its optimality gap decays to zero exponentially fast in the lifetime and in demand population size, and it converges to zero as the unit penalty or outdating cost goes to infinity. We further extend most of these results to a system under the LIFO or a general issuance policy and a backlogging system with positive lead times. We conduct a numerical study to investigate the performances of base-stock policies in these systems, and offer observations and insights based on our numerical results. This paper contributes to the literature by providing theoretical justifications for the near-optimal numerical performances of base-stock policies in the classic perishable inventory system reported in the literature, and establishing the first theoretical results of base-stock policies for the other two perishable inventory systems.

We conclude this paper by mentioning several directions for future research. First, it is worthwhile to further study the perishable inventory system under the LIFO or a general issuance policy considered in §4. For this system, among others, it remains unknown whether the best base-stock policy is asymptotically optimal with large unit penalty costs under bounded demand. Moreover, §7.2 shows that base-stock policies do not always perform well in the LIFO system. Thus, it is desirable to design more effective heuristic replenishment policies for this system. Second, it is important to develop effective heuristic policies for backlogging and lost-sales perishable inventory systems with positive lead times. For the backlogging system, §7.3 shows that base-stock policies perform poorly with large unit backlogging costs or lead times; hence more effective heuristic replenishment policies need to be developed for both asymptotic regimes. Third, it is valuable to develop effective and asymptotically optimal heuristic policies for perishable inventory systems with fixed ordering cost. For these systems, Nahmias (1978) shows that the class of (s, S) policies (which raises the total inventory level to S whenever it drops below s) performs close to optimal. It is helpful to explore their asymptotic optimality property. Finally, it is also valuable to consider the systems under the discounted cost criterion over a finite/infinite planning horizon. In particular, it will be interesting to investigate to what extent the results established for the systems under the average cost criterion can be extended to systems under discounted cost criterion.

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Online Appendix for "Asymptotic Optimality of Base-Stock Policies for Perishable Inventory Systems"

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Appendix A. Proof of Lemma 1 in Section 2

We prove the lemma by considering the case when $\mathbb{P}(D \geq S/m) > 0$ and the case when $\mathbb{P}(D \geq S/m) = 0$ separately. For convenience, we denote the state space under base-stock policy π_S as

$$\mathcal{X} = \{(x_1, x_2, \ldots, x_{m-1}) : 0 \leq x_1 \leq x_2 \leq \ldots \leq x_{m-1} \leq S\}.$$

Case 1: $\mathbb{P}(D \geq S/m) > 0$. In this case, similar to the proof of Theorem 3 in Huh et al. (2009), we first prove that the Markov chain $\{X_t(S) := (x_{t,1}^{\pi_S}, x_{t,2}^{\pi_S}, \ldots, x_{t,m-1}^{\pi_S}) : t \geq 1\}$ converges to some random vector $X_\infty(S) = (x_{\infty,1}^{\pi_S}, x_{\infty,2}^{\pi_S}, \ldots, x_{\infty,m-1}^{\pi_S})$ in distribution. Applying this result, we can prove Lemma 1 as follows. Since $X_t(S)$ converges in distribution to $X_\infty(S)$, one can verify that $\left(x_{t,1}^{\pi_S}, D_t\right)$ converges in distribution to $\left(x_{\infty,1}^{\pi_S}, D\right)$, where D is independent of $X_\infty(S)$. Since the function $(x_1 - d)^+$ is continuous in $0 \leq x_1 \leq S$, $d \geq 0$, and bounded from above by S, by applying Theorem 3.2.3 in Durrett (2010), we obtain

$$\lim_{t\to\infty} \mathbb{E}[(x_{t,1}^{\pi_S} - D_t)^+] = \mathbb{E}[(x_{\infty,1}^{\pi_S} - D)^+].$$

Therefore, Lemma 1 is established by letting $O_\infty(S) =^d (x_{\infty,1}^{\pi_S} - D)^+$, where $=^d$ denotes "equal in distribution".

We now prove that the stationary distribution of the Markov chain $\{X_t(S) : t \geq 1\}$ exists, which is also the limiting distribution that $\{X_t(S) : t \geq 1\}$ converges to. By applying Theorem 16.0.2 of Meyn and Tweedie (1993), we only need to construct a measurable subset $\mathbf{U} \subseteq \mathcal{X}$, a nontrivial measure $\nu(\cdot)$ (i.e., $\nu(\mathcal{X}) > 0$), and a positive integer $t^* \geq 1$, such that

$$\mathbb{P}(X_{t^*}(S) \in B | X_1(S) = \mathbf{x}_1) \geq \nu(B) \tag{EC.1}$$

for any $\mathbf{x}_1 \in \mathbf{U}$ and any measurable subset $B \subseteq \mathcal{X}$.

Let $\mathbf{U} = \mathcal{X}$, $t^* = 2m - 1$, and the measure $\nu(\cdot)$ be defined as follows: for any measurable set $B \subseteq \mathcal{X}$,

$$\nu(B) \triangleq \mathbb{P}\left(D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2, \left((S - \sum_{t=m}^{2m-2} D_t)^+, (S - \sum_{t=m+1}^{2m-2} D_t)^+, \ldots, (S - D_{2m-2})^+\right) \in B\right).$$

It is easy to verify that $\nu(\cdot)$ is a measure. In addition, since $\nu(\mathcal{X}) = (\mathbb{P}(D \geq S/m))^{2m-2} > 0$, it is nontrivial.

To complete the proof, it remains to verify inequality (EC.1). Note that

$$
\begin{aligned}
&\mathbb{P}(X_{2m-1}(S) \in B | X_1(S) = \mathbf{x}_1) \\
&\geq \mathbb{P}\Big(X_{2m-1}(S) \in B, D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2 | X_1(S) = \mathbf{x}_1\Big) \\
&= \mathbb{P}\Big(X_{2m-1}(S) \in B | D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2, X_1(S) = \mathbf{x}_1\Big)\mathbb{P}\Big(D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2\Big) \\
&= \mathbb{P}\Big(\big((S - \sum_{t=m}^{2m-2} D_t)^+, \ldots, (S - D_{2m-2})^+\big) \in B | D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2, X_1(S) = \mathbf{x}_1\Big)\mathbb{P}\Big(D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2\Big) \\
&= \mathbb{P}\Big(\big((S - \sum_{t=m}^{2m-2} D_t)^+, \ldots, (S - D_{2m-2})^+\big) \in B | D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2\Big)\mathbb{P}\Big(D_t \geq \frac{S}{m}, \forall 1 \leq t \leq 2m-2\Big) \\
&= \nu(B),
\end{aligned}
\tag{EC.2}
$$

where the first equality follows from the conditional probability formula and the independence between $(D_1, D_2, \ldots, D_{2m-2})$ and the initial state $X_1(S)$, the second equality follows from Corollary 2 of Cooper and Tweedie (2002), which states that if $D_t \geq S/m$ for any $1 \leq t \leq 2m-2$, then $x_{2m-1,i}^{\pi_S} = (S - \sum_{t=m+i-1}^{2m-2} D_t)^+$ for any $1 \leq i \leq m-1$ regardless of the initial state, the third equality follows from the independence between $(D_1, D_2, \ldots, D_{2m-2})$ and $X_1(S)$, and the last equality follows from the definition of $\nu(\cdot)$. The proof of Lemma 1 for Case 1 is complete.

Case 2: $\mathbb{P}(D \geq S/m) = 0$. In this case, the Markov chain $\{X_t(S) : t \geq 1\}$ may not have a stationary distribution, since one can construct a cyclic Markov chain similar to that in §3.2 of Huh et al. (2009). So, we prove $\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[(x_{t,1}^{\pi_S} - D_t)^+] = S/m - \mathbb{E}[D]$ directly by showing that

$$
\liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[(x_{t,1}^{\pi_S} - D_t)^+] \geq \frac{S}{m} - \mathbb{E}[D] \geq \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[(x_{t,1}^{\pi_S} - D_t)^+]. \tag{EC.3}
$$

By letting $O_\infty(S) =^d S/m - D$, we prove Lemma 1 under Case 2.

We first prove the first inequality in (EC.3). From the recursion (19), one can verify that

$$
\sum_{i=t-m+1}^{t} \mathbb{E}[o_i^{\pi_S}] \geq S - m\mathbb{E}[D], \quad \forall t \geq m. \tag{EC.4}
$$

Suppose $T = km + l$, where $k \geq 1$ and $1 \leq l < m$. Then, we have

$$
\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[o_t^{\pi_S}] \geq \frac{1}{km+l}\sum_{s=0}^{k-1}\sum_{t=sm+1}^{(s+1)m} \mathbb{E}[o_t^{\pi_S}] \geq \frac{k}{km+l}(S - m\mathbb{E}[D]),
$$

where the second inequality follows from (EC.4). By taking $\liminf_{T\to\infty}$ on both sides of the above inequalities, we obtain the first inequality in (EC.3).

We next prove the second inequality in (EC.3). For any vector $\mathbf{x} = (x_1, \ldots, x_{m-1}) \in \mathcal{X}$, consider the following three systems under the same base-stock policy π_S: System 1 starts with the initial state $\mathbf{x}_1^1 = \mathbf{x}$; System 2 starts with $\mathbf{x}_1^2 = (x_{m-1}, x_{m-1}, \ldots, x_{m-1})$; and System 3 starts with $\mathbf{x}_1^3 = (S/m, 2S/m, \ldots, (m-1)S/m)$. For $1 \le k \le 3$, let $\mathbf{x}_t^k$ and o_t^k denote the system state and the amount of outdates in System k in period t, respectively. Then, we have

$$\sum_{t=1}^{T} o_t^1 \le \sum_{t=1}^{T} o_t^2 \le o_1^2 + \sum_{t=2}^{T} o_t^3, \quad \forall T \ge 2, \tag{EC.5}$$

under any given demand sample path. Note that $x_{1,i}^1 \le x_{1,i}^2$ for all $1 \le i \le m-1$ and $x_{t,m}^1 = x_{t,m}^2 = S$ for any $t \ge 1$. In addition, System 2 will be empty at the beginning of period 2 (i.e., $\mathbf{x}_2 = \mathbf{0}$). Then, $x_{2,i}^2 \le x_{2,i}^3$ for all $1 \le i \le m-1$ and $x_{t,m}^2 = x_{t,m}^3 = S$ for any $t \ge 2$. Then, the two inequalities in (EC.5) can be proven by using the recursion (19) and induction.

