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A Robust Markowitz Mean-Variance Portfolio Selection Model with an Intractable Claim*

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Abstract. This paper studies a robust Markowitz mean-variance model where an intractable claim is involved in the terminal wealth. The term "intractable claim" refers to claims (rewards or losses) that are completely irrelevant to the underlying market. The payoffs of such claims cannot be predicted or hedged based on the underlying financial market even if the information of the financial market is increasingly available to the investor over time. The target of the investor is to minimize the variance in the worst scenario over all the possible realizations of the underlying intractable claim. Because of the time-inconsistent nature of the problem, both the standard penalization approach and the duality method used to tackle robust stochastic control problems fail in solving our problem. Instead, the quantile formulation approach is adopted to tackle the problem and an explicit closed-form solution is obtained. The properties of the mean-variance frontier are also discussed.

Key words. continuous-time mean-variance problem, intractable claim, background risk, quantile formulation, behavioral finance model, insurance, robust control problem

AMS subject classifications. 91G10, 91G80, 60H30

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1. Introduction.

Since Markowitz [31, 32] published his seminal work on mean-variance portfolio selection in the middle of the last century, the mean-variance portfolio selection framework has been one of the most predominant investment decision rules in financial portfolio selection theory. This framework is concerned with the allocation of wealth among a variety of financial securities so as to achieve a trade-off between the return and the risk at the end of the investment horizon. The return is measured by the expected terminal wealth, while the risk is measured by the variance of the terminal wealth.

Abundant research has been conducted in studying the dynamic mean-variance portfolio problem in both discrete- and continuous-time settings. Models with constraints on wealth process, investment strategies, and market parameters have been addressed. Recent contributions include Li and Ng [26], Zhou, and Li [42], Li, Zhou, and Lim [28], Goldfarb and Iyengar [15], Lim [29], Zhu, Li, and Wang [43], Hu and Zhou [19], Bielecki et al. [2], Sun and Wang [35], Labbé and Heunis [24], Xiong and Zhou [37], Dai, Xu, and Zhou [10], Cui, Li, and Li [7] Czichowsky and Schweizer [9], Heunis [18], Cui et al. [6], and Li and Xu [27]. In these works, the terminal wealth is defined as the investment payoff in the underlying financial market at maturity.

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At the same time, the mean-variance hedging problem has also been widely studied in the literature. Recent contributions on this problem include, among many others, Duffie and Richardson [13], Schweizer [34, 5], Lim [30], Czichowsky and Schweizer [8], and Jeanblanc et al. [20]. The goal of the mean-variance hedging problem is to minimize the distance between the investment payoff in the underlying financial market and a (stochastic or deterministic) claim.

In the existing literature, the payoff of a claim is typically determined by the investor's investment strategy as well as the state of the underlying financial market, that is, a complete specification of all relevant variables describing the financial market over the relevant time horizon. Those claims (rewards or losses) determined by the state of the underlying financial market are called *market-state-tractable contingent claims* or simply *tractable claims* in this paper. We classify them into the following categories:

Known claims: those claims whose future cash flows are known at the initial time; U.S. treasuries are examples of such claims.

Replicatable claims: those claims that can be perfectly replicated in the financial market; the vanilla European options in the Black–Scholes market setting are examples of such claims.

Other state tractable claims: all the other market-state-tractable contingent claims; for instance, options that are not replicable in an incomplete market belong to this category.

On the other hand, in financial and insurance industries, investors often face claims that are irrelevant or less related to the underlying financial market but determined by another (nontradable) "market" (for instance, traffic accidents or lottery market). We call such claims *market-state-intractable contingent claims* or simply *intractable claims*. The intractable claims can also be regarded as one kind of *background risk*; see, e.g., [21, 25, 14] for the models with background risk. It is natural to consider investment models with intractable claims. In this paper, we consider a Markowitz mean-variance problem with an intractable claim. In our model, the terminal wealth consists of an investment payoff X in the financial market and an intractable claim θ. Because, except for some trivial cases, no information is available on the relationship between X and θ, which is needed to determine the value of $\mathbf{Var}(X+\theta)$, we propose an alternative model leading to the following robust cost functional:

$$\inf_{Y\in\mathcal{R}_\theta} \mathbf{Var}(X+Y),$$

where $\mathcal{R}_\theta$ denotes the set of all the possible realizations of the underlying intractable claim θ. Similar problems were considered in the literature; for example, Li [25] examined the demand for a risky asset in the presence of two risks; a financial risk and a background risk, while Jiang, Ma, and An [21] and Franke, Schlesinger, and Stapleton [14] examined the effects of background risks on optimal portfolio choice.

Different forms of $\mathcal{R}_\theta$ lead to different models. In this paper we only focus on one case: $\mathcal{R}_\theta$ is the set of all the random variables having the same distribution function as θ, corresponding to the so-called full information intractable claim. Because this risk measure is a robust version of variance, we call our model the robust Markowitz mean-variance portfolio selection model.

Robustness is concerned with the stability of the estimators of parameters in a given model when model misspecification exists, in particular, in the presence of outlying observations. An abundance of research has been conducted on robust control problems. We refer to Hansen and Sargent [16], Goldfarb and Iyengar [15], Bordigoni, Matoussi, and Schweizer [3], An and Øksendal [1], Jin and Zhou [22], and the references therein on the robust stochastic control problems. In these robust models, the investor's knowledge about each possible scenario increases with time so that the standard penalization approach and duality method can be applied to solve them. In contrast, our knowledge about an intractable claim stays the same before it is realized at the maturity, so the existing standard penalization and duality approaches, both of which heavily depend on the updating of knowledge, cannot be directly applied to tackle the problem. Our model is in this sense significantly different from the classical robust ones and new methodologies are called for in order to analyze and solve it.

A key observation for our problem is that because the distribution or moments of the payoff of the intractable claim are known, the optimal value

$$\inf_{Y\in\mathcal{R}_\theta} \mathbf{Var}(X+Y)$$

is expected to depend only on the distribution of *controllable* financial payoff X. This suggests that we may turn the dynamic stochastic control problem into a static optimization problem and may adopt the so-called quantile formulation to solve the latter. The main idea of the quantile formulation is to change the decision variable from the terminal random payoff to its quantile function, and then apply the convex optimization or the calculus of variations techniques to solve the optimization problem. This approach has attracted great attention in recent years, for it has been proved to be a powerful tool in solving quantitative behavioral finance models under, for example, cumulative prospect theory and rank-dependent utility theory. We refer to He and Zhou [17], Xia and Zhou [36], Xu [38, 39], and Xu and Zhou [41] for the latest development of quantile formulation. In this paper, we adopt Xu's [38] relaxation method to solve the problem and obtain an explicit closed-form solution.

The rest of this paper is organized as follows. In section 2, we formulate a robust Markowitz mean-variance portfolio selection model with an intractable claim. Then, the problem is turned into a quantile formulation problem in section 3. In section 4, we adopt the relaxation method to solve the quantile formulation problem. In section 5, we present our main result, which gives the complete solution of the problem. Section 6 discusses the properties of the mean-variance frontier. Section 7 concludes the paper. Some technical results and proofs are presented in the appendix.

Notation. Throughout this paper, we make use of the following notation:
M^T, the transpose of a matrix or vector M;
$\|M\|, = \sqrt{\sum_{i,j} m_{ij}^2}$ for a matrix or vector $M=(m_{ij})$;
$\mathbb{R}^m$, m-dimensional real Euclidean space.

Throughout this paper, let $T>0$ be a fixed (deterministic) terminal time. Let $(\Omega,\mathcal{F},\mathbf{P},$ $\mathbb{F}=(\mathcal{F}_t : t\in[0,T]))$ be a fixed filtered complete probability space on which is defined a standard m-dimensional Brownian motion

$$W(t)\equiv(W^1(t),W^2(t),\ldots,W^m(t))^T,\quad t\in[0,T],$$

with $W(0) = 0$. It is also assumed that $\mathcal{F}_t$ is equal to $\sigma\{W(s) : 0 \leqslant s \leqslant t\}$ augmented by all the null sets and $\mathcal{F}_T \subsetneq \mathcal{F}$.

Remark 1.1. We remark that $\mathcal{F}_T$ is a proper subset of $\mathcal{F}$. An intractable claim considered below should be $\mathcal{F}$-measurable but may not be $\mathcal{F}_T$-measurable.

The set of all square integrable $\mathcal{F}_T$-measurable random variables is defined as

$$L^2_{\mathcal{F}_T} := \left\{ X \,\middle|\, X \text{ is an } \mathcal{F}_T\text{-measurable random variable, and } \mathbf{E}[X^2] < +\infty \right\},$$

and $L^2_{\mathcal{F}}$ is defined in a similar way.

Given a Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|_{\mathcal{H}}$, we define a Banach space

$$L^2_{\mathbb{F}}([0,T];\mathcal{H}) := \left\{ f(\cdot) \,\middle|\, \begin{array}{l} f(\cdot) \text{ is an } \mathbb{F}\text{-adapted, } \mathcal{H}\text{-valued progressively measurable} \\ \text{process on } [0,T] \text{ and } \|f(\cdot)\|_{\mathbb{F}} < +\infty \end{array} \right\}$$

with the norm

$$\|f(\cdot)\|_{\mathbb{F}} := \left(\mathbf{E}\left[\int_0^T \|f(t,\omega)\|^2_{\mathcal{H}} \,\mathrm{d}t \right] \right)^{\frac{1}{2}}.$$

The quantile $Q_X(\cdot)$ of a real-valued $\mathcal{F}$-measurable random variable X is defined as the right-continuous inverse function of its cumulative distribution function $F_X(\cdot)$, that is,

$$Q_X(t) = \sup\{s \in \mathbb{R} : F_X(s) \leqslant t\} \quad \forall\, t \in (0,1),$$

with convention $\sup\emptyset = -\infty$. We call a function a quantile if it is the quantile of some $\mathcal{F}$-measurable random variable.

