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Cutting Planes for Security-Constrained Unit Commitment with Regulation Reserve

Jianqiu Huang
Department of Industrial and Systems Engineering, University of Florida, Gainesville, FL 32611, jianqiuhuang@ufl.edu

Kai Pan
Department of Logistics and Maritime Studies, Faculty of Business, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, kai.pan@polyu.edu.hk

Yongpei Guan
Department of Industrial and Systems Engineering, University of Florida, Gainesville, FL 32611, guan@ise.ufl.edu

With significant economic and environmental benefits, renewable energy is increasingly used to generate electricity. To hedge against the uncertainty due to the increasing penetration of renewable energy, an ancillary service market was introduced to maintain reliability and efficiency, in addition to day-ahead and real-time energy markets. To co-optimize these two markets, a unit commitment problem with regulation reserve (the most common ancillary service product) is solved for daily power system operations, leading to a large-scale and computationally challenging mixed-integer program. In this paper, we analyze the polyhedral structure of the co-optimization model to speed up the solution process by deriving problem-specific strong valid inequalities. Convex hull results for certain special cases (i.e., two- and three-period cases) with rigorous proofs are provided, and strong valid inequalities covering multiple periods under the most general setting are derived. We also develop efficient polynomial-time separation algorithms for the inequalities that are in the exponential size. We further tighten the formulation by deriving an extended formulation for each generator in a higher-dimensional space. Finally, we conduct computational experiments to apply our derived inequalities as cutting planes in a branch-and-cut algorithm. Significant improvement from our inequalities over commercial solvers demonstrates the effectiveness of our approach, leading to practical usefulness to enhance the co-optimization of energy and ancillary service markets.

Key words: unit commitment; regulation reserve; cutting planes; convex hull

1. Introduction

1.1. Motivation

Renewable energy is increasingly penetrating into the power system due to its economic and environmental benefits (Dincer 2000). Many countries, such as the U.S. (U.S. EIA 2016, Gibbens 2017) and China (China Energy Portal 2017), have been investing to substantially increase electricity generation from renewable energy in the last decade. Nevertheless, due to the intermittent nature (e.g., wind output fluctuates and solar energy relies on sunny weather) of renewable energy, it creates huge uncertainties for power system operations.

To ensure reliable and cost-effective operations of the power system, system operators schedule electric generators that use traditional fuels like coal and natural gas to generate electricity at an amount equal to the electricity load, as electricity cannot be stored at a large scale. To that end, a large-scale security-constrained unit commitment (UC) problem needs to be efficiently solved to obtain the corresponding power generation schedule. This problem is difficult to solve though because a large number of generators that are geographically distributed over the power network are involved and each generator has complex physical characteristics. The significant uncertainty due to intermittent renewable energy further worsens this situation. To manage such uncertainty, an ancillary service market is crucial to different independent system operators (ISOs) to ensure power system resilience. The most common commodity in the ancillary service market is named *regulation reserve*, which is represented as certain power generation capacity reserved to handle future fluctuations like moment-to-moment changes on both electricity generation (i.e., supply) and load (i.e., demand) sides and a sudden loss of a generator or a transmission line. The capacity reserved to increase (resp. decrease) power generation when needed is called *regulation-up* (resp. *regulation-down*) *reserve*. Through the ancillary service market, real-time electricity generation outputs can be easily adjusted to satisfy load balance requirement in each time period. Despite the benefits from the ancillary service market, a challenging question arises to call for effective coordination between the existing energy markets and the ancillary service market. In this paper, we aim to help coordinate both markets effectively and efficiently.

1.2. Literature Review

The ISOs are responsible for clearing both the energy and ancillary service markets to (i) maintain the load balance at the minimum operational cost through the former, and (ii) protect the power system against disturbances from both electricity supply and demand sides through the latter. Through the energy markets, the ISOs coordinate the electric power generators to determine their generation schedules at the minimum cost by respecting physical and security restrictions. Through the ancillary service market, specific levels of regulation reserves are determined to hedge against the system uncertainties. There exist different practices among ISOs to clear both markets. Traditionally, to lower the computational burden, these two markets are cleared sequentially (Kirsch and Singh 1995, Hirst and Kirby 1997, Singh and Papalexopoulos 1999). However, sequential clearing cannot guarantee global optimality for the power generation scheduling considering ancillary services (Cheung 2008). To deal with this challenge, the ISOs are increasingly exploring the co-optimization of energy and ancillary service markets (Street et al. 2017), by which the global optimality is ensured. For instance, Midcontinent ISO (MISO) has used a co-optimization model to simultaneously clear energy and ancillary service markets since 2009 (Carlson et al. 2012). As the regulation reserve is the most commonly used ancillary service, in this paper, we focus on the co-optimization model that formulates a security-constrained UC problem co-optimizing power generation and regulation reserve. Traditional UC models that consist of different types of reserve requirements have been proposed by Li and Shahidehpour (2005), Ostrowski et al. (2012), and Morales-España et al. (2013), among others.

Due to the large scale involved and physical constraints complexity, the UC problem is difficult to solve by itself. Several optimization techniques have been proposed during the past decades to solve the UC problem, including dynamic programming algorithms (Lowery 1966, Frangioni and Gentile 2006), Lagrangian relaxation (LR) / decomposition (Muckstadt and Koenig 1977, Dubost et al. 2005, Sagastizábal 2012), and heuristic approaches (Mantawy et al. 1998, Dang and Li 2007). Note that LR was widely used in industry, but it is unable to guarantee optimality or even feasibility. To tackle this drawback, all the wholesale electricity markets in the U.S. (Carlson et al. 2012) have transitioned to adopt mixed-integer linear programming (MILP) approaches, which have advantages in obtaining an optimal solution (Streiffert et al. 2005, Nemhauser 2013). As a result, hundreds of million dollars are saved (Bixby 2010).

As cutting planes are efficient approaches to strengthen the MILP formulation and speed up the corresponding branch-and-cut algorithm, significant research progress has been made to derive cutting planes and thereby provide strong formulations via polyhedral studies for the traditional UC problem, as any small improvement in the computational performance can result in huge cost savings for the UC problem. For instance, Lee et al. (2004) develop alternating up/down inequalities to strengthen the minimum-up/-down time polytope; Rajan and Takriti (2005) and Malkin (2003) develop convex hull representation of the minimum-up/-down time polytope with start-up costs; Queyranne and Wolsey (2017) develop tight formulations for bounded up/down times and interval-dependent start-up costs; Morales-España et al. (2013) and Gentile et al. (2017) tighten the generation capacity constraints; Damcı-Kurt et al. (2016) and Ostrowski et al. (2012) develop strong valid inequalities to strengthen the ramping polytope; Pan et al. (2016) provide strong valid inequalities for the UC problem with gas-fired generators; and Pan and Guan (2016) derive convex hulls and strong valid inequalities for the integrated minimum-up/-down time and ramping polytope; among others. In addition, Frangioni and Gentile (2015), Knueven et al. (2018), and Guan et al. (2018) present extended formulations of the single-UC problem, without considering regulation reserve, in higher-dimensional space. However, there are very limited studies considering co-optimizing the energy and ancillary service markets, which is more computationally challenging than solving the energy market problem alone (Carlson et al. 2012). In this paper, we provide one of the first studies to analyze the polyhedral structure of the UC model that co-optimizes power generation and regulation reserve, leading to significant computational performance enhancement.

Table 1 Literature on Strong UC MILP Formulations

Literature	Polyhedral Study Result	Polyhedral Study Focus					
		Min-Up /-Down Time	Gener- -ation Limit	Stable Ramping	Startup Shutdown Ramping	Extended Formul- -ation	Ancillary Service
Lee et al. (2004)	convex hull	√	-	-	-	-	-
Malkin (2003)	convex hull	√	-	-	-	-	-
Rajan and Takriti (2005)	convex hull	√	-	-	-	-	-
Queyranne and Wolsey (2017)	convex hull	√	-	-	-	-	-
Morales-España et al. (2013)	cutting plane	-	√	-	√	-	√
Gentile et al. (2017)	convex hull, cutting plane	√	√	-	√	-	-
Ostrowski et al. (2012)	cutting plane	√	√	√	√	-	-
Damcı-Kurt et al. (2016)	cutting plane	-	√	√	√	-	-
Pan et al. (2016)	cutting plane	√	√	√	√	-	-
Pan and Guan (2016)	convex hull, cutting plane	√	√	√	√	-	-
Frangioni and Gentile (2015)	convex hull	-	-	-	-	√	-
Knueven et al. (2018)	convex hull	-	-	-	-	√	-
Guan et al. (2018)	convex hull	-	-	-	-	√	-
This paper	convex hull, cutting plane	√	√	√	√	√	√

We summarize the literature above on strong UC MILP formulations as well as our paper in Table 1 in terms of two perspectives: polyhedral study focus and result. For the first perspective, we consider whether a paper focuses on a specific physical constraint (i.e., minimum-up/-down, generation, and ramping limits) of a generator, whether it studies the extended formulation, and whether it considers ancillary service. Here we use "stable ramping" to represent the generation difference limits between any two consecutive online periods. For the second perspective, we consider whether a paper derives the convex hull of the studied polytope or several families of cutting planes (i.e., strong valid inequalities).

1.3. Contributions

We analyze the polyhedral structure of the UC problem with regulation reserve in two major steps: (i) in the original space and (ii) in a higher-dimensional space. First, we study a polytope obtained from co-optimizing power generation and regulation reserve in the original space, which is more complicated than the traditional UC polytope studied in the literature, and accordingly derive new families of strong valid inequalities to strengthen the original formulation. Second, we derive an extended formulation for each generator in a higher-dimensional space.

For the original polytope, we investigate its structure by first considering certain special cases and then exploring the most general case. Through considering special problem features (e.g., regulation-reserve-concerned ramping restrictions) and their interrelationships, we derive strong valid inequalities that are strong enough to describe the convex hulls of the two-period and three-period cases under various parameter settings. We then extend the study to the most general case that covers multiple periods by generalizing our insights based on the problem features. Thus we derive several families of strong valid inequalities that are facet-defining for the original multi-period polytope under mild conditions. For the inequalities that are in the exponential size of the total periods, we develop efficient polynomial-time separation algorithms to speed up the branch-and-cut process.

To derive the extended formulation in a higher-dimensional space, we start with developing an efficient dynamic programming algorithm to optimally schedule a single generator while respecting the same set of physical constraints in the original polytope. We can equivalently transform the dynamic program into a linear program that enables an integral optimal solution, leading to an alternative extended formulation. Thus, the constraints in the extended formulation can equivalently replace the original physical constraints for each generator.

Finally, we perform extensive computational experiments to show the benefits of our approach. Specifically, we demonstrate the efficiency of our proposed strong valid inequalities used as cutting planes on co-optimizing the power generation and regulation reserve under different data sets.

The remainder of this paper is organized as follows. We first provide an MILP formulation to co-optimize power generation and regulation reserve in Section 2. Then, we provide the convex hull results under different settings in Section 3, and we derive multi-period strong valid inequalities in Section 4. In Section 5, we develop an extended formulation in a higher-dimensional space for each single generator. In Section 6, we report computational results under different data sets to demonstrate the effectiveness of our proposed approaches. This paper concludes in Section 7.

2. The Co-Optimization Model

In this section, we describe the security-constrained UC model that co-optimizes power generation and regulation reserve. The objective of this model is to determine an optimal schedule of both the power generation and regulation reserve for operating the power system, which can be represented as shown in Figure 1 for a six-bus example.

First, we use $\mathcal{K}$, $\mathcal{B}$, and $\mathcal{A}$ to denote the sets of generators, buses, and transmission lines, respectively. For each generator $k \in \mathcal{K}$, we use SU^k (resp. SD^k) to denote its start-up (resp. shut-down) cost, RU^k (resp. RD^k) to denote the cost of regulation-up (resp. regulation-down) reserve, $\overline{C}^k$ (resp. $\underline{C}^k$) to denote its maximum (resp. minimum) power output when it is online, L^k (resp. ℓ^k) to denote its minimum-up (resp. minimum-down) time limit, $\overline{V}^k$ to denote the ramping rate limit when it starts up or shuts down, V^k to denote the ramping-up/-down rate limit when it is in the

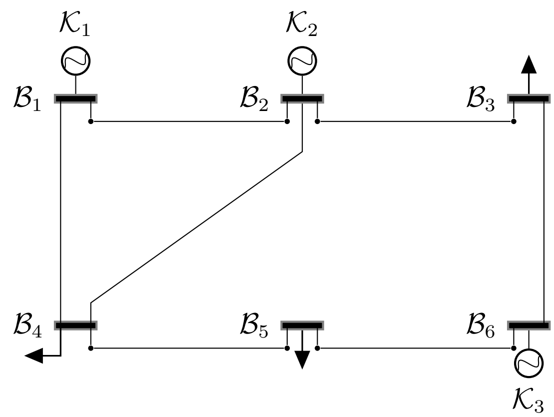

Figure 1 IEEE Six-Bus System

stable operation, and $f^k(\cdot)$ to denote its generation cost function, where $\overline{C}^k > \max\{\underline{C}^k, \overline{V}^k\}$. We let T represent the total number of time periods. For each bus $b \in \mathcal{B}$, we use D_t^b to denote the load at this bus in period t and $\mathcal{K}_b$ to denote the set of generators at this bus. For each transmission line $(m,n) \in \mathcal{A}$, we use A_{mn} to denote its transmission capacity and ρ_{mn}^b to denote the line flow distribution factor for the power flow on this line due to the net injection at each bus $b \in \mathcal{B}$. In each period t, the minimum regulation-up (resp. regulation-down) reserve requirement is denoted by R_t^+ (resp. R_t^-). For instance, Figure 1 presents a power grid with six buses (i.e., $\mathcal{B}_i$, $i = 1, \ldots, 6$), where generators (i.e., $\mathcal{K}_i$, $i = 1, 2, 3$) or loads (indicated by outgoing arrows) are located at each bus and transmission lines connect different buses. For simplicity, we use $[a,b]_{\mathbb{Z}}$ to denote the set of integer numbers between integers a and b, i.e., $\{a, a+1, \ldots, b-1, b\}$ with $[a,b]_{\mathbb{Z}} = \emptyset$ if $a > b$.

Next, for the decision variables, corresponding to each generator $k \in \mathcal{K}$ in period t, we use binary variable y_t^k to denote whether it is online (i.e., $y_t^k = 1$) or offline (i.e., $y_t^k = 0$), binary variable u_t^k to denote whether it starts up in period t (i.e., $u_t^k = 1$) or not (i.e., $u_t^k = 0$), continuous variable p_t^k to denote its power generation amount above the minimum power output $\underline{C}^k$, and continuous variable r_t^{k+} (resp. r_t^{k-}) to represent the regulation-up (resp. regulation-down) reserve amount.