Since $D_t < S/m$ for any $t \ge 1$, one can verify that

$$\mathbf{x}_t^3 = \left(\frac{S}{m}, \frac{2S}{m}, \ldots, \frac{(m-1)S}{m} \right)$$

and $o_t^3 = S/m - D_t$ for any $t \ge 1$. Applying (EC.5), we obtain

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[o_t^1] \le \limsup_{T\to\infty} \left\{ \frac{1}{T}\mathbb{E}[o_1^2] + \frac{1}{T} \sum_{t=2}^{T} \mathbb{E}[o_t^3] \right\} = \frac{S}{m} - \mathbb{E}[D].$$

Thus, the second inequality in (EC.3) is also satisfied. The proof of Lemma 1 under Case 2 is then complete. Q.E.D.

Appendix B. Proofs of Theorems 2 to 4 in Section 3

B.1. Proof of Theorem 2

For convenience, let $\mu \triangleq \mathbb{E}[\hat{D}_1]$, $\sigma \triangleq \sqrt{\mathrm{Var}[\hat{D}_1]}$ and $\sigma_X \triangleq \sqrt{\mathrm{Var}[X_1]}$ (recall the definitions of $\hat{D}_1$ and X_1 from §3.1). To highlight the dependence on n, we use the functional form $D(n)$ to represent the single-period demand. We only consider the case $\sigma_X > 0$, and similar arguments apply to the case $\sigma_X = 0$, i.e., when X_1 is deterministic. We divide the proof of Theorem 2 into three major steps.

First, we prove that for any fixed $k > 1$, there exists $n_1(k) > 0$ such that $S_n^{NP} \le k\mu n$, for any $n \ge n_1(k)$. To this end, we first prove the following inequality:

$$S_n^{NP} \le \mathbb{E}[D(n)] + \sqrt{p\mathrm{Var}[D(n)]/h}. \tag{EC.6}$$

Let $\mathcal{M}$ be the following set of distributions: $\mathcal{M} = \big\{ \widetilde{F} \text{ is a cdf of } D:\ \mathbb{E}_{\widetilde{F}}[D] = \mathbb{E}[D(n)], \mathrm{Var}_{\widetilde{F}}[D] = \mathrm{Var}[D(n)] \big\}$, where $\mathbb{E}_{\widetilde{F}}[\cdot]$ denotes the expectation taken with respect to cdf $\widetilde{F}(\cdot)$. Then

$$\begin{aligned} h(S_n^{NP} - \mathbb{E}[D(n)]) &\le (h+p)\mathbb{E}[(D(n) - S_n^{NP})^+] + h\big(S_n^{NP} - \mathbb{E}[D(n)]\big) \\ &= \min_{S\ge 0} \Big\{ h\mathbb{E}[(S - D(n))^+] + p\mathbb{E}[(D(n) - S)^+] \Big\} \\ &\le \min_{S\ge 0} \max_{\widetilde{F}\in\mathcal{M}} \Big\{ h\mathbb{E}_{\widetilde{F}}[(S - D)^+] + p\mathbb{E}_{\widetilde{F}}[(D - S)^+] \Big\} \\ &= \sqrt{ph\mathrm{Var}[D(n)]}, \end{aligned}$$

where the last equality follows from Scarf (1958). The above inequality directly implies (EC.6).

Recall that $\hat{D}_j$'s are independent of $N(n)$. From Wald's equation and the law of total variance,

$$\mathbb{E}[D(n)] = \mu\mathbb{E}[N(n)], \quad \text{and} \quad \mathrm{Var}[D(n)] = \sigma^2\mathbb{E}[N(n)] + \mu^2\mathrm{Var}[N(n)]. \tag{EC.7}$$

Moreover, from the classic central limit theorem for the renewal process (see e.g., Theorem 4.3.2 in Gallager 2012), we have the random variable

$$\frac{N(n) - \frac{1}{\mathbb{E}[X_1/n]}}{\sqrt{\mathrm{Var}[X_1/n]}\cdot\sqrt{\frac{1}{(\mathbb{E}[X_1/n])^3}}}$$

converges in distribution to the standard normal random variable when $n \to \infty$. Since $\mathbb{E}[X_1] = 1$ and $\mathrm{Var}[X_1] = \sigma_X^2$, it follows from the convergence properties that

$$\lim_{n\to\infty} \frac{\mathbb{E}[N(n)]}{n} = 1, \quad \text{and} \quad \lim_{n\to\infty} \frac{\mathrm{Var}[N(n)]}{n} = \sigma_X^2.$$

Then, it follows from (EC.6) and (EC.7) that $\limsup_{n\to\infty} S_n^{NP}/n \le \mu$. As a result, for any $k > 1$, there exists some $n_1(k)$ such that $S_n^{NP} \le k\mu n$ when $n \ge n_1(k)$.

Second, we prove that there exists some constant $\nu > 0$, which only depends on the distribution of X_1, such that for any fixed $0 < \delta < \nu$ and $n \ge 2(1+\delta)/\delta$,

$$\mathbb{E}\Big[\exp\Big(-\lambda\sum_{j=1}^{N(n)}\hat{D}_j\Big)\Big] \le \exp\Big(-\frac{\delta^2 n}{8(1+\delta)^2\nu^2}\Big) + \big(\mathbb{E}[e^{-\lambda\hat{D}_1}]\big)^{\frac{n}{1+\delta}}. \tag{EC.8}$$

For any $\delta > 0$, we first note that

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(-\lambda\sum_{j=1}^{N(n)}\hat{D}_j\Big)\Big] &= \mathbb{E}\Big[\exp\Big(-\lambda\sum_{j=1}^{N(n)}\hat{D}_j\Big)\Big|N(n) < \lceil\frac{n}{1+\delta}\rceil\Big]\mathbb{P}\big(N(n) < \lceil\frac{n}{1+\delta}\rceil\big) \\
&\quad + \mathbb{E}\Big[\exp\Big(-\lambda\sum_{j=1}^{N(n)}\hat{D}_j\Big)\Big|N(n) \ge \lceil\frac{n}{1+\delta}\rceil\Big]\mathbb{P}\big(N(n) \ge \lceil\frac{n}{1+\delta}\rceil\big) \\
&\le \mathbb{P}\Big(N(n) < \lceil\frac{n}{1+\delta}\rceil\Big) + \mathbb{E}\Big[\exp\Big(-\lambda\sum_{j=1}^{\lceil\frac{n}{1+\delta}\rceil}\hat{D}_j\Big)\Big] \\
&\le \mathbb{P}\Big(N(n) < \frac{n}{1+\delta}\Big) + \big(\mathbb{E}[e^{-\lambda\hat{D}_1}]\big)^{\frac{n}{1+\delta}}.
\end{aligned} \tag{EC.9}$$

To show (EC.8), it suffices to bound $\mathbb{P}\big(N(n) < \frac{n}{1+\delta}\big)$ from above by $\exp(-\frac{\delta^2 n}{8(1+\delta)^2\nu^2})$. For convenience, let $k(n,\delta) = \lceil\frac{n}{1+\delta}\rceil$. Note that

$$\mathbb{P}\Big(N(n) < \frac{n}{1+\delta}\Big) = \mathbb{P}\big(N(n) < k(n,\delta)\big) = \mathbb{P}\big(\sum_{i=1}^{k(n,\delta)} X_i > n\big), \tag{EC.10}$$

where the second equality follows from the definition of $N(n)$. Since X_1 is a nonnegative r.v. and $\mathbb{E}[e^{sX_1}] < \infty$ for some $s > 0$ by our assumption, $X_1 - \mathbb{E}[X_1]$ is a sub-exponential r.v. (see

Proposition 2.7.1 of Vershynin (2018) for equivalent definitions of a sub-exponential r.v.). Then, we apply Bernstein's inequality for sub-exponential random variables (see e.g., Theorem 1.13 in Rigollet and Hütter 2015) to obtain the following result: there exists some constant $\nu > 0$ (which only depends on the distribution of X_1) such that when $\delta < \nu$ and $n \geq 2(1+\delta)/\delta$,

$$\begin{aligned}
\mathbb{P}\Big(\sum_{i=1}^{k(n,\delta)} X_i > n\Big) &= \mathbb{P}\Big(\frac{1}{k(n,\delta)}\sum_{i=1}^{k(n,\delta)}(X_i - 1) > \frac{n}{k(n,\delta)} - 1\Big) \\
&\leq \exp\Big(-\frac{k(n,\delta)}{2}\big((\frac{\frac{n}{k(n,\delta)}-1}{\nu})^2 \wedge \frac{\frac{n}{k(n,\delta)}-1}{\nu}\big)\Big) \\
&= \exp\Big(-\frac{(n-k(n,\delta))^2}{2k(n,\delta)\nu^2}\Big) \\
&\leq \exp\Big(-\frac{\delta^2 n}{8(1+\delta)^2\nu^2}\Big),
\end{aligned} \tag{EC.11}$$

where the first inequality follows from Bernstein's inequality and $n \geq \frac{n}{1+\delta} + 1 > k(n,\delta)$, the second equality holds since $\frac{n}{k(n,\delta)} - 1 \leq 1 + \delta - 1 < \nu$ when $\delta < \nu$, and the second inequality holds since

$$\frac{(n-k(n,\delta))^2}{k(n,\delta)} > \frac{(n-(\frac{n}{1+\delta}+1))^2}{\frac{n}{1+\delta}+1} = \frac{(\delta n - (1+\delta))^2}{(1+\delta)(n+(1+\delta))} \geq \frac{\delta^2 n}{4(1+\delta)^2},$$

where the last inequality holds since $1+\delta \leq n\delta/2$. Then, (EC.8) follows from (EC.9), (EC.10) and (EC.11).

