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ORIGINAL RESEARCH PAPER

Consensus of semi-Markov multi-agent systems with stochastically unmatched topologies

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Abstract
This paper addresses the consensus problem of semi-Markov multi-agent systems, where the topology switching is stochastically unmatched to the original switching. The unmatched property is described by a stochastically conditional probability and quantized by a given transition probability density function (PDF). Then a consensus protocol based on the aforementioned topology is proposed and includes mode-dependent and -independent control inputs as special cases. Based on the Lyapunov function approach and some enlarging techniques, sufficient LMI conditions are presented to solve the controller directly. All the features such as conditional probability, dwell time and transition probability are involved. An improved method but needing nonsingular expectation of the truncated PDF matrix is also proposed. A practical example is offered so as to verify the effectiveness and superiority of the methods proposed in this study.

1 | INTRODUCTION

In recent years, a lot of studies on multi-agent systems have emerged, due to their widely applications in unmanned aerial vehicle, traffic control, cyber-physical system, network system etc. [1–5]. A main issue about multi-agent systems is consensus problem. It means that all the agents should reach an agreement on a concrete quantity while only using their local information. Up to now, a lot of researches on consensus have been given and divided into several categories according to the research review [6], such as consensus with constraints [7–9], event-based consensus [10–13], consensus over signed network [14–16], consensus of heterogeneous agents [17–19], and application of consensus algorithms [20–22].

Meanwhile, it can be seen from the above mentioned references that most network topologies are fixed. However, the network topology of many multi-agent systems [23–26] usually has switching characteristics due to the change of environment and all kinds of errors. Compared with random switching topologies, Markovian switching is suitable to describe a topology switching with a probability distribution property [27–31]. When more information of operation mode belonging to a stochastic switching topology described by the transition rate or probability matrix, or probability distribution, is used, it could lead to better performance compared with random or deterministic switching topologies. An important feature of traditionally Markovian switching is that the dwell time of each mode follows a memoryless exponential distribution. Moreover, the transition rate or probability from one mode to another is usually constant. Then, there are some problems about its practical application. One problem is that the switching governed by the above traditional Markov process may switch many times during a very small time interval. Due to dwell time τ of the Markov process being an exponential distribution, it is obvious that $F(t) \triangleq \Pr\{\tau \leq t\} = 1 - \mathrm{e}^{-\lambda t}$. Without loss of generality, when $\lambda = 10$ and $t = 0.05$ s, it is computed that $\Pr\{\tau \leq 0.05\} = 0.3935$. It means that the probability of small dwell time such as $\tau \leq 0.05$ s is high. Because the switching frequence on a small time interval is high, it will be harmful to hardware, and no equipment could suffer such fast switchings among subsystems. On the other hand, it is known that semi-Markovian switching [32–36] has advantages compared with deterministic (random) and Markovian switchings. One of them is that its dwell time can be an arbitrary probability distribution, and its mode could sustain for a longer period. In other words, the restriction about distribution constraint on dwell time of Markovian switching could be relaxed to any other distribution instead of only exponential distribution. Since the dwell time can be any distribution, the high occurring probability of small τ such as $\tau \leq 0.05$ s could be avoided. Two practical examples such as end stage renal disease and maternity service models are described in detail in[37] and [38], respectively. Based on these examples, it can be known that sometimes the Markov process that has dwell time belonging to an exponential distribution or switches rapidly in terms of being a small value, is impracticable. Instead, a semi-Markov process is very suitable to handle this problem. And the above mentioned fast switching among modes on short time intervals could be avoided to a certain extent. Though a fast semi-Markov switching was mentioned in [39], it seems to be an contradiction. Based on the descriptions given in reference [39] and here, in our opinions, the so-called contradiction just shows that the semi-Markov process is general and has superiorities in terms of dwell time being any distribution and including fast and slow switchings as special situations. Second, the transition probability among different modes could also be considered in a semi-Markov process. Thus, the superiority of Markovian switching over deterministic switching is also retained. Based on these aspects, it can be said that a semi-Markovian switching could balance deterministic switching and Markov switching well. However, compared with traditionally Markov process, the introduction of semi-Markov process will bring some big difficulties in the system analysis and synthesis. For example, when a Lyapunov function approach is used to study a semi-Markovian jump system, it is not easy to present the conditions with easily solvable forms. Particularly, when some control problems of semi-Markovian jump systems are considered such as [33, 34], it can be seen that the proposed results are not easy to be computed. The main reason is that the transition rate is varying with dwell time and explicitly contained in the given results, while the transition probability between adjacent switching modes plays important roles and should be considered carefully. Therefore, it is necessary to propose concise conditions being tested easily, particularly for multi-agent systems with a Markovian switching topology. When the network topology experiences a semi-Markov switching, some references about consensus could be available, such as [40–43]. By investigating the above mentioned references, it is found that the each agent was deterministic, and only its topology belongs to a Markov or semi-Markov process. When both agent and topology belong to a stochastic process, particularly semi-Markov process, it will be more general not only in theory but also in application. It will contain the above models as special ones. To the present, very few references about multi-agent system having semi-Markov switchings are available. Very recently, the containment control of stochastic multi-agent systems with semi-Markov switching topologies was considered in [44]. Similarly, in this reference, it is also found that the switching signals between agents and topologies are assumed to be same or synchronous. This will be ideal in practical applications sometimes, when the data is transmitted through unreliable communication networks. It may be very hard or impossible to guarantee all the switching signals synchronous all the time, especially for multi-agent systems with many jump agents and some switching topologies. In other words, the above mentioned methods could be referred to be as mode-dependent ones, whose assumption about synchronous switching signals is said to be an ideal one. Though a mode-independent method [45–48] for Markovian jump systems could be applied to deal with the above general case, it will be an absolute method and more conservative. The main reason is that the information about operation mode is totally neglected, even if they are available sometimes. In order to bridge the above two methods, a kind of partially mode-dependent method was proposed in [49]. Though it could balance the above methods suitably in terms of having larger application scope and less conservatism, it still have some limitations. For example, the related Bernoulli variable in addition to switching signal are still fast changed variables. In order to relax the above constraint on switching signal, some improved results were developed in [50]. Meanwhile, some other methods such as disorder method [51], unmatched method [52], were presented to enrich the results about MJSs. Particularly, it can be seen that disorder switching case could be contained in unmatched switching situation. Though many results or methods about various kinds of MJSs emerged, there are still many problems to be considered. Moreover, it can be inferred from the aforesaid references that disorder or unmatched switching could lead to negative effects to the considered system. Motivated by the above discussions, some similar problems about semi-Markov multi-agent systems but with stochastically unmatched topologies are necessary to be addressed. In detail, the switchings between agent and topology are not synchronous but stochastically unmatched. In order to deal with this problem, the first but essential problem is how to model such an unmatched property. Up to now, no reference considers about this issue for semi-Markov multi-agent systems, and a new description method should be proposed. Second, how to make the consensus analysis and develop LMI conditions are also necessarily studied. It is also a big difficulty, since the switching signal is a semi-Markov one. Third, but not the last, how to present the condition with small computation complexity is an inevitable problem and also difficult. The main reason is that so many agents and stochastically unmatched topologies exist simultaneously. All these problems or factors will make the multi-agent systems under stochastically unmatched topologies more difficult and challenging to reach consensus. Particularly, some novel techniques are needed to solve the above mentioned problems. To our best knowledge, very few results are available to design a consensus protocol for such a system. All the facts motivate the current research.

In this paper, the consensus problem of semi-Markov multi-agent systems with stochastically unmatched topologies is studied. The main contributions of this paper are as follows: (1) A stochastically conditional probability is applied to describe the switching mismatch between each agent and network topology, which is further quantized by an PDF; (2) Compared with the traditional methods for switching or Markov jump multi-agent systems, a general consensus protocol is developed to strike a balance between the original and exploited signals; (3) A

Lyapunov function approach as well as some enlarging techniques is applied to make the analysis and obtain LMI conditions ultimately. All the effects of conditional probability, dwell time and transition probability of semi-Markov process are taken into account; 4) When the expectation of the truncated PDF matrix is nonsingular, an improved result could be obtained by applying another different technique. Since all the results are presented with solvable forms, the key idea of this study could be extended other systems or problems directly, such as similar problems with uncertain or partially unknown PDF, pinning control of complexity networks with similar topologies, stabilization of semi-Markovian jump systems with stochastically unmatched controllers, and filter or observer design.

Notation: $\mathscr{E}[\cdot]$ denotes the expectation operator. $\mathbb{N}$ represents the set of natural number. $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the smallest and largest eigenvalues of a symmetric matrix M. "$*$" denotes an ellipsis for the terms induced by symmetry, diag $\{\cdots\}$ for a block-diagonal matrix, and $(M)^{\star} \triangleq M + M^T$.

2 | PROBLEM FORMULATION

Consider a type of semi-Markov multi-agent systems with q agents and defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The dynamic of agent i is described as

$$\dot{x}_i(t) = A_{\eta(t)}x_i(t) + B_{\eta(t)}u_i(t),\ i = 1, 2, \dots, q, \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ is the system state and $u_i(t) \in \mathbb{R}^m$ is the control input. $A_{\eta(t)}$ and $B_{\eta(t)}$ are known matrices of compatible dimensions. $\{\eta(t), t \geq 0\}$ is a semi-Markov process and takes values in a finite set $\mathcal{N} \triangleq \{1, 2, \dots, N\}$. Then, one has

C_1 (Markov property)

$$\Pr\{\eta(t_{k+1}) = j, \tau_k \leq t | \eta(t_k), \dots, \eta(t_0), t_k, \dots, t_0\},$$

$$= \Pr\{\eta(t_{k+1}) = j, \tau_k \leq t | \eta(t_k)\},$$

C_2 (Time homogeneity)

$$\Pr\{\eta(t_{k+1}) = j, \tau_k \leq t | \eta(t_k) = i\},$$

is independent of k, $\forall k \in \mathbb{N}$.

where t_k is the k^{th} jump point of $\{\eta(t), t \geq 0\}$ such that $0 = t_0 < t_1 < t_2 < \dots < t_k < t_{k+1} < \dots, \forall k \in \mathbb{N}$, and $\tau_k \triangleq t_{k+1} - t_k$ is the dwell time between the k-th and $(k+1)^{\text{th}}$ jump points. Then, $\{(\eta(t_k), t_k)\}_{k=0}^{\infty}$ is a time-homogeneous Markov renewal process, and $\{\eta(t_k)\}_{k=0}^{\infty}$ is a time-homogeneous Markov chain. Meanwhile, $\{t_k\}_{k=0}^{\infty}$ is a stochastic process and non-decreasing. The evolution of a semi-Markov process with transition rate matrix $\Lambda \triangleq (\upsilon_{ij}(h(t))) \in \mathbb{R}^{N\times N}$, with $h(t) \triangleq t - \sup\{t_k : t_k \leq t, k \geq 0\}$, is described by

$$\Pr\{\eta(t + h(t)) = j | \eta(t) = i\},$$
$$= \begin{cases} \upsilon_{ij}(h(t))h(t) + o(h(t)), & \text{if } i \neq j \\ 1 + \upsilon_{ii}(h(t))h(t) + o(h(t)), & \text{if } i = j \end{cases}, \tag{2}$$

where $h(t) > 0$, $\lim_{h(t)\to 0}(o(h(t))/h(t)) = 0$, $\upsilon_{ij}(h(t))$, for $i \neq j$, is the transition rate from mode i at time t to mode j at time $t + h(t)$, and $\upsilon_{ii}(h(t)) = -\sum_{j=1,j\neq i}^{N} \upsilon_{ij}(h(t))$. Define $F_i(t) \triangleq \Pr\{\tau_i \leq t\} = \Pr\{t_{k+1} - t_k \leq t | \eta(t_k) = i\}$ and $F_{ij}(t) \triangleq \Pr\{t_{k+1} - t_k \leq t | \eta(t_{k+1}) = j, \eta(t_k) = i\}$, it is known from the semi-Markov process that $F_{ij}(t)$ doesn't depend on j and k and $F_{ij}(t) = F_i(t)$. Then, it is concluded that