2. Problem formulation.

2.1. Market model.

Following Karatzas and Shreve [23], we consider a continuous-time arbitrage-free underlying financial market where $m+1$ assets are traded continuously on $[0,T]$. One of the assets is the *bond*, whose price $S_0(\cdot)$ evolves according to an ordinary differential equation:

$$\begin{cases} \mathrm{d}S_0(t) = r(t)S_0(t)\,\mathrm{d}t, & t \in [0,T], \\ S_0(0) = s_0 > 0, \end{cases}$$

where $r(t)$ is the appreciation rate of the bond at time t. The remaining m assets are *stocks*, and their prices are driven by a system of stochastic differential equations (SDEs):

$$\begin{cases} \mathrm{d}S_i(t) = S_i(t)\{\beta_i(t)\,\mathrm{d}t + \sum_{j=1}^m \sigma_{ij}(t)\,\mathrm{d}W^j(t)\}, & t \in [0,T], \\ S_i(0) = s_i > 0, \end{cases}$$

where $\beta_i(t)$ is the appreciation rate of the stock i and $\sigma_{ij}(t)$ is the volatility coefficient at time $t, i = 1,\ldots,m$. Denote the appreciation rate vector process $\beta(t) := (\beta_1(t),\ldots,\beta_m(t))^T$ and the volatility matrix process $\sigma(t) := (\sigma_{ij}(t))_{m\times m}$. We also define the excess return rate vector process:

$$B(t) := \beta(t) - r(t)\mathbf{1}, \quad t \in [0,T],$$

where $\mathbf{1} = (1, 1, \dots, 1)^T$ is an m-dimensional vector.

We impose the following basic assumptions on the market parameters in this paper:

- The processes of $r(\cdot)$, $\beta(\cdot)$, $B(\cdot)$, and $\sigma(\cdot)$ are all uniformly bounded $\mathbb{F}$-progressively measurable stochastic processes on $[0, T]$; and
- The process $\sigma(\cdot)$ is nonsingular and the price of risk process defined by

$$\theta(t) := \sigma(t)^{-1} B(t), \quad t \in [0, T],$$

  is uniformly bounded and not identically zero on $[0, T]$.

Under these assumptions, the market is complete.

2.2. Investment problem. Now consider an investor in the market and assume that her transactions have no influence on the prices of the assets in the market. Suppose she has an initial wealth $x > 0$ to invest in the underlying financial market over the time period $[0, T]$. Denote by $\pi_i(t)$ the total market value of her wealth invested in stock i at time t, $i = 1, \dots, m$. We assume that short selling is allowed in market so that $\pi_i(t)$ can take negative values. We also assume that the trading of shares takes place continuously in a self-financing fashion (i.e., there is no consumption or income) and the market is frictionless (i.e., the transactions do not incur any fees or costs). We call any

$$\pi(\cdot) := (\pi_1(\cdot), \dots, \pi_m(\cdot))^T \in L^2_{\mathbb{F}}([0, T]; \mathbb{R}^m)$$

an admissible portfolio. The investor's total wealth at time $t \geqslant 0$ corresponding to a portfolio $\pi(\cdot)$ is denoted by $X^\pi(t)$. Then the wealth process $X^\pi(\cdot)$ evolves according to an SDE (see, e.g., Karatzas and Shreve [23]):

$$\text{(1)} \qquad \begin{cases} \mathrm{d}X^\pi(t) = [r(t)X^\pi(t) + \pi(t)^T B(t)]\,\mathrm{d}t + \pi(t)^T \sigma(t)\,\mathrm{d}W(t), \quad t \in [0, T], \\ X^\pi(0) = x. \end{cases}$$

For any admissible portfolio $\pi(\cdot)$, the above SDE admits a unique solution $X^\pi(\cdot) \in L^2_{\mathbb{F}}([0, T]; \mathbb{R}^m)$. We call $X^\pi(\cdot)$ an admissible wealth process and $(X^\pi(\cdot), \pi(\cdot))$ an admissible pair. We also assume that bankruptcy is allowed so that the wealth process may take negative values. Note that the SDE (1) is linear in $(X^\pi(\cdot), \pi(\cdot))$, so the set of all admissible pairs is convex.

Markowitz's mean-variance portfolio selection refers to the problem of, given a favorable targeted mean return ϖ, finding an allowable investment policy $\pi(\cdot)$ (i.e., an admissible portfolio), such that the expected investment payoff $\mathbf{E}[X^\pi(T)]$ is ϖ, while the risk measured by the variance of the terminal wealth, $\mathbf{Var}(X^\pi(T))$ is minimized. Mathematically, it is formulated as follows.

Definition 2.1. *The classical continuous-time Markowitz mean-variance portfolio selection problem, parameterized by ϖ, is*

$$\text{(2)} \qquad \begin{aligned} \inf_{\pi(\cdot)} \quad & \mathbf{Var}(X^\pi(T)), \\ \text{subject to} \quad & \begin{cases} \mathbf{E}[X^\pi(T)] = \varpi, \\ (X^\pi(\cdot), \pi(\cdot)) \textit{ is an admissible pair.} \end{cases} \end{aligned}$$

For simplicity, we also call it the classical problem. We note here that any admissible payoff $X^{\pi}(T)$ must be $\mathcal{F}_T$-measurable.

The problem (2) can be solved by the martingale approach: first find the optimal solution X^* of a static optimization problem

$$\begin{aligned} \inf_{X} \quad & \mathbf{Var}(X), \\ \text{subject to} \quad & \mathbf{E}[X] = \varpi, \quad X \in \mathcal{A}, \end{aligned} \tag{3}$$

where we denote by $\mathcal{A}$ the set of all the admissible payoffs,

$$\mathcal{A} = \{X^{\pi}(T) : (X^{\pi}(\cdot), \pi(\cdot)) \text{ is an admissible pair}\}, \tag{4}$$

and then derive the optimal portfolio $\pi(\cdot)$ by replicating the optimal payoff $X^{\pi}(T) = X^*$. The second step is accomplished by the theory of backward stochastic differential equation (BSDE) developed by Peng [33]. Refer to Bielecki et al. [2] for more details of this martingale approach.

In the classical Markowitz mean-variance portfolio selection model, only the investor's payoff in the underlying financial market is considered. In practice, however, as stated in the introduction, investors often face claims that are irrelevant or less related to the underlying financial market but depend on allother nonunderlying market. The payoffs of such claims are not predictable, replicable, or tractable in the underlying financial market even if all the information of the underlying financial market is available to the investor. On the other hand, the distributions of the payoffs of such claims can be precisely determined and are indeed the only thing available to the investor. We call such claims *market-state-intractable-contingent claims* or simply *intractable claims.*

Intractable claims: those claims about which the only thing we know is their moments and/or distributions.

They have *full information* if their distributions are known or *partial information* if not distributions but only a few moments (such as mean and variance) are known. Lotteries are good examples of intractable claims as nobody could predict their outcomes until they are opened. The moments or distributions of the payoffs of intractable claims may be time-dependent or time-independent. We call the former *time-variant* and the latter *time-invariant* intractable claims. The total payments of car insurance contracts are time-variant, while the payoffs of lotteries are typically time-invariant.

The aforementioned practical examples motivate us to consider a Markowitz mean-variance problem with an intractable claim in the present paper. In our model, the terminal wealth of the investor consists of two parts: an investment payoff $X^{\pi}(T)$ in the underlying financial market and an intractable claim θ (which could be regarded as a reward when its realization is positive, a loss when negative). If we follow the classical Markowitz mean-variance framework, at the outset it seems natural to minimize $\mathbf{Var}(X^{\pi}(T) + \theta)$. However, it is impossible to evaluate this objective because, except for some trivial cases, no information is available on the relationship between $X^{\pi}(T)$ and θ, which is needed to determine the value of $\mathbf{Var}(X^{\pi}(T) + \theta)$. A natural alternative modeling approach is to apply the idea of robust decision making, namely, to minimize the variance in the worst scenario, leading to the following objective functional:

$$\sup_{Y\in\mathcal{R}_\theta} \mathbf{Var}(X^\pi(T)+Y),$$

where $\mathcal{R}_\theta$ denotes the set of all the possible realizations of the underlying intractable claim θ. In this paper, $\mathcal{R}_\theta$ is chosen as the set of all the random variables having the same distribution function as θ, i.e.,

$$\mathcal{R}_\theta = \{Y \mid Y \text{ is an } \mathcal{F}\text{-measurable random variable}, Y \sim \theta\},$$

corresponding to a full information intractable claim. This new risk measure reflects the risk-averse attitude of the investor. Because it is a robust version of variance, we call our model the robust Markowitz mean-variance portfolio selection model. It should be noticed that any possible realization of the underlying intractable claim $Y \in \mathcal{R}_\theta$ must be $\mathcal{F}$-measurable but may not be $\mathcal{F}_T$-measurable.

If the claim θ is deterministic, then $\mathcal{R}_\theta = \{\theta\}$, and consequently,

$$\sup_{Y\in\mathcal{R}_\theta} \mathbf{Var}(X^\pi(T)+Y) = \mathbf{Var}(X^\pi(T)).$$

The new risk measure hence reduces to the standard risk measure of variance, and our model reduces to the classical one. Therefore, our model is a natural extension of the classical one.

Now we state our model as follows.

Definition 2.2. *The continuous-time robust Markowitz mean-variance portfolio selection problem with an intractable claim θ, parameterized by ϖ, is*

$$(5) \qquad \begin{array}{ll} \inf\limits_{\pi(\cdot)} \sup\limits_{Y\in\mathcal{R}_\theta} & \mathbf{Var}(X^\pi(T)+Y), \\ \text{subject to} & \begin{cases} \mathbf{E}[X^\pi(T)+\theta] = \varpi, \\ (X^\pi(\cdot),\pi(\cdot)) \textit{ is an admissible pair.} \end{cases} \end{array}$$

Mathematically, this leads to a stochastic differential game between the investor who chooses the best financial investment portfolio $\pi(\cdot)$ and the "intractable market" who chooses the worst payoff Y:

$$\inf_{\pi(\cdot)} \sup_{Y\in\mathcal{R}_\theta} \mathbf{Var}(X^\pi(T)+Y).$$

To ensure the problem is well-posed, we assume that the intractable claim θ is square integrable throughout the paper, that is,

$$(6) \qquad \mathbf{E}[\theta^2] \equiv \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t < \infty,$$

where $\mathscr{Q}(\cdot)$ denotes the quantile of θ.[1] And consequently, Y is square integrable whenever $Y \in \mathcal{R}_\theta$.

Remark 2.1. In this paper, bankruptcy is allowed. The model can be similarly formulated when bankruptcy is prohibited in the market. Our argument below, with slight modifications, also works for the bankruptcy prohibited case. Refer to Bielecki et al. [2] for more details. We encourage interested readers to give the details of the solution.

[1] Since the distribution of θ is given, its moments are thus determined. Hence, the notation $\mathbf{E}[\theta]$ and $\mathbf{E}[\theta^2]$ would not be vague, although the payoff of θ in each market scenario is unknown.