Finally, we bound the RHS of inequality (23). Combining the results established in the previous two steps, we obtain that for any $k > 1$, $0 < \delta < \nu$ and $n \geq n_1(k) \vee (2(1+\delta)/\delta)$,

$$e^{\frac{1}{m}\lambda S_n^{NP}}\mathbb{E}[e^{-\lambda\sum_{j=1}^{N(n)}\hat{D}_j}] \leq e^{f_1(\lambda)n} + e^{f_2(\lambda)n},$$

where $f_1(\lambda)$ and $f_2(\lambda)$ are defined as

$$f_1(\lambda) \triangleq \frac{\lambda k\mu}{m} - \frac{\delta^2}{8(1+\delta)^2\nu^2}, \quad \text{and} \quad f_2(\lambda) \triangleq \frac{\lambda k\mu}{m} + \frac{1}{1+\delta}\log(\mathbb{E}[e^{-\lambda\hat{D}_1}]).$$

Note that $f_1(\lambda) < 0$ when λ is sufficiently small. When $m \geq 2$ (otherwise the RHS of (23) equals zero), $f_2'(0) = (k/m - 1/(1+\delta))\mu < 0$ when $1 < k < 2/(1+\delta)$ and $0 < \delta < 1$. Since $f_2(0) = 0$, there exists some $\lambda_0(k,\delta) > 0$ such that $f_2(\lambda) < 0$ when $0 < \lambda < \lambda_0(k,\delta)$. Thus, by choosing δ, k and λ satisfying the following three conditions: (i) $0 < \delta < 1 \wedge \nu$; (ii) $1 < k < 2/(1+\delta)$; and (iii) $0 < \lambda < \lambda_0(k,\delta) \wedge [\delta^2 m/(8k\mu(1+\delta)^2\nu^2)]$, we have

$$C_n(\widetilde{S}_n) - \mathrm{OPT}_n \leq K_1 e^{-K_2 n}, \quad \forall n \geq n_1(k) \vee 2(1+\delta)/\delta,$$

where $K_1 = 2^m(m-1)\theta/(\lambda m e) \geq 0$ and $K_2 = -\max\{f_1(\lambda), f_2(\lambda)\}m > 0$. Q.E.D.

B.2. Proof of Theorem 3

From the sketched proof of Theorem 3 in §6.2, it suffices to prove inequalities (25) and (26).

We first prove inequality (25). Note that

$$\begin{aligned} C(\widetilde{S}) - \mathrm{OPT} &\le \theta\Big(\mathbb{E}\Big[\Big(\frac{\widetilde{S}}{m} - D\Big)^+\Big] - \mathbb{E}\Big[\Big(\frac{\widetilde{S}}{m} - \frac{1}{m}\sum_{i=1}^m D_i\Big)^+\Big]\Big) \\ &= \theta\Big(\mathbb{E}\Big[\Big(D - \frac{\widetilde{S}}{m}\Big)^+\Big] - \mathbb{E}\Big[\Big(\frac{1}{m}\sum_{i=1}^m D_i - \frac{\widetilde{S}}{m}\Big)^+\Big]\Big), \end{aligned} \tag{EC.12}$$

where the inequality follows from Lemmas 1 and 2 and Proposition 2, and the equality follows from $x^+ = x + (-x)^+$ and $\mathbb{E}[D] = \mathbb{E}[\frac{1}{m}\sum_{i=1}^m D_i]$. Thus, inequality (25) holds.

We next prove inequality (26). We first provide an upper bound on $C(\widetilde{S})$. Note that

$$p\mathbb{E}[(D-\widetilde{S})^+] = p\mathbb{E}[(D-\widetilde{S})\cdot 1_{\{D>\widetilde{S}\}}] \le p\big(\bar{D}-\widetilde{S}\big)\cdot \mathbb{P}(D>\widetilde{S}) \le (h+\theta)\big(\bar{D}-\widetilde{S}\big), \tag{EC.13}$$

where the first inequality follows from $D \le \bar{D}$, a.s., and the second one follows from inequality (5). Since $\widetilde{S} \ge \underline{S}$, we have

$$\begin{aligned} C(\widetilde{S}) &\le h\mathbb{E}\big[(\widetilde{S}-D)^+\big] + \theta\mathbb{E}\big[O_\infty(\widetilde{S})\big] + (h+\theta)\big(\bar{D}-\widetilde{S}\big) \\ &\le h\mathbb{E}\big[(\underline{S}-D)^+\big] + \theta\mathbb{E}\big[O_\infty(\underline{S})\big] + (h+\theta)\big(\bar{D}-\underline{S}\big), \end{aligned} \tag{EC.14}$$

where the first inequality follows from (EC.13), and the second one follows from $a^+ - b^+ \le (a-b)^+$, and the following inequality: for any $0 \le S_1 \le S_2$,

$$\mathbb{E}[O_\infty(S_2)] \le \mathbb{E}[O_\infty(S_1)] + S_2 - S_1. \tag{EC.15}$$

The inequality (EC.15) can be proved as follows. First, by using the system dynamics, we can prove by induction that $x_{t,1}^{\pi_{S_1}} \le x_{t,1}^{\pi_{S_2}}$ for any $t \ge 1$. Second, note that for any $t \ge 1$, $x_{t+1,i}^{\pi_S} = (x_{t,i+1}^{\pi_S} - x_{t,1}^{\pi_S} \vee D_t)^+$ for $1 \le i \le m-1$ and $x_{t,m}^{\pi_S} = S$. We can prove by induction that $x_{t,i}^{\pi_{S_2}} \le x_{t,i}^{\pi_{S_1}} + S_2 - S_1$ for any $t \ge 1$ and $1 \le i \le m$. Then, by the definition of o_t^π, we obtain $o_t^{\pi_{S_2}} \le o_t^{\pi_{S_1}} + S_2 - S_1$ for any $t \ge 1$. Finally, the inequality (EC.15) follows by applying the definition of $\mathbb{E}[O_\infty(S)]$. Thus, inequality (26) holds. Q.E.D.

B.3. Proof of Theorem 4

From the sketched proof of Theorem 4 in §6.2, it suffices to prove equality (24). Recall that $\widetilde{S}_\theta$ is the minimizer of $\widetilde{C}_\theta(S)$ over $[0,\infty)$. One can easily show that $\widetilde{S}_\theta$ decreases in θ. Thus, the limit $\lim_{\theta\to\infty}\widetilde{S}_\theta$ exists and we denote it by $\widetilde{S}_\infty$. Next, we show that $\widetilde{S}_\infty \le S^0$, where $S^0 \triangleq \sup\{S \ge 0 : \mathbb{E}[(S - \sum_{i=1}^m D_i)^+] = 0\}$. One can verify that $S^0 = m\underline{D}$ (recall that $\underline{D} = \inf\{x : F(x) > 0\}$). By the definition of $\widetilde{C}_\theta(S)$ in (4), for any $\theta \ge 0$, we have

$$\begin{aligned} \widetilde{C}_\theta(\widetilde{S}_\theta) &= h\mathbb{E}[(\widetilde{S}_\theta - D)^+] + p\mathbb{E}[(D-\widetilde{S}_\theta)^+] + \frac{\theta}{m}\mathbb{E}\Big[\Big(\widetilde{S}_\theta - \sum_{i=1}^m D_i\Big)^+\Big] \\ &\le \min_{0\le S\le m\underline{D}} \widetilde{C}_\theta(S) = \min_{0\le S\le m\underline{D}}\big\{h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+]\big\}. \end{aligned} \tag{EC.16}$$

Note that inequality (EC.16) holds for all θ, and its RHS is a constant. This implies $\widetilde{S}_\infty \le S^0$. Thus,

$$\begin{aligned}\lim_{\theta\to\infty}\left\{h\mathbb{E}[(\widetilde{S}_\theta - D)^+] + p\mathbb{E}[(D-\widetilde{S}_\theta)^+]\right\} =& h\mathbb{E}[(\widetilde{S}_\infty - D)^+] + p\mathbb{E}[(D-\widetilde{S}_\infty)^+] \\ \ge& \min_{0\le S\le m\underline{D}}\left\{h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+]\right\}. \qquad \text{(EC.17)}\end{aligned}$$

After combining inequalities (EC.16) and (EC.17), we obtain equality (24). Q.E.D.

Appendix C. Proofs of Statements in Section 4

C.1. Proof of Lemma 3

First, we prove that $C^{LIFO}(S) \ge \hat{C}_L(S)$. From the system dynamics under the LIFO issuance policy described in §4.1, we have the following recursion on the outdating process under base-stock policy π_S: under any demand sample path,

$$o_{t+m-1}^{\pi_S} = \min_{i\in\{0,\ldots,m-1\}}\left\{(S-D_{t+i})^+ - \sum_{j=t+i}^{t+m-2} o_j^{\pi_S}\right\}, \quad \forall t\ge 1. \qquad \text{(EC.18)}$$

Then, it follows that

$$\sum_{j=t}^{t+m-1} o_j^{\pi_S} \ge \min_{i\in\{0,\ldots,m-1\}}\{(S-D_{t+i})^+\} = (S-\max\{D_t, D_{t+1},\ldots,D_{t+m-1}\})^+, \quad \forall t\ge 1.$$

After taking the expectation on both sides of the above inequality, we obtain

$$\sum_{j=t}^{t+m-1}\mathbb{E}[o_j^{\pi_S}] \ge \mathbb{E}[(S-\max\{D_1,D_2,\ldots,D_m\})^+], \quad \forall t\ge 1. \qquad \text{(EC.19)}$$

By the definition of $C^{LIFO}(S)$, we have

$$\begin{aligned}C^{LIFO}(S) = \limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[C_t^{\pi_S}] &= h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] + \limsup_{T\to\infty}\frac{\theta}{mT}\sum_{t=1}^{mT}\mathbb{E}[o_t^{\pi_S}] \\ &\ge h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] + \frac{\theta}{m}\mathbb{E}[(S-\max\{D_1,D_2,\ldots,D_m\})^+],\end{aligned}$$

where the inequality follows from inequality (EC.19). By the definition of $\hat{C}_L(S)$, we conclude that $C^{LIFO}(S) \ge \hat{C}_L(S)$.

Next, we prove that $C^{LIFO}(S) \le \hat{C}_U(S)$. Under policy π_S, the total leftover inventory in each period t after satisfying demand D_t is $(S-D_t)^+$. Note that the outdated inventories in periods $t, t+1, \ldots, t+m-1$ are part of this inventory. Then, we obtain

$$\sum_{j=t}^{t+m-1} o_j^{\pi_S} \le (S-D_t)^+, \quad \forall t\ge 1.$$

The rest of the proof is similar to that of the first part and we omit it for brevity. Q.E.D.

C.2. Proof of Theorem 5

First, we prove part (a). Since $\widetilde{S} \le S^{NP}$ and $\hat{S} \le S^{NP}$, it follows from inequality (27) that

$$C^{LIFO}(\hat{S}) - \mathrm{OPT}^{LIFO} \le \frac{3\theta}{2m}\mathbb{E}[(S^{NP} - D)^+].$$

Thus, part (a) holds.