Based on the notation above, the security-constrained UC model that co-optimizes power generation and regulation reserve, denoted as UCR, can be described as follows:

$$\min \quad \sum_{k \in \mathcal{K}} \left(\sum_{t=2}^{T} \left(\mathrm{SU}^k u_t^k + \mathrm{SD}^k \left(y_{t-1}^k - y_t^k + u_t^k \right) \right) + \sum_{t=1}^{T} \left(f^k \left(p_t^k + \underline{C} y_t^k \right) + \mathrm{RU}^k r_t^{k+} + \mathrm{RD}^k r_t^{k-} \right) \right) \tag{1a}$$

$$\text{s.t.} \quad \sum_{i=t-L^k+1}^{t} u_i^k \le y_t^k, \ \forall t \in [L^k+1, T]_{\mathbb{Z}}, \forall k \in \mathcal{K}, \tag{1b}$$

$$(\text{UCR}) \quad \sum_{i=t-\ell^k+1}^{t} u_i^k \le 1 - y_{t-\ell^k}^k, \ \forall t \in [\ell^k+1, T]_{\mathbb{Z}}, \forall k \in \mathcal{K}, \tag{1c}$$

$$-y_{t-1}^k + y_t^k - u_t^k \leq 0,\ \forall t \in [2,T]_{\mathbb{Z}}, \forall k \in \mathcal{K}, \tag{1d}$$

$$p_t^k - r_t^{k-} \geq 0,\ \forall t \in [1,T]_{\mathbb{Z}}, \forall k \in \mathcal{K}, \tag{1e}$$

$$p_t^k + r_t^{k+} \leq (\overline{C}^k - \underline{C}^k) y_t^k,\ \forall t \in [1,T]_{\mathbb{Z}}, \forall k \in \mathcal{K}, \tag{1f}$$

$$\left(p_t^k + \underline{C}^k y_t^k + r_t^{k+}\right) - \left(p_{t-1}^k + \underline{C}^k y_{t-1}^k\right) \leq V^k y_{t-1}^k + \overline{V}^k \left(1 - y_{t-1}^k\right),\ \forall t \in [2,T]_{\mathbb{Z}}, \forall k \in \mathcal{K}, \tag{1g}$$

$$\left(p_{t-1}^k + \underline{C}^k y_{t-1}^k\right) - \left(p_t^k + \underline{C}^k y_t^k - r_t^{k-}\right) \leq V^k y_t^k + \overline{V}^k \left(1 - y_t^k\right),\ \forall t \in [2,T]_{\mathbb{Z}}, \forall k \in \mathcal{K}, \tag{1h}$$

$$\sum_{k \in \mathcal{K}} r_t^{k+} \geq R_t^+,\ \forall t \in [1,T]_{\mathbb{Z}}, \tag{1i}$$

$$\sum_{k \in \mathcal{K}} r_t^{k-} \geq R_t^-,\ \forall t \in [1,T]_{\mathbb{Z}}, \tag{1j}$$

$$\sum_{k \in \mathcal{K}} \left(p_t^k + \underline{C}^k y_t^k\right) = \sum_{b \in \mathcal{B}} D_t^b,\ \forall t \in [1,T]_{\mathbb{Z}}, \tag{1k}$$

$$-A_{mn} \leq \sum_{b \in \mathcal{B}} \rho_{mn}^b \left(\sum_{k \in \mathcal{K}_b} (p_t^k + \underline{C}^k y_t^k) - D_t^b \right) \leq A_{mn},\ \forall t \in [1,T]_{\mathbb{Z}}, \forall (m,n) \in \mathcal{A}, \tag{1l}$$

$$r_t^{k\pm} \geq 0, y_t^k \in \{0,1\},\ \forall t \in [1,T]_{\mathbb{Z}};\ u_t^k \in \{0,1\},\ \forall t \in [2,T]_{\mathbb{Z}}, \forall k \in \mathcal{K}. \tag{1m}$$

The objective function (1a) is to minimize the total cost including start-up, shut-down, generation, and regulation-up/-down reserve costs. Constraints (1b) and (1c) describe the minimum-up and minimum-down time limits, respectively. That is, if generator k starts up (resp. shuts down) in period t, then it has to stay online (resp. offline) until time $t + L^k - 1$ (resp. $t + \ell^k - 1$) and is allowed to shut down (resp. start up) in period $t + L^k$ (resp. $t + \ell^k$). Constraints (1d) describe the relationship between the online/offline status y and start-up decision u and help set the shut-down cost term (i.e., $\mathrm{SD}^k(y_{t-1}^k - y_t^k + u_t^k)$) in (1a). Note that (1d) enforces $u_t^k = 1$ when $y_t^k = 1$ and $y_{t-1}^k = 0$. Meanwhile, constraints (1b) - (1d), together with the minimization objective, ensure that the shut-down cost SD^k is incurred for generator k in period t (i.e., $y_{t-1}^k - y_t^k + u_t^k = 1$) if and only if $y_{t-1}^k = 1$ and $y_t^k = u_t^k = 0$. Thus, if generator k starts up in period t (i.e., $u_t^k = 1$), then it implies that $y_t^k = 1$ and $y_{t-1}^k = 0$; and if generator k shuts down in period t, then it implies that $y_{t-1}^k = 1$ and $y_t^k = 0$. Constraints (1e) and (1f) describe the minimum and maximum power generation amounts that allow the generator to additionally provide regulation-down and regulation-up reserves, respectively. Constraints (1g) and (1h) describe the ramping-up and ramping-down rate restrictions, i.e., the maximum generation increment and decrement between two consecutive periods, respectively. They ensure the power generation does not increase/decrease too significantly, while guaranteeing the regulation reserve requirements to be satisfied. Constraints (1i) and (1j) describe the system-wide regulation-up and regulation-down reserve requirements in each period t, respectively. Constraints (1k) represent the load balance restriction. Finally, constraints (1l) represent the transmission line capacity limit. The generation cost function $f^k(\cdot)$ is a quadratic function in general (defined as $f^k(p_t^k + \underline{C}^k y_t^k) = a^k(p_t^k + \underline{C}^k y_t^k)^2 + b^k(p_t^k + \underline{C}^k y_t^k) + c^k y_t^k$) and is commonly approximated by a piecewise linear function (Carrión and Arroyo 2006). Accordingly, UCR is an MILP formulation.

The consideration of regulation reserve makes model UCR significantly different from the traditional UC model without regulation reserve. Technically, as compared to the traditional model, two more continuous variables, i.e., r_t^{k+} and r_t^{k-}, are integrated with power generation variables (i.e., p_t^k) in constraints (1e) - (1h) due to the co-optimization purpose. Specifically, constraints (1e) ensure that each generator k should schedule enough power generation (i.e., $p_t^k + \underline{C}^k$) so that the power generation will be above the minimum power output $\underline{C}^k$ (i.e., $p_t^k - r_t^{k-} \geq 0$) when it is called to provide regulation-down reserve (i.e., r_t^{k-}) in the real-time operations. Constraints (1f) ensure that each generator k cannot plan too much power generation so that the power generation will be below the maximum power output $\overline{C}^k$ (i.e., $p_t^k + \underline{C}^k + r_t^{k+} \leq \overline{C}^k$) when it is called to provide regulation-up reserve (i.e., r_t^{k+}) in the real-time operations. Similarly, constraints (1g) and (1h) ensure that the ramping-up and ramping-down rate restrictions can be satisfied when each generator k is called to provide regulation-up and regulation-down reserves, respectively. The inclusion of regulation reserve leads to significantly increased complexity in scheduling the power generation and regulation reserves to decide the optimal online/offline status, power generation amount, and regulation reserve amount. More importantly, it leads to high difficulty in solving the resulting MILP model UCR by particularly complicating the choices of optimal integral solutions. In the following, we provide an example to show that the inclusion of regulation reserve leads to different optimal integral solutions between the UC model with and without regulation reserve.

Example 1. We consider two generators (i.e., $\mathcal{K} = \{1, 2\}$) at the same bus without transmission line (i.e., $\mathcal{B} = \{1\}$). We let $T = 3$ and the total loads in each period $d_1 = 10$, $d_2 = 15$, and $d_3 = 25$, with the minimum reserve requirements $R_t^+ = R_t^- = 0.1 d_t, \forall t = 1, 2, 3$. We provide the detailed physical parameters of these two generators in Table 2. We solve the UC model in two cases: (i)

Table 2 Example 1 Data

Generator	$\overline{C}^k$	$\underline{C}^k$	V^k	$\overline{V}^k$	L^k	ℓ^k	SU^k	SD^k	RU^k	RD^k	a^k	b^k	c^k
$k = 1$	45	6	7	10	1	1	180	0	56	56	0.07	30	30
$k = 2$	60	9	10	14	1	1	350	0	46	46	0.01	20	60

with reserve, i.e., our model (1); and (ii) without reserve, i.e., a traditional UC model. The solution results are reported in Table 3, from which we can observe (i) when the reserve requirements are not considered, only generator 2 will be online to satisfy the loads with the minimum total cost; and (ii) when the reserve requirements are included, the system has to coordinate both generators to

Table 3 Example 1 Result

Case	Generator 1			Generator 2		
	y_1^1	y_2^1	y_3^1	y_1^2	y_2^2	y_3^2
Without Reserve	0	0	0	1	1	1
With Reserve	1	1	1	0	0	1

be online. Such difference is due to the complex physical characteristics. More specifically, the load difference between periods 3 and 2 is $d_3 - d_2 = 10$, which requires the online generator(s) to ramp up (i.e., increase the total generation amount) by 10 from periods 2 to 3. Both generators 1 and 2 have their stable ramping rates no larger than 10. When no reserve is considered, using generator 2 with $V^2 = 10$ can satisfy this ramping requirement. However, when the reserve requirements are included, the online generator(s) need to prepare excessive capacity as regulation-up/-down reserve besides satisfying the loads. It follows that using either single generator only is not enough anymore, and thus both generators are online in period 3. Therefore, we can observe that considering reserve requirements leads to a different generator online/offline schedule compared to the traditional case without reserve requirements.

In this paper, we improve the branch-and-cut algorithm for model UCR by performing a polyhedral study of the corresponding MILP formulation and providing strong valid inequalities as cutting planes under different settings. More specifically, we focus on constraints (1b) - (1h), which include all the physical restrictions of every single generator. Thus, without loss of generality, we remove superscript k and collect all the physical constraints into the following set with $\mathbb{B} := \{0,1\}$:

$$P := \Big\{ (p, r^+, r^-, y, u) \in \mathbb{R}_+^T \times \mathbb{R}_+^T \times \mathbb{R}_+^T \times \mathbb{B}^T \times \mathbb{B}^{T-1} :$$

$$\sum_{i=t-L+1}^{t} u_i \le y_t, \ \forall t \in [L+1, T]_{\mathbb{Z}}, \tag{2a}$$

$$\sum_{i=t-\ell+1}^{t} u_i \le 1 - y_{t-\ell}, \ \forall t \in [\ell+1, T]_{\mathbb{Z}}, \tag{2b}$$

$$y_t - y_{t-1} - u_t \le 0, \ \forall t \in [2, T]_{\mathbb{Z}}, \tag{2c}$$

$$p_t - r_t^- \ge 0, \ \forall t \in [1, T]_{\mathbb{Z}}, \tag{2d}$$

$$p_t + r_t^+ \le (\overline{C} - \underline{C}) y_t, \ \forall t \in [1, T]_{\mathbb{Z}}, \tag{2e}$$

$$p_t + r_t^+ - p_{t-1} \le \overline{V} + (\underline{C} + V - \overline{V}) y_{t-1} - \underline{C} y_t, \ \forall t \in [2, T]_{\mathbb{Z}}, \tag{2f}$$

$$p_{t-1} - p_t + r_t^- \le \overline{V} + (\underline{C} + V - \overline{V}) y_t - \underline{C} y_{t-1}, \ \forall t \in [2, T]_{\mathbb{Z}} \Big\}. \tag{2g}$$

In the following sections, we strengthen the original formulation by deriving convex hull representations for conv(P) for certain cases in Section 3 and general strong valid inequalities for conv(P) in Section 4, where conv(P) is defined as the convex hull description of set P. We also derive the extended formulation for every single generator in a higher-dimensional space in Section 5.

REMARK 1. Since we focus on the physical characteristics of every single generator, any improvement for set P can benefit every problem with P embedded because strong valid inequalities for $\text{conv}(P)$ are also valid for any problem with P embedded. Therefore, our derived strong valid inequalities can help solve other power system problems with P embedded more efficiently, regardless of which market settings.

3. Convex Hulls

We provide the convex hull results for two cases: two-period case in Section 3.1 and certain three-period cases in Section 3.2.

3.1. Two-Period Convex Hull

In this section, we focus on a two-period case of set P and tighten constraints for the original multi-period formulation. To that end, we derive the corresponding strong valid inequalities, which can be applied to any two consecutive periods of polytope $\text{conv}(P)$. Without loss of generality, we assume the minimum-up/-down time limit to be one. We also assume $\underline{C} < \overline{V} < \underline{C} + V$ throughout this paper to reflect the general physical characteristics of a thermal generator. Since the setting we consider is general and accordingly, our derived inequalities can be applied to describe the convex hulls for other parameter settings. The corresponding original constraint set with two periods, denoted by P_2, can be described as follows:

$$
\begin{aligned}
P_2 := \Big\{ & (p, r^+, r^-, y, u) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathbb{B}^2 \times \mathbb{B} : \\
& u_2 \le y_2, && \text{(3a)} \\
& u_2 \le 1 - y_1, && \text{(3b)} \\
& y_1 \ge y_2 - u_2, && \text{(3c)} \\
& p_t \ge r_t^-, \ \forall t = 1, 2, && \text{(3d)} \\
& p_1 + r_1^+ \le (\overline{C} - \underline{C}) y_1, && \text{(3e)} \\
& p_2 + r_2^+ \le (\overline{C} - \underline{C}) y_2, && \text{(3f)} \\
& p_2 + r_2^+ - p_1 \le \overline{V} + (\underline{C} + V - \overline{V}) y_1 - \underline{C} y_2, && \text{(3g)} \\
& p_1 - p_2 + r_2^- \le \overline{V} + (\underline{C} + V - \overline{V}) y_2 - \underline{C} y_1 \Big\}. && \text{(3h)}
\end{aligned}
$$

Note that our assumptions on the physical characteristics of a thermal generator are very mild and fit well with the industrial practices. When a generator starts up in a period, its start-up ramping rate limit $\overline{V}$ should be (in general slightly) larger than the minimum generation output $\underline{C}$ in order for the generator to be online in this period. Meanwhile, the generator can start up to generate over $\underline{C}$ in the current period and project to ramp up in the next period by V to generate over $\underline{C}+V$. Thus, $\overline{V}$ is smaller than $\underline{C}+V$ in general because the generator starts up in the current period and does not bypass a two-period ramping up process.

In the following, we provide the linear description of $\text{conv}(P_2)$. We first prove our derived inequalities are valid for $\text{conv}(P_2)$ in Proposition 1 and show that they are also facet-defining for $\text{conv}(P_2)$ in Proposition 2. Then, we prove that the derived inequalities with additional nonnegative restrictions of decision variables can describe the convex hull in Theorem 1.

Proposition 1. *The inequalities*

$$p_1 \le (\overline{V}-\underline{C})y_1 + (\overline{C}-\overline{V})(y_2-u_2), \tag{4}$$

$$p_2 + r_2^+ \le (\overline{C}-\underline{C})y_2 - (\overline{C}-\overline{V})u_2, \tag{5}$$

$$p_2 + r_2^+ - p_1 \le V y_2 - (\underline{C}+V-\overline{V})u_2, \tag{6}$$

$$p_1 - p_2 + r_2^- \le (\overline{V}-\underline{C})y_1 + (\underline{C}+V-\overline{V})(y_2-u_2), \tag{7}$$

$$r_2^+ + r_2^- \le 2V y_2 - (\underline{C}+2V-\overline{V})u_2, \tag{8}$$

are valid for conv(P_2) *when* $\overline{C}-\underline{C}-2V>0$.