3. Static problem and its quantile formulation.

Similar to the classical case, the new problem (5) can be solved by the martingale approach: first find the optimal solution X^* of a static optimization problem

$$\begin{aligned} &\inf_{X} \sup_{Y\in\mathcal{R}_\theta} \quad \mathbf{Var}(X+Y), \\ &\text{subject to} \quad \mathbf{E}[X+\theta]=\varpi, \quad X\in\mathcal{A}, \end{aligned} \tag{7}$$

and then derive the optimal portfolio $\pi(\cdot)$ by replicating the optimal payoff $X^\pi(T)=X^*$ by the standard theory of BSDE. From now on, we focus on solving the problem (7) and leave the second step to the interested readers.

Remark 3.1. The model (7) in fact encompasses single-period models as well. For example, it is a single-period model when $\mathcal{A}=\{\sum\lambda_i\xi_i\}$, where ξ_i is the possible payoff of one unit of asset i at the end of the investment horizon and λ_i is the number of shares invested in it.

Let us first reformulate the problem (7) into a more tractable form. Observing that for any feasible solution X of the problem (7) and any $Y\in\mathcal{R}_\theta$, we have

$$\mathbf{E}[X+\theta]=\mathbf{E}[X]+\mathbf{E}[\theta]=\mathbf{E}[X]+\int_0^1\mathscr{Q}(t)\,\mathrm{d}t=\varpi$$

and

$$\begin{aligned}\mathbf{Var}(X+Y)&=\mathbf{E}[(X+Y)^2]-(\mathbf{E}[X+Y])^2=\mathbf{E}[(X+Y)^2]-(\mathbf{E}[X]+\mathbf{E}[Y])^2\\&=\mathbf{E}[(X+Y)^2]-(\mathbf{E}[X]+\mathbf{E}[\theta])^2=\mathbf{E}[(X+Y)^2]-\varpi^2,\end{aligned}$$

the problem (7) is equivalent to[2]

$$\begin{aligned} &\inf_{X} \sup_{Y\in\mathcal{R}_\theta} \quad \mathbf{E}[(X+Y)^2], \\ &\text{subject to} \quad \mathbf{E}[X]=d, \quad X\in\mathcal{A}, \end{aligned} \tag{8}$$

where

$$d:=\varpi-\int_0^1\mathscr{Q}(t)\,\mathrm{d}t$$

represents the targeted mean return at the end of the investment horizon in the underlying financial market.

To solve the problem (8), we need to express the set $\mathcal{A}$ in a more tractable form. The following result is well-known (see, e.g., Bielecki et al. [2]).

Lemma 3.1. *The set of all the admissible payoffs, $\mathcal{A}$ can be expressed as*

$$\mathcal{A}=\{X\in L^2_{\mathcal{F}_T}:\mathbf{E}[\rho X]=x\},$$

where

$$\rho:=\exp\left(-\int_0^T\left(r(s)+\frac{1}{2}\|\theta(s)\|^2\right)\mathrm{d}s+\int_0^T\theta(s)^T\,\mathrm{d}W(s)\right).$$

[2] In this paper, problems are called equivalent if they admit the same optimal solution(s).

We remark that $\rho > 0$ is not a constant as $\theta(\cdot)$ is not identically zero. Moreover, ρ is $\mathcal{F}_T$-measurable.

Now the static the problem (8) is equivalent to

$$\inf_{X\in L^2_{\mathcal{F}_T}} \quad J_0(X), \qquad \text{subject to} \quad \mathbf{E}[X]=d, \quad \mathbf{E}[\rho X]=x, \tag{9}$$

where

$$J_0(X) := \sup_{Y\in\mathcal{R}_\theta} \mathbf{E}[(X+Y)^2].$$

Let us show some properties of the cost function $J_0(\cdot)$.

Lemma 3.2. *The function $J_0(\cdot)$ is finite and convex on $L^2_{\mathcal{F}_T}$.*

Proof. For any $X \in L^2_{\mathcal{F}_T}$, we have

$$J_0(X) = \sup_{Y\in\mathcal{R}_\theta} \mathbf{E}[(X+Y)^2] \leqslant \sup_{Y\in\mathcal{R}_\theta} \mathbf{E}[2(X^2+Y^2)] \\ = \mathbf{E}[2X^2] + \sup_{Y\in\mathcal{R}_\theta} \mathbf{E}[Y^2] = 2\mathbf{E}[X^2] + \mathbf{E}[\theta^2] < \infty,$$

where we used the assumption (6). Thus the function $J_0(\cdot)$ is finite. For any $X_1,\ X_2 \in L^2_{\mathcal{F}_T}$ and $\alpha \in (0,1)$, we have by the convexity of square function

$$\begin{aligned} J_0(\alpha X_1 + (1-\alpha)X_2) &= \sup_{Y\in\mathcal{R}_\theta} \mathbf{E}[(\alpha X_1 + (1-\alpha)X_2 + Y)^2] \\ &\leqslant \sup_{Y\in\mathcal{R}_\theta} \mathbf{E}[\alpha(X_1+Y)^2 + (1-\alpha)(X_2+Y)^2] \\ &\leqslant \alpha \sup_{Y\in\mathcal{R}_\theta} \mathbf{E}[(X_1+Y)^2] + (1-\alpha) \sup_{Y\in\mathcal{R}_\theta} \mathbf{E}[(X_2+Y)^2] \\ &= \alpha J_0(X_1) + (1-\alpha) J_0(X_2). \end{aligned}$$

Therefore, the function $J_0(\cdot)$ is convex. ■

In the subsequent argument, we will frequently use the following well-known result, which is often called the Hoeffding–Frechet bounds or Hardy–Littlewood inequality. Recalling that the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is atomless, we have the following.

Lemma 3.3. *Let $Q_X(\cdot)$ and $Q_Y(\cdot)$ denote the quantiles of $X \in L^2_{\mathcal{F}}$ and $Y \in L^2_{\mathcal{F}}$, respectively. Then X and Y are comonotonic*[3] *if and only if*

$$\sup_{\psi\sim Y} \mathbf{E}[X\psi] = \mathbf{E}[XY] = \int_0^1 Q_X(t)Q_Y(t)\,\mathrm{d}t.$$

Similarly, X and Y are anticomonotonic if and only if

$$\inf_{\psi\sim Y} \mathbf{E}[X\psi] = \mathbf{E}[XY] = \int_0^1 Q_X(t)Q_Y(1-t)\,\mathrm{d}t.$$

Now we are ready to give an explicit expression for $J_0(\cdot)$.

[3]See, e.g., [11, 12] for the properties of comonotonic and anticomonotonic random variables.

Lemma 3.4. *The function $J_0(\cdot)$ is law-invariant*[4] *on $L^2_{\mathcal{F}_T}$ and*

$$J_0(X) = \int_0^1 (Q_X(t) + \mathscr{Q}(t))^2 \,\mathrm{d}t, \tag{10}$$

where $Q_X(\cdot)$ denotes the quantile of X.

Proof. Observing that for any $X \in L^2_{\mathcal{F}_T}$,

$$\begin{aligned} J_0(X) = \sup_{Y\in\mathcal{R}_\theta} \mathbf{E}[(X+Y)^2] &= \sup_{Y\in\mathcal{R}_\theta} (\mathbf{E}[X^2] + \mathbf{E}[Y^2] + 2\mathbf{E}[XY]) \\ &= \mathbf{E}[X^2] + \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t + 2\sup_{Y\in\mathcal{R}_\theta} \mathbf{E}[XY]; \end{aligned}$$

and applying Lemma 3.3 to the last term, we deduce that

$$\begin{aligned} J_0(X) &= \mathbf{E}[X^2] + \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t + 2\sup_{Y\in\mathcal{R}_\theta} \mathbf{E}[XY] \\ &= \int_0^1 Q_X(t)^2\,\mathrm{d}t + \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t + 2\int_0^1 Q_X(t)\mathscr{Q}(t)\,\mathrm{d}t = \int_0^1 (Q_X(t)+\mathscr{Q}(t))^2\,\mathrm{d}t. \end{aligned}$$

Furthermore, it follows that $J_0(\cdot)$ is law-invariant. ∎

Remark 3.2. It seems very hard to show the convexity of $J_0(\cdot)$ via the expression (10). It is also interesting to investigate whether $J_0(\cdot)$ will be still convex if $\mathscr{Q}(\cdot)$ in (10) is replaced by a general nonmonotone function and the financial meaning of such a cost function.

Because $J_0(\cdot)$ is convex, applying the Lagrangian method, the problem (9) is equivalent to

$$\begin{aligned} &\inf_{X\in L^2_{\mathcal{F}_T}} && J_\lambda(X), \\ &\text{subject to} && \mathbf{E}[\rho X] = x, \end{aligned} \tag{11}$$

for some real scalar λ, where

$$\begin{aligned} J_\lambda(X) := J_0(X) - \lambda(\mathbf{E}[X]-d) &= \sup_{Y\in\mathcal{R}_\theta} \mathbf{E}[(X+Y)^2] - \lambda(\mathbf{E}[X]-d) \\ &= \int_0^1 (Q_X(t)+\mathscr{Q}(t))^2\,\mathrm{d}t - \lambda\int_0^1 Q_X(t)\,\mathrm{d}t + \lambda d. \end{aligned} \tag{12}$$

The next lemma then follows immediately from the above results.

Lemma 3.5. *The function $J_\lambda(\cdot)$ is finite, convex, and law-invariant on $L^2_{\mathcal{F}_T}$.*

Let $V_\lambda(x)$ denote the optimal value of the problem (11).

Lemma 3.6. *The function $V_\lambda(\cdot)$ is finite and convex on $(0, +\infty)$.*

[4]A functional is called law-invariant if it gives the same value for any two identically distributed random variables.