Next, we prove part (b). From inequality (27), we also have

$$\begin{aligned} C^{LIFO}(\hat{S}) - \mathrm{OPT}^{LIFO} \le& \frac{\theta}{2m}\Big(2(m-1)\mathbb{E}[D] + \mathbb{E}[(D-\hat{S})^+] + \mathbb{E}[(D-\widetilde{S})^+] - \mathbb{E}[(\max\{D_1,\ldots,D_m\} - \hat{S})^+] \\ & + \mathbb{E}[(\max\{D_1,\ldots,D_m\} - \widetilde{S})^+] - 2\mathbb{E}[(\sum_{i=1}^m D_i - \widetilde{S})^+]\Big) \\ \le& \frac{(m-1)\theta}{m}\mathbb{E}[D], \end{aligned} \tag{EC.20}$$

where the two inequalities follow from $x^+ = x + (-x)^+$ for any x and that $D_1, \ldots, D_m$ and D are *i.i.d.* r.v.'s. When demand D is unbounded, one can easily verify that $\lim_{p\to\infty} \widetilde{C}_p(\widetilde{S}_p) = \infty$. Since $\mathrm{OPT}^{LIFO} \ge \widetilde{C}(\widetilde{S})$ by Proposition 2, it follows that $\lim_{p\to\infty} \mathrm{OPT}_p^{LIFO} = \infty$. Combining this with inequality (EC.20), we obtain part (b).

Finally, we prove part (c). From Proposition 2 and the proof of Theorem 4, we have

$$\lim_{\theta\to\infty} \mathrm{OPT}_\theta^{LIFO} \ge \lim_{\theta\to\infty} \widetilde{C}_\theta(\widetilde{S}_\theta) = \min_{0\le S\le m\underline{D}} \{h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+]\}. \tag{EC.21}$$

By a similar proof to that of Theorem 4, we can also prove that $\lim_{\theta\to\infty} \hat{S}_\theta = \underline{D}$ and

$$\lim_{\theta\to\infty} \theta(\mathbb{E}[(\hat{S}_\theta - D)^+] + \mathbb{E}[(\hat{S}_\theta - \max\{D_1,\ldots,D_m\})^+]) = 0.$$

Since $C^{LIFO}(S) \le \hat{C}_U(S)$ for any $S \ge 0$ by Lemma 3, it follows that

$$\limsup_{\theta\to\infty} C_\theta^{LIFO}(\hat{S}_\theta) \le \limsup_{\theta\to\infty} \hat{C}_{U,\theta}(\hat{S}_\theta) = p(\mathbb{E}[D] - \underline{D}) + \frac{1}{m}\lim_{\theta\to\infty}(\theta\mathbb{E}[(\hat{S}_\theta - D)^+]) = p(\mathbb{E}[D] - \underline{D}). \tag{EC.22}$$

When $\underline{D} = 0$, after combining inequalities (EC.21) and (EC.22), we obtain $\lim_{\theta\to\infty}(C_\theta^{LIFO}(\hat{S}_\theta) - \mathrm{OPT}_\theta^{LIFO}) = 0$. Thus, part (c) holds. Q.E.D.

C.3. Proof of Proposition 1

From Lemma 3, we have $C^{LIFO}(S^{LIFO,*}) \ge \min_{S\ge 0} \hat{C}_L(S)$. In addition, by a similar proof to that of Theorem 4, we can prove that $\lim_{\theta\to\infty}\big(\min_{S\ge 0} \hat{C}_{L,\theta}(S)\big) = p(\mathbb{E}[D] - \underline{D})$. Combining these results with inequality (EC.22) and the definition of $S^{LIFO,*}$, we obtain $\lim_{\theta\to\infty} C_\theta(S_\theta^{LIFO,*}) = p(\mathbb{E}[D] - \underline{D})$.

Next, we construct an admissible policy and show that its long-run average cost is strictly less than $p(\mathbb{E}[D] - \underline{D})$ and independent of θ when $m \ge 2$ and $S^{NP} > \underline{D} > 0$. Without loss of generality, suppose that the system is initially empty. Consider the following admissible policy:

order $\min\{S^{NP}, m\underline{D}\}$ units in periods 1, $m+1$, $2m+1$, ..., whereas use the base-stock policy with level $\underline{D}$ in all other periods. One can easily check that 1) there is no inventory outdating under this policy; 2) the expected holding and penalty cost in each of periods 1, $m+1$, $2m+1$, ... is

$$h\mathbb{E}[(\min\{S^{NP}, m\underline{D}\} - D)^+] + p\mathbb{E}[(D - \min\{S^{NP}, m\underline{D}\})^+],$$

which is strictly less than $p(\mathbb{E}[D] - \underline{D})$ when $m \geq 2$ and $S^{NP} > \underline{D} > 0$ by the definition of S^{NP}; and 3) the expected holding and penalty cost in each of all other periods is at most $p(\mathbb{E}[D] - \underline{D})$ because by the definition of S^{NP}, we have

$$h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] \leq p(\mathbb{E}[D] - \underline{D}), \ \forall S \in [\underline{D}, \min\{S^{NP}, m\underline{D}\}].$$

Combining these results together, we conclude that the long-run average cost under this policy is strictly less than $p(\mathbb{E}[D] - \underline{D})$ and independent of θ.

Finally, by the definition of OPT^{LIFO}, we obtain $\lim_{\theta\to\infty} \text{OPT}_\theta^{LIFO} < p(\mathbb{E}[D] - \underline{D})$. As a result, $\lim_{\theta\to\infty}(C_\theta^{LIFO}(S_\theta^{LIFO,*}) - \text{OPT}_\theta^{LIFO}) > 0$. Q.E.D.

C.4. Proof of Lemma 4

Note that the long-run average holding and penalty cost under base-stock policy π_S always equals $h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+]$, regardless of the inventory issuance policy. From the definitions of $C^{FIFO}(S)$, $C^{LIFO}(S)$ and $C^{GI}(S)$, it suffices to prove the following inequalities under any demand sample path:

$$\sum_{i=1}^t o_i^{FIFO,\pi_S} \leq \sum_{i=1}^t o_i^{GI,\pi_S} \leq \sum_{i=1}^t o_i^{LIFO,\pi_S}, \quad \forall t \geq 1, \tag{EC.23}$$

where o_t^{FIFO,π_S}, o_t^{GI,π_S}, and o_t^{LIFO,π_S} denote the amounts of outdates in period t under base-stock policy π_S, when the FIFO, general, and LIFO inventory issuance policies are adopted respectively.

Since the system is initially empty, $o_t^{FIFO,\pi_S} = o_t^{GI,\pi_S} = o_t^{LIFO,\pi_S} = 0$ for any $t = 1, \dots, m-1$. Thus, the inequalities in (EC.23) hold when $1 \leq t \leq m-1$. Now consider a general period $t \geq m$. Suppose inductively that the inequalities in (EC.23) hold for each period $s = 1, 2, \dots, t-1$. In the following, we prove that the inequalities in (EC.23) hold for period t. Then, by induction, the inequalities in (EC.23) hold for any period $t \geq 1$.

First, we prove the following inequalities under a general issuance policy: for any $t \geq m$:

$$\left(S - \sum_{i=t-m+1}^{t} D_i - \sum_{i=t-m+1}^{t-1} o_i^{GI,\pi_S}\right)^+ \leq o_t^{GI,\pi_S} \leq \min_{t-m+1\leq i\leq t}\left\{(S-D_i)^+ - \sum_{j=i}^{t-1} o_j^{GI,\pi_S}\right\}. \tag{EC.24}$$

To see the first inequality in (EC.24), we note that the S units of total inventory at the beginning of period $t-m+1$ are either used to satisfy demands or outdated in periods $t-m+1, \dots, t$.

Since the total demand in periods $t-m+1,\ldots,t$ is $\sum_{i=t-m+1}^{t} D_i$, we have $\sum_{i=t-m+1}^{t} o_i^{GI,\pi_S} \geq (S-\sum_{i=t-m+1}^{t} D_i)^+$, leading to the first inequality in (EC.24) due to $o_t^{GI,\pi_S} \geq 0$. To see the second inequality in (EC.24), we note that for each $t-m+1 \leq i \leq t$, the outdated inventory in periods $i, i+1, \ldots, t$ are part of the leftover inventory at the end of period i after satisfying demand in period i, whose amount equals $(S-D_i)^+$. Thus, for each $t-m+1 \leq i \leq t$, we have $\sum_{j=i}^{t} o_j^{GI,\pi_S} \leq (S-D_i)^+$, leading to the second inequality in (EC.24).

Applying the first inequality in (EC.24), we obtain

$$\begin{aligned}
\sum_{i=1}^{t} o_i^{GI,\pi_S} &\geq \max\left\{S - \sum_{i=t-m+1}^{t} D_i + \sum_{i=1}^{t-m} o_i^{GI,\pi_S}, \quad \sum_{i=1}^{t-1} o_i^{GI,\pi_S}\right\} \\
&\geq \max\left\{S - \sum_{i=t-m+1}^{t} D_i + \sum_{i=1}^{t-m} o_i^{FIFO,\pi_S}, \quad \sum_{i=1}^{t-1} o_i^{FIFO,\pi_S}\right\} \\
&= \sum_{i=1}^{t} o_i^{FIFO,\pi_S}, \qquad \text{(EC.25)}
\end{aligned}$$

where the second inequality follows from the inductive assumption and the identity follows from the recursion for the outdated inventory under the FIFO issuance policy in equation (19). Similarly, applying the second inequality in (EC.24), the inductive assumption and the recursion for the outdated inventory under the LIFO issuance policy in equation (EC.18), we obtain $\sum_{i=1}^{t} o_i^{GI,\pi_S} \leq \sum_{i=1}^{t} o_i^{LIFO,\pi_S}$, which, together with (EC.25), leads to (EC.23). Q.E.D.