Proof. See Online Supplement EC.1.2 for the detailed proof. □

Our proposed inequalities are derived through investigating the specific problem features (i.e., minimum-up/-down time, regulation-reserve-concerned ramping, and capacity lower/upper limits) that are involved in the original constraint set. We illustrate two approaches that we use to derive them and thereby provide the corresponding intuition behind the inequalities. We choose two inequalities (i.e., (4) and (6)) in Proposition 1 for illustration.

First, we derive an inequality (e.g., (4)) by combing possible disjunctive cases for operating a generator in a period. For instance, we focus on the generation amount above the minimum power output in period 1, i.e., p_1, which is potentially affected by the generator's online/offline status in periods 1 and 2 because we consider only two periods in P_2. Thus, we design an inequality in the form of

$$p_1 \le M_1 y_1 + M_2 y_2 + M_3 u_2 \tag{9}$$

to designate a variable upper bound. We aim to carefully choose the right coefficients M_1, M_2, and M_3 so that this inequality can be as strong as possible (e.g., facet-defining). We consider all the possible choices for (y_1, y_2, u_2) in the original constraints and the corresponding upper bound for p_1 under each choice. That is, there are three possible cases: (i) If $y_1 = 0$, i.e., the generator is offline, then $p_1 = 0$ due to (3e), where we also have $y_2 - u_2 = 0$ due to (3a) and (3c); (ii) If $y_1 = 1$ and $y_2 = 0$, i.e., the generator shuts down in period 2, then $p_1 \le \overline{V} - \underline{C}$ due to the shut-down ramping constraint (3h), where we also have $u_2 = 0$ due to constraint (3a); (iii) If $y_1 = y_2 = 1$ and $u_2 = 0$, i.e., the generator stays online throughout both periods 1 and 2, then the generation amount is simply bounded above by the maximum power output due to constraint (3e), i.e., $p_1 \leq \overline{C} - \underline{C}$. Then, we plug the values of (y_1, y_2, u_2) under each case above into the right-hand-side (RHS) of (9), and enforce this RHS to be tight at the corresponding upper bound that we obtain in the above three cases. That is, we let $M_2 + M_3 = 0$ due to case (i), $M_1 = \overline{V} - \underline{C}$ due to case (ii), and $M_1 + M_2 = \overline{C} - \underline{C}$ due to case (iii). It follows that

$$M_1 = \overline{V} - \underline{C},\ M_2 = \overline{C} - \overline{V},\ \text{and } M_3 = -\overline{C} + \overline{V},$$

which derive (9) as inequality (4) in Proposition 1.

Second, we derive an inequality (i.e., (6)) by tightening an existing constraint while maintaining its validity. For instance, we choose the original constraint (3g) and tighten (i.e., reduce) its RHS by replacing the first term in its RHS $\overline{V}$ with $\overline{V}y_2$, as $y_2 \leq 1$, leading to the following inequality:

$$p_2 + r_2^+ - p_1 \leq \overline{V}y_2 + (\underline{C} + V - \overline{V})y_1 - \underline{C}y_2. \tag{10}$$

Inequality (10) is valid when $y_2 = 1$ because (10) is the same as (3g) in this case. When $y_2 = 0$, $p_2 + r_2^+ = 0$ following constraint (3f), and (10) simply becomes $p_1 \geq -(\underline{C} + V - \overline{V})y_1$, which is also valid because $\overline{V} < \underline{C} + V$. We further tighten inequality (10) by replacing y_1 in its RHS with $y_2 - u_2$, as $y_1 \geq y_2 - u_2$ due to (3c). It leads to the following inequality:

$$p_2 + r_2^+ - p_1 \leq \overline{V}y_2 + (\underline{C} + V - \overline{V})(y_2 - u_2) - \underline{C}y_2, \tag{11}$$

which is clearly valid when $y_1 = y_2 - u_2$ because (11) is the same as (10) in this case. When $y_1 > y_2 - u_2$, it implies that $y_1 = 1$ and $y_2 = u_2 = 0$ following constraints (3a) and (3b). Due to constraint (3f), we have $p_2 + r_2^+ = 0$. The derived inequality (10) becomes $p_1 \geq 0$, which is also valid. Therefore, by rearranging the RHS in (11), we obtain inequality (6) in Proposition 1.

Adding nonnegative restrictions of the variables u_t, r_t^+, and r_t^-, we have a polytope Q_2, which is then proved to be the convex hull representation of P_2 in Theorem 1.

$$Q_2 := \Big\{(p, r^+, r^-, y, u) \in \mathbb{R}^9 : (3a) - (3e), (4) - (8),$$
$$u_2 \geq 0, \tag{12a}$$
$$r_t^+ \geq 0,\ \forall t = 1, 2, \tag{12b}$$
$$r_t^- \geq 0,\ \forall t = 1, 2\Big\}. \tag{12c}$$

To show that $Q_2 = \text{conv}(P_2)$, we further prove that (i) all the inequalities in Q_2 are facet-defining for conv(P_2); (ii) the polytope Q_2 is full-dimensional; (iii) all the inequalities that describe P_2 are dominated by those in Q_2; and (iv) every extreme point in Q_2 is integral in y and u.

PROPOSITION 2. *Inequalities (4) - (8) are facet-defining for conv(P_2) when $\overline{C} - \underline{C} - 2V > 0$.*

Proof. See Online Supplement EC.1.2 for the detailed proof. □

PROPOSITION 3. *The polytope Q_2 is full-dimensional.*

Proof. See Online Supplement EC.1.3 for the detailed proof. □

PROPOSITION 4. *All inequalities of P_2 are dominated by inequalities of Q_2.*

Proof. See Online Supplement EC.1.4 for the detailed proof. □

PROPOSITION 5. *Every extreme point in Q_2 is integral in y and u.*

Proof. See Online Supplement EC.1.5 for the detailed proof. □

According to Propositions 1 - 5, we can complete the proof of Theorem 1 as follows:

THEOREM 1. *When $\overline{C} - \underline{C} - 2V > 0$, we have $Q_2 = conv(P_2)$.*

Proof. Due to the formulation representations of Q_2 and P_2, we know that Q_2 and P_2 are bounded. From Propositions 1 and 2, we have all the inequalities in Q_2 are valid and facet-defining and thus $Q_2 \supseteq \text{conv}(P_2)$. According to Proposition 5, all extreme points in Q_2 are integral in y and u. Therefore, we conclude that $Q_2 = \text{conv}(P_2)$. □

EXAMPLE 2. We consider a generator with $\overline{C} = 10$, $\underline{C} = 2$, $\overline{V} = 4$, and $V = 3$. The original constraint set

$$
\begin{aligned}
P_2 := \Big\{ & (p, r^+, r^-, y, u) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathbb{B}^2 \times \mathbb{B} : \\
& u_2 \le y_2,\ u_2 \le 1 - y_1,\ y_1 \ge y_2 - u_2,\ p_1 \ge r_1^-,\ p_2 \ge r_2^-, \\
& p_1 + r_1^+ \le 8y_1,\ p_2 + r_2^+ \le 8y_2, \\
& p_2 + r_2^+ - p_1 \le 4 + y_1 - 2y_2,\ p_1 - p_2 + r_2^- \le 4 + y_2 - 2y_1 \Big\}.
\end{aligned}
\tag{13}
$$

Following Theorem 1, the convex hull of P_2 can be described as follows:

$$
\begin{aligned}
\text{conv}(P_2) := \Big\{ & (p, r^+, r^-, y, u) \in \mathbb{R}^9 :\ (13) \\
& p_1 \le 2y_1 + 6(y_2 - u_2),\ p_2 + r_2^+ \le 8y_2 - 6u_2, \\
& p_2 + r_2^+ - p_1 \le 3y_2 - u_2,\ p_1 - p_2 + r_2^- \le 2y_1 + y_2 - u_2, \\
& r_2^+ + r_2^- \le 6y_2 - 4u_2 \Big\}.
\end{aligned}
$$

Although the strong valid inequalities proposed in Proposition 1 are only valid for conv(P_2) when $\overline{C} - \underline{C} - 2V > 0$, they are sufficient to describe conv(P_2) when this condition does not hold, as shown in the following theorem. It also supports our motivation to study the general physical setting in the main analyses here.

THEOREM 2. *When* $\underline{C} + V < \overline{C} \leq \underline{C} + 2V$, *we have conv*($P_2$) $= \{(p, r^+, r^-, y, u) \in \mathbb{R}^9 : (3a) - (3e), (4) - (7), (12a) - (12c)\}$. *When* $\overline{V} < \overline{C} \leq \underline{C} + V$, *we have conv*($P_2$) $= \{(p, r^+, r^-, y, u) \in \mathbb{R}^9 : (3a) - (3e), (4) - (5), (12a) - (12c)\}$.

Proof. The proof is similar to that for Theorem 1 and thus it is omitted. □

Theorem 1 indicates that when optimizing a linear cost function over P_2, we only need to solve a linear program with all the constraints in Q_2 to obtain the integral optimal solutions, which will reduce the computational burden significantly. In addition, our derived inequalities (4) - (8) can be applied for any two consecutive periods to strengthen the problem with P embedded. Furthermore, Theorem 2 shows that when the generator has a small generation capacity, a subset of inequalities in Q_2 are sufficient to provide the corresponding convex hull.

3.2. Three-Period Convex Hulls

In this section, we investigate the three-period formulation, i.e., $T = 3$ in set P, and provide convex hull descriptions for various cases with different minimum-up/-down time limits and parameter settings. We consider the following two parameter settings: (i) $\overline{C} - \overline{V} - 2V \geq 0$ and (ii) $\overline{C} - \overline{V} - 2V < 0$ and $\overline{C} - \underline{C} - 2V \geq 0$. For each parameter setting, we discuss the following four possible cases in terms of minimum-up/-down time limit: (i) $L = \ell = 1$; (ii) $L = 1$, $\ell = 2$; (iii) $L = 2$, $\ell = 1$; and (iv) $L = \ell = 2$. As the derived inequalities are different for different cases, here we consider the most representative case in which $\overline{C} - \overline{V} - 2V \geq 0$ and $L = \ell = 2$, with the results for other cases provided in Online Supplement EC.3. The corresponding original constraint set can be described as follows:

$$P_3^2 := \Big\{ (p, r^+, r^-, y, u) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{B}^3 \times \mathbb{B}^2 :$$

$$u_2 + u_3 \leq y_3, \tag{14a}$$

$$y_1 + u_2 + u_3 \leq 1, \tag{14b}$$

$$u_2 \geq y_2 - y_1, \; u_3 \geq y_3 - y_2, \tag{14c}$$

$$r_t^+ \geq 0, \; r_t^- \geq 0, \; \forall t = 1, 2, 3, \tag{14d}$$

$$p_t \geq r_t^-, \; \forall t = 1, 2, 3, \tag{14e}$$

$$p_t + r_t^+ \leq (\overline{C} - \underline{C}) y_t, \; \forall t = 1, 2, 3, \tag{14f}$$

$$p_t + r_t^+ - p_{t-1} \leq \overline{V} + (\underline{C} + V - \overline{V}) y_{t-1} - \underline{C} y_t, \; \forall t = 2, 3, \tag{14g}$$

$$p_{t-1} - p_t + r_t^- \leq \overline{V} + (\underline{C} + V - \overline{V}) y_t - \underline{C} y_{t-1}, \; \forall t = 2, 3 \Big\}. \tag{14h}$$

We follow a similar way to derive strong valid inequalities for conv(P_3^2) as we do for conv(P_2) in Section 3.1. For brevity, we present the complete convex hull description, denoted by $Q_3^2 :=$ conv(P_3^2), together with rigorous proofs in Online Supplement EC.2. Here we choose two inequalities from Q_3^2, i.e., (15) and (16), in the following part to illustrate the insights behind the derived inequalities.

First, the ramping-down constraints (14h) with $t=3$ can be strengthened as follows:

$$p_2 + r_2^+ - p_3 + r_3^- \leq (\overline{V} - \underline{C})y_2 + 2V(y_2 - u_2) + (\underline{C} + V - \overline{V})(y_3 - u_3 - u_2). \tag{15}$$

Inequality (15) can better represent the generator physical ramping restriction by incorporating both the power generation and regulation reserve variables. To show the insights and accordingly the validity of (15), we consider the case where the generator is online in all of the three periods, i.e., $y_1 = y_2 = y_3 = 1$ and $u_2 = u_3 = 0$. In this case, inequality (15) enforces $p_2 + r_2^+ - p_3 + r_3^- \leq 3V$, whereas the original (14h) with $t = 3$ tells us $p_2 - p_3 + r_3^- \leq V$. In fact, when the generator is running in all of the three periods, the physical restrictions of the generator provide an upper bound for the variable r_2^+, as shown in Figure 2. For instance, in period $t=2$, the generator can increase or decrease its power generation output based on the generation output in period 1 (i.e., $|p_2 - p_1|$) by ramping rate limit V. If the generator is scheduled to increase its generation output by its maximum ramping rate limit V (i.e., $p_2 = p_1 + V$), then there is no capacity for regulation-up reserve. In contrast, if the generator is scheduled to decrease the power generation output by V (i.e., $p_2 = p_1 - V$), then there exists at most $2V$ of capacity for regulation-up reserve. It follows that $r_2^+ \leq 2V$. Thus, as compared to (14h) with $t=3$, our derived inequality (15) can additionally incorporate the upper bound of regulation-up reserve r_2^+, thereby tightening (14h).