Proof. Since $\frac{x}{\mathbf{E}[\rho]}$ is a feasible solution of the problem (11), by definition, we have $V_\lambda(x) \leqslant J_\lambda(\frac{x}{\mathbf{E}[\rho]}) < +\infty$. For any $\alpha \in (0,1)$, $X_1 \in L^2_{\mathcal{F}_T}$, and $X_2 \in L^2_{\mathcal{F}_T}$ such that $\mathbf{E}[\rho X_1] = x_1 > 0$, $\mathbf{E}[\rho X_2] = x_2 > 0$, we have $\mathbf{E}[\rho(\alpha X_1 + (1-\alpha)X_2)] = \alpha x_1 + (1-\alpha)x_2$ and consequently,

$$V_\lambda(\alpha x_1 + (1-\alpha)x_2) \leqslant J_\lambda(\alpha X_1 + (1-\alpha)X_2) \leqslant \alpha J_\lambda(X_1) + (1-\alpha)J_\lambda(X_2),$$

where the last inequality is due to the convexity of $J_\lambda(\cdot)$. Because X_1 and X_2 are arbitrarily chosen, the convexity of $V_\lambda(\cdot)$ is thus proved. ■

Remark 3.3. Because $J_\lambda(\frac{x}{\mathbf{E}[\rho]})$ is a quadratic function in x, $V_\lambda(\cdot)$ is at most quadratic growth.

If we apply the Lagrangian method to solve the problem (11), then we need to consider

$$\inf_{X \in L^2_{\mathcal{F}_T}} J_\lambda(X) - \mu(\mathbf{E}[\rho X] - x).$$

The sign of μ thus plays an important role in this problem. To determine the sign of μ, we have the following key observation. As is well-known, μ is in fact equal to $V'_\lambda(x)$, so the sign of μ is the same as that of $V'_\lambda(x)$. On the other hand, since $V_\lambda(\cdot)$ is convex, to determine the sign of $V'_\lambda(x)$, it suffices to determine the minimizer of $V_\lambda(\cdot)$. This will be done in the following section.

3.1. Minimizer of $V_\lambda(\cdot)$. We want to determine the minimizer of $V_\lambda(\cdot)$ on $\mathbb{R}$. To this end, consider the unconstrained version of the problem (11):

$$\inf_{X \in L^2_{\mathcal{F}_T}} J_\lambda(X). \tag{13}$$

The optimal value of this problem clearly provides a lower bound for that of the problem (11). Let us show that this optimal value is in fact the infimum of $V_\lambda(\cdot)$.

Suppose the problem (13) admits an optimal solution X^*; we then set

$$x^* := \mathbf{E}[\rho X^*]. \tag{14}$$

Clearly, X^* is also an optimal solution of problem

$$\inf_{X \in L^2_{\mathcal{F}_T}} J_\lambda(X), \quad \text{subject to} \quad \mathbf{E}[\rho X] = x^*,$$

whose optimal value is $V_\lambda(x^*)$ by definition. Therefore,

$$V_\lambda(x^*) = J_\lambda(X^*) = \inf_{X \in L^2_{\mathcal{F}_T}} J_\lambda(X) \leqslant V_\lambda(y) \quad \forall\, y \in \mathbb{R}. \tag{15}$$

This means x^* is a minimizer of $V_\lambda(\cdot)$. If the problem (13) does not admit an optimal solution, an approximation argument leads to the same conclusion.

Remark 3.4. We do not know the existence and uniqueness of the solution of the problem (13). We may take any one if the solution is not unique, though we will show below that the solution in fact exists and is unique.

Now let us focus on the problem (13). Recall (12),

$$\inf_{X\in L^2_{\mathcal{F}_T}} J_\lambda(X) = \int_0^1 (Q_X(t)+\mathscr{Q}(t))^2\,\mathrm{d}t - \lambda\int_0^1 Q_X(t)\,\mathrm{d}t + \lambda d$$
$$= \int_0^1 \left(Q_X(t)+\mathscr{Q}(t)-\tfrac{1}{2}\lambda\right)^2 \mathrm{d}t - \frac{1}{4}\lambda^2 + \lambda\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \lambda d. \tag{16}$$

Observe that the objective functional in (16) only depends on the quantile of X. Denote by $\mathcal{Q}$ the set of all quantiles generated by $X \in L^2_{\mathcal{F}_T}$:

$$\mathcal{Q} := \left\{Q(\cdot)\ :\ Q(\cdot) \text{ is the quantile of some } X \in L^2_{\mathcal{F}_T}\right\}.$$

It is not hard to show that

$$\mathcal{Q} = \left\{Q(\cdot)\ :\ Q(\cdot) \text{ is a quantile with } \int_0^1 Q^2(t)\,\mathrm{d}t < \infty\right\}.$$

We now consider the quantile formulation[5] of the problem (16), that is,

$$\inf_{Q\in\mathcal{Q}} \int_0^1 \left(Q(t)+\mathscr{Q}(t)-\tfrac{1}{2}\lambda\right)^2 \mathrm{d}t - \frac{1}{4}\lambda^2 + \lambda\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \lambda d. \tag{17}$$

Observe that the quantile of any optimal solution of the problem (16) solves the above problem.

Theorem 3.5. *The unique optimal solution of the problem* (17) *is given by*

$$Q_0(t) := -\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s + \tfrac{1}{2}\lambda \quad \forall\, t \in (0,1).$$

A proof for this result is given in the appendix.

Because the unique solution of the problem (17) is a constant, the optimal solution of the problem (16) (as well as the problem (13)) is also unique and a constant given by

$$X^* = -\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s + \tfrac{1}{2}\lambda.$$

And consequently, the minimizer x^* of $V_\lambda(\cdot)$ on $\mathbb{R}$ is uniquely given by

$$x^* = \mathbf{E}[\rho X^*] = -\mathbf{E}[\rho]\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s + \tfrac{1}{2}\lambda\mathbf{E}[\rho],$$

and the corresponding minimum value is

$$V_\lambda(x^*) = J_\lambda(X^*) = \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t - \left(\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t\right)^2 - \frac{1}{4}\lambda^2 + \lambda\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \lambda d.$$

[5]See Xu [39] for more about the quantile formulation problem.

Because $V_\lambda(\cdot)$ is a convex function and admits a unique minimizer x^*, we conclude as follows.

Corollary 3.6. *Let*

$$x^* = -\mathbf{E}[\rho]\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s + \tfrac{1}{2}\lambda\mathbf{E}[\rho]. \tag{18}$$

Then the function $V_\lambda(\cdot)$ is strictly increasing on $[x^*, +\infty)$ and strictly decreasing on $(-\infty, x^*]$.

3.2. Quantile formulation.

Because $J_\lambda(\cdot)$ is convex, applying the Lagrangian method, we conclude as follows.

Proposition 3.7. *The problem* (11) *is equivalent to*

$$\inf_{X\in L^2_{\mathcal{F}_T}} J_\lambda(X) - \mu(\mathbf{E}[\rho X] - x) \tag{19}$$

for some μ satisfying

$$\mu \begin{cases} > 0 & \text{if } x > x^*; \\ = 0 & \text{if } x = x^*; \\ < 0 & \text{if } x < x^*, \end{cases}$$

where

$$x^* = -\mathbf{E}[\rho]\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s + \tfrac{1}{2}\lambda\mathbf{E}[\rho].$$

Proof. It follows from Corollary 3.6. ∎

Recalling (12), the problem (19) is

$$\inf_{X\in L^2_{\mathcal{F}_T}} \int_0^1 (Q_X(t)+\mathscr{Q}(t))^2\,\mathrm{d}t - \lambda\int_0^1 Q_X(t)\,\mathrm{d}t - \mu\mathbf{E}[\rho X] + \lambda d + \mu x, \tag{20}$$

and this is equal to

$$\begin{aligned} &\inf_{X\in L^2_{\mathcal{F}_T}} \inf_{\psi\sim X} \int_0^1 (Q_\psi(t)+\mathscr{Q}(t))^2\,\mathrm{d}t - \lambda\int_0^1 Q_\psi(t)\,\mathrm{d}t - \mu\mathbf{E}[\rho\psi] + \lambda d + \mu x \\ &\quad= \inf_{X\in L^2_{\mathcal{F}_T}} \inf_{\psi\sim X} \int_0^1 (Q_X(t)+\mathscr{Q}(t))^2\,\mathrm{d}t - \lambda\int_0^1 Q_X(t)\,\mathrm{d}t - \mu\mathbf{E}[\rho\psi] + \lambda d + \mu x \\ &\quad= \inf_{X\in L^2_{\mathcal{F}_T}} \left(\int_0^1 (Q_X(t)+\mathscr{Q}(t))^2\,\mathrm{d}t - \lambda\int_0^1 Q_X(t)\,\mathrm{d}t - \sup_{\psi\sim X}\ \mu\mathbf{E}[\rho\psi] + \lambda d + \mu x\right). \end{aligned} \tag{21}$$

The inner optimization problem is solved by Lemma 3.3,

$$\sup_{\psi\sim X}\ \mu\mathbf{E}[\rho\psi] = \begin{Bmatrix} \mu\int_0^1 Q_X(t)Q_\rho(t)\,\mathrm{d}t & \text{if } \mu \geqslant 0 \\ \mu\int_0^1 Q_X(t)Q_\rho(1-t)\,\mathrm{d}t & \text{if } \mu < 0 \end{Bmatrix} = |\mu|\int_0^1 Q_X(t)\eta(t)\,\mathrm{d}t, \tag{22}$$

where

$$\eta(t) := Q_\rho(t)\mathbb{1}_{\{\mu\geqslant 0\}} - Q_\rho(1-t)\mathbb{1}_{\{\mu<0\}} = Q_\rho(t)\mathbb{1}_{\{x\geqslant x^*\}} - Q_\rho(1-t)\mathbb{1}_{\{x<x^*\}}, \tag{23}$$

and the last equation is due to Proposition 3.7.

Hence the problem (20) reduces to

$$\inf_{X\in L^2_{\mathcal{F}_T}} \int_0^1 (Q_X(t)+\mathscr{Q}(t))^2 \,\mathrm{d}t - \lambda\int_0^1 Q_X(t)\,\mathrm{d}t - |\mu|\int_0^1 Q_X(t)\eta(t)\,\mathrm{d}t + \lambda d + \mu x$$

or

$$\inf_{X\in L^2_{\mathcal{F}_T}} \int_0^1 (Q_X(t)-\zeta(t))^2\,\mathrm{d}t + C_0, \tag{24}$$

where

$$\zeta(t) := \frac{1}{2}\lambda + \frac{1}{2}|\mu|\eta(t) - \mathscr{Q}(t), \tag{25}$$

and C_0 does not depend on X and is given by

$$C_0 = -\frac{\mu^2}{4}\int_0^1 \eta^2(t)\,\mathrm{d}t - \frac{1}{4}\lambda^2 + \lambda d + \mu x.$$

We remark that the optimal solution of the problem (24) and ρ are comonotonic if $\mu\geqslant 0$ and anticomonotonic if $\mu<0$.