$$\Pr\{\eta(t_{k+1}) = j, t_{k+1} - t_k \leq t | \eta(t_k) = i\} = \pi_{ij}F_i(t), \tag{3}$$

where $\pi_{ij} \triangleq \Pr\{\eta(t_{k+1}) = j | \eta(t_k) = i\} \in [0, 1]$, $i \neq j$ and $\forall k \geq 0$, is the transition probability from state i to state j, while $\pi_{ii} = 0$ and satisfied $\sum_{j=1}^{N} \pi_{ij} = 1$. Define $\tau_i^{\min}$ and $\tau_i^{\max}$ such as $\tau_i^{\min} \leq \tau_i \leq \tau_i^{\max}$, $\tau_{\min} = \min_{i\in\mathcal{N}}\{\tau_i^{\min}\}$ and $\tau_{\max} = \max_{i\in\mathcal{N}}\{\tau_i^{\max}\}$, it is obvious that $\tau_{\min} \leq \tau_i^{\min} \leq \tau_i \leq \tau_i^{\max} \leq \tau_{\max}$ for any $i \in \mathcal{N}$. When $i \neq j$, it is obtained that

$$\begin{aligned}
\Pr\{\eta(t) = i\} &= \sum_{k=0}^{\infty} \Pr\{\eta(t) = i, \eta(t_k) = i, t \in [t_k, t_{k+1})\}, \\
&= \sum_{k=0}^{\infty} \Pr\{\tau_i \geq t - t_k, \eta(t_k) = i, t \in [t_k, t_{k+1})\}, \\
&= \sum_{k=0}^{\infty} \Pr\{\tau_i \geq h(t), \eta(t_k) = i, t \in [t_k, t_{k+1})\}, \\
&= \sum_{k=0}^{\infty} \Pr\{\tau_i \geq h(t) | \eta(t_k) = i, t \in [t_k, t_{k+1})\} \\
&\quad \times \Pr\{\eta(t_k) = i, t \in [t_k, t_{k+1})\}, \\
&= \Pr\{\tau_i \geq h(t)\} = 1 - F_i(h(t)),
\end{aligned} \tag{4}$$

while

$$\begin{aligned}
&\Pr\{\eta(t) = i, \eta(t + \Delta t) = j\} \\
&= \sum_{k=0}^{\infty} \Pr\{\eta(t) = i, \eta(t + \Delta t) = j, \\
&\quad \eta(t_k) = i, \eta(t_{k+1}) = j, t \in [t_k, t_{k+1})\}, \\
&= \sum_{k=0}^{\infty} \Pr\{h(t) \leq \tau_i < h(t) + \Delta t, \eta(t_{k+1}) = j
\end{aligned}$$

$$|\eta(t_k) = i, t \in [t_k, t_{k+1})\}\Pr\{\eta(t_k) = i, t \in [t_k, t_{k+1})\},$$
$$= \sum_{k=0}^{\infty} \pi_{ij}\Pr\{h(t) \le \tau_i < h(t) + \Delta t$$
$$|\eta(t_k) = i, t \in [t_k, t_{k+1})\}\Pr\{\eta(t_k) = i, t \in [t_k, t_{k+1})\},$$
$$= \pi_{ij}\Pr\{h(t) \le \tau_i < h(t) + \Delta t\},$$
$$= \pi_{ij}(F_i(h(t) + \Delta t) - F_i(h(t))). \tag{5}$$

Based on the above conditions, it is concluded that

$$\begin{aligned} v_{ij}(h(t)) &= \lim_{\Delta t\to 0} \frac{\Pr\{\eta(t+\Delta t) = j|\eta(t) = i\}}{\Delta t}, \\ &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}\frac{\Pr\{\eta(t+\Delta t) = j, \eta(t) = i\}}{\Pr\{\eta(t) = i\}}, \\ &= \frac{\pi_{ij}}{1 - F_i(h(t))}\lim_{\Delta t\to 0}\frac{F_i(h(t)+\Delta t) - F_i(h(t))}{\Delta t}, \\ &= \pi_{ij}\frac{f_i(h(t))}{1 - F_i(h(t))}, \ \forall i \ne j \in \mathcal{N}, \end{aligned} \tag{6}$$

where $f_i(t)$ is the continuous differentiable bounded density function of $F_i(t)$.

On the other hand, let $\mathcal{V} = \{1, 2, \dots, q\}$ be the set of q agents. The topology graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ describes the interaction among agents, where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the agents' edge set. If there exists an edge between agent i and agent j, then $(i, j) \in \mathcal{E}$. $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is a null graph if its edge set is empty. The structure of $\mathcal{G}$ can be represented by an adjacency matrix $G = [g_{ij}] \in \mathbb{R}^{q\times q}$, and g_{ij} represent the interconnection topology among agents. If agent j is connected with agent i, then $g_{ij} \ne 0$; otherwise $g_{ij} = 0$. When $\mathcal{G}$ is undirected, one gets that $G = G^T$. In this paper, the multi-agent system is assumed to have another random protocol. In detail, $\mathcal{G}(\delta(t))$ switches in a topology set $\{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_M\}$ randomly, where $\mathcal{M} \triangleq \{1, 2, \dots, M\}$. The adjacency matrix is $G^{[\delta(t)]} = (g_{ij}^{[\delta(t)]})_{q\times q}$. Then, the control input of system (1) is designed as

$$u_i(t) = K^{[\delta(t)]}\sum_{j\in\mathcal{V}} g_{ij}^{[\delta(t)]}(x_j(t) - x_i(t)), \forall t \in [t_k, t_{k+1}), \tag{7}$$

where control gain $K^{[\delta(t)]}$ is to be determined. Switching signal $\delta(t)$ here represents which topology to be selected currently, since there are totally M topologies added to system (1). Without loss of generality, it is assumed that $M \le N$. More importantly, $\{\delta(t), t \ge 0\}$ cannot be obtained directly but depends on $\{\eta(t), t \ge 0\}$. Therefore, it is stochastically unmatched to $\{\eta(t), t \ge 0\}$ such as

$$\alpha_{i\ell}^{\xi(t)} \triangleq \Pr\{\delta(t) = \ell\,|\eta(t) = i, \xi(t)\}, \forall i \in \mathcal{N}, \forall \ell \in \mathcal{M}, \forall t \ge 0, \tag{8}$$

where $0 \le \alpha_{i\ell}^{\xi(t)} \le 1$, and stochastic process $\{\xi(t), t \ge 0\}$ has some probability distribution. Moreover, $\alpha_{i\ell}^{\xi(t)}$ is a continuous and random variable. Similar to the truncated Gaussian PDF in [53] and based on definition in [54], the truncated PDF of $\alpha_{i\ell}^{\xi(t)}$ could be defined and computed as

$$\begin{aligned} p(\alpha_{i\ell}^{\xi(t)}) &\triangleq \Pr\left\{\alpha_{i\ell}^{\xi(t)}|0 \le \alpha_{i\ell}^{\xi(t)} \le 1\right\}, \\ &= \frac{\Pr\left\{\alpha_{i\ell}^{\xi(t)}, 0 \le \alpha_{i\ell}^{\xi(t)} \le 1\right\}}{\Pr\left\{0 \le \alpha_{i\ell}^{\xi(t)} \le 1\right\}}, \\ &= \frac{\Pr\left\{\alpha_{i\ell}^{\xi(t)}\right\}}{\Pr\left\{0 \le \alpha_{i\ell}^{\xi(t)} \le 1\right\}}, \\ &= \frac{f\left(\alpha_{i\ell}^{\xi(t)}\right)}{F(1) - F(0)}, \end{aligned} \tag{9}$$

where $f(\cdot) \triangleq \Pr\{\cdot\}$ is the PDF without truncation, and $F(\cdot)$ is the cumulative distribution function of $f(\cdot)$. Then, the truncated PDF matrix is constructed as

$$Q^{\xi(t)} = \begin{bmatrix} p\left(\alpha_{11}^{\xi(t)}\right) & p\left(\alpha_{12}^{\xi(t)}\right) & \cdots & p\left(\alpha_{1M}^{\xi(t)}\right) \\ p\left(\alpha_{21}^{\xi(t)}\right) & p\left(\alpha_{22}^{\xi(t)}\right) & \cdots & p\left(\alpha_{2M}^{\xi(t)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ p\left(\alpha_{N1}^{\xi(t)}\right) & p\left(\alpha_{N2}^{\xi(t)}\right) & \cdots & p\left(\alpha_{NM}^{\xi(t)}\right) \end{bmatrix}, \tag{10}$$

where $p(\alpha_{i\ell}^{\xi(t)})$ is given beforehand. Then, $\mathscr{E}[\alpha_{i\ell}^{\xi(t)}]$ is solved as

$$\begin{aligned} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right] &= \int_0^1 \alpha_{i\ell}^{\xi(t)} p(\alpha_{i\ell}^{\xi(t)}) d\alpha_{i\ell}^{\xi(t)}, \\ &= \int_0^1 \alpha_{i\ell}^{\xi(t)} \frac{f(\alpha_{i\ell}^{\xi(t)})}{F(1) - F(0)} d\alpha_{i\ell}^{\xi(t)}. \end{aligned} \tag{11}$$

Matrix $\mathscr{E}[Q^{\xi(t)}]$ is given to be

$$\mathscr{E}\left[Q^{\xi(t)}\right] \triangleq \begin{bmatrix} \mathscr{E}\left[\alpha_{11}^{\xi(t)}\right] & \mathscr{E}\left[\alpha_{12}^{\xi(t)}\right] & \cdots & \mathscr{E}\left[\alpha_{1M}^{\xi(t)}\right] \\ \mathscr{E}\left[\alpha_{21}^{\xi(t)}\right] & \mathscr{E}\left[\alpha_{22}^{\xi(t)}\right] & \cdots & \mathscr{E}\left[\alpha_{2M}^{\xi(t)}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{E}\left[\alpha_{N1}^{\xi(t)}\right] & \mathscr{E}\left[\alpha_{N2}^{\xi(t)}\right] & \cdots & \mathscr{E}\left[\alpha_{NM}^{\xi(t)}\right] \end{bmatrix}, \tag{12}$$

where $\sum_{\ell=1}^{M} \mathscr{E}[\alpha_{i\ell}^{\xi(t)}] = 1, \forall i \in \mathcal{N}$.

Remark 1. It is said that conditional probability (8) is more general and contains some special cases. First, it includes mode-dependent case as a special one, when $\alpha_{ii}^{\xi(t)} \equiv 1$ and $\mathcal{M} \equiv \mathcal{N}$, $\forall t \ge 0$, $\forall i \in \mathcal{N}$. Then, signals $\eta(t)$ and $\delta(t)$ are always synchronous. Second, mode-independent case is obtained by letting $\alpha_{ik}^{\xi(t)} \equiv 1, \forall t \ge 0, \forall i \in \mathcal{N}, \exists k \in \mathcal{M}$, and k is the only one integer. There will be only one element in set $\mathcal{M}$. Third, when

$\xi(t)$ has nothing to do with $\delta(t)$, $\alpha_{i\ell}^{\xi(t)}$ will be a constant $\alpha_{i\ell}$. A special case similar to [55] could be obtained. Fourth, stochastic process $\xi(t)$ could be specialized as a Gaussian one. Then, the truncated PDF is computed to be $p(\alpha_{i\ell}^{\xi(t)}) = \frac{\frac{1}{\sigma_{i\ell}} f(\frac{\alpha_{i\ell}^{\xi(t)} - \mu_{i\ell}}{\sigma_{i\ell}})}{F(\frac{1-\mu_{i\ell}}{\sigma_{i\ell}}) - F(\frac{0-\mu_{i\ell}}{\sigma_{i\ell}})}$, where $\mu_{i\ell}$ and $\sigma_{i\ell}$ are mean and variance of Gaussian PDF distribution, respectively. The expectation of $\alpha_{i\ell}^{\xi(t)}$ becomes $\mathscr{E}[\alpha_{i\ell}^{\xi(t)}] = \mu_{i\ell} + \frac{f(\frac{0-\mu_{i\ell}}{\sigma_{i\ell}}) - f(\frac{1-\mu_{i\ell}}{\sigma_{i\ell}})}{F(\frac{0-\mu_{i\ell}}{\sigma_{i\ell}}) - F(\frac{1-\mu_{i\ell}}{\sigma_{i\ell}})} \sigma_{i\ell}$.