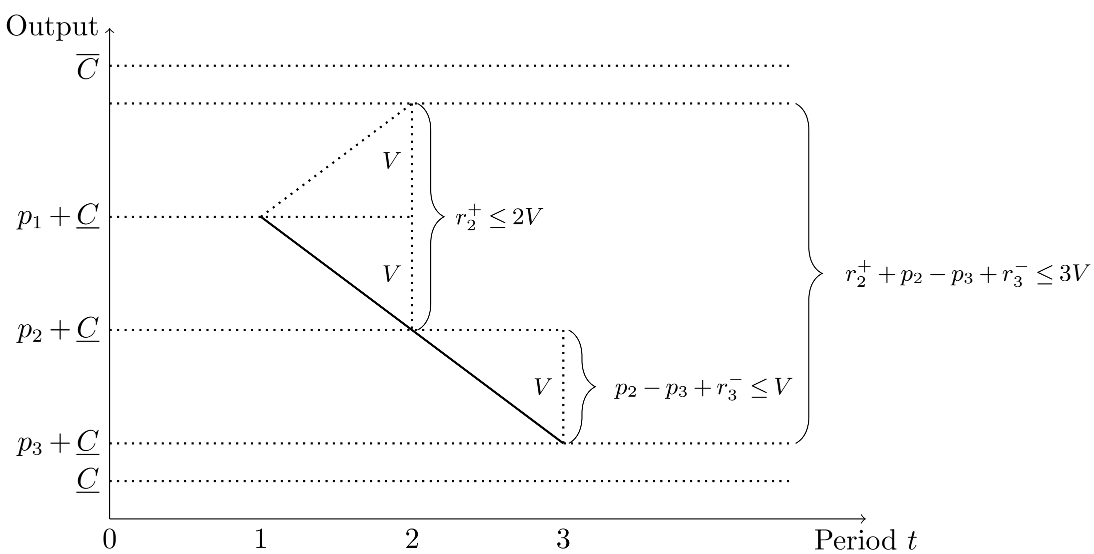

Figure 2 Generator Online in Three Periods for Inequality (15)

Next, we provide a strong valid inequality to bound the regulation-up/-down reserves as follows:

$$r_2^+ + r_3^+ + r_3^- \leq 2Vy_2 + 2Vy_3 - (\underline{C} + 3V - \overline{V})u_2 - (\underline{C} + 2V - \overline{V})u_3. \tag{16}$$

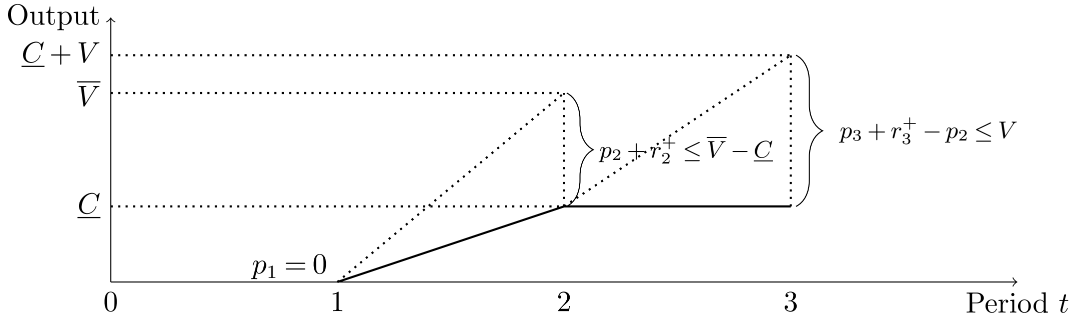

Figure 3 Generator Starting Up in Period $t=2$ for Inequality (16)

Similar to the above analysis, we show how inequality (16) can capture the physical characteristics that are not explicitly represented by a single original constraint in P_3^2. To avoid a replication of the previous analysis for inequality (15), we consider a different case where the generator starts up in period $t=2$, i.e., $y_1=0$, $y_2=y_3=1$, $u_2=1$, and $u_3=0$. In this case, inequality (16) becomes $r_2^+ + r_3^+ + r_3^- \leq \overline{V} - \underline{C} + V$, which is a combined representation of generator physical restrictions, as shown in Figure 3. In fact, if the generator is scheduled to start up in period $t=2$, then by the original ramping-up constraints (14g), the power generation output in period 2 is bounded above by the start-up ramping rate limit $\overline{V}$ (i.e., $p_2+r_2^+\leq\overline{V}-\underline{C}$). Furthermore, the ramping constraint between $t=2$ and $t=3$ implies that $p_3+r_3^+-p_2\leq V$. Therefore, by original constraints, we have $r_2^+ + r_3^+ + p_3 \leq \overline{V} - \underline{C} + V$. Note that $r_3^- \leq p_3$, then we conclude that the derived inequality (16) $r_2^+ + r_3^+ + r_3^- \leq \overline{V} - \underline{C} + V$ holds. We can observe that inequality (16) combines these complicated relationships in one single inequality explicitly, which enables us to tighten the linear programming (LP) relaxation of the original formulation. In addition, the case considered here happens in realistic applications. That is, if a fast-ramping generator is planned to provide regulation reserve, then it can be scheduled to start up in some time period, maintain the minimum power generation output, and use the ramping capability V as regulation-up reserve.

Note that inequalities (15) and (16), as well as the other nontrivial inequalities in Q_3^2, are valid and facet-defining for $\text{conv}(P_3^2)$ and thus they are tighter than the original physical constraints in P_3^2 (i.e., (14f) - (14h)). Our explanations on the intuition behind (15) and (16) also imply that they are tighter than the original constraints because each of (15) and (16) incorporates more complicated characteristics in a single inequality than each original constraint.

Remark 2. The derived inequalities in each convex hull description for the three-period cases can be applied as cutting planes in a branch-and-cut algorithm for any three consecutive periods of the general multi-period case. As the amount of derived inequalities in each convex hull description is a constant number, we conclude that the number of cutting planes added to the branch-and-cut algorithm is polynomial in the total number of periods T.

4. Multi-Period Strong Valid Inequalities

In this section, we further derive strong valid inequalities by considering multiple periods and the most general parameter setting. We continue to focus on the specific problem features and derive the inequalities as strong as possible. Our derived inequalities strengthen the variable upper bound in Section 4.1 and the ramping rate in Section 4.2, respectively. For notational brevity, we define $\sum_{s=a}^{b} y_s = \sum_{s=a}^{b} u_s = 0$ if $a > b$.

4.1. Variable-Upper-Bound Inequalities

We derive strong valid inequalities to tighten the upper bounds of continuous variables by considering the effects of minimum-up/-down time and ramping constraints in P, in addition to the original capacity lower/upper bound constraints (2d) - (2e). For example, if a generator is online in period $t-k$ and shuts down in period t, then the generation amount $p_{t-k} + \underline{C} y_{t-k}$ is bounded above by $\overline{V} + kV$, instead of $\overline{C}$, due to ramping-down constraints (2g) if $\overline{V} + kV < \overline{C}$. In addition, by considering the minimum-up/-down time and ramping rate limits, we can derive a family of inequalities in the following proposition, which provide better upper bounds for p_{t-k}.

Proposition 6. *For each $k \in \{[2, T-2]_{\mathbb{Z}} : \overline{C} - \overline{V} - (k-1)V > 0\}$ and each $t \in [k + \min\{2, L\}, T]_{\mathbb{Z}}$, the inequality*

$$p_{t-k} \leq \left(\overline{V} - \underline{C}\right) y_{t-k} + V \sum_{s=1}^{k-1} \left(y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i} \right) + \left(\overline{C} - \overline{V} - (k-1)V\right) \left(y_t - \sum_{i=0}^{\min\{k,L-1\}} u_{t-i} \right) \tag{17}$$

is valid for conv(P). Furthermore, it is facet-defining for conv(P) when one of the following conditions is satisfied: (1) $L \leq 3$ and $k = \lfloor \frac{\overline{C} - \overline{V}}{V} \rfloor + 1$ for all $t \in [k + \min\{2, L\}, T]_{\mathbb{Z}}$; (2) $L \leq 3$ and $t = T$.

Proof. See Online Supplement EC.4.1 for the detailed proof. □

Note that in Proposition 6, the number of inequalities is polynomial in the order of T^2, i.e., $\mathcal{O}(T^2)$. Thus, we can add all of them into the model to solve the problem efficiently without performing a selective scheme to choose only some of them. In the following, we propose a family of inequalities whose sizes are exponential in T, where we do need to selectively choose some of them that are more effective than others.

Proposition 7. *For each $k \in \{[1, T-2]_{\mathbb{Z}} : \overline{C} - \overline{V} - (k+1)V > 0\}$, $t \in [2, T-k]_{\mathbb{Z}}$, and $S \subseteq [1, k-1]_{\mathbb{Z}}$, the inequalities*

$$r_t^- \leq \left(\overline{V} - \underline{C}\right) y_t + \left(\underline{C} + (k+1)V - \overline{V}\right) \left(y_{t+k} - \sum_{s=\max\{2,s'\}}^{t+k} u_s \right) + \sum_{s=\max\{2,s'\}}^{t} (t-s) V u_s \quad (\text{if } k = 1), \tag{18}$$

$$r_t^- \leq \left(\overline{V} - \underline{C}\right) y_t + \sum_{i \in S \cup \{1\}} (d_i - i) V \left(y_{t+i} - \sum_{s=\max\{2, t+i-L+1\}}^{t+i} u_s \right) + \left(\underline{C} + V - \overline{V}\right) \left(y_{t+k} - \sum_{s=\max\{2, s'\}}^{t+k} u_s \right)$$
$$+ \sum_{s=\max\{2, t+q-L+1\}}^{t} (t-s) V u_s \quad (\textit{if } k \geq 2 \textit{ and } S \neq \emptyset), \tag{19}$$

are valid for conv(P), where $s' = t + k - L + 1$*,* $d_i = \min\{a \in S \cup \{k\} : a > i\}$*, and* $q = \max\{a \in S\}$ *if* $L = 1$ *and* $q = 0$ *if* $L \geq 2$*. Furthermore, inequality (18) is facet-defining for conv(P) when* $t + k = T$*.*

Proof. See Online Supplement EC.4.2 for the detailed proof. □

Since there are an exponential number of choices for set S, the size of inequalities (19) is exponential in T. It follows that adding all of these inequalities to the model will significantly increase the model size and accordingly decrease the computational performance. Thus, we perform a separation procedure to find the most effective inequalities during the process of solving the original problem. That is, during the branch-and-bound process of solving a mixed-integer program and given a solution of the linear programming (LP) relaxation of the original problem corresponding to one branch-and-bound node, we check which inequality in the family (19) is violated with the largest violation, namely separating the given solution. If one such inequality is found, then we add it to the original model to continue the solution process. In the following, we provide a detailed separation procedure for inequalities (19). Meanwhile, we show that the most violated inequality can be efficiently found in polynomial time.

Separation: To find the most violated inequality (19), we construct a shortest path problem on an acyclic digraph $\mathbb{G} = (\mathbb{V}, \mathbb{A})$ for any given solution $(\hat{p}, \hat{r}^+, \hat{r}^-, \hat{y}, \hat{u}) \in \mathbb{R}_+^{5n-1}$, as shown in Figure 4. By letting $k' = \max\{k \in [1, T-2]_{\mathbb{Z}} : \overline{C} - \overline{V} - (k+1)V > 0\}$, we describe the graph as follows:

1. Node set $\mathbb{V} = \{o, d\} \cup \mathbb{V}'$ with o representing the origin, d representing the destination, and $\mathbb{V}' = \{t, t+1, \ldots, t+k'\}$ representing the set of time periods from t to $t+k'$ in inequalities (19).
2. Arc set $\mathbb{A} = \{a_{ot}, a_{(t+k')d}\} \cup \mathbb{A}'$ with $\mathbb{A}' = \cup_{t \leq t_1 < t_2 \leq t+k'} a_{t_1 t_2}$. The weight of each arc (i, j), denoted by ω_{ij}, is defined as follows.
   (a) $\omega_{ot} = \left(\overline{V} - \underline{C}\right) \hat{y}_t - \hat{r}_t^-$;
   (b) $\omega_{(t+k')d} = \left(\underline{C} + V - \overline{V}\right) \left(\hat{y}_{t+k'} - \sum_{s=\max\{2, t+k'-L+1\}}^{t+k'} \hat{u}_s\right) + \sum_{s=\max\{2, t+q-L+1\}}^{t-1} (t-s) V \hat{u}_s$;
   (c) $\omega_{t_1 t_2} = (t_2 - t_1) V \left(\hat{y}_{t_1} - \sum_{s=\max\{2, t_1-L+1\}}^{t_1} \hat{u}_s\right)$ for any $t_1, t_2 \in [t, t+k']_{\mathbb{Z}}$ and $t_1 < t_2$.

Based on the construction of graph $\mathbb{G}$, the optimal value of the shortest path from o to d represents the largest violation of inequalities (19) corresponding to a given t when the value is negative. The corresponding optimal solution determines a set S, which includes all the nodes on the shortest path. It follows that the corresponding inequality (19) with this set S will be added into the original model to cut off the solution $(\hat{p}, \hat{r}^+, \hat{r}^-, \hat{y}, \hat{u})$ and strengthen the model. Since $\mathbb{G}$

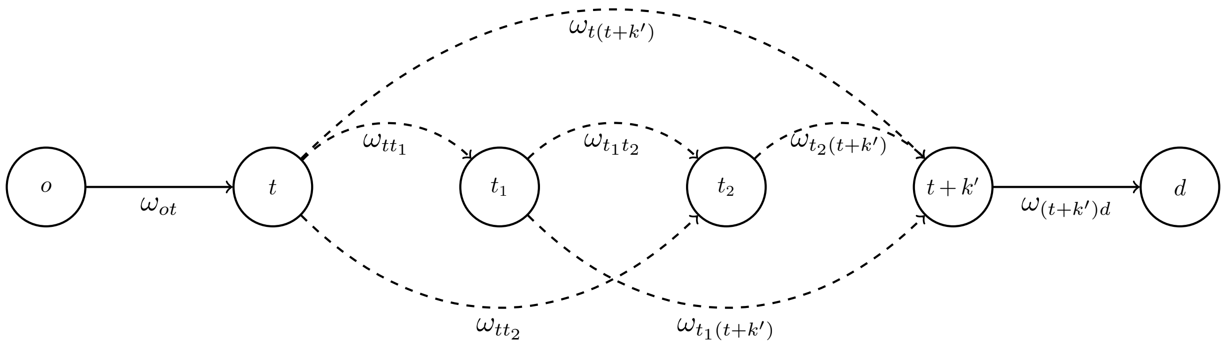

Figure 4 Acyclic Digraph of Separation Scheme for Inequalities (19)

is acyclic with $\mathcal{O}(T^2)$ arcs and $\mathcal{O}(T)$ nodes, finding the shortest path takes $\mathcal{O}(T^2)$ time for each given t by using the topological sorting algorithm. Therefore, we have an $\mathcal{O}(T^3)$ algorithm to solve the shortest path problem for all t.

PROPOSITION 8. *Given a point $(\hat{p},\hat{r}^+,\hat{r}^-,\hat{y},\hat{u})\in\mathbb{R}_+^{5n-1}$, there exists an $\mathcal{O}(T^3)$ algorithm to find the most violated inequality* (19)*, if any.*

Besides providing tighter upper bounds for single variables, e.g., p_{t-k} in Proposition 6 and r_t^- in Proposition 7, we also develop a family of inequalities to strengthen the summation of regulation-up and regulation-down reserves, as shown in the following proposition.

PROPOSITION 9. *For each $t\in[3,T]_{\mathbb{Z}}$, the inequality*

$$r_t^+ + r_t^- \le 2Vy_t - \left(\underline{C} + 2V - \overline{V}\right)u_t - \left(\underline{C} + V - \overline{V}\right)u_{t-1} \tag{20}$$

is valid when $\overline{C}-\underline{C}-2V>0$. Furthermore, it is facet-defining for conv(P) when $L\in[2,T-1]_{\mathbb{Z}}$.

Proof. See Online Supplement EC.4.3 for the detailed proof. □

In each period t, the original constraint set P does not specify the relationship between the regulation-up reserve (i.e., r_t^+) and regulation-down reserve (i.e., r_t^-) in one single constraint. It is interesting to observe that their direct relationship can be represented by one family of strong valid inequalities (i.e., (20)) through combining the effects from ramping rate limits and capacity lower/upper bounds. Furthermore, those inequalities are facets of the convex hull, which means that they are called for in the branch-and-cut algorithm. By explicitly providing them in our customized branch-and-cut scheme, the computational performance can be significantly improved through reducing both the search space and the branch-and-bound tree size.

REMARK 3. When $L=1$ or $t=2$, we have another family of strong valid inequalities, i.e., $r_t^+ + r_t^- \le 2Vy_t - (\underline{C}+2V-\overline{V})u_t$, which is very similar to (20). It is also valid and facet-defining for conv(P). The corresponding proofs are similar and thus are omitted here.