Appendix D. Proof of Lemma 5 in Section 5

We first prove inequality (13). From the system dynamics under policy π_S, one can easily verify that for $t \geq m+L$,

$$\begin{aligned}
x_{t,m}^{\pi_S} &= S - \sum_{i=t-L}^{t-1} D_i - \sum_{i=t-L}^{t-1} o_i^{\pi_S}, \\
o_t^{\pi_S} &= \Big(S - \sum_{i=t-m-L+1}^{t} D_i - \sum_{i=t-m-L+1}^{t-1} o_i^{\pi}\Big)^+.
\end{aligned}$$

Since $o_i^{\pi_S}$ is non-negative for any period i and $\{D_t : t \geq 1\}$ are *i.i.d.* random variables, it follows from the above two identities that, for $t \geq m+L$,

$$\mathbb{E}[(x_{t,m}^{\pi_S} - D_t)^+] \leq \mathbb{E}\Big[\Big(S - \sum_{i=1}^{L+1} D_i\Big)^+\Big], \qquad \text{(EC.26)}$$

$$\mathbb{E}[o_t^{\pi_S}] \leq \mathbb{E}\Big[\Big(S - \sum_{i=1}^{m+L} D_i\Big)^+\Big]. \qquad \text{(EC.27)}$$

In addition, for $t \geq m+2L$, we have

$$\begin{aligned}\mathbb{E}[(D_t - x^{\pi_S}_{t,m})^+] &\leq \mathbb{E}\Big[\Big(\sum_{i=1}^{L+1} D_i - S\Big)^+\Big] + \sum_{i=t-L}^{t-1}\mathbb{E}[o_i^{\pi_S}] \\ &\leq \mathbb{E}\Big[\Big(\sum_{i=1}^{L+1} D_i - S\Big)^+\Big] + L\mathbb{E}\Big[\Big(S - \sum_{i=1}^{m+L} D_i\Big)^+\Big], \end{aligned} \tag{EC.28}$$

where the second inequality follows from inequality (EC.27).

Recall that $C_t^{\pi_S} = h(x^{\pi_S}_{t,m} - D_t)^+ + b(D_t - x^{\pi_S}_{t,m})^+ + \theta o_t^{\pi_S}$. From the definition of $C^L(S)$, we have

$$\begin{aligned} C^L(S) &= \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[C_t^{\pi_S}] = \limsup_{T\to\infty}\frac{1}{T}\sum_{t=m+2L}^{T}\mathbb{E}[C_t^{\pi_S}] \\ &\leq h\mathbb{E}\Big[\Big(S - \sum_{i=1}^{L+1} D_i\Big)^+\Big] + b\mathbb{E}\Big[\Big(\sum_{i=1}^{L+1} D_i - S\Big)^+\Big] + (\theta + bL)\mathbb{E}\Big[\Big(S - \sum_{i=1}^{m+L} D_i\Big)^+\Big], \end{aligned}$$

where the inequality follows from inequalities (EC.26) to (EC.28). Therefore, inequality (13) holds.

We next prove inequality (14). For any admissible policy $\pi \in \Pi$, one can easily verify that

$$x^{\pi}_{t+L,m} = x^{\pi}_{t,m+L} - \sum_{i=t}^{t+L-1} D_i - \sum_{i=t}^{t+L-1} o_i^{\pi}, \quad \forall t \geq 1, \tag{EC.29}$$

$$\sum_{i=t}^{t+m+L-1} o_i^{\pi} \geq \big(x^{\pi}_{t,m+L} - \sum_{i=t}^{t+m+L-1} D_i\big)^+, \quad \forall t \geq 1. \tag{EC.30}$$

For any $t \geq 1$, it follows from (EC.29) and (EC.30) that

$$\begin{aligned} & h(x^{\pi}_{t+L,m} - D_{t+L})^+ + b(D_{t+L} - x^{\pi}_{t+L,m})^+ \\ =& h\big(x^{\pi}_{t,m+L} - \sum_{i=t}^{t+L} D_i - \sum_{i=t}^{t+L-1} o_i^{\pi}\big)^+ + b\big(\sum_{i=t}^{t+L} D_i + \sum_{i=t}^{t+L-1} o_i^{\pi} - x^{\pi}_{t,m+L}\big)^+ \\ \geq& h\big(x^{\pi}_{t,m+L} - \sum_{i=t}^{t+L} D_i\big)^+ + b\big(\sum_{i=t}^{t+L} D_i - x^{\pi}_{t,m+L}\big)^+ - h\sum_{i=t}^{t+L-1} o_i^{\pi}, \end{aligned} \tag{EC.31}$$

and

$$(m+L)\sum_{t=1}^{T+m+L-1} o_t^{\pi} \geq \sum_{t=1}^{T}\sum_{i=t}^{t+m+L-1} o_i^{\pi} \geq \sum_{t=1}^{T}(x^{\pi}_{t,m+L} - \sum_{i=t}^{t+m+L-1} D_i)^+. \tag{EC.32}$$

From the definition of C_t^{π}, we further have

$$\begin{aligned} \sum_{t=1}^{T+m+L} C_t^{\pi} &\geq \sum_{t=1}^{T+m}\big(h(x^{\pi}_{t+L,m} - D_{t+L})^+ + b(D_{t+L} - x^{\pi}_{t+L,m})^+\big) + \theta\sum_{t=1}^{T+m+L} o_t^{\pi} \\ &\geq \sum_{t=1}^{T+m}\left\{h\big(x^{\pi}_{t,m+L} - \sum_{i=t}^{t+L} D_i\big)^+ + b\big(\sum_{i=t}^{t+L} D_i - x^{\pi}_{t,m+L}\big)^+\right\} + (\theta - hL)\sum_{t=1}^{T+m+L-1} o_t^{\pi} \\ &\geq \sum_{t=1}^{T}\left\{h\big(x^{\pi}_{t,m+L} - \sum_{i=t}^{t+L} D_i\big)^+ + b\big(\sum_{i=t}^{t+L} D_i - x^{\pi}_{t,m+L}\big)^+ + \frac{\theta - hL}{m+L}(x^{\pi}_{t,m+L} - \sum_{i=t}^{t+m+L-1} D_i)^+\right\}, \end{aligned} \tag{EC.33}$$

where the first inequality is obtained from dropping the holding and backlogging costs in the first L periods, the second one follows from inequality (EC.31) and the inequality $\sum_{t=1}^{T+m}\sum_{i=t}^{t+L-1} o_i^\pi \leq L\sum_{t=1}^{T+m+L-1} o_i^\pi$, and the last inequality follows from $\theta \geq hL$ and inequality (EC.32).

Since $x_{t,m+L}^{\pi_S} = S$ for each $t \geq 1$, inequality (14) follows directly after applying inequality (EC.33) to policy π_S. Q.E.D.

Appendix E. Proofs of Lemma 6 and Proposition 4 in Section 6

E.1. Proof of Lemma 6

Since the generic demand D has a finite upper support $\bar{D}$, we can define the state space as

$$\mathcal{X} = \{\mathbf{x} = (x_1, x_2, \ldots, x_{m-1}) : 0 \leq x_1 \leq x_2 \leq \ldots \leq x_{m-1} \leq \bar{D}\}.$$

Based on the discussion after Lemma 6, it remains to prove that there exist an optimal policy, denoted by π^*, for our system under the average-cost criterion, and a sequence of discount factors $\{\alpha_n \in (0,1) : n \geq 1\}$ converging to one such that

$$x_{t,m}^{\pi^*}(\mathbf{x}) = \lim_{n\to\infty} x_{t,m}^{\alpha_n,*}(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{X}, \forall t \geq 1,$$

where for any system state $\mathbf{x} \in \mathcal{X}$, $x_{t,m}^{\pi^*}(\mathbf{x})$ is the order-up-to level in period t under policy π^* and $x_{t,m}^{\alpha_n,*}(\mathbf{x})$ is the order-up-to level under the optimal policy in period t for the system under the discounted-cost criterion with discount factor $\alpha_n \in (0,1)$.

According to Theorem 3.8 in Schäl (1993), the above statement is proven once we verify a set of three conditions (i.e., a general assumption, and conditions (S) and (B) stated in Schäl (1993)) for our perishable inventory system. For brevity, we omit their detailed statements. Among the three conditions, the general assumption and condition (S) can be easily verified, and we omit the details for brevity. To verify condition (B), we need to prove the following inequality:

$$\sup_{\alpha<1}\left(J_\alpha^*(\mathbf{x}) - \inf_{\mathbf{x}'\in\mathcal{X}} J_\alpha^*(\mathbf{x}')\right) < \infty, \quad \forall \mathbf{x} \in \mathcal{X}, \tag{EC.34}$$

where $J_\alpha^*(\mathbf{x}) \triangleq \inf_\pi J_\alpha^\pi(\mathbf{x})$ and $J_\alpha^\pi(\mathbf{x})$ denotes the expected total discounted cost under an admissible policy π with the initial state $\mathbf{x}$ and discount factor $\alpha \in (0,1)$.

We now prove inequality (EC.34). For an initial state $\mathbf{x}' \in \mathcal{X}$, let $\mathbf{x}_m^{\alpha*}(\mathbf{x}')$ be the system state in period m under the optimal policy for the system under the discounted-cost criterion with discount factor $\alpha \in (0,1)$. From Theorem 1 in Nandakumar and Morton (1993), there exists a simple, compact positive ordering region (P.O.R.) of the system state $\mathbf{x}$ including $\mathbf{x} = \mathbf{0}$ such that it is optimal not to order outside the P.O.R. and once the system enters the P.O.R., it can never leave it. Since the product has a fixed lifetime of m periods, $\mathbf{x}_m^{\alpha*}(\mathbf{x}')$ must be in the P.O.R. under any demand sample path regardless of the initial state $\mathbf{x}'$. Then, we have

$$J_\alpha^*(\mathbf{x}') \ge \alpha^{m-1}\mathbb{E}[J_\alpha^*(\mathbf{x}_m^{\alpha*}(\mathbf{x}'))] \ge \alpha^{m-1}J_\alpha^*(\mathbf{0}), \tag{EC.35}$$

where the first inequality follows from the definition of $J_\alpha^*(\cdot)$, and the second one holds since $J_\alpha^*(\mathbf{x})$ is increasing in $\mathbf{x}$ in the P.O.R. by Theorem 1 in Nandakumar and Morton (1993).

For any initial state $\mathbf{x} \in \mathcal{X}$, we define a feasible policy π as follows: it does not place any order in the first $m-1$ periods, and then orders optimally under the discounted-cost criterion with the discount factor $\alpha \in (0,1)$ from period m onwards. Then, we have

$$\begin{aligned} J_\alpha^*(\mathbf{x}) - \inf_{\mathbf{x}' \in \mathcal{X}} J_\alpha^*(\mathbf{x}') &\le J_\alpha^\pi(\mathbf{x}) - \alpha^{m-1}J_\alpha^*(\mathbf{0}) \\ &\le (m-1)\cdot\big((h+\theta)x_{m-1} + p\mathbb{E}[D]\big), \end{aligned}$$

where the first inequality follows from the definition of $J_\alpha^*(\cdot)$ and (EC.35), and the second inequality follows from the definition of policy π and the fact that the inventory levels in the first $m-1$ periods under policy π do not exceed x_{m-1}, and $\alpha < 1$. Since $J_\alpha^*(\mathbf{x}) - \inf_{\mathbf{x}' \in \mathcal{X}} J_\alpha^*(\mathbf{x}')$ is uniformly bounded for all $\alpha \in (0,1)$, inequality (EC.34) holds. Q.E.D.