The quantile formulation of the problem (24) is

$$\inf_{Q\in\mathcal{Q}} \int_0^1 (Q(t)-\zeta(t))^2\,\mathrm{d}t + C_0. \tag{26}$$

We will use the relaxation method introduced by Xu [38] to solve this problem. This can also be solved by the calculus of variations method used in Xia and Zhou [36]. Remark that if $\overline{Q}(\cdot)$ is an optimal solution of the problem (26), then the optimal solution of the problem (24) is given by

$$\overline{X} = \begin{cases} \overline{Q}(U) & \text{if } \mu\geqslant 0; \\ \overline{Q}(1-U) & \text{if } \mu<0, \end{cases}$$

where U is any $\mathcal{F}_T$-measurable random variable uniformly distributed on $(0,1)$ and is comonotonic with ρ.

4. Relaxation method.

Rewrite the problem (26) as

$$\begin{aligned}\inf_{Q\in\mathcal{Q}} \int_0^1 (Q(t)-\zeta(t))^2\,\mathrm{d}t + C_0 &= \int_0^1 \Big(Q^2(t) - 2Q(t)\zeta(t)\Big)\,\mathrm{d}t + \int_0^1 \zeta^2(t)\,\mathrm{d}t + C_0 \\ &= \int_0^1 \Big(Q^2(t) + 2Q(t)\varphi'(t)\Big)\,\mathrm{d}t + C_1,\end{aligned} \tag{27}$$

where

$$\varphi(t) := \int_t^1 \zeta(s)\,\mathrm{d}s, \tag{28}$$

and C_1 does not depend on $Q \in \mathcal{Q}$ and is given by

$$C_1 = \int_0^1 \zeta^2(t)\,\mathrm{d}t + C_0 = \int_0^1 \zeta^2(t)\,\mathrm{d}t - \frac{\mu^2}{4}\int_0^1 \eta^2(t)\,\mathrm{d}t - \frac{1}{4}\lambda^2 + \lambda d + \mu x.$$

The key idea of applying the relaxation method to solve the problem (27) is replacing $\varphi(\cdot)$ by some function $\delta(\cdot)$ so that

(i) the new Lagrangian gives a lower bound to that in the problem (27);
(ii) the new problem can be solved by pointwise minimizing the new Lagrangian; and
(iii) there is no gap between the new and old Lagrangians in the pointwise solution.

This approach allows us to solve the problem completely without making any assumptions on the function $\varphi(\cdot)$.

Theorem 4.1. *The unique optimal solution of the problem* (26) *is given by*

$$\overline{Q}(t) := -\delta'(t), \quad t \in (0,1),$$

where $\delta(\cdot)$ is the concave envelope of $\varphi(\cdot)$ on $[0,1]$, that is,

$$\delta(t) := \sup_{0\leqslant a\leqslant t\leqslant b\leqslant 1} \frac{(b-t)\varphi(a) + (t-a)\varphi(b)}{b-a}, \quad t \in [0,1].$$

Furthermore, the optimal solution of the problem (24) *as well as the problem* (19) *is given by*

$$\overline{X} = \begin{cases} \overline{Q}(U) & \text{if } \mu \geqslant 0; \\ \overline{Q}(1-U) & \text{if } \mu < 0, \end{cases} \tag{29}$$

where U is any $\mathcal{F}_T$-measurable random variable uniformly distributed on $(0,1)$ and is comonotonic with ρ.

Its proof is given in the appendix.

5. Lagrangian multipliers and optimal solution. Observe that the problem (19) and the problem (9) are equivalent if there exist λ and μ such that

$$\mathbf{E}[\rho\overline{X}] = x, \qquad \mathbf{E}[\overline{X}] = d,$$

where $\overline{X}$ is defined in (29). We next show the existence of such λ and μ.

Observing that

$$\begin{aligned} \int_0^1 \eta(t)\,\mathrm{d}t &= \int_0^1 \Big(Q_\rho(t)\mathbb{1}_{\{\mu\geqslant 0\}} - Q_\rho(1-t)\mathbb{1}_{\{\mu<0\}}\Big)\,\mathrm{d}t \\ &= \int_0^1 Q_\rho(t)\,\mathrm{d}t\mathbb{1}_{\{\mu\geqslant 0\}} - \int_0^1 Q_\rho(1-t)\,\mathrm{d}t\mathbb{1}_{\{\mu<0\}} \\ &= \int_0^1 Q_\rho(t)\,\mathrm{d}t\mathbb{1}_{\{\mu\geqslant 0\}} - \int_0^1 Q_\rho(t)\,\mathrm{d}t\mathbb{1}_{\{\mu<0\}} \\ &= \mathrm{sgn}(\mu)\int_0^1 Q_\rho(t)\,\mathrm{d}t = \mathrm{sgn}(\mu)\mathbf{E}[\rho], \end{aligned} \tag{30}$$

where $\mathrm{sgn}(\mu) := \mathbb{1}_{\{\mu\geqslant 0\}} - \mathbb{1}_{\{\mu<0\}}$, we have

$$\begin{aligned}(31)\quad \mathbf{E}[\overline{X}] &= \int_0^1 \overline{Q}(t)\,\mathrm{d}t = -\int_0^1 \delta'(t)\,\mathrm{d}t = \delta(0)-\delta(1) = \varphi(0)-\varphi(1) = \int_0^1 \zeta(t)\,\mathrm{d}t \\ &= \frac{1}{2}\lambda + \frac{1}{2}|\mu|\int_0^1 \eta(t)\,\mathrm{d}t - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t = \frac{1}{2}\lambda + \frac{1}{2}|\mu|\mathrm{sgn}(\mu)\mathbf{E}[\rho] - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t \\ &= \frac{1}{2}\lambda + \frac{1}{2}\mu\mathbf{E}[\rho] - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t.\end{aligned}$$

Hence, $\mathbf{E}[\overline{X}] = d$ if and only if

$$(32)\qquad \lambda = 2d + 2\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \mu\mathbf{E}[\rho],$$

which is henceforth assumed. In this case, by (31) and (32), we have

$$(33)\qquad \delta(0) = \varphi(0) = \varphi(0) - \varphi(1) = d.$$

Now our problem reduces to finding μ such that $\mathbf{E}[\rho\overline{X}] = x$ under the condition (32).

We need some properties of the function $\delta(\cdot)$ for further discussion; their proofs are given in the appendix.

Lemma 5.1. *The function δ is continuous and increasing with respect to μ on $[0,+\infty)$ and decreasing on $(-\infty,0]$.*

Lemma 5.2. *We have*

$$\lim_{|\mu|\to\infty} \delta(t) = +\infty \quad \forall\, t\in(0,1).$$

Lemma 5.3. *We have*

$$\lim_{|\mu|\to 0} \delta(t) = (1-t)d \quad \forall\, t\in[0,1].$$

In particular, when $\mu = 0$,

$$\delta(t) = (1-t)d \quad \forall\, t\in[0,1].$$

Now we are ready to show the following.

Proposition 5.1. *For any $x>0$ and $d\in\mathbb{R}$, there exist λ and μ such that $\mathbf{E}[\rho\overline{X}] = x$ and $\mathbf{E}[\overline{X}] = d$. Furthermore,*

$$\lambda = 2d + 2\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \mu\mathbf{E}[\rho].$$

Proof. It suffices to show that there exists μ such that $\mathbf{E}[\rho\overline{X}] = x$ under the condition $\lambda = 2d + 2\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \mu\mathbf{E}[\rho]$.

We consider three different cases. They correspond to the lower, middle, and upper parts of the mean-variance frontier.

(†) $d < \frac{1}{\mathbf{E}[\rho]}x$. We will show that there exists $\mu > 0$ such that $\mathbf{E}[\rho\overline{X}] = x$. Observe that if $\mu > 0$, then

$$\mathbf{E}[\rho\overline{X}] = \int_0^1 \overline{Q}(t)Q_\rho(t)\,\mathrm{d}t = -\int_0^1 \delta'(t)Q_\rho(t)\,\mathrm{d}t = \int_0^1 \delta(t)\,\mathrm{d}Q_\rho(t),$$

where we used Fubini's theorem and $Q_\rho(0) = \delta(1) = 0$. Applying Lemmas 5.1, 5.2, and 5.3 and using the monotone convergence theorem,

$$\lim_{\mu\to+\infty} \mathbf{E}[\rho\overline{X}] = \lim_{\mu\to+\infty} \int_0^1 \delta(t)\,\mathrm{d}Q_\rho(t) = \int_0^1 \lim_{\mu\to+\infty} \delta(t)\,\mathrm{d}Q_\rho(t) = +\infty$$

and

$$\begin{aligned}\lim_{\mu\to 0+} \mathbf{E}[\rho\overline{X}] &= \lim_{\mu\to 0+} \int_0^1 \delta(t)\,\mathrm{d}Q_\rho(t) = \int_0^1 \lim_{\mu\to 0+} \delta(t)\,\mathrm{d}Q_\rho(t)\\ &= \int_0^1 (1-t)d\,\mathrm{d}Q_\rho(t) = d\int_0^1 Q_\rho(t)\,\mathrm{d}t = d\mathbf{E}[\rho] < x,\end{aligned}$$

where we used Fubini's theorem to get the second to last equality. Therefore, there exists $\mu > 0$ such that $\mathbf{E}[\rho\overline{X}] = x$.