4.2. Ramping Inequalities

We strengthen the ramping-up/-down constraints by additionally considering other physical characteristics, as shown in the following proposition.

PROPOSITION 10. *For each* $k \in \{[1, T-1]_{\mathbb{Z}} : \overline{C} - \overline{V} - (k+1)V > 0\}$ *and each* $t \in [k+2, T]_{\mathbb{Z}}$, *the inequality*

$$p_t + r_t^+ - p_{t-k} + r_{t-k}^- \leq (k+2)Vy_t - \sum_{s=0}^{\min\{k, L-1\}} \left(\underline{C} + (k+2-s)V - \overline{V}\right) u_{t-s} \qquad (21)$$

is valid and facet-defining for conv(P).

Proof. See Online Supplement EC.4.4 for the detailed proof. □

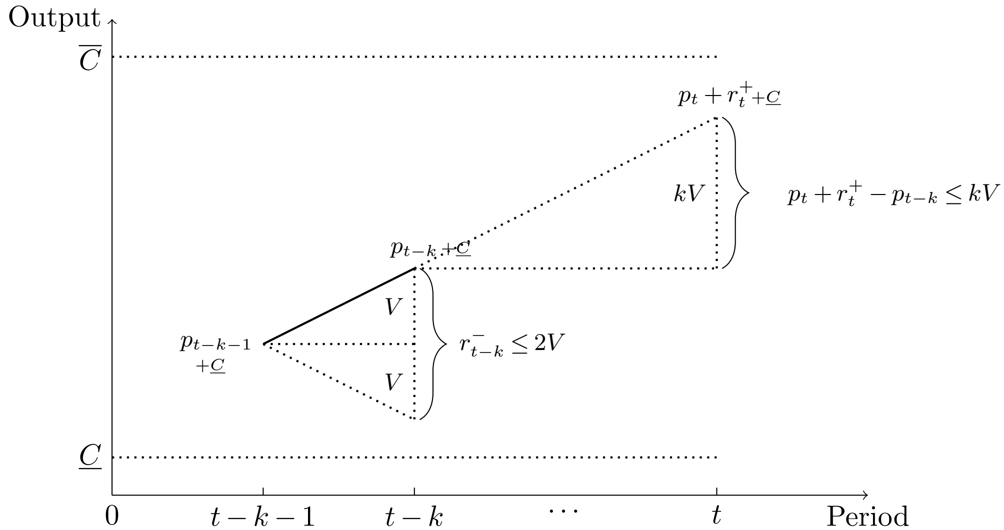

Figure 5 Generator Online in k Periods for Inequality (21)

The insight of inequality (21) is similar to that of (15), which represents the physical restrictions of the generator by incorporating both power generation and regulation reserve variables. For example, as shown in Figure 5, if the generator is online from periods $t-k-1$ to t, then the ramping restriction between periods t and $t-k$ is bounded from above by k times of ramping rate limit V, i.e., $p_t + r_t^+ - p_{t-k} \leq kV$. In addition, the regulation-down reserve variable r_{t-k}^- is also physically bounded by $2V$. Thus, we have inequality (21) indicates that $p_t + r_t^+ - p_{t-k} + r_{t-k}^- \leq (k+2)V$.

5. An Extended Formulation

In this section, we further tighten the set P by deriving an extended formulation in a higher-dimensional space for a single generator with regulation reserve. To that end, we remove the superscript k and the coupling constraints (1i) - (1l) in (1) and consider the following single-generator problem:

$$\min \quad \sum_{t=2}^{T}\Big(\mathrm{SU}u_t + \mathrm{SD}(y_{t-1} - y_t + u_t)\Big) + \sum_{t=1}^{T}\Big(g_t\left(p_t + \underline{C}y_t\right) + \mathrm{RU}r_t^+ + \mathrm{RD}r_t^-\Big) \tag{22a}$$

$$\text{s.t.} \quad (2a) - (2g), \tag{22b}$$

where $g_t(p_t + \underline{C}y_t) = f(p_t + \underline{C}y_t) - \rho_t(p_t + \underline{C}y_t)$ with ρ_t representing the electricity price in period t. The objective function (22a) is to minimize the total cost minus the revenue, i.e., net cost, where the total cost includes the start-up, shut-down, power generation, regulation-up reserve, and regulation-down reserve costs. Meanwhile, we lower approximate $g_t(\cdot)$ using a piecewise linear function with N pieces, from which ν_j^t and φ_j are the slope and intercept of the jth piece in period t, respectively. We also define $\mathcal{TK} = \{(t,k) \in \mathbb{Z}^2 : t \in [2,T]_{\mathbb{Z}}, k \in [\min\{t+L-1, T-1\}, T-1]_{\mathbb{Z}} \cup \{T\}\} \cup \{(t,k) \in \mathbb{Z}^2 : t = 1, k \in [1,T]_{\mathbb{Z}}\}$ to represent the set of all possible combinations of each t and k to construct a time interval $[t,k]_{\mathbb{Z}}$ in which the generator is online. Note that problem (22) respects the same set of constraints in P and thus its extended formulation can help strengthen the set P.

First, we introduce decision variables. For each period $t \in [1,T]_{\mathbb{Z}}$, we use binary variable α_t to denote if the generator starts up for the first time in period t (i.e., $\alpha_t = 1$) or not (i.e., $\alpha_t = 0$), and binary variable θ_t to denote if the generator shuts down in period $t+1$ and stays offline to the end (i.e., $\theta_t = 1$) or not (i.e., $\theta_t = 0$). For each $(t,k) \in \mathcal{TK}$, we use binary variable β_{tk} to denote if the generator starts up in period t and shuts down in period $k+1$ (i.e., $\beta_{tk} = 1$) or not (i.e., $\beta_{tk} = 0$), binary variable γ_{tk} to denote if the generator shuts down in period $t+1$ and starts up in period k (i.e., $\gamma_{tk} = 1$) or not (i.e., $\gamma_{tk} = 0$). For each 3-tuple $(t,k,s) \in \{(t,k,s) \in \mathbb{Z}^3 : (t,k) \in \mathcal{TK}, s \in [t,k]_{\mathbb{Z}}\}$, corresponding to each period s in the interval $[t,k]_{\mathbb{Z}}$, we use continuous variable q_{tk}^s to denote the power generation amount above the minimum power output, continuous variable m_{tk}^s (resp. n_{tk}^s) to denote the regulation-up (resp. regulation-down) reserve amount, and continuous variable w_{tk}^s to denote a part of the net cost.

Next, we describe the following formulation:

$$\min \quad \sum_{t=2}^{T}\mathrm{SU}\alpha_t + \sum_{k=1}^{T-1}\mathrm{SD}\beta_{1k} + \sum_{t=2}^{T}\sum_{k=t+L-1}^{T-1}\mathrm{SD}\beta_{tk} + \sum_{t=1}^{T-\ell-1}\sum_{k=t+\ell+1}^{T}\mathrm{SU}\gamma_{tk} + \sum_{(t,k)\in\mathcal{TK}}\sum_{s=t}^{k} w_{tk}^s \tag{23a}$$

$$\text{s.t.} \quad p_s = \sum_{(t,k)\in\mathcal{TK}, t\le s\le k} q_{tk}^s, \; r_s^+ = \sum_{(t,k)\in\mathcal{TK}, t\le s\le k} m_{tk}^s, \; r_s^- = \sum_{(t,k)\in\mathcal{TK}, t\le s\le k} n_{tk}^s,$$

$$y_s = \sum_{(t,k)\in\mathcal{TK}, t\le s\le k} \beta_{tk}, \; u_s = \alpha_s + \sum_{(t,k)\in\mathcal{TK}, k=s} \gamma_{tk}, \; \forall s \in [1,T]_{\mathbb{Z}} \tag{23b}$$

$$\sum_{t=1}^{T}\alpha_t \le 1, \tag{23c}$$

$$-\alpha_t + \sum_{k=1}^{T} \beta_{tk} = 0, \ t = 1, \tag{23d}$$

$$-\alpha_t + \sum_{k=\min\{t+L-1,T\}}^{T} \beta_{tk} = 0, \ \forall t \in [2, \ell+1]_{\mathbb{Z}}, \tag{23e}$$

$$-\alpha_t + \sum_{k=\min\{t+L-1,T\}}^{T} \beta_{tk} - \sum_{k=1}^{t-\ell-1} \gamma_{kt} = 0, \ \forall t \in [\ell+2, T]_{\mathbb{Z}}, \tag{23f}$$

$$-\sum_{k=1}^{t-L+1} \beta_{kt} + \sum_{k=t+\ell+1}^{T} \gamma_{tk} \leq 0, \ \forall t \in [1, T-\ell-1]_{\mathbb{Z}}, \tag{23g}$$

$$\theta_t - \sum_{k=1}^{t-L+1} \beta_{kt} = 0, \ \forall t \in [T-\ell, T]_{\mathbb{Z}}, \tag{23h}$$

$$q_{tk}^s + m_{tk}^s \leq (\overline{C} - \underline{C})\beta_{tk}, \ \forall (t,k) \in \mathcal{TK}, \forall s \in [t,k]_{\mathbb{Z}}, \tag{23i}$$

$$-q_{tk}^s + n_{tk}^s \leq 0, \ \forall (t,k) \in \mathcal{TK}, \forall s \in [t,k]_{\mathbb{Z}}, \tag{23j}$$

$$q_{tk}^t + m_{tk}^t \leq (\overline{V} - \underline{C})\beta_{tk}, \ \forall (t,k) \in \mathcal{TK} \text{ with } t \geq 2, \tag{23k}$$

$$q_{tk}^k \leq (\overline{V} - \underline{C})\beta_{tk}, \ \forall (t,k) \in \mathcal{TK} \text{ with } k \leq T-1, \tag{23l}$$

$$q_{tk}^{s-1} - q_{tk}^s + n_{tk}^s \leq V\beta_{tk}, \ \forall (t,k) \in \mathcal{TK}, \forall s \in [t+1,k]_{\mathbb{Z}}, \tag{23m}$$

$$q_{tk}^s + m_{tk}^s - q_{tk}^{s-1} \leq V\beta_{tk}, \ \forall (t,k) \in \mathcal{TK}, \forall s \in [t+1,k]_{\mathbb{Z}}, \tag{23n}$$

$$w_{tk}^s - \nu_j^s q_{tk}^s - \mathrm{RU} m_{tk}^s - \mathrm{RD} n_{tk}^s \geq (\nu_j^s \underline{C} + \varphi_j)\beta_{tk}, \ \forall (t,k) \in \mathcal{TK}, \forall s \in [t,k]_{\mathbb{Z}}, \forall j \in [1,N]_{\mathbb{Z}}, \tag{23o}$$

$$\alpha, \beta, \gamma, m, n \geq 0. \tag{23p}$$

The objective (23a) is to minimize the net cost, where the first and fourth terms represent the start-up cost, the second and third terms represent the shut-down cost, and the final term represents the generation and reserve costs minus the revenue. Constraints (23b) map the decision variables $(\alpha, \beta, \gamma, \theta, q, m, n)$ defined for the extended formulation to the decision variables (p, r^+, r^-, y, u) defined for the original formulation. Constraints (23c) - (23h) describe the synchronous relationships among the binary variables $(\alpha, \beta, \gamma, \theta)$. Specifically, constraint (23c) describes that there is at most one first-time start-up. Constraints (23d) - (23e) describe that given a period $t \in [1, \ell+1]_{\mathbb{Z}}$, if there is an online interval $[t,k]_{\mathbb{Z}}$ for the first time, then $\alpha_t = 1$. Note that when $\alpha_1 = 1$, it implies that the generator has been online prior to period 1 because there is no start-up cost in period 1 by model (22) as well as model (23). Thus here we allow the generator to shut down immediately as indicated in (23d). Constraints (23f) describe that given a period $t \in [\ell+2, T]_{\mathbb{Z}}$, if the generator shuts down in period $k+1$, followed by a coming start-up in period t such that $k \in [1, t-\ell-1]_{\mathbb{Z}}$, then it leads to an online interval $[t, k']_{\mathbb{Z}}$ with $k' \in [\min\{t+L-1, T\}, T]_{\mathbb{Z}}$. Note that here we also have $\alpha_t = 0$ because the first start-up should be before period $k+1$. Constraints (23g) describe that given a period $t \in [1, T-\ell-1]_{\mathbb{Z}}$, if there is no online interval since k until t with $k \in [1, t-L+1]_{\mathbb{Z}}$, then there would be no shut-down in period $t+1$ and accordingly no offline interval since $t+1$ until k with $k \in [t+\ell+1, T]_{\mathbb{Z}}$. Constraints (23h) describe that given a period $t \in [T-\ell, T]_{\mathbb{Z}}$, if there is an online interval since k until t, i.e., the generator shuts down in period $t+1$, then there will be no more start-up allowed due to the minimum-down time requirement, i.e., $\theta_t = 1$. Constraints (23i) and (23j), similar to (1f) and (1e), describe the maximum and minimum power generation amounts, while considering that the generator may additionally provide regulation-up and regulation-down reserves, respectively. Constraints (23k) (resp. (23l)) describe the start-up (resp. shut-down) ramping rate limits. Constraints (23m) (resp. (23n)) describe the ramping-down (resp. ramping-up) rate limits. Note that constraints (23k) - (23n) also incorporate the regulation reserve. Constraints (23o) are used to approximate the generation cost function as a piecewise linear function.

Theorem 3. *Formulation (23) is an extended formulation for the single-generator unit commitment problem with regulation reserve, in which an integral optimal solution can be obtained by solving (23) as a linear program. Meanwhile, if $(p^*, r^{+*}, r^{-*}, y^*, u^*, \alpha^*, \beta^*, \gamma^*, \theta^*, q^*, m^*, n^*)$ is an optimal solution to (23), then $(p^*, r^{+*}, r^{-*}, y^*, u^*)$ is an optimal solution to the problem in the original space.*

Proof. See Online Supplement EC.5 for the detailed proof. □

The extended formulation of problem (22) provides the convex hull representation for set P in a higher dimensional space by including more decision variables and constraints. Therefore, for any given generator $k \in \mathcal{K}$, its original physical constraints (i.e., (1b) - (1h) or (2a) - (2g)) can be replaced by the constraints (with physical meanings) in the extended formulation, i.e., (23b) - (23n) and (23p), when solving problem (1). This can help us obtain a further tighter LP lower bound for model UCR (i.e., problem (1)).

6. Computational Experiments

In this section, we conduct computational experiments to verify the effectiveness of our proposed strong valid inequalities in solving the security-constrained UC problem that co-optimizes power generation and regulation reserve as described in Section 2. Two different data sets, i.e., the power system data in Carrión and Arroyo (2006) and Ostrowski et al. (2012) and a modified IEEE 118-bus system data available at `http://motor.ece.iit.edu/data/SCUC_118/`, are used. We use a computer node with two AMD Opteron 2378 Quad Core Processors at 2.4 GHz and 4 GB memory to perform the experiments. The time limit is set as one hour per run except when specified. We compare our strong valid inequalities added as user cuts with default CPLEX 12.3 settings.