E.2. Proof of Proposition 4

Note that equation (EC.29) and inequality (EC.30) in the proof of Lemma 5 hold for any admissible policy π under any issuance policy. Then, it follows that inequality (EC.33) holds for any admissible policy π under any issuance policy. From the definition of C_t^π in (2), we have

$$\begin{aligned} \sum_{t=1}^{T+m+L} \mathbb{E}[C_t^\pi] &\ge \sum_{t=1}^{T}\Big\{h\big(\mathbb{E}[x_{t,m+L}^\pi] - \sum_{i=t}^{t+L} D_i\big)^+ + b\big(\sum_{i=t}^{t+L} D_i - \mathbb{E}[x_{t,m+L}^\pi]\big)^+ + \frac{\theta - hL}{m+L}(\mathbb{E}[x_{t,m+L}^\pi] - \sum_{i=t}^{t+m+L-1} D_i)^+\Big\} \\ &\ge T\widetilde{C}^L(\widetilde{S}^L), \end{aligned} \tag{EC.36}$$

where the first inequality follows from inequality (EC.33), the independence between $x_{t,m+L}^\pi$ and $(D_t, \dots, D_{t+m+L-1})$ for each period t and the conditional Jensen's inequality, and the second inequality follows from the definition of $\widetilde{C}^L(\cdot)$ and $\widetilde{S}^L$. Then, it follows from the definition of C^π and inequality (EC.36) that

$$C^\pi = \limsup_{T\to\infty} \frac{1}{T+m+L} \sum_{t=1}^{T+m+L} \mathbb{E}[C_t^\pi] \ge \widetilde{C}^L(\widetilde{S}^L).$$

Since the above inequality holds for any admissible policy $\pi \in \Pi$ and under any issuance policy, we obtain $\mathrm{OPT}^L \ge \widetilde{C}^L(\widetilde{S}^L)$ for any $L \ge 1$ under any issuance policy. Q.E.D.

Appendix F. Asymptotic Convergence Rate in Large Penalty Costs

In this appendix, we characterize the convergence rate for the optimality gap of policy $\pi_{\widetilde{S}}$ in the classic system with large unit penalty cost p under four classes of demands. The results are presented in the following proposition, in which for non-negative functions $f(x)$ and $g(x)$, the notation $f(x)=\mathcal{O}(g(x))$ means that $\limsup_{x\to\infty} f(x)/g(x)<\infty$. These results show that the asymptotic rate at which the optimality gap of policy $\pi_{\widetilde{S}}$ converges to zero in the unit penalty cost depends on the specific demand distribution and differs for different demand distributions. Similar results can be established for other classes of continuous demand distributions.

PROPOSITION EC.1. *Suppose that demand D is a continuous random variable with probability density function $f(\cdot)$. Then, the following results hold:*

(a) If D follows a Weibull distribution with shape parameter $\beta>0$, then $C_p^{FIFO}(\widetilde{S}_p)-\mathrm{OPT}_p^{FIFO}=\mathcal{O}\big(p^{-\frac{1}{m\beta}}\cdot(\log p)^{\frac{1}{\beta}-1}\big)$;

(b) If D follows a fat-tailed distribution with parameter $\alpha>1$, then $C_p^{FIFO}(\widetilde{S}_p)-\mathrm{OPT}_p^{FIFO}=\mathcal{O}(p^{-(1-\frac{1}{\alpha})})$;

(c) If D is bounded and $f(\bar{D})>0$, then $C_p^{FIFO}(\widetilde{S}_p)-\mathrm{OPT}_p^{FIFO}=\mathcal{O}(p^{-1})$;

(d) If D follows a triangular distribution, then $C_p^{FIFO}(\widetilde{S}_p)-\mathrm{OPT}_p^{FIFO}=\mathcal{O}(p^{-\frac{1}{2}})$.

Proof of Proposition EC.1. The proofs of parts (a)-(b) are based on inequality (25) and the proofs of parts (c)-(d) are based on inequality (26). For convenience, let $\bar{F}(x)\triangleq 1-F(x)$ for a c.d.f. $F(\cdot)$.

Proof of part (a). Since $\widetilde{S}_p\ge \underline{S}_p$ and from inequality (25), it suffices to prove

$$\mathbb{E}\Big[\big(D-\frac{1}{m}\underline{S}_p\big)^+\Big]\sim p^{-\frac{1}{m\beta}}(\log p)^{\frac{1}{\beta}-1}, \tag{EC.37}$$

where the notation $f(p)\sim g(p)$ represents the limit $\lim_{p\to\infty} f(p)/g(p)$ exists and is positive. Since both $\mathbb{E}[(D-\frac{1}{m}\underline{S}_p)^+]$ and $p^{-\frac{1}{m\beta}}(\log p)^{\frac{1}{\beta}-1}$ converge to zero as $p\to\infty$, by applying L'Hospital's rule, we obtain

$$\lim_{p\to\infty}\frac{\mathbb{E}[(D-\frac{1}{m}\underline{S}_p)^+]}{p^{-\frac{1}{m\beta}}(\log p)^{\frac{1}{\beta}-1}}=\lim_{p\to\infty}\frac{\frac{1}{m}\underline{S}_p'\bar{F}(\frac{1}{m}\underline{S}_p)}{p^{-\frac{1}{m\beta}-1}(\log p)^{\frac{1}{\beta}-1}\big(\frac{1}{m\beta}-(\frac{1}{\beta}-1)(\log p)^{-1}\big)},$$

where $\underline{S}_p'$ denotes the derivative of $\underline{S}_p$ with respect to p. Therefore, it suffices to derive the order of $\underline{S}_p'$ and $\bar{F}(\frac{1}{m}\underline{S}_p)$.

One can verify that $\underline{S}_p=\alpha\big(\log(\frac{p+h}{h+\theta})\big)^{\frac{1}{\beta}}$ under a Weibull distribution for $\bar{F}(\cdot)$ with scale parameter α (>0) and shape parameter β (>0). By taking the derivative with respect to p, we obtain

$$\underline{S}_p'=\frac{\alpha}{\beta(p+h)}\left(\log\Big(\frac{p+h}{h+\theta}\Big)\right)^{\frac{1}{\beta}-1}\sim p^{-1}(\log p)^{\frac{1}{\beta}-1}.$$

In addition, it follows that $\bar{F}(\frac{1}{m}\underline{S}_p) = (\frac{p+h}{h+\theta})^{-\frac{1}{m^\beta}} \sim p^{-\frac{1}{m^\beta}}$. Therefore,

$$\frac{\underline{S}_p'\bar{F}(\frac{1}{m}\underline{S}_p)}{p^{-\frac{1}{m^\beta}-1}(\log p)^{\frac{1}{\beta}-1}\left(\frac{1}{m^\beta}-(\frac{1}{\beta}-1)(\log p)^{-1}\right)} \sim \frac{p^{-1}(\log p)^{\frac{1}{\beta}-1}p^{-\frac{1}{m^\beta}}}{p^{-\frac{1}{m^\beta}-1}(\log p)^{\frac{1}{\beta}-1}\left(\frac{1}{m^\beta}-(\frac{1}{\beta}-1)(\log p)^{-1}\right)} \sim 1,$$

which implies (EC.37).

Proof of part (b). Since $\widetilde{S}_p \geq \underline{S}_p$ and from inequality (25), it suffices to prove

$$\mathbb{E}\Big[\Big(D - \frac{1}{m}\underline{S}_p\Big)^+\Big] \sim p^{-(1-\frac{1}{\alpha})}. \tag{EC.38}$$

Since both $\mathbb{E}\big[(D-\frac{1}{m}\underline{S}_p)^+\big]$ and $p^{-(1-\frac{1}{\alpha})}$ converge to zero as $p\to\infty$, by applying L'Hospital's rule, we obtain

$$\lim_{p\to\infty}\frac{\mathbb{E}[(D-\frac{1}{m}\underline{S}_p)^+]}{p^{-(1-\frac{1}{\alpha})}} = \lim_{p\to\infty}\frac{\frac{1}{m}\underline{S}_p'\bar{F}(\frac{1}{m}\underline{S}_p)}{(1-\frac{1}{\alpha})p^{-(2-\frac{1}{\alpha})}}.$$

Hence again, we only need to derive the order of $\underline{S}_p'$ and $\bar{F}(\frac{1}{m}\underline{S}_p)$.

By definition, the tail function of a fat-tailed distribution satisfies $\bar{F}(x)\sim x^{-\alpha}$ for some $\alpha>1$. Therefore, $\bar{F}(\underline{S}_p) = (h+\theta)/(p+h) \sim \underline{S}_p^{-\alpha}$, which implies that $\underline{S}_p \sim p^{\frac{1}{\alpha}}$. From L'Hospital's rule, we obtain $\underline{S}_p' \sim p^{\frac{1}{\alpha}-1}$. In addition, $\bar{F}(\frac{1}{m}\underline{S}_p)\sim(\frac{1}{m}\underline{S}_p)^{-\alpha}\sim(\underline{S}_p)^{-\alpha}\sim p^{-1}$. Therefore,

$$\frac{\underline{S}_p'\bar{F}(\frac{1}{m}\underline{S}_p)}{p^{-(2-\frac{1}{\alpha})}} \sim \frac{p^{\frac{1}{\alpha}-1}p^{-1}}{p^{-(2-\frac{1}{\alpha})}} \sim 1,$$

which implies (EC.38).

Proof of part (c). From inequality (26), it suffices to prove that $\bar{D}-\underline{S}_p \sim p^{-1}$. Since $F(\underline{S}_p) = (p-\theta)/(p+h)$, by taking the derivative with respect to p on both sides, we obtain $\underline{S}_p'\cdot f(\underline{S}_p) = \frac{h+\theta}{(p+h)^2}$. Since $f(\bar{D})>0$, we have

$$\lim_{p\to\infty}\frac{\bar{D}-\underline{S}_p}{p^{-1}} = \lim_{p\to\infty}\frac{\underline{S}_p'}{p^{-2}} = \lim_{p\to\infty}\frac{(h+\theta)}{p^{-2}(p+h)^2 f(\underline{S}_p)} = \frac{h+\theta}{f(\bar{D})} \in (0,\infty),$$

which implies that $\bar{D}-\underline{S}_p\sim p^{-1}$.