Define $x = [x_1^T \quad x_2^T \quad \dots \quad x_q^T]^T$, the dynamic system is equal to be

$$\Sigma_{\eta(t)} : \begin{cases} \dot{x}(t) = (I_q \otimes A_{\eta(t)} - L^{[\delta(t)]} \otimes B_{\eta(t)} K^{[\delta(t)]}) x(t), \\ \qquad t \in [t_k, t_{k+1}) \\ x(t_k) = x(t_k^-), k \in \mathbb{N}, \end{cases} \tag{13}$$

where $L^{[\delta(t)]}$ is the Laplacian matrix and defined as $L^{[\delta(t)]} \triangleq D^{[\delta(t)]} - G^{[\delta(t)]}$, and $D^{[\delta(t)]} \triangleq \text{diag}\{d_1^{[\delta(t)]}, \dots, d_N^{[\delta(t)]}\}$, where $d_i^{[\delta(t)]} = \sum_{j \in \mathcal{V}} g_{ij}^{[\delta(t)]}$ is the inner degree of agent i. An important feature of $L^{[\delta(t)]}$ is that all the row sums of $L^{[\delta(t)]}$ are zero, and $e_0 = (1, 1, \dots, 1)^T \in \mathbb{R}^q$ is an eigenvector of $L^{[\delta(t)]}$ associated with the eigenvalue $\lambda_0^{[\delta(t)]} = 0$. Moreover, the Laplacian matrix $L^{[\delta(t)]}$ is positive semi-definite such as $0 = \lambda_1^{[\delta(t)]} < \lambda_2^{[\delta(t)]} \leq \dots \leq \lambda_q^{[\delta(t)]}$. As an undirected topology, the inner degree is equal to the outer degree, then $\mathcal{G}^{[\delta(t)]}$ is symmetric.

Definition 1 ([31]). The multi-agent system (13) is synchronized if its every solution satisfies

$$\lim_{t \to \infty} \mathscr{E}\left[x_j(t) - x_i(t)\right] = 0, \forall i, j \in \mathcal{V}.$$

Define the average state $\bar{x}(t)$ and error vector $\tilde{x}_i(t)$ as follows:

$$\bar{x}(t) = \frac{1}{q} \sum_{j=1}^{q} x_j(t), \tag{14}$$

$$\tilde{x}_i(t) = x_i(t) - \bar{x}(t), i \in \mathcal{V}, \tag{15}$$

it can be said that $\tilde{x}_i(t)$ could be used to measure the state disagreement of $x_i(t)$ to the average state of all agents. Then, one could get $\lim_{t\to\infty} \mathscr{E}[x_j(t) - x_i(t)] = 0$, $\forall i, j \in \mathcal{V}$, if $\lim_{t\to\infty} \mathscr{E}[\tilde{x}_i(t)] = 0, \forall i \in \mathcal{V}$, that is, multi-agent system (13) is consensus. Thus, the consensus problem is transformed to analyze the stability of corresponding system. The control input (7) is rewritten as

$$\tilde{u}_i(t) = K^{[\delta(t)]} \sum_{j \in \mathcal{V}} g_{ij}^{[\delta(t)]} (\tilde{x}_j(t) - \tilde{x}_i(t)), \forall t \in [t_k, t_{k+1}). \tag{16}$$

Define matrix $\tilde{L} \triangleq (\tilde{l}_{ij})_{q \times q} \in \mathbb{R}^{q \times q}$ with its entries given by

$$\tilde{l}_{ij} = \begin{cases} \frac{q-1}{q}, & i = j, \\ -\frac{1}{q}, & i \neq j. \end{cases}$$

The error dynamic system is given to be

$$\tilde{\Sigma}_{\eta(t)} : \begin{cases} \dot{\tilde{x}}(t) = (I_q \otimes A_{\eta(t)} - \tilde{L} L^{[\delta(t)]} \otimes B_{\eta(t)} K^{[\delta(t)]}) \tilde{x}(t), \\ \qquad t \in [t_k, t_{k+1}) \\ \tilde{x}(t_k) = \tilde{x}(t_k^-), k \in \mathbb{N}. \end{cases} \tag{17}$$

For matrix $\tilde{L}$, it is known that there exists an orthogonal matrix $U \in \mathbb{R}^{q \times q}$ such that

$$U^T \tilde{L} U = \begin{bmatrix} I_{q-1} & \mathbf{0}_{q-1} \\ & 0 \end{bmatrix},$$

and

$$U^T L^{[\delta(t)]} U = \begin{bmatrix} \tilde{\mathfrak{L}}^{[\delta(t)]} & \mathbf{0}_{q-1} \\ & 0 \end{bmatrix},$$

where $\tilde{\mathfrak{L}}^{[\delta(t)]} \in \mathbb{R}^{(q-1) \times (q-1)}$ is positive definite. Let $e(t) = (U^T \otimes I_n) \tilde{x}(t)$, it is obtained from (17) that

$$\hat{\Sigma}_{\eta(t)} : \begin{cases} \dot{e}(t) = (I_q \otimes A_{\eta(t)} - \mathfrak{L}^{[\delta(t)]} \otimes B_{\eta(t)} K^{[\delta(t)]}) e(t), \\ \qquad t \in [t_k, t_{k+1}) \\ e(t_k) = e(t_k^-), k \in \mathbb{N}, \end{cases} \tag{18}$$

where

$$\mathfrak{L}^{[\delta(t)]} = \begin{bmatrix} \tilde{\mathfrak{L}}^{[\delta(t)]} & \mathbf{0}_{q-1} \\ & 0 \end{bmatrix}.$$

Let $e(t) = \left[\tilde{e}^T(t) \; e_q^T(t)\right]^T$ and $\tilde{e}(t) = \left[e_1^T(t) \cdots e_{q-1}^T(t)\right]^T$, system (18) is equivalent to the following subsystems:

$$\hat{\Sigma}^1_{\eta(t)} : \begin{cases} \dot{\tilde{e}}(t) = (I_{q-1} \otimes A_{\eta(t)} - \tilde{\mathfrak{L}}^{[\delta(t)]} \otimes B_{\eta(t)} K^{[\delta(t)]}) \tilde{e}(t), \\ \qquad t \in [t_k, t_{k+1}) \\ \tilde{e}(t_k) = \tilde{e}(t_k^-), k \in \mathbb{N}, \end{cases} \tag{19}$$

$$\hat{\Sigma}^2_{\eta(t)} : \begin{cases} \dot{e}_q(t) = A_{\eta(t)} e_q(t), & t \in [t_k, t_{k+1}), \\ e_q(t_k) = e_q(t_k^-), & k \in \mathbb{N}. \end{cases} \tag{20}$$

Then, the above synchronization condition could be guaranteed by $\mathscr{E}[\tilde{e}(t)] \to 0$, such that

$$\mathscr{E}\left[x_i(t)\right] \to \mathscr{E}\left[\bar{x}(t) = \frac{1}{q}\sum_{j=1}^{q} x_j(t)\right], \forall i \in \bar{\mathcal{V}} \triangleq \mathcal{V} \setminus \{q\}. \quad (21)$$

It can be inferred that $\mathscr{E}[\sum_{i=1}^{q} x_i(t)] \to \mathscr{E}[(q-1)\bar{x}(t) + x_q(t)]$. Since $\mathscr{E}[\sum_{i=1}^{q} x_i(t)] \to \mathscr{E}[q\bar{x}(t)]$ holds, it is concluded that $\mathscr{E}[x_q(t)] \to \mathscr{E}[\bar{x}(t)]$. It means that the synchronization of system (17) only needs $\mathscr{E}[x_i(t)] \to \mathscr{E}[\bar{x}(t)]$, $\forall i \in \bar{\mathcal{V}}$, which could be guaranteed by system (18) stochastically stable.

Definition 2 ([36]). System (18) is stochastically stable, if for any initial conditions $\tilde{e}_0 \in \mathbb{R}^{q-1}$, $\eta_0 \in \mathcal{N}$ and $\delta_0 \in \mathcal{M}$, there exists a constant $M(\tilde{e}_0, \eta_0, \delta_0)$ such that

$$\mathscr{E}\left[\int_0^{\infty} \|\tilde{e}(t)\|^2 dt | \tilde{e}_0, \eta_0, \delta_0\right] \le M(\tilde{e}_0, \eta_0, \delta_0).$$

Lemma 1 ([56]). *Let $T_1 = T_1^T$, T_2, T_3 and T_4 be given matrices with appropriate dimensions, where $\underline{\Delta} = \mathrm{diag}\{\underline{\Delta}_1, \dots, \underline{\Delta}_p\}$ and $\overline{\Delta} = \mathrm{diag}\{\overline{\Delta}_1, \dots, \overline{\Delta}_p\}$ with $\underline{\Delta}_i, \overline{\Delta}_i \in \mathbb{R}^{Ni\times Ni}$. Define $\underline{T}_4 = I - T_4\underline{\Delta}$ and $\overline{T}_4 = I - T_4\overline{\Delta}$, where $\det(\underline{T}_4) \ne 0$ and $\det(\overline{T}_4) \ne 0$ should be satisfied. Consider the following inequality*

$$\mathbb{T}(\Delta) = T_1 + \{T_2\Delta(I - T_4\Delta)^{-1}T_3\}^{\star} < 0,$$

holds for any $\Delta \in \Lambda \triangleq \{\mathrm{diag}\{\Delta_1, \dots, \Delta_p\} : \Delta_i \in \{\underline{\Delta}_i, \overline{\Delta}_i\}, i = 1, 2, \dots, p\}$, if there exists matrices S and W such that

$$\begin{bmatrix} \Pi_1 & * \\ \Pi_2 & \Pi_3 \end{bmatrix} < 0,$$

where

$$\Pi_1 = T_1 - T_2\left(\underline{\Delta}S\overline{\Delta}^T\right)^{\star} T_2^T,$$

$$\Pi_2 = T_3 + \left(\underline{T}_4 S\overline{\Delta}^T + \overline{T}_4 S^T \underline{\Delta}^T - W\right) T_2^T,$$

$$\Pi_3 = -\left(\underline{T}_4 S \overline{T}_4^T + W T_4^T\right)^{\star},$$

and matrices $S \in \mathbb{R}^{Np\times Np}$ and $W \in \mathbb{R}^{Np\times Np}$ satisfying $\Delta W + W^T\Delta^T = 0$.