6.1. Power System Data in Carrión and Arroyo (2006) and Ostrowski et al. (2012)

In this experiment, we test twenty cases (see Table V in Ostrowski et al. (2012)) that are created using eight different types of generators (as shown in Table 4). We set the total number of periods $T = 24$ (e.g., leading to one-day schedule with each period being one hour) and the minimum required regulation-up/-down reserve amount in each period as 3% of the total demand, i.e., $R_t^+ = R_t^- = 3\% \sum_{b\in\mathcal{B}} D_t^b,\ \forall t \in [1,T]_{\mathbb{Z}}$. Hourly demand $\sum_{b\in\mathcal{B}} D_t^b$ is given in Table VI in Ostrowski et al. (2012) for each $t \in [1,T]_{\mathbb{Z}}$. Due to lack of transmission data in Carrión and Arroyo (2006) and Ostrowski et al. (2012), transmission constraints (11) are not included here. Meanwhile, the optimality gap tolerance is set as 0.05%. The problem sizes of the original co-optimization model (1) for each of the twenty cases are described in Table 5, where the number of generators (labeled as " # Gen."), the number of binary variables in problem (1) (labeled as " # Bin."), and the number of continuous variables in problem (1) (labeled as " # Cont.") are reported.

Table 4 Generator Data (Carrión and Arroyo 2006, Ostrowski et al. 2012)

Generators	$\underline{C}$ (MW)	$\overline{C}$ (MW)	L/ℓ (h)	V (MW/h)	$\overline{V}$ (MW/h)	SU ($/h)	a ($/MW^{2}h)	b ($/MWh)	c ($/h)
1	150	455	8	91	180	2000	0.00048	16.19	1000
2	150	455	8	91	180	2000	0.00031	17.26	970
3	20	130	5	26	35	500	0.002	16.6	700
4	20	130	5	26	35	500	0.00211	16.5	680
5	25	162	6	32.4	40	700	0.00398	19.7	450
6	20	80	3	16	28	150	0.00712	22.26	370
7	25	85	3	17	33	200	0.00079	27.74	480
8	10	55	1	11	15	60	0.00413	25.92	660

Table 5 Problem Sizes with Data in Carrión and Arroyo (2006) and Ostrowski et al. (2012)

Case	# Gen.	# Bin.	# Cont.	Case	# Gen.	# Bin.	# Cont.
1	28	1344	2016	11	132	6336	9504
2	35	1680	2520	12	156	7488	11232
3	44	2112	3168	13	156	7488	11232
4	45	2160	3240	14	165	7920	11880
5	49	2352	3528	15	167	8016	12024
6	50	2400	3600	16	172	8256	12384
7	51	2448	3672	17	182	8736	13104
8	51	2448	3672	18	182	8736	13104
9	52	2496	3744	19	183	8784	13176
10	54	2592	3888	20	187	8976	13464

In each case, we compare three formulations: "MILP", "Strong" and "Extend", where "MILP" represents the original MILP formulation (1), "Strong" represents the original MILP formulation (1) with our derived strong valid inequalities in Sections 3 and 4 added as user cuts, and "Extend" represents the original MILP formulation (1) with constraints (1b) - (1h) replaced by (23b) - (23n) and (23p) from the extended formulation for each given $k \in \mathcal{K}$. In "Strong", all of the inequalities derived in Section 3 and all of the inequalities (17), (18), (20), and (21) in Section 4 are added as a whole because the total number of these inequalities is a polynomial function of T. For (19), as the total number of inequalities is an exponential function of T, we limit the number of inequalities added because it is well known that adding too many inequalities will increase the problem size significantly and thus potentially reduce the computational performance. Thus, we select the inequities (19) with $|S| \in [2,5]_{\mathbb{Z}}$, and $k \in [3, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, by which we heuristically enforce a more restrictive validity condition and obtain a good computational performance. The time limit set for "Extend" is five hours per run.

We report the computational results in Table 6. The column "Integer Obj. (\$)" represents the objective value (i.e., Z_{Int}) corresponding to the best integer solution obtained from the three formulations. The column "IGap(%)" represents the integrality gap of each formulation. We define $\text{IGap}(\%)_{(\cdot)} = (Z_{\text{Int}} - Z_{\text{LP}(\cdot)})/Z_{\text{Int}} \times 100\%$, where $Z_{\text{LP}(\cdot)}$ denotes the optimal value of the LP relaxation of each formulation. For example, $\text{IGap}(\%)_{(\text{Extend})} = (Z_{\text{Int}} - Z_{\text{LP(Extend)}})/Z_{\text{Int}} \times 100\%$. For the "Strong" formulation, $Z_{\text{LP(Strong)}}$ is obtained by solving the LP relaxation problem with our strong valid inequalities added to the model as constraints. The column "Red. IGap(%)" represents how the "Strong" and "Extend" formulations reduce the integrality gap compared to the "MILP" formulation, and we define it as $(\text{IGap}_{(\text{MILP})} - \text{IGap}_{(\cdot)})/\text{IGap}_{(\text{MILP})} \times 100\%$. More specifically, it shows that how much the "Strong" and "Extend" formulations tighten the LP relaxation of the original problem (1). The column "Time (s) (TGap(%))" provides the computational time for solving each case. If a formulation cannot be solved to the optimality gap tolerance (i.e., 0.05%) within the time limit (i.e., one hour for "MILP" and "Strong" and five hours for "Extend"), we label it by "***" and report the terminating gap, i.e., the relative gap between the objective value and the best lower bound, in parentheses. We use "-" to indicate the cases where no feasible integer solution can be found within the time limit. The column "# Nodes" represents the number of nodes explored in the branch-and-bound process, where we use "-" to indicate that only the root node has been explored after reaching the time limit. The column "# User Cuts" shows the number of our derived strong valid inequalities added as cutting planes when solving the "Strong" formulation for each case. Overall, the "Extend" formulation tightens the LP relaxation but performs poorly due to the large-size of the formulation. Thus, we omit the comparisons with it in the next case study. In addition, by adding our proposed valid inequalities as cutting planes, the "Strong" approach solves most cases into optimality, while the "MILP" approach fails for most cases. For the instances that cannot be solved into optimality by both approaches, "Strong" provides smaller terminating gaps than "MILP" does. More specifically, although the "MILP" formulation may provide fairly good solutions with relatively small integrality gaps, the "Strong" formulation can obtain even better lower bounds faster because our derived inequalities help reduce the LP relaxation of the original problem significantly. That is, our "Strong" approach results in a better solution closer to the optimal one.

Table 6 Computational Results for Data in Carrión and Arroyo (2006) and Ostrowski et al. (2012)

Case	Integer Obj. ($)	IGap(%)			Red. IGap(%)		Time (s) (TGap(%))			# Nodes			# User Cuts
		MILP	Strong	Extend	MILP	Extend	MILP	Strong	Extend	MILP	Strong	Extend	
1	3806802	0.66	0.18	0.16	72.82	76.27	***(0.12)	***(0.07)	***(0.18)	256829	227823	549	216
2	4785647	0.63	0.28	0.14	55.56	77.34	***(0.16)	***(0.07)	***(0.09)	194662	180616	539	349
3	5101670	0.71	0.29	0.14	58.73	80.71	***(0.10)	1060.1	***(0.21)	139593	37627	224	385
4	4776350	0.70	0.24	0.13	65.10	81.07	411.3	78.3	-	16403	5742	177	287
5	5381203	0.88	0.42	0.16	52.27	81.51	***(0.11)	231.0	***(0.06)	102329	11858	426	554
6	4407806	1.08	0.50	0.09	54.18	91.83	334.8	17.2	14500	13271	545	513	672
7	5809788	0.67	0.27	0.13	59.56	80.56	***(0.08)	***(0.06)	-	134500	332664	114	497
8	5164631	0.91	0.42	0.16	53.99	85.53	***(0.07)	261.8	-	146975	13226	114	385
9	5612092	0.88	0.43	0.17	51.53	80.57	***(0.12)	1672.3	***(0.09)	79193	51118	355	570
10	5072376	1.04	0.49	0.15	53.17	85.53	***(0.05)	40.6	14690	258801	554	659	655
11	15741354	0.60	0.21	0.09	65.55	84.74	***(0.15)	1519.2	-	20634	12288	-	787
12	17152330	0.68	0.27	0.11	60.87	83.11	***(0.18)	971.0	-	28273	2907	-	1600
13	16833130	0.73	0.29	0.13	59.76	82.91	***(0.17)	775.3	-	18973	3526	-	1509
14	20058101	0.67	0.27	0.11	60.66	83.86	***(0.16)	542.1	-	12227	499	-	2097
15	17329467	0.90	0.40	0.15	55.15	83.76	***(0.16)	1128.1	-	14035	2399	-	1472
16	19425755	0.64	0.25	0.10	60.98	84.52	***(0.13)	1403.6	-	28694	3720	-	1789
17	19628597	0.81	0.35	0.13	57.29	83.46	***(0.15)	392.9	-	12971	561	-	1757
18	19547980	0.82	0.35	0.13	56.99	83.72	***(0.14)	543.7	-	16109	589	-	1718
19	20050001	0.73	0.28	0.12	61.06	83.63	***(0.15)	608.4	-	17751	552	-	1653
20	19665461	0.81	0.33	0.13	59.82	84.41	***(0.13)	361.1	-	26446	548	-	1754

6.2. Modified IEEE 118-Bus System

The IEEE 118-bus system is a widely-used test system for power system operations. There are 54 generators, 118 buses, 186 transmission lines, and 91 load buses in this system, where the parameters are slightly modified here. We set the total number of periods $T = 24$ and the minimum required regulation-up/-down reserve amount as 3% of the total demand in each period. With available transmission data, we include transmission constraints (1l) in the experiments. We following the setting in Section 6.1 to compare the formulations "MILP" and "Strong", and we set the optimality gap tolerance as 0.01%. Based on this data setting, there are $2,592$ binary variables and $3,888$ continuous variables in the original co-optimization model (1).

First, we report the computational results for the system with different load profiles in five intervals: $\overline{d}_t^n \in [0.5d_t^n, 0.7d_t^n]$, $[0.7d_t^n, 0.9d_t^n]$, $[0.9d_t^n, 1.1d_t^n]$, $[1.1d_t^n, 1.3d_t^n]$, and $[1.3d_t^n, 1.5d_t^n]$ for each nominal load d_t^n at bus n in period t, respectively. We randomly generate ten cases for each interval and provide their average performance in Table 7. The header of Table 7 is the same as that of

Table 6 except the first three columns. The column "Interval" represents the aforementioned five load profile intervals. The column "Integer Obj. ($)" provides the average optimal value over the ten cases for each load profile interval. The standard deviations of the optimal values of these ten cases are reported in the next column "Std. Dev." for each interval. Table 7 shows that adding our derived inequalities as cutting planes can generally solve LP relaxation problems faster and tighten the integrality gap. Our approach also reduces the computational time, terminating gap, and branch-and-bound nodes significantly.

Table 7 Computational Results for Modified IEEE 118-Bus System with Different Load Intervals

Interval	Integer Obj. ($)	Std. Dev.	IGap(%)		Red. IGap(%)	Time (s) (TGap(%))		# Nodes		# User Cuts
			MILP	Strong		MILP	Strong	MILP	Strong	
0.5-0.7	977917	2266	1.44	0.63	56.13	1957 (0.013)	979.5	28109	25626	308
0.7-0.9	1301176	2339	1.15	0.33	71.15	2576 (0.036)	1767 (0.020)	90747	85574	373
0.9-1.1	1629153	2392	1.03	0.25	76.31	3516 (0.742)	1795 (0.014)	59578	58719	423
1.1-1.3	1960402	3261	1.05	0.24	77.46	3600 (0.131)	2287 (0.014)	70008	42203	488
1.3-1.5	2298222	3004	1.13	0.16	85.50	3600 (0.095)	1224	77625	54585	643

Table 8 Computational Results for Modified IEEE 118-Bus System with Load $\overline{d}_t^n \in [0.7d_t^n, 1.3d_t^n]$

Case	Integer Obj. ($)	IGap(%)		Red. IGap(%)	Time (s) (TGap(%))		# Nodes		# User Cuts
		MILP	Strong		MILP	Strong	MILP	Strong	
1	1628494	1.05	0.32	69.19	***(0.12)	1450.3	43757	44374	409
2	1629569	1.01	0.25	75.24	***(0.12)	2589.8	49267	119970	427
3	1627370	1.04	0.26	75.29	***(0.03)	747.0	29801	13355	469
4	1622951	1.04	0.25	75.91	2948.7	402.1	48656	5822	392
5	1627893	1.04	0.25	75.96	***(0.05)	2506.3	77735	148725	457
6	1624446	1.06	0.25	76.62	***(0.02)	1338.7	60034	26710	415
7	1631721	1.12	0.26	76.69	***(0.10)	3201.2	79413	113967	463
8	1623505	1.13	0.26	76.83	***(0.06)	1576.2	76414	49305	462
9	1646628	1.06	0.24	77.53	1265.9	240.8	22998	7761	402
10	1628788	1.05	0.24	77.59	2048.8	1239.4	45329	23566	388
11	1620066	1.10	0.24	78.21	***(0.05)	2232.9	71448	114850	428
12	1621846	1.21	0.26	78.22	***(0.11)	1795.4	46201	53522	435
13	1620859	1.11	0.23	79.18	***(0.07)	399.4	58649	12827	419
14	1633056	1.20	0.23	81.19	***(0.05)	2450.4	71103	95808	394
15	1628582	1.24	0.20	84.07	1590.9	194.7	31032	4681	388

Next, we provide fifteen cases under the same load profile to further demonstrate the effectiveness of our derived inequalities. That is, we generate random load $\overline{d}_t^n \in [0.7d_t^n, 1.3d_t^n]$ fifteen times for each nominal load d_t^n at bus n in period t in the system. The computational results for all the cases are reported in Table 8. Although the problem for the modified IEEE 118-bus system is generally hard to solve, by adding our derived inequalities, it is significantly faster to solve the LP

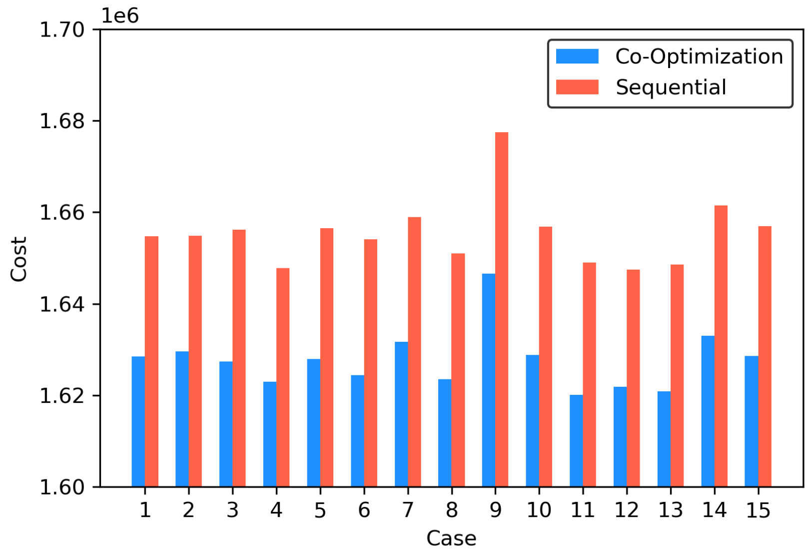

Figure 6 Co-Optimization and Sequential Model Comparison

relaxation problem and achieve a better lower bound, which leads to more chances for our "Strong" formulation to solve the problem within the time limit quickly.