Proof of part (d). From inequality (26), it suffices to prove that $\bar{D}-\underline{S}_p\sim p^{-\frac{1}{2}}$. Let $[\underline{D},\bar{D}]$ be the support of the triangular distribution and a be the mode. Then, the tail distribution function $\bar{F}(\cdot)$ on $[\underline{D},\bar{D}]$ is given by

$$\bar{F}(x) = \begin{cases} 1-\frac{(x-\underline{D})^2}{(\bar{D}-\underline{D})(a-\underline{D})}, & \underline{D}\leq x\leq a; \\ \frac{(\bar{D}-x)^2}{(\bar{D}-\underline{D})(\bar{D}-a)}, & a\leq x\leq \bar{D}. \end{cases}$$

When p is sufficiently large, $(h+\theta)/(p+h)$ is very close to zero and therefore, $\underline{S}_p$ satisfies

$$\bar{F}(\underline{S}_p) = \frac{(\bar{D}-\underline{S}_p)^2}{(\bar{D}-\underline{D})(\bar{D}-a)} = \frac{h+\theta}{p+h}.$$

Therefore, $\bar{D}-\underline{S}_p\sim p^{-\frac{1}{2}}$. Q.E.D.

Appendix G. A Class of Asymptotically Optimal Base-Stock Policies

In this appendix, we consider a class of base-stock policies for the classic system under the FIFO issuance policy, denoted as $\{\pi_{\widetilde{S}^{\alpha,\beta}} : \alpha \geq 0, \beta \geq 0\}$, and extend some of our results established for base-stock policy $\pi_{\widetilde{S}}$ in §3 to this class of policies. For any $\alpha \geq 0$ and $\beta \geq 0$, we first define the approximate cost function $\widetilde{C}^{\alpha,\beta}(S)$ as

$$\widetilde{C}^{\alpha,\beta}(S) \triangleq h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+] + \theta UL^{\alpha,\beta}(S), \quad \forall S \geq 0, \tag{EC.39}$$

where

$$UL^{\alpha,\beta}(S) \triangleq \frac{\alpha}{m}\mathbb{E}\left[\left(S - \sum_{i=1}^{m} D_i\right)^+\right] + \beta\mathbb{E}\left[\left(\frac{S}{m} - D\right)^+\right].$$

That is, we construct the approximate cost function $\widetilde{C}^{\alpha,\beta}(S)$ by approximating the term $\mathbb{E}[O_\infty(S)]$ by a non-negative linear combination of its lower bound $\frac{1}{m}\mathbb{E}[(S - \sum_{i=1}^{m} D_i)^+]$ and its upper bound $\mathbb{E}[(\frac{S}{m} - D)^+]$ (see Lemma 2). Then, we define $\widetilde{S}^{\alpha,\beta}$ as the minimizer of function $\widetilde{C}^{\alpha,\beta}(S)$, i.e.,

$$\widetilde{S}^{\alpha,\beta} = \inf \arg\min_{S \geq 0} \widetilde{C}^{\alpha,\beta}(S).$$

Similar to inequality (5), one can easily verify that

$$F^{-1}\left(\frac{p}{p+h+\theta(\alpha+\beta)}\right) \leq \widetilde{S}^{\alpha,\beta} \leq F^{-1}\left(\frac{p}{p+h}\right) \tag{EC.40}$$

The following proposition extends Theorem 1 to 4 to the class of base-stock policies $\{\pi_{\widetilde{S}^{\alpha,\beta}} : \alpha \geq 0, \beta \geq 0\}$ under certain conditions.

Proposition EC.2. *(a) When $\beta = 0$, the optimality gap of policy $\pi_{\widetilde{S}^{\alpha,0}}$ converges to zero exponentially fast in the lifetime m;*

(b) Under the assumption of Theorem 2, the optimality gap of policy $\pi_{\widetilde{S}^{\alpha,\beta}}$ converges to zero exponentially fast in the demand population size n;

(c) When $\alpha + \beta = 1$ or demand D is bounded, $\lim_{p\to\infty}(C_p(\widetilde{S}_p^{\alpha,\beta}) - \mathrm{OPT}_p) = 0$*;*

(d) When $\alpha + \beta > 0$, $\lim_{\theta\to\infty}(C_\theta(\widetilde{S}_\theta^{\alpha,\beta}) - \mathrm{OPT}_\theta) = 0$.

Part (a) shows that Theorem 1 holds for policy $\pi_{\widetilde{S}^{\alpha,\beta}}$ when $\beta = 0$. When $\beta > 0$, it remains unknown whether the optimality gap of $\pi_{\widetilde{S}^{\alpha,\beta}}$ converges to zero exponentially fast in the lifetime m. Part (b) extends Theorem 2 to policy $\pi_{\widetilde{S}^{\alpha,\beta}}$ with arbitrary $\alpha \geq 0$ and $\beta \geq 0$. Part (c) reveals two cases under which $\pi_{\widetilde{S}^{\alpha,\beta}}$ is asymptotically optimal with large p. When $\alpha + \beta = 1$, the difference between $UL^{\alpha,\beta}(S)$ and $\frac{1}{m}\mathbb{E}[(S - \sum_{i=1}^{m} D_i)^+]$ converges to zero when $S \to \infty$. Therefore, these two approximations of $\mathbb{E}[O_\infty(S)]$ are roughly the same when p is large. On the other hand, when D is bounded, both $\widetilde{S}$ and $\widetilde{S}^{\alpha,\beta}$ converge to $\bar{D}$. Thus, in both cases, $\pi_{\widetilde{S}^{\alpha,\beta}}$ is asymptotically optimal with large p. Part (d) can be explained as follows. When either α or β is positive, base-stock level $\widetilde{S}^{\alpha,\beta}$ converges to $\widetilde{S}_\infty$ and the long-run average outdating costs under policy $\widetilde{S}^{\alpha,\beta}$ converges to zero as $\theta \to \infty$. Thus, from the discussion in §3.3, $\pi_{\widetilde{S}^{\alpha,\beta}}$ is asymptotically optimal with large unit outdating costs. Note that when $\alpha = \beta = 0$, $\pi_{S^{NP}}$ is asymptotically optimal with large unit outdating costs only when $S^{NP} \le m\underline{D}$. This is because S^{NP} is independent of θ, and when $S^{NP} > m\underline{D}$, by Lemma 2(b), the long-run average outdating cost under $\pi_{\widetilde{S}^{NP}}$ is positive and increases linearly in θ. Thus, $\pi_{\widetilde{S}^{NP}}$ is *not* asymptotically optimal with large θ when $S^{NP} > m\underline{D}$.

Similar to Proposition EC.1, we can also characterize the convergence rate of the optimality gap for base-stock policy $\pi_{\widetilde{S}^{\alpha,\beta}}$ as p increases, and prove that $\pi_{\widetilde{S}^{\alpha,\beta}}$ satisfies Proposition EC.1 (a)-(b) when $\alpha+\beta=1$ and Proposition EC.1 (c)-(d), established for $\pi_{\widetilde{S}}$. The details are omitted.

Proof of Proposition EC.2. First, we prove parts (a) and (b). Similar to the proofs of Theorems 1 and 2, if we can establish the following upper bound on the optimality gap of policy $\pi_{\widetilde{S}^{\alpha,\beta}}$:

$$C(\widetilde{S}^{\alpha,\beta}) - \mathrm{OPT} \le \theta\bigg(\frac{m-\alpha}{m}\mathbb{E}\Big[\Big(\widetilde{S}^{\alpha,\beta} - \sum_{i=1}^{m} D_i\Big)^+\Big] - \beta\mathbb{E}\Big[\Big(\frac{1}{m}\widetilde{S}^{\alpha,\beta} - D\Big)^+\Big] + \frac{\alpha-1}{m}\mathbb{E}\Big[\Big(\widetilde{S} - \sum_{i=1}^{m} D_i\Big)^+\Big] + \beta\mathbb{E}\Big[\Big(\frac{1}{m}\widetilde{S} - D\Big)^+\Big]\bigg), \tag{EC.41}$$

then the results in Proposition EC.2 can be proven easily using the similar arguments in the proofs of Theorems 1 and 2.

Now, we show inequality (EC.41). From Proposition 2 and the optimality of $\widetilde{S}^{\alpha,\beta}$, we obtain

$$C(\widetilde{S}^{\alpha,\beta}) - \mathrm{OPT} \le C(\widetilde{S}^{\alpha,\beta}) - \widetilde{C}^{\alpha,\beta}(\widetilde{S}^{\alpha,\beta}) + \widetilde{C}^{\alpha,\beta}(\widetilde{S}) - \widetilde{C}(\widetilde{S}). \tag{EC.42}$$

From Lemma 2(a), and the definitions of $C(\cdot)$, $\widetilde{C}^{\alpha,\beta}(\cdot)$ and $UL^{\alpha,\beta}(\cdot)$, we have

$$\begin{aligned} C(\widetilde{S}^{\alpha,\beta}) - \widetilde{C}^{\alpha,\beta}(\widetilde{S}^{\alpha,\beta}) &\le \theta\mathbb{E}\Big[\big(\widetilde{S}^{\alpha,\beta} - \sum_{i=1}^{m} D_i\big)^+\Big] - \theta UL^{\alpha,\beta}(\widetilde{S}^{\alpha,\beta}) \\ &= \theta\Big(1-\frac{\alpha}{m}\Big)\mathbb{E}\Big[\big(\widetilde{S}^{\alpha,\beta} - \sum_{i=1}^{m} D_i\big)^+\Big] - \theta\beta\mathbb{E}\Big[\Big(\frac{\widetilde{S}^{\alpha,\beta}}{m} - D\Big)^+\Big]. \end{aligned} \tag{EC.43}$$

From the definitions of $\widetilde{C}^{\alpha,\beta}(\cdot)$ and $\widetilde{C}(\cdot)$, we also have

$$\widetilde{C}^{\alpha,\beta}(\widetilde{S}) - \widetilde{C}(\widetilde{S}) = \theta\frac{\alpha-1}{m}\mathbb{E}\Big[\big(\widetilde{S} - \sum_{i=1}^{m} D_i\big)^+\Big] + \theta\beta\mathbb{E}\Big[\Big(\frac{\widetilde{S}}{m} - D\Big)^+\Big]. \tag{EC.44}$$

Combining (EC.42)-(EC.44), we obtain inequality (EC.41).