3 | MAIN RESULTS

Theorem 1. *There is a controller (7) such that the multi-agent system (13) could reach consensus, if for given scalars $\mu \ge 1$ and $\gamma_i \in \mathbb{R}^-$, there exist matrices $X_i > 0$, S and $Y^{[\ell]}$, $i \in \mathcal{N}$, $\ell \in \mathcal{M}$, such that*

$$\begin{bmatrix} \bar{\Omega}_{i\kappa} & \hat{\Omega}_{i\kappa} \\ & (-S)^{\star} \end{bmatrix} \le 0, \forall \kappa \in \bar{\mathcal{V}}, \quad (22)$$

$$(1 - 2\mu)X_i + \mu X_j \le 0, \forall j \in \mathcal{N}, \quad (23)$$

$$\mu \sum_{\ell \in \mathcal{N}} \mathscr{E}[e^{\gamma_\ell \tau_\ell}]\pi_{i\ell} - 1 < 0, \quad (24)$$

where

$$\bar{\Omega}_{i\kappa} = \left(A_i S - \sum_{\ell=1}^{M} \lambda_\kappa^{[\ell]} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right] B_i Y^{[\ell]}\right)^{\star} - \gamma_i X_i,$$

$$\hat{\Omega}_{i\kappa} = A_i S - \sum_{\ell=1}^{M} \lambda_\kappa^{[\ell]} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right] B_i Y^{[\ell]} + X_i - S^T,$$

and $\lambda_\kappa^{[\ell]}$, $\forall \kappa \in \bar{\mathcal{V}}$, $\ell \in \mathcal{M}$, is given in (29). Then, one could get the control gain by

$$K^{[\ell]} = Y^{[\ell]} S^{-1} \quad (25)$$

Proof. A Lyapunov function for system (19) is selected to be

$$V(\tilde{e}(t), t, \eta(t)) = \tilde{e}^T(t)\tilde{P}_{\eta(t)}\tilde{e}(t), \quad (26)$$

where $\tilde{P}_{\eta(t)} = I_{q-1} \otimes P_{\eta(t)} \in \mathbb{R}^{(q-1)n\times(q-1)n}$. Weak infinitesimal generator $\mathscr{L}$ is defined for stochastic process $\{\tilde{e}(t), \eta(t), \xi(t)\}$. For each $\eta(t) = i \in \mathcal{N}$, $\forall t \in [t_k, t_{k+1})$, and belong to a semi-Markov process, it is computed based on (8) that

$$\begin{aligned}
&\mathcal{L}V(\tilde{e}(t), t, i) \\
&= \lim_{\varsigma\to 0^+} \frac{1}{\varsigma}\left\{\mathcal{E}\left[V(\tilde{e}(t+\varsigma), \eta(t+\varsigma), t+\varsigma)|\tilde{e}(t), \eta(t) = i, \xi(t)\right]\right. \\
&\quad \left.- V(\tilde{e}(t), \eta(t), t)\right\}, \\
&= \lim_{\varsigma\to 0^+} \frac{1}{\varsigma}\left\{\mathcal{E}\left[\tilde{e}^T(t)(I_{q-1} \otimes A_{\eta(t)} - \tilde{\mathfrak{L}}^{[\delta(t)]} \otimes B_{\eta(t)}K^{[\delta(t)]})^T\right.\right. \\
&\quad \times \tilde{P}_{\eta(t+\varsigma)}\tilde{e}(t)\varsigma + \tilde{e}^T(t)\tilde{P}_{\eta(t+\varsigma)} \\
&\quad \times (I_{q-1} \otimes A_{\eta(t)} - \tilde{\mathfrak{L}}^{[\delta(t)]} \otimes B_{\eta(t)}K^{[\delta(t)]})\tilde{e}(t)\varsigma \\
&\quad \left.\left.+ \tilde{e}^T(t)\tilde{P}_{\eta(t+\varsigma)}\tilde{e}(t)|\tilde{e}(t), \eta(t) = i, \xi(t)\right] - \tilde{e}^T(t)\tilde{P}_{\eta(t)}\tilde{e}(t)\right\}, \\
&= \lim_{\varsigma\to 0^+} \frac{1}{\varsigma}\{\tilde{e}^T(t)(I_{q-1} \otimes A_i^T \mathcal{E}\left[\tilde{P}_{\eta(t+\varsigma)}|\tilde{e}(t), \eta(t) = i, \xi(t)\right] \\
&\quad - \mathcal{E}\left[(\tilde{\mathfrak{L}}^{[\delta(t)]} \otimes B_i K^{[\delta(t)]})^T \tilde{P}_{\eta(t+\varsigma)}|\tilde{e}(t), \eta(t) = i, \xi(t)\right]) \\
&\quad \times \tilde{e}(t)\varsigma + \tilde{e}^T(t)(\mathcal{E}\left[\tilde{P}_{\eta(t+\varsigma)}|\tilde{e}(t), \eta(t) = i, \xi(t)\right] I_{q-1} \otimes A_i
\end{aligned}$$

$$- \mathcal{E}\left[\tilde{P}_{\eta(t+\varsigma)}\tilde{\mathfrak{L}}^{[\delta(t)]} \otimes B_i K^{[\delta(t)]} | \tilde{e}(t), \eta(t) = i, \xi(t)\right])\tilde{e}(t)\varsigma$$

$$+ \tilde{e}^T(t)\mathcal{E}\left[\tilde{P}_{\eta(t+\varsigma)} | \tilde{e}(t), \eta(t) = i, \xi(t)\right]\tilde{e}(t) - \tilde{e}^T(t)\tilde{P}_i\tilde{e}(t)\},$$

$$= \tilde{e}^T(t)[I_{q-1} \otimes (A_i^T P_i)^\star$$

$$- \mathcal{E}\left[\tilde{\mathfrak{L}}^{[\delta(t)]} \otimes (P_i B_i K^{[\delta(t)]})^\star | \tilde{e}(t), \eta(t) = i, \xi(t)\right]]\tilde{e}(t),$$

$$= \tilde{e}^T(t)[I_{q-1} \otimes (P_i A_i)^\star$$

$$- \sum_{\ell=1}^{M} \mathcal{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\tilde{\mathfrak{L}}^{[\ell]} \otimes (P_i B_i K^{[\ell]})^\star]\tilde{e}(t),$$

$$\leq \gamma_i \tilde{e}^T(t) I_{q-1} \otimes P_i \tilde{e}(t),$$

$$= \gamma_i V(\tilde{e}(t), t, i), \forall t \in [t_k, t_{k+1}). \tag{27}$$

It is guaranteed by

$$I_{q-1} \otimes (P_i A_i)^\star - \sum_{\ell=1}^{M} \mathcal{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\tilde{\mathfrak{L}}^{[\ell]} \otimes (P_i B_i K^{[\ell]})^\star - \gamma_i I_{q-1} \otimes P_i \leq 0. \tag{28}$$

Because of matrix $\tilde{\mathfrak{L}}^{[i]}$ connected and undirected, it is known that there exists an orthogonal matrix $U_i \in \mathbb{R}^{(q-1)\times(q-1)}$ such that

$$\Lambda^{[i]} \triangleq U_i^T \tilde{\mathfrak{L}}^{[i]} U_i = \text{diag}\{\lambda_1^{[i]}, \lambda_2^{[i]}, \dots, \lambda_{q-1}^{[i]}\}. \tag{29}$$

Then, inequality (28) is equivalent to

$$I_{q-1} \otimes (P_i A_i)^\star - \sum_{\ell=1}^{M} \mathcal{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\Lambda^{[\ell]} \otimes (P_i B_i K^{[\ell]})^\star - \gamma_i I_{q-1} \otimes P_i \leq 0. \tag{30}$$

It could be guaranteed by

$$(A_i X_i)^\star - \sum_{\ell=1}^{M} \mathcal{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\lambda_\kappa^{[\ell]}\left(B_i K^{[\ell]} X_i\right)^\star - \gamma_i X_i \leq 0, \kappa \in \bar{\mathcal{V}}, \tag{31}$$

where $X_i = P_i^{-1}$. Under representation (25), it is concluded that the above inequality could be implied by condition (22) with pre- and post-multiplying condition its both side with $[I \quad A_i - \sum_{\ell=1}^{M} \mathcal{E}[\alpha_{i\ell}^{\xi(t)}]\lambda_\kappa^{[\ell]} B_i K^{[\ell]}]$ and its transposition. Then, it is obvious that

$$\mathcal{L}V(\tilde{e}(t), i, t) \leq \gamma_i V(\tilde{e}(t), i, t), \forall t \in [t_k, t_{k+1}). \tag{32}$$

For any $t \geq t_0$ and $k \geq 1$, it is obtained based on inequality (27) that

$$\begin{aligned} &\mathcal{E}\left[V(\tilde{e}(t), \eta(t), t)\right] \\ &= \sum_{k=0}^{\infty} \mathcal{E}\left[V(\tilde{e}(t), \eta(t), t) I(N_{\eta(t)} = k)\right], \\ &\leq \max_{i \in \mathcal{N}}\{\mathcal{E}[e^{\gamma_i \tau_i}], 1\} \sum_{k=0}^{\infty} \mathcal{E}\left[V(\tilde{e}(t_k), \eta(t_k), t_k)\right]. \end{aligned} \tag{33}$$

On the other hand, based on condition (23), it could be known that

$$\begin{aligned} X_i^T(P_i - \mu P_j)X_i &= X_i^T X_i^{-1} X_i - \mu X_i^T X_j^{-1} X_i, \\ &\leq X_i - \mu\left(X_i^T + X_i - X_j\right), \\ &= (1 - 2\mu)X_i + \mu X_j \leq 0. \end{aligned} \tag{34}$$

Then, it is obtained that

$$\begin{aligned} &V(\tilde{e}(t_k), \eta(t_k), t_k) - \mu V\left(\tilde{e}(t_k^-), \eta(t_k^-), t_k^-\right) \\ &= V\left(\tilde{e}(t_k), i_k, t_k\right) - \mu V\left(\tilde{e}(t_k), i_{k-1}, t_k\right), \\ &= \tilde{e}^T(t_k)(P_i - \mu P_j)\tilde{e}(t_k) \leq 0, \quad \forall i, j \in \mathcal{N}. \end{aligned} \tag{35}$$

Based on condition (34) and via exploiting total probability formula, it is further obtained for any $k \geq 1$ that

$$\begin{aligned} &\mathcal{E}\left[V(\tilde{e}(t_k), \eta(t_k), t_k)\right] \\ &= \mathcal{E}\left[\mathcal{E}\left[V(\tilde{e}(t_k), \eta(t_k), t_k) | \bigcap_{m=1}^{k}\{\eta(t_m) = i_m\}\right]\right], \\ &\leq \mu\mathcal{E}\left[\mathcal{E}\left[V(\tilde{e}(t_k^-), \eta(t_k^-), t_k^-) | \bigcap_{m=1}^{k}\{\eta(t_m) = i_m\}\right]\right], \\ &\leq \mu\mathcal{E}\left[\mathcal{E}\left[e^{\gamma_{i_{k-1}}\tau_{i_{k-1}}} \times V(\tilde{e}(t_{k-1}), \eta(t_{k-1}), t_{k-1}) | \bigcap_{m=1}^{k}\{\eta(t_m) = i_m\}\right]\right], \\ &\leq \mu \sum_{i_1 \in \mathcal{N}} \cdots \sum_{i_{k-1} \in \mathcal{N}} \mu^{k-1} V(\tilde{e}_0, \eta_0, t_0)\mathcal{E}\left[e^{\gamma_{\eta_0}\tau_{\eta_0}}\right] \times \prod_{m=1}^{k-1} \mathcal{E}\left[e^{\gamma_{i_m}\tau_{i_m}}\right] \prod_{m=1}^{k-1} \pi_{i_{m-1}i_m}, \\ &= V(\tilde{e}_0, \eta_0, t_0)\mathcal{E}\left[e^{\gamma_{\eta_0}\tau_{\eta_0}}\right] \end{aligned}$$

$$
\begin{aligned}
&\times \mu^k \sum_{i_1\in\mathcal{N}} \cdots \sum_{i_{k-1}\in\mathcal{N}} \prod_{m=1}^{k-1} \mathscr{E}\left[e^{\gamma_{i_m}\tau_{i_m}}\right]\pi_{i_{m-1}i_m},\\
&\le V(\tilde e_0,\eta_0,t_0)\mathscr{E}\left[e^{\gamma_{\eta_0}\tau_{\eta_0}}\right]\\
&\times \mu^k \prod_{i_1\in\mathcal{N},\ldots,i_{k-1}\in\mathcal{N}} \max_{i\in\mathcal{N}}\left\{\sum_{\ell=1}^{N}\mathscr{E}[e^{\gamma_\ell\tau_\ell}]\pi_{i\ell}\right\},\\
&= \mu V(\tilde e_0,\eta_0,t_0)\mathscr{E}\left[e^{\gamma_{\eta_0}\tau_{\eta_0}}\right]\left(\max_{i\in\mathcal{N}}\left\{\mu\sum_{\ell=1}^{N}\mathscr{E}[e^{\gamma_\ell\tau_\ell}]\pi_{i\ell}\right\}\right)^{k-1}.
\end{aligned}
\tag{36}
$$