Finally, we provide computational results to show the value of co-optimization. We compare our co-optimization model (1) with a sequential model to show the total cost difference. In the sequential model, the power generation for the energy market and the regulation reserve are scheduled separately: (i) a traditional UC model without the regulation reserve is firstly solved to determine the online/offline status and power generation amount of each generator in each period; and (ii) a reserve scheduling model is then solved to determine the regulation reserve amount of each online generator in each period, and an offline generator may be required to turn on to provide regulation reserve. Here we continue to use the data setting for Table 8 to correspondingly obtain the optimal value of the sequential model for each case, while the column "Integer Obj. ($)" in Table 8 represents the optimal value of our co-optimization model (1). We show the cost difference between these two models in Figure 6. We can observe that the sequential model leads to a larger cost for any case than the co-optimization model does, with the gap between them being around $1\% - 2\%$, which can be significantly large due to the large daily operational cost involved in the power system in practice. This demonstrates the value of co-optimization, and also echoes the industry practice from MISO who implemented such a co-optimization model since 2009 and thereby obtained at least $60 million annual savings (MISO 2019).

7. Conclusions

In this paper, we investigated the co-optimization model of energy and ancillary service markets, which leads to the security-constrained UC problem with regulation reserve. The problem is computationally challenging because the UC problem is difficult by itself and the inclusion of regulation reserve complicates it further. To improve the co-optimization performance for clearing both energy and ancillary service markets, we improved the branch-and-cut algorithm by performing polyhedral studies on the integrated polytope of minimum-up/-down time, ramping, power generation upper/lower bound, and regulation-up/-down reserve restrictions. We provided two-period and three-period convex hull descriptions with rigorous proofs, and then developed strong valid inequalities for the general multi-period case. In addition, we showed that under mild conditions, our derived inequalities are facet-defining for the polytope and efficient separation schemes were proposed for those exponential-sized inequalities. We also developed an extended formulation for every single generator that can provide integral solutions for single-generator problems and further tighten the lower bound. Finally, we conducted computational experiments by using two different data sets and the results demonstrate the significant efficiency of applying our derived inequalities to a branch-and-cut algorithm as cutting planes.

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References

Bixby, R. E. (2010). Mixed-integer programming: It works better than you may think. In *FERC Conference*.

Carlson, B., Chen, Y., Hong, M., Jones, R., Larson, K., Ma, X., Nieuwesteeg, P., Song, H., Sperry, K., Tackett, M., Taylor, D., Wan, J., and Zak, E. (2012). MISO unlocks billions in savings through the application of operations research for energy and ancillary services markets. *Interfaces*, 42(1):58–73.

Carrión, M. and Arroyo, J. (2006). A computationally efficient mixed-integer linear formulation for the thermal unit commitment problem. *IEEE Transactions on Power Systems*, 21(3):1371–1378.

Cheung, K. W. (2008). Ancillary service market design and implementation in North America: from theory to practice. In *Third International Conference on Electric Utility Deregulation and Restructuring and Power Technologies*, pages 66–73. IEEE.

China Energy Portal (2017). 2016 detailed electricity statistics. https://chinaenergyportal.org/en/2016-detailed-electricity-statistics/. Accessed: June 27, 2020.

Damcı-Kurt, P., Küçükyavuz, S., Rajan, D., and Atamtürk, A. (2016). A polyhedral study of production ramping. *Mathematical Programming*, 158(1-2):175–205.

Dang, C. and Li, M. (2007). A floating-point genetic algorithm for solving the unit commitment problem. *European Journal of Operational Research*, 181(3):1370–1395.

Dincer, I. (2000). Renewable energy and sustainable development: a crucial review. *Renewable and Sustainable Energy Reviews*, 4(2):157–175.

Dubost, L., Gonzalez, R., and Lemaréchal, C. (2005). A primal-proximal heuristic applied to the French unit-commitment problem. *Mathematical Programming*, 104(1):129–151.

Frangioni, A. and Gentile, C. (2006). Solving nonlinear single-unit commitment problems with ramping constraints. *Operations Research*, 54(4):767–775.

Frangioni, A. and Gentile, C. (2015). An extended MIP formulation for the single-unit commitment problem with ramping constraints. In *17th British-French-German Conf. on Optimization, London, June*, volume 1517.

Gentile, C., Morales-España, G., and Ramos, A. (2017). A tight MIP formulation of the unit commitment problem with start-up and shut-down constraints. *EURO Journal on Computational Optimization*, 5(1-2):177–201.

Gibbens, S. (2017). Renewable energy record set in U.S. https://news.nationalgeographic.com/2017/06/solar-wind-renewable-energy-record/. Accessed: June 27, 2020.

Guan, Y., Pan, K., and Zhou, K. (2018). Polynomial time algorithms and extended formulations for unit commitment problems. *IISE Transactions*, 50(8):735–751.

Hirst, E. and Kirby, B. (1997). *Creating Competitive Markets for Ancillary Services*. Oak Ridge National Laboratory.

Kirsch, L. D. and Singh, H. (1995). Pricing ancillary electric power services. *The Electricity Journal*, 8(8):28–36.

Knueven, B., Ostrowski, J., and Wang, J. (2018). The ramping polytope and cut generation for the unit commitment problem. *INFORMS Journal on Computing*, 30(4):739–749.

Lee, J., Leung, J., and Margot, F. (2004). Min-up/min-down polytopes. *Discrete Optimization*, 1(1):77–85.

Li, Z. and Shahidehpour, M. (2005). Security-constrained unit commitment for simultaneous clearing of energy and ancillary services markets. *IEEE Transactions on Power Systems*, 20(2):1079–1088.

Lowery, P. (1966). Generating unit commitment by dynamic programming. *IEEE Transactions on Power Apparatus and Systems*, PAS-85(5):422–426.

Malkin, P. (2003). Minimum runtime and stoptime polyhedra. *Report, CORE, Université Catholique de Louvain*.

Mantawy, A. H., Abdel-Magid, Y. L., and Selim, S. Z. (1998). A simulated annealing algorithm for unit commitment. *IEEE Transactions on Power Systems*, 13(1):197–204.

MISO (2019). MISO Energy and Ancillary Services Co-optimization. http://www.ercot.com/content/wcm/key_documents_lists/187700/4._MISO_Energy_and_Ancillary_Service_Co-optimization_091819.pdf.

Morales-España, G., Latorre, J. M., and Ramos, A. (2013). Tight and compact MILP formulation for the thermal unit commitment problem. *IEEE Transactions on Power Systems*, 28(4):4897–4908.

Muckstadt, J. A. and Koenig, S. A. (1977). An application of Lagrangian relaxation to scheduling in power-generation systems. *Operations Research*, 25(3):387–403.

Nemhauser, G. L. (2013). Integer Programming: Global Impact. https://smartech.gatech.edu/bitstream/handle/1853/49829/presentation.pdf?sequence=1.

Ostrowski, J., Anjos, M. F., and Vannelli, A. (2012). Tight mixed integer linear programming formulations for the unit commitment problem. *IEEE Transactions on Power Systems*, 1(27):39–46.

Pan, K. and Guan, Y. (2016). A polyhedral study of the integrated minimum-up/-down time and ramping polytope. Available online http://arxiv.org/pdf/1604.02184v1.pdf.

Pan, K., Guan, Y., Watson, J.-P., and Wang, J. (2016). Strengthened MILP formulation for certain gas turbine unit commitment problems. *IEEE Transactions on Power Systems*, 31(2):1440–1448.

Queyranne, M. and Wolsey, L. A. (2017). Tight MIP formulations for bounded up/down times and interval-dependent start-ups. *Mathematical Programming*, 164(1-2):129–155.

Rajan, D. and Takriti, S. (2005). Minimum up/down polytopes of the unit commitment problem with start-up costs. Research Report RC23628, IBM.

Sagastizábal, C. (2012). Divide to conquer: decomposition methods for energy optimization. *Mathematical Programming*, 134(1):187–222.

Singh, H. and Papalexopoulos, A. (1999). Competitive procurement of ancillary services by an independent system operator. *IEEE Transactions on Power Systems*, 14(2):498–504.

Street, A., Brigatto, A., and Valladao, D. M. (2017). Co-optimization of energy and ancillary services for hydrothermal operation planning under a general security criterion. *IEEE Transactions on Power Systems*, 32(6):4914–4923.

Streiffert, D., Philbrick, R., and Ott, A. (2005). A mixed integer programming solution for market clearing and reliability analysis. In *IEEE Power Engineering Society General Meeting 2005*, pages 2724–2731. IEEE.

U.S. EIA (2016). Electric Power Monthly. https://www.eia.gov/electricity/monthly/epm_table_grapher.cfm?t=epmt_1_01. Accessed: June 27, 2020.

Online Supplement for "Cutting Planes for Security-Constrained Unit Commitment with Regulation Reserve"

The detailed proofs for the theoretical results in Sections 3 - 5 are provided in this online supplement as follows.

EC.1. Proofs of Two-Period Convex Hull

EC.1.1. Proof of Proposition 1

Proof. We prove the validity of each inequality as follows.

For **inequality** (4), we discuss the following three cases: 1) if $y_1 = 0$, then $p_1 = 0$ due to (2e). Inequality (4) becomes $0 \le (\overline{C} - \overline{V})(y_2 - u_2)$, which is valid due to (2a); 2) if $y_1 = 1$ and $y_2 = 0$, then inequality (4) converts to $p_1 \le \overline{V} - \underline{C}$, which is valid due to constraints (2g); 3) if $y_1 = y_2 = 1$ and $u_2 = 0$, then (4) converts to $p_1 \le \overline{C} - \underline{C}$, which is valid due to constraints (2e).

For **inequality** (5), we discuss the following three cases: 1) if $y_2 = 0$, then $p_2 = r_2^+ = 0$ and thus inequality (5) is valid; 2) if $y_2 = 1$ and $u_2 = 0$, then inequality (5) converts to $p_2 + r_2^+ \le \overline{C} - \underline{C}$, which is valid due to constraints (2e); 3) if $y_2 = u_2 = 1$, then inequality (5) converts to $p_2 + r_2^+ \le \overline{V} - \underline{C}$, which is valid due to ramping-up constraints (2f).

For **inequality** (6), we discuss the following three cases: 1) if $y_2 = 0$, then $p_2 = r_2^+ = 0$ and thus inequality (6) converts to $p_1 \ge 0$, which is valid; 2) if $y_2 = 1$ and $u_2 = 0$, then inequality (6) converts to $p_2 + r_2^+ - p_1 \le V$, which is valid due to ramping-up constraints (2f); 3) if $y_2 = u_2 = 1$, then inequality (6) converts to $p_2 + r_2^+ - p_1 \le \overline{V} - \underline{C}$, which is valid due to ramping-up constraints (2f).

For **inequality** (7), we discuss the following three cases: 1) if $y_1 = 0$, then $p_1 = 0$ due to (2e). Inequality (7) converts to $-p_2 + r_2^- \le (\underline{C} + V - \overline{V})(y_2 - u_2)$, which is valid because $p_2 \ge r_2^-$ due to constraints (2d), and $y_2 - u_2 \ge 0$ due to constraints (2a); 2) if $y_1 = 1$ and $y_2 = 0$, then inequality (7) converts to $p_1 - p_2 + r_2^- \le \overline{V} - \underline{C}$, which is valid due to ramping-down constraints (2g); 3) if $y_1 = y_2 = 1$ and $u_2 = 0$, then inequality (7) converts to $p_1 - p_2 + r_2^- \le V$, which is valid due to ramping-down constraints (2g).

For **inequality** (8), we discuss the following three cases: 1) if $y_2 = 0$, then $u_2 = r_2^+ = r_2^- = 0$ and thus inequality (8) converts to $0 \le 0$, which is clearly valid; 2) if $y_2 = 1$ and $u_2 = 0$, then inequality (8) converts to $r_2^+ + r_2^- \le 2V$, which is valid because in this case (2f) becomes $p_2 + r_2^+ - p_1 \le V$, (2g) becomes $p_1 - p_2 + r_2^- \le V$, and their summation leads to $r_2^+ + r_2^- \le 2V$; 3) if $y_2 = u_2 = 1$, then inequality (8) converts to $r_2^+ + r_2^- \le \overline{V} - \underline{C}$, which is valid because $p_2 + r_2^+ \le \overline{V} - \underline{C}$ due to constraints (2f), and $r_2^- \le p_2$ due to constraints (2d). $\square$

EC.1.2. Proof of Proposition 2

Proof. For inequalities (4) - (8), we provide nine affinely independent points in $\text{conv}(P_2)$ that satisfy each inequality at equality. Since $\vec{0} \in \text{conv}(P_2)$, we only need to provide eight linearly independent points for each inequality, as shown in Table EC.1 - Table EC.5.

Table EC.1 Linearly Independent Points for Inequality (4)

p_1	p_2	r_1^+	r_1^-	r_2^+	r_2^-	y_1	y_2	u_2
$\overline{V}-\underline{C}$	0	0	0	0	0	1	0	0
$\overline{V}-\underline{C}$	0	0	$\overline{V}-\underline{C}$	0	0	1	0	0
$\overline{V}-\underline{C}$	0	$\overline{C}-\overline{V}$	0	0	0	1	0	0
$\overline{C}-\underline{C}$	$\overline{C}-\underline{C}-V$	0	0	0	0	1	1	0
0	0	0	0	0	0	0	1	1
0	0	0	0	$\overline{V}-\underline{C}$	0	0	1	1
0	$\overline{V}-\underline{C}$	0	0	0	0	0	1	1
0	$\overline{V}-\underline{C}$	0	0	0	$\overline{V}-\underline{C}$	0	1	1

Table EC.2 Linearly Independent Points for Inequality (5)

p_1	p_2	r_1^+	r_1^-	r_2^+	r_2^-	y_1	y_2	u_2
0	0	0	0	0	0	1	0	0
0	0	$\overline{C}-\underline{C}$	0	0	0	1	0	0
$\overline{V}-\underline{C}$	0	0	0	0	0	1	0	0
$\overline{V}-\underline{C}$	0	0	$\overline{V}-\underline{C}$	0	0	1	0	0
0	0	0	0	$\overline{V}-\underline{C}$	0	0	1	1
0	$\overline{V}-\underline{C}$	0	0	0	0	0	1	1
0	$\overline{V}-\underline{C}$	0	0	0	$\overline{V}-\underline{C}$	0	1	1
$\overline{C}-\underline{C}-V$	$\overline{C}-\underline{C}-2V$	0	0	$2V$	0	1	1	0

Table EC.3 Linearly Independent Points for Inequality (6)

p_1	p_2	r_1^+	r_1^-	r_2^+	r_2^-	y_1	y_2	u_2
0	0	0	0	0	0	1	0	0
0	0	$\overline{C}-\underline{C}$	0	0	0	1	0	0
0	0	0	0	$\overline{V}-\underline{C}$	0	0	1	1
0	$\overline{V}-\underline{C}$	0	0	0	0	0	1	1
0	$\overline{V}-\underline{C}$	0	0	0	$\overline{V}-\underline{C}$	0	1	1
0	0	0	0	V	0	1	1	0
V	0	0	0	$2V$	0	1	1	0
V	0	0	V	$2V$	0	1	1	0

By performing Gaussian elimination on each of Tables EC.1 - EC.5, we can easily obtain lower triangular matrices in each table. Thus, we conclude that the points in each of Table EC.1 - EC.5 are linearly independent. □