Then, we prove part (c). First, we consider the case with $\alpha+\beta=1$ and unbounded demand. Following the proof of inequality (EC.41) while replacing the use of the upper bound from Lemma 2(a) with that from Lemma 2(a) in inequality (EC.43), we can obtain the following inequality:

$$C_p(\widetilde{S}_p^{\alpha,\beta}) - \mathrm{OPT}_p \le \theta\mathbb{E}\Big[\Big(\frac{\widetilde{S}_p^{\alpha,\beta}}{m} - D\Big)^+\Big] - \theta UL^{\alpha,\beta}(\widetilde{S}_p^{\alpha,\beta}) + \theta UL^{\alpha,\beta}(\widetilde{S}_p) - \frac{\theta}{m}\mathbb{E}\Big[\big(\widetilde{S}_p - \sum_{i=1}^{m} D_i\big)^+\Big].$$

Then, by the definition of $UL^{\alpha,\beta}(\cdot)$ and since $\beta = 1-\alpha$ and $x^+ = x + (-x)^+$ for any $x \in \Re$, after some simple algebra we obtain that

$$C_p(\widetilde{S}_p^{\alpha,\beta}) - \mathrm{OPT}_p \leq \alpha\theta\Bigg(\mathbb{E}\Big[\Big(D - \frac{\widetilde{S}_p^{\alpha,\beta}}{m}\Big)^+\Big] - \frac{1}{m}\mathbb{E}\Big[\big(\sum_{i=1}^m D_i - \widetilde{S}_p^{\alpha,\beta}\big)^+\Big]\Bigg) + (1-\alpha)\theta\Bigg(\mathbb{E}\Big[\Big(D - \frac{\widetilde{S}_p}{m}\Big)^+\Big] - \frac{1}{m}\mathbb{E}\Big[\Big(\sum_{i=1}^m D_i - \widetilde{S}_p\Big)^+\Big]\Bigg). \tag{EC.45}$$

When the demand D is unbounded, one can easily verify that $\lim_{p\to\infty} \widetilde{S}_p^{\alpha,\beta} = \lim_{p\to\infty} \widetilde{S}_p = \infty$, and thus, the RHS of inequality (EC.45) converges to zero as $p \to \infty$. Therefore, $\lim_{p\to\infty}(C_p(\widetilde{S}_p^{\alpha,\beta}) - \mathrm{OPT}_p) = 0$ when $\alpha + \beta = 1$ and demand D is unbounded.

Next, we consider the case with bounded demand (i.e., $\bar{D} < \infty$). In this case, we prove the following inequality:

$$C(\widetilde{S}^{\alpha,\beta}) - \mathrm{OPT} \leq (h+\theta)(\bar{D} - \underline{S}) + (h + \theta(\alpha+\beta))(\bar{D} - \widetilde{S}^{\alpha,\beta}). \tag{EC.46}$$

Together with $\lim_{p\to\infty} \underline{S}_p = \lim_{p\to\infty} \widetilde{S}_p^{\alpha,\beta} = \bar{D}$, this directly leads to $\lim_{p\to\infty}(C(\widetilde{S}_p^{\alpha,\beta}) - \mathrm{OPT}_p) = 0$.

The proof of inequality (EC.46) is analogous to that of inequality (26) for base-stock policy $\pi_{\widetilde{S}}$. The following are the differences. First, since $\widetilde{S}^{\alpha,\beta} \geq F^{-1}(\frac{p}{p+h+\theta(\alpha+\beta)})$ from (EC.40), the inequality $p\mathbb{E}[(D-\widetilde{S})^+] \leq (h+\theta)\big(\bar{D} - \widetilde{S}\big)$ should be modified to $p\mathbb{E}[(D - \widetilde{S}^{\alpha,\beta})^+] \leq (h + \theta(\alpha+\beta))\big(\bar{D} - \widetilde{S}^{\alpha,\beta}\big)$. Based on the above inequality, $\widetilde{S}^{\alpha,\beta} \leq \bar{D}$, and the fact that $\mathbb{E}[O_\infty(S)]$ increases in S (which can be easily proven from the recursion (19)), inequality (EC.14) should be modified accordingly to

$$C(\widetilde{S}^{\alpha,\beta}) \leq h(\bar{D} - \mathbb{E}[D]) + \theta\mathbb{E}\big[O_\infty(\bar{D})\big] + (h + \theta(\alpha+\beta))\big(\bar{D} - \widetilde{S}^{\alpha,\beta}\big).$$

Combining this inequality with the inequality in Proposition 3 and since $\mathbb{E}[O_\infty(S)] - S$ is decreasing in S, we obtain inequality (EC.46). The proof of part (c) is complete.

Finally, we prove part (d). Suppose $\alpha + \beta > 0$. By applying similar arguments to those in the proof of Theorem 4, we can easily show that $\lim_{\theta\to\infty} \theta UL^{\alpha,\beta}(\widetilde{S}_\theta^{\alpha,\beta}) = 0$ and

$$\lim_{\theta\to\infty} h\mathbb{E}[(\widetilde{S}_\theta^{\alpha,\beta} - D)^+] + p\mathbb{E}[(D - \widetilde{S}_\theta^{\alpha,\beta})^+] = \min_{0 \leq S \leq m\underline{D}}\{h\mathbb{E}[(S-D)^+] + p\mathbb{E}[(D-S)^+]\}. \tag{EC.47}$$

Since $\mathbb{E}[O_\infty(S)] \leq \mathbb{E}[(S - \sum_{i=1}^m D_i)^+]$ by Lemma 2(a) and $\mathbb{E}[O_\infty(S)] \leq \mathbb{E}[(S/m - D)^+]$ by Lemma 2(b), it follows from $\lim_{\theta\to\infty} \theta UL^{\alpha,\beta}(\widetilde{S}_\theta^{\alpha,\beta}) = 0$ that $\lim_{\theta\to\infty} \theta\mathbb{E}[O_\infty(\widetilde{S}_\theta^{\alpha,\beta})] = 0$. Therefore,

$$\lim_{\theta\to\infty}(C_\theta(\widetilde{S}_\theta^{\alpha,\beta}) - \mathrm{OPT}_\theta) = \lim_{\theta\to\infty} \theta\mathbb{E}[O_\infty(\widetilde{S}_\theta^{\alpha,\beta})] = 0,$$

where the first equality follows from Lemma 1, (EC.47), and the proof of Theorem 4. Q.E.D.

Appendix H. Figures in Section 7

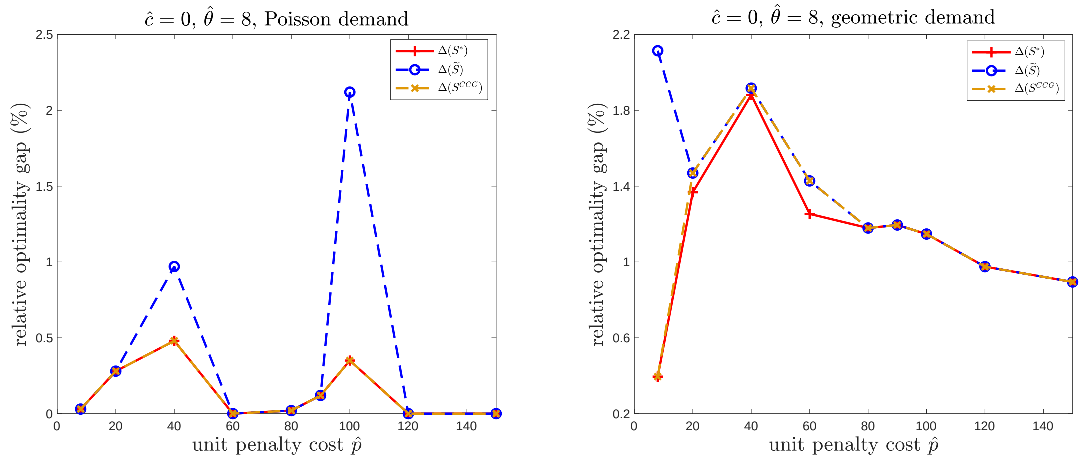

Figure EC.1 **Performances of base-stock policies in the lost-sales system under FIFO issuance policy and different unit penalty costs** $\hat{p} \in \{8, 20, 40, 60, 80, 90, 100, 120, 150\}$**, or correspondingly, under different service levels** $\hat{p}/(\hat{p} + \hat{h}) \times 100\% \in \{88.89\%, 95.24\%, 97.56\%, 98.36\%, 98.77\%, 98.90\%, 99.01\%, 99.17\%, 99.34\%\}$

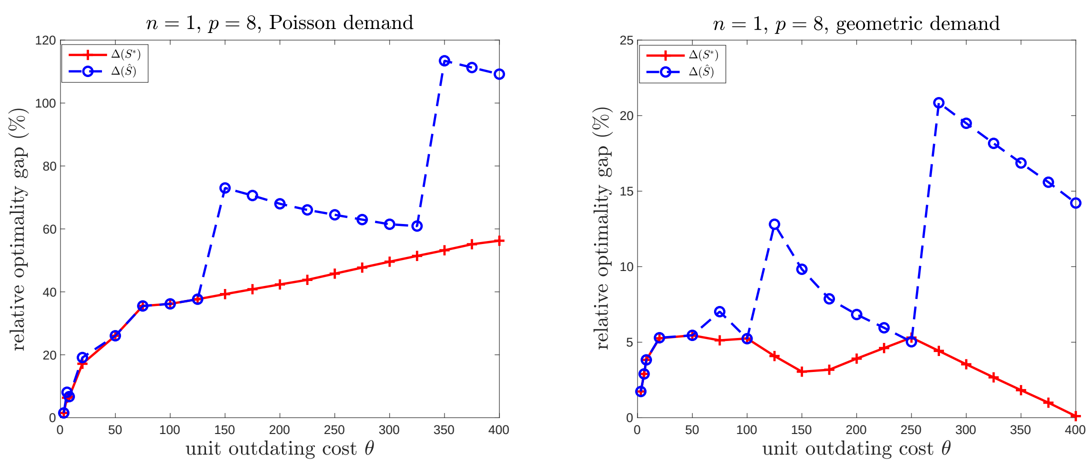

Figure EC.2 **Performances of base-stock policies in the lost-sales system under LIFO issuance policy and different unit outdating costs** $\hat{\theta} \in \{3, 6, 8, 20, 50, 75, 100, 125, 150, 175, 200, 225, 250, 275, 300, 325, 350, 375, 400\}$

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