(††) $d = \frac{1}{\mathbf{E}[\rho]}x$. In this case, we take $\mu = 0$. Then by Lemma 5.3,

$$\overline{X} = \overline{Q}(U) = -\delta'(U) = d, \quad \mathbf{E}[\rho\overline{X}] = d\mathbf{E}[\rho] = x.$$

(†††) $d > \frac{1}{\mathbf{E}[\rho]}x$. Observe that if $\mu < 0$, then

$$\begin{aligned}\mathbf{E}[\rho\overline{X}] &= \int_0^1 \overline{Q}(t)Q_\rho(1-t)\,\mathrm{d}t = -\int_0^1 \delta'(t)Q_\rho(1-t)\,\mathrm{d}t\\ &= -\int_0^1 \delta'(1-t)Q_\rho(t)\,\mathrm{d}t = \int_0^1 (\delta(1-t)-\delta(0))'Q_\rho(t)\,\mathrm{d}t\\ &= \int_0^1 (\delta(0)-\delta(1-t))\,\mathrm{d}Q_\rho(t) = \int_0^1 (d-\delta(1-t))\,\mathrm{d}Q_\rho(t),\end{aligned}$$

where we used the fact that $Q_\rho(0) = 0$ and Fubini's theorem to get the second to last equality. Applying again Lemmas 5.1, 5.2, and 5.3 and using the monotone convergence theorem, we have

$$\begin{aligned}\lim_{\mu\to-\infty} \mathbf{E}[\rho\overline{X}] &= \lim_{\mu\to-\infty} \int_0^1 (d-\delta(1-t))\,\mathrm{d}Q_\rho(t)\\ &= \int_0^1 \lim_{\mu\to-\infty} (d-\delta(1-t))\,\mathrm{d}Q_\rho(t) = -\infty\end{aligned}$$

and

$$\lim_{\mu\to 0-} \mathbf{E}[\rho\overline{X}] = \lim_{\mu\to 0-} \int_0^1 (d-\delta(1-t))\,\mathrm{d}Q_\rho(t)$$
$$= \int_0^1 \lim_{\mu\to 0-} (d-\delta(1-t))\,\mathrm{d}Q_\rho(t)$$
$$= \int_0^1 (d-td)\,\mathrm{d}Q_\rho(t) = d\int_0^1 Q_\rho(t)\,\mathrm{d}t = d\mathbf{E}[\rho] > x,$$

where we used Fubini's theorem to get the second to last equality. Therefore, there exists $\mu<0$ such that $\mathbf{E}[\rho\overline{X}] = x$.

The proof is complete. ∎

From the proof, we can see the following.

Corollary 5.2. *The implied μ is increasing with respect to x on $\mathbb{R}$.*

This is useful for numerical calculation of the optimal solution and the mean-variance frontier. We also note the following:

- When $d<\frac{1}{\mathbf{E}[\rho]}x$, $\overline{X}$ and ρ are comonotonic. In this case, the initial target d is too low compared with the initial wealth x, and this forces the investor to take unnecessary risk. This property is consistent with the classical model and the optimal portfolio is not efficient.
- When $d>\frac{1}{\mathbf{E}[\rho]}x$, $\overline{X}$ and ρ are anticomonotonic. In this case, the initial target d is relatively high compared with the initial wealth x, and the investor would get a higher return if she would like to take a higher risk. This is also consistent with the classical model and the optimal portfolio is efficient.

Putting all of the results obtained thus far together, we conclude as follows.

Theorem 5.3. *The optimal solution of the problem* (7) *is given by*

$$\overline{X} = \begin{cases} -\delta'(U) & \text{if } d<\frac{1}{\mathbf{E}[\rho]}x;\\ \frac{1}{\mathbf{E}[\rho]}x & \text{if } d=\frac{1}{\mathbf{E}[\rho]}x;\\ -\delta'(1-U) & \text{if } d>\frac{1}{\mathbf{E}[\rho]}x, \end{cases}$$

where U is any $\mathcal{F}_T$-measurable random variable uniformly distributed on $(0,1)$ *and is comonotonic with ρ, the function $\delta(\cdot)$ is given by*

$$\delta(t) = \sup_{0\leqslant a\leqslant t\leqslant b\leqslant 1} \frac{(b-t)\varphi(a)+(t-a)\varphi(b)}{b-a}, \quad t\in[0,1];$$

the function $\varphi(\cdot)$ *is given by*

$$\begin{aligned}\varphi(t) = (1-t)d + (1-t)\left(\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \frac{1}{1-t}\int_t^1 \mathscr{Q}(s)\,\mathrm{d}s\right)\\ - \frac{1}{2}\mu(1-t)\left(\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{1-t}\int_t^1 Q_\rho(s)\,\mathrm{d}s\right)\mathbb{1}_{\{d\leqslant \frac{1}{\mathbf{E}[\rho]}x\}}\\ - \frac{1}{2}\mu(1-t)\left(\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{1-t}\int_0^{1-t} Q_\rho(s)\,\mathrm{d}s\right)\mathbb{1}_{\{d> \frac{1}{\mathbf{E}[\rho]}x\}}, \quad t\in[0,1];\end{aligned}$$

and the implied constant μ *exists and is determined by* $\mathbf{E}[\rho\overline{X}] = x$.

Corollary 5.4. *Under the same assumption of Theorem* 5.3*, if* θ *is a constant, then* $\varphi(t) = \delta(t)$ *for all* $t \in [0,1]$.

Their proofs are given in the appendix. If the intractable claim is a constant, our model as well as the optimal solution reduces to the classical one.

6. Mean-variance frontier.

In this part, we study the properties of the mean-variance frontier of the model. Similar to the classical model, the mean-variance frontier of the robust model is defined as a set in the $\mathbb{R}^2$ plane:

$$\left\{\left(\sup_{Y\in\mathcal{R}_\theta}\sqrt{\mathbf{Var}(\overline{X}+Y)}, \mathbf{E}[\overline{X}+\theta]\right) : \varpi\in\mathbb{R}\right\},$$

where $\overline{X}$ is defined in Theorem 5.3.

Theorem 6.1. *If*

$$\frac{1}{\mathbf{Var}(\rho)}\left(x - \varpi\mathbf{E}[\rho] + \int_0^1 \mathscr{Q}(t)Q_\rho(1-t)\,\mathrm{d}t\right) \leqslant -\sup_{t\in(0,1)}\frac{\mathscr{Q}'(t)}{Q_\rho'(1-t)},$$

then the corresponding mean-variance frontier is linear and given by

$$\left(\frac{1}{\sqrt{\mathbf{Var}(\rho)}}\left(\varpi\mathbf{E}[\rho] - x - \int_0^1 \mathscr{Q}(t)Q_\rho(1-t)\,\mathrm{d}t\right), \varpi\right).$$

If

$$\frac{1}{\mathbf{Var}(\rho)}\left(x - \varpi\mathbf{E}[\rho] + \int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t\right) \geqslant \sup_{t\in(0,1)}\frac{\mathscr{Q}'(t)}{Q_\rho'(t)},$$

then the corresponding mean-variance frontier is linear and given by

$$\left(\frac{1}{\sqrt{\mathbf{Var}(\rho)}}\left(x - \varpi\mathbf{E}[\rho] + \int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t\right), \varpi\right).$$

The financial meaning of this result is as follows. As is well-known, the mean-variance frontier consists of two half-lines when no intractable claim is involved. The above result states that the intractable claim has no impact on the shape of the frontier for extremely high and low target levels of the targeted mean return.

By contrast, it changes the shape of the frontier for the nonextreme case. This can be seen from the following remark.

Remark 6.2. We see that $\varphi''(t) = \mathscr{Q}'(t) - \frac{1}{2}\mu Q'_\rho(t) > 0$ as $\mu \to 0+$ from the proof of Theorem 6.1. This is also true when $\mu \to 0-$. This means φ is not concave when μ is close to zero or, equivalently, d is close to $\frac{1}{\mathbf{E}[\rho]}x$. The corresponding part of the mean-variance frontier is no longer linear.[6]

The assumptions in Theorem 6.1 can be satisfied. The following gives an example in the Black–Scholes market setting.

Corollary 6.3. *If ρ is log-normal distributed and $\mathscr{Q}'(t)$ is bounded,[7] then the mean-variance frontier is linear for big and small ϖ.*

Proof. By Theorem 6.1, it suffices to show that both $\frac{\mathscr{Q}'(t)}{Q'_\rho(1-t)}$ and $\frac{\mathscr{Q}'(t)}{Q'_\rho(t)}$ are bounded on $(0,1)$. By assumption, $\mathscr{Q}'(t)$ is bounded, so it reduces to showing $Q'_\rho(t)$ is uniformly bounded away from zero. In fact,

$$
\begin{aligned}
Q'_\rho(F_\rho(t)) = \frac{1}{F'_\rho(t)} &= t\sqrt{2\pi\,\mathbf{Var}(\rho)}e^{\frac{1}{2\,\mathbf{Var}(\rho)}(\ln(t)-\mathbf{E}[\rho])^2} \\
&= \sqrt{2\pi\,\mathbf{Var}(\rho)}e^{\frac{1}{2\,\mathbf{Var}(\rho)}(\ln(t)-\mathbf{E}[\rho])^2+\ln(t)} \geqslant \sqrt{2\pi\,\mathbf{Var}(\rho)}e^{\mathbf{E}[\rho]-\frac{1}{2}\mathbf{Var}(\rho)}.
\end{aligned}
$$

The right-hand side is a positive constant, so the claim follows. ∎

7. Concluding remarks.

In this paper, we formulate a robust Markowitz mean-variance model where an intractable claim is involved in the final wealth in a continuous-time complete market setting. The occurrence of the intractable claim makes the standard penalization approach and duality method fail in solving the model. Instead, we adopt the quantile formulation technique to tackle the problem and an explicit closed-form solution is eventually obtained. Futhurmore, we also compare the shape of the mean-variance frontier to that of the classical model.

In insurance practice, intractable claims such as car insurance contracts are often priced by multiplying the expected losses by a safety loading factor. The theoretical foundation of this pricing rule is the law of large numbers. In the case of a small number of insureds, such a pricing rule becomes inapplicable or at least questionable. In such a situation, it is natural to consider the worst possible scenario when pricing the contract. On the other hand, insurance companies also need to consider the *best* number of insurance contracts to sell so as to achieve a trade-off between the insurance premiums received from selling these contracts and the risk arising from execution of these contracts. If the insurance companies believe in the so-called utility indifference pricing rule (see Carmona [4]) and adopt our robust variance as their utility, the results obtained in this paper will be able to analytically (and thus numerically) price the intractable claims and determine the best number to sell.

In this paper, we have only considered the mean-variance model with a full information intractable claim. In a series of forthcoming papers, we will study the mean-variance model

[6] We leave the proof for the interested readers.

[7] For example, when θ is uniformly distributed, $\mathscr{Q}'(t)$ is bounded.

with partial information intractable claims (see Xu [40]) as well as the (expected utility or rank-dependent utility) indifference pricing models with full or partial information intractable claims.

Appendix A. One lemma.