Based on condition (24) and inequality (36), one has

$$
\begin{aligned}
&\mathscr{E}\left[\int_0^\infty \alpha\|\tilde e(t)\|^2 dt\,|\tilde e_0,\eta_0\right]\\
&\le \mathscr{E}\left[\int_0^\infty V(\tilde e(t),\eta(t),t)dt\,|\tilde e_0,\eta_0\right],\\
&\le \sum_{k=0}^{\infty}\mathscr{E}\left[\int_0^\infty e^{\gamma_{i_k}(t-t_k)}V(\tilde e(t_k),\eta(t_k),t_k)\right.\\
&\left.\times I(N_{\eta(t)}=k)dt\,|\tilde e_0,\eta_0\right],\\
&\le \sum_{k=1}^{\infty}\mu V(\tilde e_0,\eta_0,t_0)\mathscr{E}\left[e^{\gamma_{\eta_0}\tau_{\eta_0}}\right]\left(\max_{i\in\mathcal{N}}\left\{\mu\sum_{\ell=1}^{N}\mathscr{E}[e^{\gamma_\ell\tau_\ell}]\pi_{i\ell}\right\}\right)^{k-1}\\
&\times \mathscr{E}\left[\int_{t_k}^{t_{k+1}^-} e^{\gamma_{i_k}(t-t_k)}dt\right] + V(\tilde e_0,\eta_0,t_0)\mathscr{E}\left[\int_0^{t_1^-} e^{\gamma_{\eta_0}t}dt\right],\\
&= V(\tilde e_0,\eta_0,t_0)\max_{i\in\mathcal{N}}\left\{\mathscr{E}\left[\frac{e^{\gamma_i\tau_i}-1}{\gamma_i}\right]\right\}\\
&\times\left[1+\mu\mathscr{E}\left[e^{\gamma_{\eta_0}\tau_{\eta_0}}\right]\sum_{k=1}^{\infty}\left(\max_{i\in\mathcal{N}}\left\{\mu\sum_{\ell=1}^{N}\mathscr{E}[e^{\gamma_\ell\tau_\ell}]\pi_{i\ell}\right\}\right)^{k-1}\right],\\
&\le V(\tilde e_0,\eta_0,t_0)\max_{i\in\mathcal{N}}\left\{\mathscr{E}\left[\frac{e^{\gamma_i\tau_i}-1}{\gamma_i}\right]\right\}\\
&\times\frac{1-\max_{i\in\mathcal{N}}\left\{\mu\sum_{\ell=1}^{N}\mathscr{E}[e^{\gamma_\ell\tau_l}]\pi_{i\ell}\right\}+\mu\mathscr{E}\left[e^{\gamma_{\eta_0}\tau_{\eta_0}}\right]}{1-\max_{i\in\mathcal{N}}\left\{\mu\sum_{\ell=1}^{N}\mathscr{E}[e^{\gamma_\ell\tau_\ell}]\pi_{i\ell}\right\}},\\
&= M(\tilde e_0,\eta_0),
\end{aligned}
\tag{37}
$$

where $\alpha = \min_{i\in\mathcal{N}}\{\lambda_{\min}(P_i)\}$. This completes the proof.

Remark 2. Since the presented conditions are given in terms of LMIs, they could be solved by applying standard softwares directly. Though the given conditions are presented without Kronecker product and have smaller dimensions, they are essentially the same as the references with Kronecker product. And because the considered system is a semi-Markov multi-agent system, the computation complexity of given conditions will be huge, especially the quantities of operation mode and agent increase very large. In detail, there are $N+1+M$ matrix variables to be computed by solving N^2+qN LMIs, while $N+1$ scalars are given beforehand. Then, it is necessary to consider the reduction of computation complexity. Moreover, there are some other limitations about the proposed conditions. For example, it is seen from conditions (22) and (24) that the expectation computation such as $\mathscr{E}[\alpha_{i\ell}^{\xi(t)}]$ and $\mathscr{E}[e^{\gamma_\ell\tau_\ell}]$ needs to know the distribution of related random variables. But, it is difficult to achieve this aim in practice, since they are stochastic variables. Though an alternative method based on a robust approach could be used, some assumptions or restrictions about the aforementioned variables' bounds still exist and are not satisfied in practice. Then, some improved method or results having easy implementation are necessarily studied and will be our future work.

When switching signals $\eta(t)$ and $\delta(t)$ are synchronous, system (19) is equivalent to

$$
\hat\Sigma^1_{\eta(t)}:\begin{cases}\dot{\tilde e}(t) = (I_{q-1}\otimes A_{\eta(t)} - \tilde{\mathfrak{L}}_{\eta(t)}\otimes B_{\eta(t)}K_{\eta(t)})\tilde e(t),\\ \qquad t\in[t_k,t_{k+1})\\ \tilde e(t_k) = \tilde e(t_k^-),\ k\in\mathbb{N}.\end{cases}
\tag{38}
$$

The control input becomes to be

$$
u_i(t) = K_{\eta(t)}\sum_{j\in\mathcal{V}} g_{ij}(\eta(t))(x_j(t)-x_i(t)),\forall t\in[t_k,t_{k+1}). \tag{39}
$$

Then, one has the following corollary.

Corollary 1. *There is a controller (7) such that the resulting multi-agent system is consensus, if for given scalars $\mu\ge 1$ and $\gamma_i\in\mathbb{R}^-$, there exist matrices $X_i>0$, Y_i, $\forall i\in\mathcal{N}$, $\forall j\in\mathcal{N}$, satisfying conditions (23), (24) and*

$$
(A_iX_i-\lambda_{i\ell}B_iY_i)^\star-\gamma_iX_i\le 0,\ell\in\bar{\mathcal{V}}, \tag{40}
$$

where $\lambda_{i\ell}$, $i\in\mathcal{N}$, $\forall\ell\in\bar{\mathcal{V}}$, is given in (45). Then, the control gain is computed by

$$
K_i = Y_iX_i^{-1}. \tag{41}
$$

Proof. Then, the computation of $\mathscr{L}V(\tilde e(t),i,t)$ system (38) based on (26) is computed that

$$
\begin{aligned}
\mathscr{L}V(\tilde e(t),i,t) &= \tilde e^T(t)[I_{q-1}\otimes(P_iA_i)^\star-\tilde{\mathfrak{L}}_i\otimes(P_iB_iK_i)^\star]\tilde e(t),\\
&\forall\ t\in[t_k,t_{k+1}).
\end{aligned}
\tag{42}
$$

It is guaranteed by

$$
I_{q-1}\otimes(P_iA_i)^\star-\tilde{\mathfrak{L}}_i\otimes(P_iB_iK_i)^\star-\gamma_iI_{q-1}\otimes P_i\le 0, \tag{43}
$$

which is equivalent to

$$I_{q-1} \otimes (A_i X_i)^\star - \tilde{\mathfrak{L}}_i \otimes (B_i K_i X_i)^\star - \gamma_i I_{q-1} \otimes X_i \leq 0, \quad (44)$$

with $X_i = P_i^{-1}$. Meanwhile, there exists an orthogonal matrix $U_i \in \mathbb{R}^{(q-1)\times(q-1)}$ such that

$$\Lambda_i = U_i^T \tilde{\mathfrak{L}}_i U_i = \mathrm{diag}\{\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{i(q-1)}\}. \quad (45)$$

Then inequality (44) with representation (41) is equivalent to

$$I_{q-1} \otimes (A_i X_i)^\star - \Lambda_i \otimes (B_i Y_i)^\star - \gamma_i I_{q-1} \otimes X_i \leq 0. \quad (46)$$

It could be guaranteed by condition (40). The other steps are same to ones in Theorem 1 and omitted. This completes the proof.

Moreover, the control input could also be selected to be

$$u_i(t) = K \sum_{j\in\mathcal{V}} g_{ij}^{[\eta(t)]}(x_j(t) - x_i(t)), \forall t \in [t_k, t_{k+1}), \quad (47)$$

while some other special cases could been similarly developed and omitter here. Then, a similar corollary could be obtained.

Corollary 2. *There is a controller (47) such that the resulting multi-agent system is consensus, if for given scalars $\mu \geq 1$ and $\gamma_i \in \mathbb{R}^-$, there exist matrices $X_i > 0$, Y, $\forall i \in \mathcal{N}, \forall j \in \mathcal{N}$, satisfying conditions (23), (24) and*

$$\begin{bmatrix} \tilde{\Omega}_{i\kappa} & \check{\Omega}_{i\kappa} \\ & (-S)^\star \end{bmatrix} \leq 0, \forall \kappa \in \bar{\mathcal{V}}, \quad (48)$$

where

$$\tilde{\Omega}_{i\kappa} = \left(A_i S - \sum_{\ell=1}^{M} \lambda_\kappa^{[\ell]} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right] B_i Y\right)^\star - \gamma_i X_i,$$

$$\check{\Omega}_{i\kappa} = A_i S - \sum_{\ell=1}^{M} \lambda_\kappa^{[\ell]} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right] B_i Y + X_i - S^T.$$

Then, the control gain is computed by

$$K = YS^{-1}. \quad (49)$$

Proof. Based on the proof of Theorem 1, the proof of this theorem could be obtained easily and omitted here. This completes the proof.

Since matrix S is selected to be a common one, it will lead to more conservatism in the above conditions. When matrix $\mathscr{E}[Q^{\xi(t)}]$ is nonsingular, a less conservative result could be gotten.

Theorem 2. *When $\mathscr{E}[Q^{\xi(t)}]$ is nonsingular, there is a controller (7) such that the multi-agent system (13) could reach consensus, if for given scalars $\mu \geq 1$ and $\gamma_i \in \mathbb{R}^-$, there exist matrices $X_i > 0$, S_i and F_i, $i \in \mathcal{N}$, $j \in \mathcal{N}$, satisfying conditions (23), (24) and*

$$\begin{bmatrix} (A_i X_i - \bar{\lambda} B_i F_i)^\star - \gamma_i X_i + (S_i)^\star & \hat{\lambda} F_i^T B_i^T + S_i - S_i^T \\ & (-S_i)^\star \end{bmatrix} < 0, \quad (50)$$

where scalars $\bar{\lambda}$ and $\hat{\lambda}$ are defined as

$$\bar{\lambda} = \frac{\lambda_{\max} + \lambda_{\min}}{2}, \hat{\lambda} = \frac{\lambda_{\max} - \lambda_{\min}}{2},$$

with $\lambda_{\max} = \max_{i\in\bar{\mathcal{V}}, \ell\in\mathcal{M}} \{\lambda_i^{[\ell]}\}$ and $\lambda_{\min} = \min_{i\in\bar{\mathcal{V}}, \ell\in\mathcal{M}} \{\lambda_i^{[\ell]}\}$. Then, one could get the control gain by

$$\begin{bmatrix} K_1 \\ \vdots \\ K_N \end{bmatrix} = \left[(\mathscr{E}\left[Q^{\xi(t)}\right])^{-1} \otimes I_m\right] \begin{bmatrix} F_1 X_1^{-1} \\ \vdots \\ F_N X_N^{-1} \end{bmatrix}. \quad (51)$$

Proof. It is known from (30) that

$$\begin{aligned} & I_{q-1} \otimes (P_i A_i)^\star - \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right] (\bar{\lambda} I_{q-1} + \hat{\lambda}\Delta) \\ & \otimes (P_i B_i K^{[\ell]})^\star - \gamma_i I_{q-1} \otimes P_i \leq 0 \end{aligned}, \quad (52)$$

where $\Delta = \mathrm{diag}\{\delta_1, \dots, \delta_{q-1}\}$ with $\delta_i \in [-1, 1], i \in \bar{\mathcal{V}}$. It is equivalent to

$$\begin{aligned} & I_{q-1} \otimes (P_i A_i)^\star - \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right] (\bar{\lambda} I_{q-1} + \hat{\lambda}\Delta) \otimes (P_i B_i K^{[\ell]})^\star \\ & - \gamma_i I_{q-1} \otimes P_i, \\ & = I_{q-1} \otimes [(P_i A_i)^\star - \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right] \bar{\lambda} (P_i B_i K^{[\ell]})^\star - \gamma_i P_i] \\ & - \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right] \hat{\lambda} (\Delta \otimes I_n) \left[I_{q-1} \otimes (P_i B_i K^{[\ell]})^\star\right] \leq 0. \end{aligned} \quad (53)$$

Moreover, it is obtained that

$$\begin{aligned} & \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right] \hat{\lambda} (\Delta \otimes I_n) \left[I_{q-1} \otimes (P_i B_i K^{[\ell]})^\star\right] \\ & = \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right] \hat{\lambda} (\Delta \otimes I_n) \left[I_{q-1} \otimes P_i B_i K^{[\ell]}\right. \\ & \left. + I_{q-1} \otimes (P_i B_i K^{[\ell]})^T\right], \\ & = \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right] \hat{\lambda} (\Delta \otimes I_n)(I_{q-1} \otimes P_i B_i K^{[\ell]}) \\ & + \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right] \hat{\lambda} (\Delta \otimes I_n) \left[I_{q-1} \otimes (P_i B_i K^{[\ell]})^T\right]. \end{aligned} \quad (54)$$

Because of Δ being a diagonal matrix, it is easy to get that

$$
\begin{aligned}
&(\Delta \otimes I_n)\left[I_{q-1} \otimes (P_iB_iK^{[\ell]})^T\right] \\
=&\Delta \otimes (P_iB_iK^{[\ell]})^T, \\
=&\left[I_{q-1} \otimes (P_iB_iK^{[\ell]})^T\right](\Delta \otimes I_n).
\end{aligned} \tag{55}
$$

Based on (55), it is known from (54) that

$$
\begin{aligned}
&\sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\hat{\lambda}(\Delta \otimes I_n)\left[I_{q-1} \otimes (P_iB_iK^{[\ell]})^\star\right] \\
&= \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\hat{\lambda}(\Delta \otimes I_n)(I_{q-1} \otimes P_iB_iK^{[\ell]}) \\
&\quad + \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\hat{\lambda}(\Delta \otimes I_n)\left[I_{q-1} \otimes (P_iB_iK^{[\ell]})^T\right], \\
&= \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\hat{\lambda}(\Delta \otimes I_n)(I_{q-1} \otimes P_iB_iK^{[\ell]}) \\
&\quad + \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\hat{\lambda}\left[I_{q-1} \otimes (P_iB_iK^{[\ell]})^T\right](\Delta \otimes I_n), \\
&= \left[\sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\hat{\lambda}(\Delta \otimes I_n)(I_{q-1} \otimes P_iB_iK^{[\ell]})\right]^\star.
\end{aligned} \tag{56}
$$

Then, inequality (53) is transformed to

$$
\begin{aligned}
&I_{q-1} \otimes \left[(P_iA_i)^\star - \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\bar{\lambda}(P_iB_iK^{[\ell]})^\star - \gamma_iP_i\right] \\
&- \left[\sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\hat{\lambda}(\Delta \otimes I_n)(I_{q-1} \otimes P_iB_iK^{[\ell]})\right]^\star \le 0.
\end{aligned} \tag{57}
$$

Define

$$
\Omega = \{\Delta | \Delta = \text{diag}\{\delta_1, \dots, \delta_{q-1}\}, \delta_i \in [-1, 1], i \in \bar{\mathcal{V}}\},
$$

and

$$
\tilde{\Omega} = \{\tilde{\Delta} | \tilde{\Delta} = \text{diag}\{\tilde{\delta}_1, \dots, \tilde{\delta}_n\}, \tilde{\delta}_i \in [-1, 1], i = 1, \dots, n\}.
$$

Because of $q - 1 \ge n$, it is obtained that $\Omega \subseteq \tilde{\Omega}$. Then, inequality (57) could be guaranteed by

$$
\begin{aligned}
&I_{q-1} \otimes \left[(P_iA_i)^\star - \sum_{\ell=1}^{N} \mathcal{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\bar{\lambda}(P_iB_iK^{[\ell]})^\star - \gamma_iP_i\right] \\
&- \left[\sum_{\ell=1}^{N} \mathcal{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\hat{\lambda}(I_{q-1} \otimes \tilde{\Delta})(I_{q-1} \otimes P_iB_iK^{[\ell]})\right]^\star \\
=&I_{q-1} \otimes \left[(P_iA_i)^\star - \sum_{\ell=1}^{N} \mathcal{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\bar{\lambda}(P_iB_iK^{[\ell]})^\star - \gamma_iP_i\right] \\
&- \left[\sum_{\ell=1}^{N} \mathcal{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\hat{\lambda}I_{q-1} \otimes \tilde{\Delta}P_iB_iK^{[\ell]}\right]^\star, \\
=&I_{q-1} \otimes \left[(P_iA_i)^\star - \sum_{\ell=1}^{N} \mathcal{E}\left[\alpha_{i\ell}^{\xi(t)}\right], \bar{\lambda}(P_iB_iK^{[\ell]})^\star - \gamma_iP_i\right. \\
&\left.-\left(\sum_{\ell=1}^{N} \mathcal{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\hat{\lambda}\tilde{\Delta}P_iB_iK^{[\ell]}\right)^\star\right], \\
\le&0.
\end{aligned} \tag{58}
$$

It could be implied by

$$
\begin{aligned}
&(A_iX_i)^\star - \sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\bar{\lambda}(B_iK^{[\ell]}X_i)^\star - \gamma_iX_i \\
&- \left(\sum_{\ell=1}^{N} \mathscr{E}\left[\alpha_{i\ell}^{\xi(t)}\right]\hat{\lambda}\tilde{\Delta}B_iK^{[\ell]}X_i\right)^\star \le 0,
\end{aligned} \tag{59}
$$

where $X_i = P_i^{-1}$. Let $F_i \triangleq \sum_{\ell=1}^{N} \mathscr{E}[\alpha_{i\ell}^{\xi(t)}]K_\ell X_i$, the above inequality is equivalent to

$$
(A_iX_i - \bar{\lambda}B_iF_i)^\star - \gamma_iX_i - \hat{\lambda}(\tilde{\Delta}B_iF_i)^\star \le 0. \tag{60}
$$

Let

$$
\left[\begin{array}{c|c} T_{i1} & T_2 \\ \hline T_{i3} & T_4 \end{array}\right],
$$

where

$$
T_{i1} = (A_iX_i - \bar{\lambda}B_iF_i)^\star - \gamma_iX_i,
$$

$$
T_2 = -I_n, T_{i3} = \hat{\lambda}B_iF_i, T_4 = \mathbf{0}_{n\times n},
$$

it is known from Lemma 1 that inequality (60) is guaranteed by

$$\begin{bmatrix} \Pi_{i1} & * \\ \Pi_{i2} & \Pi_{i3} \end{bmatrix} \leq 0, \tag{61}$$

where

$$\Pi_{i1} = (A_i X_i - \bar{\lambda} B_i F_i)^\star - \gamma_i X_i + (S_i)^\star,$$
$$\Pi_{i2} = \hat{\lambda} B_i F_i + S_i^T - S_i, \Pi_{i3} = (-S_i)^\star,$$
$$\underline{T}_4 = I_n, \overline{T}_4 = I_n, \underline{\Delta} = -I_n, \overline{\Delta} = I_n.$$

As for definition F_i, it is rewritten to be

$$F_i X_i^{-1} = \left(\begin{bmatrix} \mathscr{E}\left[\alpha_{i1}^{\xi(t)}\right] & \cdots & \mathscr{E}\left[\alpha_{iN}^{\xi(t)}\right] \end{bmatrix} \otimes I_m \right) \begin{bmatrix} K_1 \\ \vdots \\ K_N \end{bmatrix}, \tag{62}$$

which is equal to

$$\begin{bmatrix} F_1 X_1^{-1} \\ \vdots \\ F_N X_N^{-1} \end{bmatrix} = \left(\begin{bmatrix} \mathscr{E}\left[\alpha_{11}^{\xi(t)}\right] & \cdots & \mathscr{E}\left[\alpha_{1N}^{\xi(t)}\right] \\ \vdots & \ddots & \vdots \\ \mathscr{E}\left[\alpha_{N1}^{\xi(t)}\right] & \cdots & \mathscr{E}\left[\alpha_{NN}^{\xi(t)}\right] \end{bmatrix} \otimes I_m \right) \begin{bmatrix} K_1 \\ \vdots \\ K_N \end{bmatrix}. \tag{63}$$

Because $\mathscr{E}[Q^{\xi(t)}]$ is nonsingular, one could compute the gains by (51). The other processes are similar to the ones in Theorem 1 and omitted here. This completes the proof.

Remark 3. Though the above results are concerned about system (1) under exact (12), its key idea could be applied to other systems or problems. The main reason is that all the conditions are established in terms of LMIs. For example, when matrix (12) is uncertain or partially unknown, one could deal with them by the methods applied to handle uncertain or partially unknown transition probability matrix of discrete-time Markovian jump systems such as [57]. Second, when $\xi(t)$ is specialized to be a Gaussian one, similar results could be developed directly. Third, in order to obtain solvable conditions, some enlarged inequalities such as (23), (24), and (59), have been applied, which bring bigger conservatism. How to give less conservatism but concise conditions is necessary to be further studied. Fourth, when the adjacency matrix G is directed, similar results could be obtained, while an additional assumption about graph $\mathcal{G}$ being a balanced one [23] is needed. Based on Lemma 2 and Theorem 1 in [24] and under the above assumption, it can be inferred that system (18) still holds. And the rest of the processes can be done via using the methods proposed for undirected topology. Fifth, but not the last, by exploiting the method in this paper, one could also consider other problems, such the pinning control of complexity networks with semi-Markov topologies, stabilization of semi-Markovian jump systems with stochastically unmatched controllers, and filter or observer design.

TABLE 1 The parameter values of $A_{\eta(t)}$ and $B_{\eta(t)}$

Airspeed	$a_{\eta(t)}$	$b_{\eta(t)}$	$c_{\eta(t)}$
135	0.3681	1.4200	3.5446
60	0.0664	0.1198	0.9775
170	0.5047	2.5460	5.1120

4 | NUMERICAL EXAMPLE

4.1 | Example 1

Consider a VTOL helicopter model partly cited from [58] and described as (1). Its parameters are given to be

$$A_{\eta(t)} = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & a_{\eta(t)} & -0.707 & b_{\eta(t)} \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B_{\eta(t)} = \begin{bmatrix} 0.4422 & 0.1761 \\ c_{\eta(t)} & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix},$$

state variables x_1, x_2, x_3 and x_4 represent the horizontal velocity, vertical velocity, pitch rate and pitch angle. The parameter values in the above matrices correspond to the airspeed shown in Table 1.