Lemma A.1. *Let $f(\cdot) : [0,1] \mapsto \mathbb{R}$ be a square integrable increasing function and let $c \in [0,1]$. Then*

$$\inf_{a\leqslant b}\int_0^c \Big(a+f(t)\Big)^2\,\mathrm{d}t+\int_c^1 \Big(b+f(t)\Big)^2\,\mathrm{d}t=\inf_a\int_0^1 \Big(a+f(t)\Big)^2\,\mathrm{d}t.$$

Proof. Denote

$$g(a)=\int_0^c \Big(a+f(t)\Big)^2\,\mathrm{d}t,\qquad h(b)=\int_c^1 \Big(b+f(t)\Big)^2\,\mathrm{d}t,$$

and the minimizers of g and h are, respectively, denoted by

$$\overline{a}=-\frac{1}{c}\int_0^c f(t)\,\mathrm{d}t,\qquad \overline{b}=-\frac{1}{1-c}\int_c^1 f(t)\,\mathrm{d}t.$$

We remark that $\overline{a}\geqslant\overline{b}$ as f is increasing. Note both g and h are strictly decreasing on $(-\infty,\overline{a}]\cap(-\infty,\overline{b}]=(-\infty,\overline{b}]$ and strictly increasing on $[\overline{a},\infty)\cap[\overline{b},\infty)=[\overline{a},\infty)$; therefore,

$$g(a)+h(b)>g(\overline{b})+h(\overline{b}),\quad a\leqslant b<\overline{b},$$

and

$$g(a)+h(b)>g(\overline{a})+h(\overline{a}),\quad \overline{a}<a\leqslant b.$$

Hence

$$\inf_{a\leqslant b}\ g(a)+h(b)=\inf_{\overline{b}\leqslant a\leqslant b\leqslant\overline{a}} g(a)+h(b).$$

On $[\overline{b},\overline{a}]$, we have g is strictly decreasing and h is strictly increasing; therefore,

$$\inf_{\overline{b}\leqslant a\leqslant b\leqslant\overline{a}} g(a)+h(b)=\inf_{\overline{b}\leqslant a=b\leqslant\overline{a}} g(a)+h(b)=\inf_{\overline{b}\leqslant a\leqslant\overline{a}} g(a)+h(a).$$

Thus,

$$\inf_{a\leqslant b} g(a)+h(b)=\inf_{\overline{b}\leqslant a\leqslant\overline{a}} g(a)+h(a)\geqslant\inf_a\ g(a)+h(a)=\inf_a\int_0^1\Big(a+f(t)\Big)^2\,\mathrm{d}t.$$

On the other hand,

$$\inf_{a\leqslant b} g(a)+h(b)\leqslant\inf_{a=b} g(a)+h(b)=\inf_a\ g(a)+h(a)=\inf_a\int_0^1\Big(a+f(t)\Big)^2\,\mathrm{d}t.$$

The desired result follows. ∎

Appendix B. Proofs.

Proof of Theorem 3.5. For any $Q \in \mathcal{Q}$, set

$$t_Q = \inf\Big\{t \in (0,1) : Q(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda \geqslant 0\Big\}.$$

Because both $Q(t)$ and $\mathscr{Q}(t)$ are increasing, we have

$$Q(t)+\mathscr{Q}(t)-\tfrac{1}{2}\lambda \leqslant Q(t_Q-)+\mathscr{Q}(t)-\tfrac{1}{2}\lambda \leqslant Q(t_Q-)+\mathscr{Q}(t_Q-)-\tfrac{1}{2}\lambda \leqslant 0 \quad \forall\, t \in (0,t_Q) \tag{34}$$

and

$$Q(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda \geqslant Q(t_Q) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda \geqslant Q(t_Q) + \mathscr{Q}(t_Q) - \tfrac{1}{2}\lambda \geqslant 0 \quad \forall\ t \in [t_Q,1) \tag{35}$$

It then follows that

$$\Big|Q(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big| \geqslant \Big|\widetilde{Q}(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big| \quad \forall\, t \in (0,1),$$

where

$$\widetilde{Q}(t) = Q(t_Q-)\mathbb{1}_{(0,t_Q)}(t) + Q(t_Q)\mathbb{1}_{[t_Q,1)}(t) \quad \forall\, t \in (0,1).$$

Observing that $\widetilde{Q}(\cdot)$ is increasing and takes at most two different values, we thus have

$$\begin{aligned}
\int_0^1 \Big(Q(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t &\geqslant \int_0^1 \Big(\widetilde{Q}(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t \\
&\geqslant \inf_{c\in(0,1),\, a\leqslant b} \int_0^1 \Big(a\mathbb{1}_{(0,c)}(t) + b\mathbb{1}_{[c,1)}(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t \\
&= \inf_{c\in(0,1)} \inf_{a\leqslant b} \int_0^c \Big(a + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t + \int_c^1 \Big(b + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t \\
&= \inf_{c\in(0,1)} \inf_{a} \int_0^1 \Big(a + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t = \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t - \Big(\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t\Big)^2,
\end{aligned}$$

where we used Lemma A.1 to get the second to last equation. We note that the lower bound on the right-hand side is achieved at Q_0. So Q_0 is an optimal solution of the problem (17). The uniqueness follows from the strict convexity of quadratic functions. ■

Proof of Theorem 4.1. Let $\delta(\cdot)$ and $\overline{Q}(\cdot)$ be defined as in the assumption. Then $\delta(\cdot)$ is an absolutely continuous function dominating $\varphi(\cdot)$ on $[0,1]$. Observing $\delta(0) = \varphi(0)$ and $\delta(1) = \varphi(1) = 0$ and using Fubini's theorem,

$$\int_0^1 Q(t)(\varphi'(t) - \delta'(t))\,\mathrm{d}t = \int_0^1 \Big(\delta(t) - \varphi(t)\Big)\,\mathrm{d}Q(t) \geqslant 0 \tag{36}$$

for every $Q(\cdot) \in \mathcal{Q}$. The above inequality leads to

$$\begin{aligned}
\int_0^1 \Big(Q^2(t) + 2Q(t)\varphi'(t)\Big)\,\mathrm{d}t &\geqslant \int_0^1 \Big(Q^2(t) + 2Q(t)\delta'(t)\Big)\,\mathrm{d}t \\
&= \int_0^1 \Big(Q(t) + \delta'(t)\Big)^2 - \Big(\delta'(t)\Big)^2 \,\mathrm{d}t \geqslant -\int_0^1 \Big(\delta'(t)\Big)^2 \,\mathrm{d}t \\
&= \int_0^1 \Big(\overline{Q}^2(t) + 2\overline{Q}(t)\delta'(t)\Big)\,\mathrm{d}t.
\end{aligned} \tag{37}$$

To make $\overline{Q}(\cdot)$ an optimal solution of the problem (11), it suffices, by (37), to have

$$\int_0^1 \Big(\overline{Q}^2(t) + 2\overline{Q}(t)\varphi'(t)\Big)\,\mathrm{d}t = \int_0^1 \Big(\overline{Q}^2(t) + 2\overline{Q}(t)\delta'(t)\Big)\,\mathrm{d}t. \tag{38}$$

Observing $\delta(0) = \varphi(0)$ and $\delta(1) = \varphi(1) = 0$ and using Fubini's theorem, the above equality is equivalent to

$$0 = \int_0^1 \delta'(t)\Big(\varphi'(t) - \delta'(t)\Big)\,\mathrm{d}t = \int_0^1 \Big(\delta(t) - \varphi(t)\Big)\,\mathrm{d}\delta'(t).$$

This is true because $\delta'(\cdot)$ is a constant on any subinterval of $\{t \in [0, 1] : \delta(t) \neq \varphi(t)\}$ by the definition of $\delta(\cdot)$. ■

Proof of Lemma 5.1. We take (32) into the definition of φ to get

$$\begin{aligned}
\varphi(t) &= \int_t^1 \zeta(s)\,\mathrm{d}s = \int_t^1 \Big(\frac{1}{2}\lambda + \frac{1}{2}|\mu|\eta(s) - \mathscr{Q}(s)\Big)\,\mathrm{d}s \\
&= \int_t^1 \Big(d + \int_0^1 \mathscr{Q}(r)\,\mathrm{d}r - \frac{1}{2}\mu\mathbf{E}[\rho] + \frac{1}{2}|\mu|\eta(s) - \mathscr{Q}(s)\Big)\,\mathrm{d}s \\
&= (1-t)d + (1-t)\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s - \frac{1}{2}(1-t)\mu\mathbf{E}[\rho] + \frac{1}{2}|\mu|\int_t^1 \eta(s)\,\mathrm{d}s - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s \\
&= (1-t)d + (1-t)\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s - \frac{1}{2}(1-t)|\mu|\int_0^1 \eta(s)\,\mathrm{d}s + \frac{1}{2}|\mu|\int_t^1 \eta(s)\,\mathrm{d}s - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s \\
&= (1-t)d + (1-t)\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s + \frac{1}{2}(1-t)|\mu|\Big(\frac{1}{1-t}\int_t^1 \eta(s)\,\mathrm{d}s - \int_0^1 \eta(s)\,\mathrm{d}s\Big) - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s.
\end{aligned} \tag{39}$$

Observe that η is an increasing function, so we have

$$\frac{1}{1-t}\int_t^1 \eta(s)\,\mathrm{d}s - \int_0^1 \eta(s)\,\mathrm{d}s \geqslant 0 \quad \forall\, t \in (0, 1).$$

And consequently, for each fixed $t \in (0, 1)$, $\varphi(t)$ is continuously increasing with respect to μ on $[0, +\infty)$ and decreasing on $(-\infty, 0]$. Since δ is the concave envelope of φ, it is also continuously increasing with respect to μ on $[0, +\infty)$ and decreasing on $(-\infty, 0]$. ■

Proof of Lemma 5.2. Suppose

$$\frac{1}{1-t}\int_t^1 \eta(s)\,\mathrm{d}s - \int_0^1 \eta(s)\,\mathrm{d}s = 0 \quad \forall\, t \in (0, 1),$$

or equivalently,

$$\int_t^1 \eta(s)\,\mathrm{d}s = (1-t)\int_0^1 \eta(s)\,\mathrm{d}s \quad \forall\, t \in (0, 1).$$

Differentiating both sides with respect to t we get that $\eta(\cdot)$ is a constant. This means ρ is a constant, which contradicts our assumption. We conclude that there exists $t_0 \in (0,1)$ such that

$$\frac{1}{1-t_0}\int_{t_0}^{1} \eta(s)\,\mathrm{d}s - \int_0^1 \eta(s)\,\mathrm{d}s > 0,$$

and by (39),

$$\lim_{|\mu|\to\infty} \varphi(t_0) = +\infty.$$

Observe that δ is concave and (33), so

$$\delta(t) \geqslant \frac{t_0 - t}{t_0}\delta(0) + \frac{t}{t_0}\delta(t_0) \geqslant \frac{t_0 - t}{t_0} d + \frac{t}{t_0}\varphi(t_0) \quad \forall\, t \in [0, t_0];$$

$$\delta(t) \geqslant \frac{1-t}{1-t_0}\delta(t_0) + \frac{t-t_0}{1-t_0}\delta(1) \geqslant \frac{1-t}{1-t_0}\varphi(t_0) \quad \forall\, t \in [t_0, 1].$$