Mode $\eta(t) \in \mathcal{N} = \{1, 2, 3\}$ belongs to be a semi-Markov process and indicates three different airspeeds. The transition probability matrix is given as

$$P = \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0.3 & 0 & 0.7 \\ 0.4 & 0.6 & 0 \end{bmatrix}.$$

Without loss of generality, it is assumed that there are four agents in the semi-Markov multi-agent system. And the coupling matrix described in (13) has 3 modes and given as follows:

$$L^{[\delta(t)=1]} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix},$$

$$L^{[\delta(t)=2]} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

$$L^{[\delta(t)=3]} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Here, mode $\delta(t)$ is another stochastic process and takes values in $\mathcal{M} = \{1, 2, 3\}$. In detail, switching signals $\eta(t)$ and $\delta(t)$ are stochastically unmatched such as (8). Without loss of generality, $\{\xi(t), t \geq 0\}$ is assumed to belong to the Bernoulli distribution, such as $\Pr\{\xi(t) = 1\} = \alpha$ and $\Pr\{\xi(t) = 2\} = 1 - \alpha, \alpha \in [0, 1]$. Here, α is given to be $\alpha = 0.5$. Then, the truncated PDF matrix $Q^{\xi(t)}$ is given to be

$$Q^{\xi(t)=1} = \begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.5 & 0.4 \end{bmatrix}, Q^{\xi(t)=2} = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}.$$

Then, it is computed that

$$\mathcal{E}\left[Q^{\xi(t)}\right] = \sum_{j=1}^{2} \Pr\{\xi(t) = j\} Q^{\xi(t)=j},$$

$$= \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}.$$

Based on the proposed methods, it is concluded that the consensus problem of system (13) could be guaranteed by the stability of system (19). The related parameters of $\tilde{L}$, U and $\tilde{\mathfrak{L}}$ are computed as

$$\tilde{L} = \begin{bmatrix} 0.75 & -0.25 & -0.25 & -0.25 \\ -0.25 & 0.75 & -0.25 & -0.25 \\ -0.25 & -0.25 & 0.75 & -0.25 \\ -0.25 & -0.25 & -0.25 & 0.75 \end{bmatrix},$$

$$U = \begin{bmatrix} -0.2113 & -0.2887 & 0.7887 & 0.5 \\ 0.7887 & -0.2887 & -0.2113 & 0.5 \\ -0.5774 & -0.2887 & -0.5774 & 0.5 \\ 0 & 0.8660 & 0 & 0.5 \end{bmatrix},$$

$$\tilde{\mathfrak{L}}^{[1]} = \begin{bmatrix} 1.1786 & 0.2440 & -0.6667 \\ 0.2440 & 1.3333 & -0.9107 \\ -0.6667 & -0.9107 & 3.4880 \end{bmatrix},$$

$$\tilde{\mathfrak{L}}^{[2]} = \begin{bmatrix} 3.4880 & -0.9107 & -0.6667 \\ -0.9107 & 1.3333 & 0.2440 \\ -0.6667 & 0.2440 & 1.1786 \end{bmatrix},$$

TABLE 2 The value of $\lambda_{\kappa}^{[\ell]}$ in Theorem 1

	$\ell = 1$	$\ell = 2$	$\ell = 3$
$\kappa = 1$	1	4	1
$\kappa = 2$	4	1	4
$\kappa = 3$	1	1	1

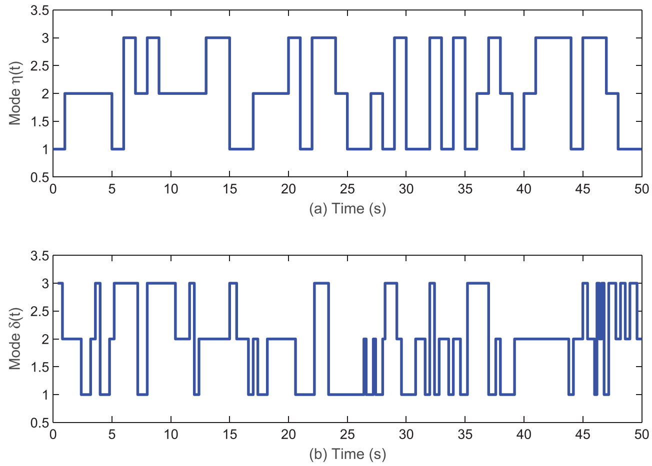

FIGURE 1 The simulations of modes $\eta(t)$ and $\delta(t)$

$$\tilde{\mathfrak{L}}^{[3]} = \begin{bmatrix} 2.3333 & 0.6667 & 1.3333 \\ 0.6667 & 1.3333 & 0.6667 \\ 1.3333 & 0.6667 & 2.3333 \end{bmatrix}.$$

Then, eigenvalue $\lambda_{\kappa}^{[\ell]}$ is computed and listed in Table 2.

Let $\gamma_1 = -0.85$, $\gamma_2 = -0.75$, $\gamma_3 = -0.55$, $\mu = 1.12$, $\mathcal{E}[e^{\gamma_1 \tau_1}] = 0.4274$, $\mathcal{E}[e^{\gamma_2 \tau_2}] = 0.2231$ and $\mathcal{E}[e^{\gamma_3 \tau_3}] = 0.1920$, the control gains of (7) are computed as

$$K_1 = \begin{bmatrix} -0.0234 & -0.1332 & -0.3149 & -0.1028 \\ -0.6820 & -0.2247 & 0.0406 & 0.8100 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.1236 & -0.0639 & -0.4068 & -0.3780 \\ -0.7627 & -0.1360 & 0.1643 & 0.9776 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 0.3994 & -0.0973 & -0.6214 & -1.1683 \\ -0.1770 & -0.1143 & -0.3593 & -0.2108 \end{bmatrix}.$$

Under the initial conditions given as $e_1(0) = \begin{bmatrix} 1 & 0 & -1 & 0.5 \end{bmatrix}^T$, $e_2(0) = \begin{bmatrix} 0 & -0.5 & -1 & 0 \end{bmatrix}^T$, $e_3(0) = \begin{bmatrix} 1 & 0 & -1 & 0.5 \end{bmatrix}^T$ and applying the above designed controllers, one has the simulations of error system (19) given in Figures 1 and 2. Moreover, another simulation based on a Monte Carlo simulation method is shown in Figure 3. Because of the considered system being a multi-agent system with 4 degrees and having 4 agents, the Monte Carlo

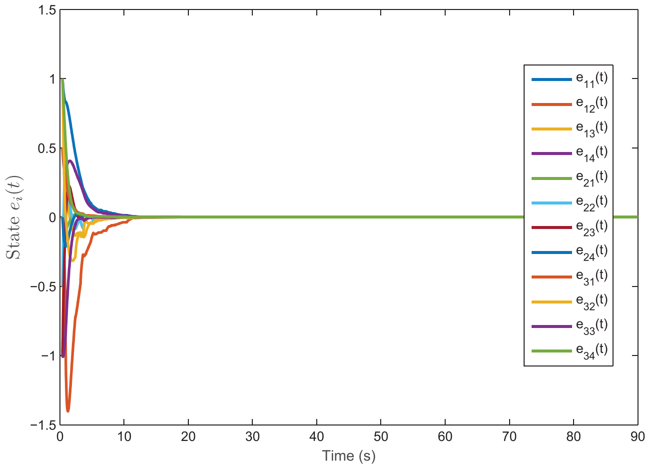

FIGURE 2 The state responses of error system (19)

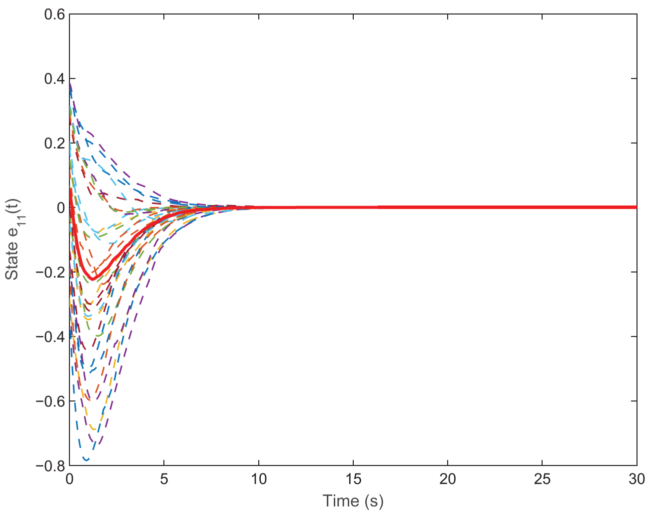

FIGURE 3 The Monte Carlo simulation of error state $e_{11}(t)$ of error system (19)

simulation will be chaotic. To make the above simulation effective but concise, without loss of generality, only error state $e_{11}(t)$ is simulated in Figure 3. Based on these simulations, it is seen that though the topology mode is stochastic and unmatched to original mode $\eta(t)$, the multi-agent system could also achieve consensus by control protocol (7). To the contrary, when modes $\eta(t)$ and $\delta(t)$ are synchronous such as [59], one could get mode-dependent controller (39) by Corollary 1. The control gains are computed as

$$K_1 = \begin{bmatrix} 10148 & 1752 & -3488 & -8783 \\ 12381 & 1664 & -4342 & -11581 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -26451 & -4508 & 8970 & 22827 \\ 25774 & -3005 & 10544 & 26497 \end{bmatrix},$$

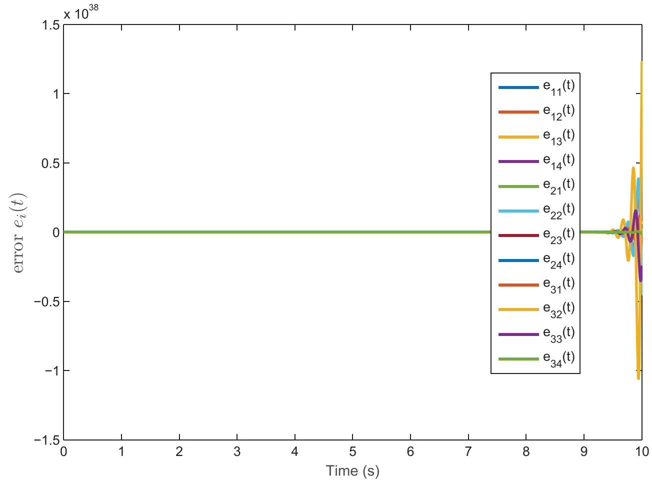

FIGURE 4 The curves of error system (19) for controller (39) under condition (8)

$$K_3 = \begin{bmatrix} 57450 & 9825 & -19312 & -49376 \\ 78905 & 11105 & -24903 & -69945 \end{bmatrix}.$$

It can be said that controller (39) is an ideal one, whose mode signal should be available online. In other words, when controller (39) suffers a situation such as (8), it could be seen from Figure 4 that it will be disabled. On the other hand, mode-independent control input (47) could also be applied to deal with (8) based on computation method [47], which is actually described in Corollary 2. It is seen from description (47) that no information on mode signal is used. Because of control input (47), neglecting all information of mode signal and its probability distribution, it is more conservative than (7) with conditional probability (8). In detail, it is concluded that under the same initial conditions, there will be no solvable solutions to Corollary 2. In this sense, it can be said that more information about stochastic unmatched probability (8) could lead to less conservative results.

When expectation $\mathscr{E}[Q^{\xi(t)}]$ is nonsingular, Theorem 2 proposes another method for (7). Under the same parameters, the corresponding gains are computed as

$$K_1 = \begin{bmatrix} -892.5 & -216.2 & 315.7 & 826.4 \\ -1441.4 & -274.0 & 417.1 & 1251.7 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -2537.5 & -619.0 & 918.4 & 2375.8 \\ -3337.1 & -569.7 & 1165.1 & 3170.2 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 4604.0 & 1121.7 & -1661.0 & -4304.2 \\ 6315.8 & 1106.0 & -2122.2 & -5885.9 \end{bmatrix}.$$

In order to make some comparisons between Theorems 1 and 2, without loss of generality, γ_i is assumed to be equal and denoted by γ. It is known from Theorem 1 that the solvable range of γ is $\gamma \in [-0.771, -0.072]$. To the contrary, there are still solvable solutions to Theorem 2, even if $\gamma = -2$. For this example, it can be said that Theorem 2 is less conservative than Theorem 1, while conditions about $\mathcal{M} = \mathcal{N}$ and nonsingular $\mathcal{E}[\mathcal{Q}^{\xi(t)}]$ should be satisfied simultaneously. Then, which one to be selected between Theorems 1 and 2 should be considered on the concrete situations.

On the other hand, it is claimed that description (8) is more general than one in [55], since similar description (8) in the reference is always constant. In other words, when the conditional probability in this reference is changed, the result designed beforehand may be disabled. For example, when the conditional probability matrix is given to be $\mathcal{Q}^{\xi(t)=1}$, it can be known that the solvable range of γ is $[-0.809, -0.072]$. Meanwhile, the similar solvable range for $\mathcal{Q}^{\xi(t)=2}$ is $[-0.650, -0.072]$. Based on these computations, it can be seen that when $\gamma \in [-0.809, -0.650)$, the method in the above reference will be disabled if conditional probability matrix varies from $\mathcal{Q}^{\xi(t)=1}$ to $\mathcal{Q}^{\xi(t)=2}$. On the other hand, a larger solvable range based on model (8) could be obtained as $[-0.771, -0.072]$ and could bear conditional probability matrix varying. In this sense, it could be said that our results based on (8) needing more information on conditional probability are less conservative than ones obtained by [55].

5 | CONCLUSIONS

In this paper, a different consensus protocol based on stochastically unmatched topologies has been introduced to address the consensus problem of semi-Markov multi-agent systems. The aforementioned unmatched property has been modeled by a stochastically conditional probability and quantized by a PDF. Sufficient conditions for the existence of such a consensus protocol have been established by using a Lyapunov function approach and some enlarging techniques. An improved algorithm based on nonsingular expectation of the truncated PDF matrix has also been presented, whose superiority has been illustrated by an example. It has been proved that the conditional probability, dwell time and transition probability are important in consensus analysis and synthesis. In the future, some problems remain to be studied. First, the findings in this study are also applicable to the investigation of other problems or systems about engineering issues such as direct topology and discrete-time systems. Second, computation complexity and conservatism of results should be further reduced, especially in the case of significant system mode and agent increase.

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REFERENCES

1. Guo, G., Wen, S.X.: Communication scheduling and control of a platoon of vehicles in VANETs. IEEE Trans. Intell. Transp. Syst. 17(6), 1551–1563 (2016)
2. Vilarinho, C., et al.: Design of a multi-agent system for real-time traffic control. IEEE Intell. Syst. 31(4), 68–80 (2016)
3. Wen, S.X., Guo, G.: Sampled-data control for connected vehicles with Markovian switching topologies and communication delay. IEEE Trans. Intell. Transp. Syst. 21(7), 2930–2942 (2020)
4. Ding, D., et al.: Observer-based event-triggering consensus control for multi-agent systems with lossy sensors and cyber-attacks. IEEE Trans. Cybern. 47(8), 1936–1947 (2017)
5. Ye, D., et al.: Stochastic coding detection scheme in cyber-physical systems against replay attack. Inf. Sci. 481, 432–444 (2019)
6. Qin, J., et al.: Recent advances in consensus of multi-agent systems: A brief survey. IEEE Trans. Ind. Electron. 64(6), 4972–4983 (2017)
7. Yang, T., et al.: Global consensus for discrete-time multi-agent systems with input saturation constraints. Automatica 50(2), 499–506 (2014)
8. Chen, S., et al.: Fault-tolerant consensus of multi-agent system with distributed adaptive protocol. IEEE Trans. Cybern. 45(10), 2142–2155 (2015)
9. Xu, J.: Adaptive iterative learning control for high-order nonlinear multi-agent systems consensus tracking. Syst. Control Lett. 89, 16–23 (2016)
10. Fan, Y., et al.: Self-triggered consensus for multi-agent systems with Zeno-free triggers. IEEE Trans. Autom. Control 60(10), 2779–2784 (2015)
11. Hu, W., et al.: Consensus of linear multi-agent systems by distributed event-triggered strategy. IEEE Trans. Cybern. 46(1), 148–157 (2016)
12. Yang, D., et al.: Decentralized event-triggered consensus for linear multi-agent systems under general directed graphs. Automatica 69, 242–249 (2016)
13. Garcia, E., et al.: Periodic event-triggered synchronization of linear multi-agent systems with communication delays. IEEE Trans. Autom. Control 62(1), 366–371 (2017)
14. Valcher, M.E., Misra, P.: On the consensus and bipartite consensus in high-order multi-agent dynamical systems with antagonistic interactions. Syst. Control Lett. 66, 94–103 (2014)
15. Jiang, Y., et al.: Sign-consensus of linear multi-agent systems over signed directed graphs. IEEE Trans. Ind. Electron. 64(6), 5075–5083 (2017)
16. Qin, J., et al.: On the bipartite consensus for generic linear multi-agent systems with input saturation. IEEE Trans. Cybern. 47(8), 1948–1958 (2017)
17. Seyboth, G.S., et al.: On robust synchronization of heterogeneous linear multi-agent systems with static couplings. Automatica 53, 392–399 (2015)
18. Mei, J., et al.: Distributed consensus of second-order multi-agent systems with heterogeneous unknown inertias and control gains under a directed graph. IEEE Trans. Autom. Control 61(8), 2019–2034 (2016)
19. Zheng, Y., et al.: Consensus of hybrid multi-agent systems. IEEE Trans. Neural Networks Learn. Syst. 29(4), 1359–1365 (2018)
20. Wen, G., et al.: Neural-network-based adaptive leader-following consensus control for second-order nonlinear multi-agent systems. IET Control Theory Appl. 9(13), 1927–1934 (2015)
21. Sun, Q., et al.: A multi-agent-based consensus algorithm for distributed coordinated control of distributed generators in the energy internet. IEEE Trans. Smart Grid 6(6), 3006–3019 (2015)
22. Li, C., et al.: Multi-agent-based distributed state of charge balancing control for distributed energy storage units in AC microgrids. IEEE Trans. Ind. Appl. 53(3), 2369–2381 (2017)
23. Saber, R.O., Murray, R.M.: Consensus problems in networks of agents with switching topology and time-delays. IEEE Trans. Autom. Control 49(9), 1520–1533 (2004)
24. Lin, P., Jia, Y.: Average consensus in networks of multi-agents with both switching topology and coupling time-delay. Physica A 387(1), 303–313 (2008)
25. Li, H., et al.: Second-order consensus seeking in multi-agent systems with nonlinear dynamics over random switching directed networks. IEEE Trans. Circuits Syst. I Regul. Pap. 60(6), 1595–1607 (2013)
26. Wu, Z., et al.: Event-triggered control for consensus of multi-agent systems with fixed/switching topologies. IEEE Trans. Syst. Man Cybern.: Systems 48(10), 1736–1746 (2018)

27. Lou, Y., Hong, Y.: Target containment control of multi-agent systems with random switching interconnection topologies. Automatica 48(5), 879–885 (2012)
28. You, K., et al.: Consensus condition for linear multi-agent systems over randomly switching topologies. Automatica 49(10), 3125–3132 (2013)
29. Ding, L., Guo, G.: Sampled-data leader-following consensus for nonlinear multi-agent systems with Markovian switching topologies and communication delay. J. Franklin Inst. 352(1), 369–383 (2015)
30. Zhang, H., et al.: H_∞ consensus of event-based multi-agent systems with switching topology. Inf. Sci. 370–371, 623–635 (2016)
31. Ge, X., Han, Q.: Consensus of multi-agent systems subject to partially accessible and overlapping Markovian network topologies. IEEE Trans. Cybern. 47(8), 1807–1819 (2017)
32. Zhang, L., et al.: Stability and stabilization of discrete-time semi-Markov jump linear systems via semi-Markov kernel approach. IEEE Trans. Autom. Control 61(2), 503–508 (2016)
33. Shen, H., et al.: Slow state variables feedback stabilization for semi-Markov jump systems with singular perturbations. IEEE Trans. Autom. Control 63(8), 2709–2714 (2018)
34. Jiang, B., et al.: Stability and stabilization for singular switching semi-Markovian jump systems with generally uncertain transition rates. IEEE Trans. Autom. Control 63(11), 3919–3926 (2018)
35. Wang, B., Zhu, Q.: Stability analysis of semi-Markov switched stochastic systems. Automatica 94, 72–80 (2018)
36. Shen, H., et al.: Reliable mixed passive and H_∞ filtering for semi-Markov jump systems with randomly occurring uncertainties and sensor failures. Int. J. Robust Nonlinear Control 25, 3231–3251 (2015)
37. Janssen, J.: Semi-Markov Models: Theory and Applications. Springer, Boston (1986)
38. Weiss, E.N., et al.: An iterative estimation and validation procedure for specification of semi-Markov models with application to hospital patient flow. Oper. Res. 30(6), 1082–1104 (1982)
39. Anisimov, V.V.: Averaging in Markov models with fast semi-Markov switches and applications. Commun. Stat. - Theory Methods 33(3), 517–531 (2004)
40. Dai, J., Guo, G.: Exponential consensus of nonlinear multi-agent systems with semi-Markov switching topologies. IET Control Theory Appl. 11(18), 3363–3371 (2017)
41. Dai, J., Guo, G.: Event-triggered leader-following consensus for multi-agent systems with semi-Markov switching topologies. Inf. Sci. 459, 290–301 (2018)
42. Shen, H., et al.: Fault-tolerant leader-following consensus for multi-agent systems subject to semi-Markov switching topologies: An event-triggered control scheme. Nonlinear Anal. Hybrid Syst. 34, 92–107 (2019)
43. Cong, M., Mu, X.: H_∞ consensus of linear multi-agent systems with semi-Markov switching network topologies and measurement noises. IEEE Access 7, 156089–156096 (2019)
44. Li, K., Mu, X.: Containment control of stochastic multi-agent systems with semi-Markovian switching topologies. Int. J. Robust Nonlinear Control 29, 4943–4955 (2019)
45. Wu, H., Cai, K.: Mode-independent robust stabilization for uncertain Markovian jump nonlinear systems via fuzzy control. IEEE Trans. Syst. Man Cybern. B, Cybern. 36(3), 509–519 (2005)
46. de Souza, C.E., et al.: Mode-independent H_∞ filters for Markovian jump linear systems. IEEE Trans. Autom. Control 51(11), 1837–1841 (2006)
47. Liu, P.H., et al.: Design of H_∞ filter for Markov jumping linear systems with non-accessible mode information. Automatica 44(10), 2655–2660 (2008)
48. Todorov, M., Fragoso, M.: New methods for mode-independent robust control of Markov jump linear systems. Syst. Control Lett. 30, 38–44 (2016)
49. Wang, G.L., et al.: A partially delay-dependent and disordered controller design for discrete-time delayed systems. Int. J. Robust Nonlinear Control 27(16), 2646–2668 (2017)
50. Wang, G., Xu, L.: Guaranteed cost control for Markovian jump systems with controller failures. IET Control Theory Appl. 13, 3139–3147 (2019)
51. Wang, G.L., et al.: Stabilization of stochastic delay systems via a disordered controller. Appl. Math. Comput. 314, 98–109 (2017)
52. Wang, G.L., et al.: Fault-tolerant control of Markovian jump systems via a partially mode-available but unmatched controller. J. Franklin Inst. 354(17), 7717–7731 (2017)
53. Luan, X., et al.: H_∞ control for discrete-time Markov jump systems with uncertain transition probabilities. IEEE Trans. Autom. Control 58(6), 1566–1572 (2013)
54. Cramér, H.: Mathematical Methods of Statistics. Princeton University Press, Princeton (1946)
55. Costa, O.L.d.V., et al.: A detector-based approach for the H_2 control of Markov jump linear systems with partial information. IEEE Trans. Autom. Control 60(5), 1219–1234 (2015)
56. Sevilla, F.R.S., et al.: A semidefinite relaxation procedure for fault-tolerant observer design. IEEE Trans. Autom. Control 60(12), 3332–3337 (2015)
57. Zhang, L., Boukas, E.: Stability and stabilization of Markovian jump linear systems with partly unknown transition probabilities. Automatica 45(2), 463–468 (2009)
58. Farias, D.P.D., et al.: Output feedback control of Markov jump linear systems in continuous time. IEEE Trans. Autom. Control 45(5), 944–949 (2000)
59. Wang, G., et al.: Stabilization of singular Markovian jump systems with time-varying switchings. Inf. Sci. 297, 254–270 (2015)

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