The claim follows immediately. ■

Proof of Lemma 5.3. By the continuity of δ in μ, it suffices to show, when $\mu = 0$,

$$\delta(t) = (1-t)d \quad \forall\, t \in [0,1].$$

Because δ is the smallest concave function dominating φ, it suffices to prove that

$$(1-t)d \geqslant \varphi(t) \quad \forall\, t \in [0,1].$$

In fact, using $\mu = 0$ and (39), we have

$$\varphi(t) - (1-t)d = (1-t)\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s \leqslant 0,$$

where the last inequality is due to $\mathscr{Q}$ begin an increasing function. ■

Proof of Theorem 5.3. We first notice that $\mu > 0$ is equivalent to $d < \frac{1}{\mathbf{E}[\rho]}x$, and $\mu < 0$ is equivalent to $d > \frac{1}{\mathbf{E}[\rho]}x$. Therefore, by (23),

$$\eta(t) = Q_\rho(t)\mathbb{1}_{\{\mu \geqslant 0\}} - Q_\rho(1-t)\mathbb{1}_{\{\mu<0\}} = Q_\rho(t)\mathbb{1}_{\{d \leqslant \frac{1}{\mathbf{E}[\rho]}x\}} - Q_\rho(1-t)\mathbb{1}_{\{d > \frac{1}{\mathbf{E}[\rho]}x\}},$$

and consequently, by (25),

$$\zeta(t) = \frac{1}{2}\lambda + \frac{1}{2}|\mu|\eta(t) - \mathscr{Q}(t) = \frac{1}{2}\lambda + \frac{1}{2}\mu Q_\rho(t)\mathbb{1}_{\{d \leqslant \frac{1}{\mathbf{E}[\rho]}x\}} + \frac{1}{2}\mu Q_\rho(1-t)\mathbb{1}_{\{d > \frac{1}{\mathbf{E}[\rho]}x\}} - \mathscr{Q}(t),$$

and by (28),

$$\begin{aligned}\varphi(t) = \int_t^1 \zeta(s)\,\mathrm{d}s &= \frac{1}{2}\lambda(1-t) + \frac{1}{2}\mu\int_t^1 Q_\rho(s)\,\mathrm{d}s\mathbb{1}_{\{d\leqslant\frac{1}{\mathbf{E}[\rho]}x\}} \\ &\quad + \frac{1}{2}\mu\int_t^1 Q_\rho(1-s)\,\mathrm{d}s\mathbb{1}_{\{d>\frac{1}{\mathbf{E}[\rho]}x\}} - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s \\ &= \left(d + \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \frac{1}{2}\mu\mathbf{E}[\rho]\right)(1-t) + \frac{1}{2}\mu\int_t^1 Q_\rho(s)\,\mathrm{d}s\mathbb{1}_{\{d\leqslant\frac{1}{\mathbf{E}[\rho]}x\}} \\ &\quad + \frac{1}{2}\mu\int_0^{1-t} Q_\rho(s)\,\mathrm{d}s\mathbb{1}_{\{d>\frac{1}{\mathbf{E}[\rho]}x\}} - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s \\ &= (1-t)d + (1-t)\left(\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \frac{1}{1-t}\int_t^1 \mathscr{Q}(s)\,\mathrm{d}s\right) \\ &\quad - \frac{1}{2}\mu(1-t)\left(\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{1-t}\int_t^1 Q_\rho(s)\,\mathrm{d}s\right)\mathbb{1}_{\{d\leqslant\frac{1}{\mathbf{E}[\rho]}x\}} \\ &\quad - \frac{1}{2}\mu(1-t)\left(\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{1-t}\int_0^{1-t} Q_\rho(s)\,\mathrm{d}s\right)\mathbb{1}_{\{d>\frac{1}{\mathbf{E}[\rho]}x\}}.\end{aligned}$$

The left is an easy exercise. ∎

Proof of Corollary 5.4. Suppose $d \leqslant \frac{1}{\mathbf{E}[\rho]}x$. Then

$$\begin{aligned}\varphi(t) &= (1-t)d + (1-t)\left(\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \frac{1}{1-t}\int_t^1 \mathscr{Q}(s)\,\mathrm{d}s\right) \\ &\quad - \frac{1}{2}\mu(1-t)\left(\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{1-t}\int_t^1 Q_\rho(s)\,\mathrm{d}s\right) \\ &= (1-t)d + (1-t)\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s - \frac{1}{2}\mu(1-t)\int_0^1 Q_\rho(s)\,\mathrm{d}s + \frac{1}{2}\mu\int_t^1 Q_\rho(s)\,\mathrm{d}s,\end{aligned}$$

and thus,

$$\varphi'(t) = -d - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \mathscr{Q}(t) + \frac{1}{2}\mu\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{2}\mu Q_\rho(t). \tag{40}$$

If θ is a constant, so is $\mathscr{Q}(\cdot)$, and consequently,

$$\varphi'(t) = -d + \frac{1}{2}\mu\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{2}\mu Q_\rho(t)$$

is decreasing as $\mu \geqslant 0$. This means $\varphi(\cdot)$ is concave, so it coincides with its concave envelope $\delta(\cdot)$. The case $d > \frac{1}{\mathbf{E}[\rho]}x$ can be treated similarly. ∎

Proof of Theorem 6.1. Suppose

$$\frac{1}{\mathbf{Var}(\rho)}\left(x - \varpi\mathbf{E}[\rho] + \int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t\right) \geqslant \sup_{t\in(0,1)}\frac{\mathscr{Q}'(t)}{Q'_\rho(t)}.$$

We will show

$$\mu := \frac{2}{\mathbf{Var}(\rho)}\left(x - \varpi\mathbf{E}[\rho] + \int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t\right)$$

satisfies the requirement in Theorem 5.3. Observing $\mu \geqslant 0$, so $d \leqslant \frac{1}{\mathbf{E}[\rho]}x$. By (40),

$$\varphi'(t) = -d - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \mathscr{Q}(t) + \frac{1}{2}\mu\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{2}\mu Q_\rho(t)$$

and consequently,

$$\varphi''(t) = \mathscr{Q}'(t) - \frac{1}{2}\mu Q_\rho'(t) \leqslant 0.$$

This means $\varphi(\cdot)$ is concave, so $\varphi(\cdot) = \delta(\cdot)$. Thus,

$$\begin{aligned}
\mathbf{E}[\rho\overline{X}] &= \int_0^1 \overline{Q}(t)Q_\rho(t)\,\mathrm{d}t = -\int_0^1 \delta'(t)Q_\rho(t)\,\mathrm{d}t = -\int_0^1 \varphi'(t)Q_\rho(t)\,\mathrm{d}t \\
&= -\int_0^1\left(-d - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \mathscr{Q}(t) + \frac{1}{2}\mu\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{2}\mu Q_\rho(t)\right)Q_\rho(t)\,\mathrm{d}t \\
&= d\int_0^1 Q_\rho(t)\,\mathrm{d}t + \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t\int_0^1 Q_\rho(t)\,\mathrm{d}t - \int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t \\
&\quad - \frac{1}{2}\mu\left(\int_0^1 Q_\rho(t)\,\mathrm{d}t\right)^2 + \frac{1}{2}\mu\int_0^1 Q_\rho^2(t)\,\mathrm{d}t \\
&= d\mathbf{E}[\rho] + \mathbf{E}[\rho]\mathbf{E}[\theta] - \int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t + \frac{1}{2}\mu\,\mathbf{Var}(\rho) \\
&= \varpi\mathbf{E}[\rho] - \int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t + \frac{1}{2}\mu\,\mathbf{Var}(\rho) = x.
\end{aligned}$$

Observing that

$$\varphi'(t) - \mathscr{Q}(t) = -d - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \frac{1}{2}\mu\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{2}\mu Q_\rho(t) = A - \frac{1}{2}\mu Q_\rho(t),$$

where A is independent of t, therefore,

$$\begin{aligned}
&\sup_{Y\in\mathcal{R}_\theta}\mathbf{Var}(\overline{X}+Y) = \sup_{Y\in\mathcal{R}_\theta}\mathbf{E}[(\overline{X}+Y)^2] - (\mathbf{E}[\overline{X}+Y])^2 = \sup_{Y\in\mathcal{R}_\theta}\mathbf{E}[(\overline{X}+Y)^2] - \varpi^2 \\
&= J_0(\overline{X}) - \left(\int_0^1 Q_{\overline{X}}(t) + \mathscr{Q}(t)\,\mathrm{d}t\right)^2 = \int_0^1 (Q_{\overline{X}}(t) + \mathscr{Q}(t))^2\,\mathrm{d}t - \left(\int_0^1 Q_{\overline{X}}(t) + \mathscr{Q}(t)\,\mathrm{d}t\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 (-\delta'(t) + \mathscr{Q}(t))^2 \,\mathrm{d}t - \left(\int_0^1 -\delta'(t) + \mathscr{Q}(t)\,\mathrm{d}t \right)^2 \\
&= \int_0^1 (-\varphi'(t) + \mathscr{Q}(t))^2 \,\mathrm{d}t - \left(\int_0^1 -\varphi'(t) + \mathscr{Q}(t)\,\mathrm{d}t \right)^2 \\
&= \int_0^1 (-A + \frac{1}{2}\mu Q_\rho(t))^2 \,\mathrm{d}t - \left(\int_0^1 -A + \frac{1}{2}\mu Q_\rho(t)\,\mathrm{d}t \right)^2 \\
&= \frac{1}{4}\mu^2 \int_0^1 Q_\rho(t)^2 \,\mathrm{d}t - \frac{1}{4}\mu^2 \left(\int_0^1 Q_\rho(t)\,\mathrm{d}t \right)^2 \\
&= \frac{1}{4}\mu^2\, \mathbf{Var}(\rho) = \frac{1}{\mathbf{Var}(\rho)} \left(x - \varpi \mathbf{E}[\rho] + \int_0^1 \mathscr{Q}(t) Q_\rho(t)\,\mathrm{d}t \right)^2 .
\end{aligned}
$$

This finishes the proof. The other case can be treated in a similar way. ∎

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