Table EC.4 Linearly Independent Points for Inequality (7)

p_1	p_2	r_1^+	r_1^-	r_2^+	r_2^-	y_1	y_2	u_2
$\overline{V}-\underline{C}$	0	0	0	0	0	1	0	0
$\overline{V}-\underline{C}$	0	0	$\overline{V}-\underline{C}$	0	0	1	0	0
$\overline{V}-\underline{C}$	0	$\overline{C}-\overline{V}$	0	0	0	1	0	0
0	0	0	0	0	0	0	1	1
0	0	0	0	$\overline{V}-\underline{C}$	0	0	1	1
0	$\overline{V}-\underline{C}$	0	0	0	$\overline{V}-\underline{C}$	0	1	1
V	0	0	0	0	0	1	1	0
$\overline{C}-\underline{C}-V$	$\overline{C}-\underline{C}-2V$	0	0	$2V$	0	1	1	0

Table EC.5 Linearly Independent Points for Inequality (8)

p_1	p_2	r_1^+	r_1^-	r_2^+	r_2^-	y_1	y_2	u_2
0	0	0	0	0	0	1	0	0
0	0	$\overline{C}-\underline{C}$	0	0	0	1	0	0
$\overline{V}-\underline{C}$	0	0	0	0	0	1	0	0
$\overline{V}-\underline{C}$	0	0	$\overline{V}-\underline{C}$	0	0	1	0	0
0	0	0	0	$\overline{V}-\underline{C}$	0	0	1	1
0	$\overline{V}-\underline{C}$	0	0	0	$\overline{V}-\underline{C}$	0	1	1
V	0	0	0	$2V$	0	1	1	0
$\overline{C}-\underline{C}-V$	$\overline{C}-\underline{C}-2V$	0	0	$2V$	0	1	1	0

EC.1.3. Proof of Proposition 3

Proof. We prove that $dim(Q_2)=9$ because there are nine variables in Q_2. We create ten affinely independent points in Q_2. Since $\vec{0}\in Q_2$, we only need to create nine more linearly independent points in Q_2 as shown in Table EC.6. □

Table EC.6 Linearly Independent Points in Q_2

p_1	p_2	r_1^+	r_1^-	r_2^+	r_2^-	y_1	y_2	u_2
$\overline{C}-\underline{C}$	0	0	0	0	0	1	0	0
$\overline{C}-\underline{C}$	0	0	$\overline{C}-\underline{C}$	0	0	1	0	0
0	0	$\overline{C}-\underline{C}$	0	0	0	1	0	0
0	0	0	0	0	V	1	1	0
0	V	0	0	0	0	1	1	0
0	$\overline{V}-\underline{C}$	0	0	0	0	0	1	1
0	0	0	0	$\overline{V}-\underline{C}$	0	0	1	1
0	$\overline{V}-\underline{C}$	0	0	0	$\overline{V}-\underline{C}$	0	1	1

EC.1.4. Proof of Proposition 4

Proof. Since (3a) - (3e) also belong to Q_2, we only need to consider inequalities (3f) - (3h).

For **inequality** (3f), it is clear that (3f) is dominated by (5) since the right-hand-side (RHS) of (5) is less than that of (3f).

For **inequality** (3g), we start with the RHS of inequality (6) and show that it is less than the RHS of inequality (3g) as follows: $Vy_2 - (\underline{C} + V - \overline{V})u_2 = \overline{V}y_2 + (\underline{C} + V - \overline{V})(y_2 - u_2) - \underline{C}y_2 \leq \overline{V}y_2 + (\underline{C} + V - \overline{V})y_1 - \underline{C}y_2 \leq \overline{V} + (\underline{C} + V - \overline{V})y_1 - \underline{C}y_2$, where the first inequality holds due to constraint (3c) and the last inequality holds because $y_2 \leq 1$. Thus, inequality (3g) is dominated by inequality (6).

For **inequality** (3h), we start with the RHS of inequality (7) and show that it is less than the RHS of inequality (3h) as follows: $(\overline{V} - \underline{C})y_1 + (\underline{C} + V - \overline{V})(y_2 - u_2) = \overline{V}y_1 + (\underline{C} + V - \overline{V})y_2 - (\underline{C} + V - \overline{V})u_2 - \underline{C}y_1 \leq \overline{V} + (\underline{C} + V - \overline{V})y_2 - \underline{C}y_1$, where the inequality holds because $y_1 \leq 1$, $\underline{C} + V - \overline{V} > 0$, and $u_2 \geq 0$. Thus, inequality (3h) is dominated by inequality (3h). □

EC.1.5. Proof of Proposition 5

Proof. It is sufficient to show that every point $z = (p_1, p_2, r_1^+, r_2^+, r_2^-, y_1, y_2, u_2) \in Q_2$ can be written as $z = \sum_{s \in S} \lambda_s z^s$ for some $\lambda_s \geq 0$ and $\sum_{s \in S} \lambda_s = 1$, where $z^s \in P_2, s \in S$ with $y(z^s)$ and $u(z^s)$ binary.

For a given $z = (\overline{p}_1, \overline{p}_2, \overline{r}_1^+, \overline{r}_2^+, \overline{r}_2^-, \overline{y}_1, \overline{y}_2, \overline{u}_2) \in Q_2$, we can pick $z^1, \ldots, z^4 \in P_2$ such that $z^1 = (\hat{p}_{11}, 0, \hat{r}_{11}^+, 0, 0, 1, 0, 0)$, $z^2 = (0, \hat{p}_{21}, 0, \hat{r}_{21}^+, \hat{r}_{21}^-, 0, 1, 1)$, $z^3 = (\hat{p}_{12}, \hat{p}_{22}, \hat{r}_{12}^+, \hat{r}_{22}^+, \hat{r}_{22}^-, 1, 1, 0)$, $z^4 = (0, 0, 0, 0, 0, 0, 0, 0)$. In addition, we have $\lambda_1 = \overline{y}_1 - \overline{y}_2 + \overline{u}_2$, $\lambda_2 = \overline{u}_2$, $\lambda_3 = \overline{y}_2 - \overline{u}_2$, and $\lambda_4 = 1 - \overline{y}_1 - \overline{u}_2$. Note that $\sum_{s=1}^4 \lambda_s = 1$. Due to (2a) - (2b), we have $\lambda_s \geq 0$ for each $s = 1, \ldots, 4$.

It is clear that $\overline{y}_i = y_i(z) = \sum_{s=1}^4 \lambda_s y_i(z^s)$ for $i = 1, 2$ and $\overline{u}_2 = u_2(z) = \sum_{s=1}^4 \lambda_s u_2(z^s)$. So we only need to determine the values of $\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-$ and show that $\overline{p}_1 = \lambda_1 \hat{p}_{11} + \lambda_3 \hat{p}_{12}$, $\overline{p}_2 = \lambda_2 \hat{p}_{21} + \lambda_3 \hat{p}_{22}$, $\overline{r}_1^+ = \lambda_1 \hat{r}_{11}^+ + \lambda_3 \hat{r}_{12}^+$, $\overline{r}_2^+ = \lambda_2 \hat{r}_{21}^+ + \lambda_3 \hat{r}_{22}^+$, and $\overline{r}_2^- = \lambda_2 \hat{r}_{21}^- + \lambda_3 \hat{r}_{22}^-$. The corresponding feasible region for $(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-)$ is described as follows:

$$A = \Big\{ (\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-) \in \mathbb{R}_+^{10} :$$

$$\hat{p}_{11} \leq \overline{V} - \underline{C}, \tag{EC.1}$$

$$\hat{p}_{11} + \hat{r}_{11}^+ \leq \overline{C} - \underline{C}, \tag{EC.2}$$

$$\hat{p}_{12}^+ \leq \overline{C} - \underline{C}, \tag{EC.3}$$

$$\hat{p}_{12} + \hat{r}_{12}^+ \leq \overline{C} - \underline{C}, \tag{EC.4}$$

$$\hat{p}_{21} + \hat{r}_{21}^+ \leq \overline{V} - \underline{C}, \tag{EC.5}$$

$$\hat{p}_{21} \geq \hat{r}_{21}^-, \tag{EC.6}$$

$$\hat{p}_{22} + \hat{r}_{22}^+ \leq \overline{C} - \underline{C}, \tag{EC.7}$$

$$\hat{r}_{21}^+ + \hat{r}_{21}^- \leq \overline{V} - \underline{C}, \tag{EC.8}$$

$$\hat{r}_{22}^+ + \hat{r}_{22}^- \leq 2V, \tag{EC.9}$$

$$\hat{p}_{22} + \hat{r}^+_{22} - \hat{p}_{12} \le V, \tag{EC.10}$$

$$\left.\hat{p}_{12} - \hat{p}_{22} + \hat{r}^-_{22} \le V\right\}. \tag{EC.11}$$

To show that $\overline{p}_i = \sum_{s=1}^4 \lambda_s p_i(z^s)$ $(\forall i = 1,2)$, $\overline{r}^+_i = \sum_{s=1}^4 \lambda_s r^+_i(z^s)$ $(\forall i = 1,2)$, and $\overline{r}^-_2 = \sum_{s=1}^4 \lambda_s r^-_2(z^s)$, it is equivalent to prove that fixing $(\overline{y}_1, \overline{y}_2, \overline{u}_2) \in B = \{(\overline{y}_1, \overline{y}_2, \overline{u}_2) \in [0,1]^3 : $ (2a) – (2b)$\}$, for each $(\overline{p}_1, \overline{p}_2, \overline{r}^+_1, \overline{r}^+_2, \overline{r}^-_2)$ in the set

$$C = \Big\{(\overline{p}_1, \overline{p}_2, \overline{r}^+_1, \overline{r}^+_2, \overline{r}^-_2) \in \mathbb{R}^5_+ : \overline{p}_1 \le (\overline{V} - \underline{C})\overline{y}_1 + (\overline{C} - \overline{V})(\overline{y}_2 - \overline{u}_2), \tag{EC.12a}$$

$$\overline{p}_1 + \overline{r}^+_1 \le (\overline{C} - \underline{C})\overline{y}_1, \tag{EC.12b}$$

$$\overline{p}_2 \ge \overline{r}^-_2, \tag{EC.12c}$$

$$\overline{p}_2 + \overline{r}^+_2 \le (\overline{C} - \underline{C})\overline{y}_2 - (\overline{C} - \overline{V})\overline{u}_2, \tag{EC.12d}$$

$$\overline{r}^+_2 + \overline{r}^-_2 \le 2V\overline{y}_2 - (\underline{C} + 2V - \overline{V})\overline{u}_2, \tag{EC.12e}$$

$$\overline{p}_1 - \overline{p}_2 + \overline{r}^-_2 \le (\overline{V} - \underline{C})\overline{y}_1 + (\underline{C} + V - \overline{V})(\overline{y}_2 - \overline{u}_2), \tag{EC.12f}$$

$$\left.\overline{p}_2 + \overline{r}^+_2 - \overline{p}_1 \le V\overline{y}_2 - (\underline{C} + V - \overline{V})\overline{u}_2\right\}, \tag{EC.12g}$$

there exists $(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}^+_{11}, \hat{r}^+_{12}, \hat{r}^+_{21}, \hat{r}^+_{22}, \hat{r}^-_{21}, \hat{r}^-_{22}) \in A$ such that $\overline{p}_1 = \lambda_1\hat{p}_{11} + \lambda_3\hat{p}_{12}$, $\overline{p}_2 = \lambda_2\hat{p}_{21} + \lambda_3\hat{p}_{22}$, $\overline{r}^+_1 = \lambda_1\hat{r}^+_{11} + \lambda_3\hat{r}^+_{12}$, $\overline{r}^+_2 = \lambda_2\hat{r}^+_{21} + \lambda_3\hat{r}^+_{22}$, and $\overline{r}^-_2 = \lambda_2\hat{r}^-_{21} + \lambda_3\hat{r}^-_{22}$. That is, the linear transformation $F : A \to C$ is surjective, where

$$F = \begin{pmatrix} \overline{y}_1 - \overline{y}_2 + \overline{u}_2 & \overline{y}_2 - \overline{u}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{u}_2 & \overline{y}_2 - \overline{u}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{y}_1 - \overline{y}_2 + \overline{u}_2 & \overline{y}_2 - \overline{u}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \overline{u}_2 & \overline{y}_2 - \overline{u}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{u}_2 & \overline{y}_2 - \overline{u}_2 \end{pmatrix}.$$

Note that any point in C can be represented as a convex combination of the extreme points in C since C is compact. Thus, it suffices to prove that for any extreme point $w \in C$, there exists a point $p \in A$ such that $Fp = w$. Nevertheless, it is not convenient to enumerate all the extreme points in C. Accordingly, instead of considering the extreme points in C, we equivalently show that this conclusion holds for any point on the faces of C instead, i.e., satisfying one of (EC.12a) - (EC.12g) at equality.

Satisfying (EC.12a) at equality. In this case, we have the corresponding set $C_{(\text{EC.12a})}$ as follows, where $C_{(\text{EC.12a})} \subseteq C$, by substituting (EC.12a) at equality into (EC.12b) - (EC.12g).

$$C_{(\text{EC.12a})} = \Big\{\ \overline{p}_1 = (\overline{V} - \underline{C})(\overline{y}_1 - \overline{y}_2 + \overline{u}_2) + (\overline{C} - \underline{C})(\overline{y}_2 - \overline{u}_2), \tag{EC.13a}$$

$$\overline{r}^+_1 \le (\overline{C} - \overline{V})(\overline{y}_1 - \overline{y}_2 + \overline{u}_2), \tag{EC.13b}$$

$$\overline{p}_2 \ge \overline{r}^-_2, \tag{EC.13c}$$

$$\overline{p}_2 + \overline{r}^+_2 \le (\overline{C} - \underline{C})\overline{y}_2 - (\overline{C} - \overline{V})\overline{u}_2, \tag{EC.13d}$$

$$\overline{p}_2 + \overline{r}_2^+ \leq (\overline{C} - \underline{C})\overline{y}_2 - (\overline{C} - \overline{V})\overline{u}_2 + (\overline{V} - \underline{C})(\overline{y}_1 - \overline{y}_2 + \overline{u}_2) + V(\overline{y}_2 - \overline{u}_2), \tag{EC.13e}$$

$$\overline{r}_2^+ + \overline{r}_2^- \leq 2V\overline{y}_2 - (\underline{C} + 2V - \overline{V})\overline{u}_2, \tag{EC.13f}$$

$$-\overline{p}_2 + \overline{r}_2^- \leq (\underline{C} + V - \overline{C})(\overline{y}_2 - \overline{u}_2).\Big\} \tag{EC.13g}$$

Note that since $(\overline{V} - \underline{C})(\overline{y}_1 - \overline{y}_2 + \overline{u}_2) \geq 0$ and $V(\overline{y}_2 - \overline{u}_2) \geq 0$, inequality (EC.13e) is dominated by (EC.13d). Since $\overline{p}_1 = (\overline{V} - \underline{C})(\overline{y}_1 - \overline{y}_2 + \overline{u}_2) + (\overline{C} - \underline{C})(\overline{y}_2 - \overline{u}_2) = (\overline{y}_1 - \overline{y}_2 + \overline{u}_2)\hat{p}_{11} + (\overline{y}_2 - \overline{u}_2)\hat{p}_{12}$, it is clear that we can pick $\hat{p}_{11} = \overline{V} - \underline{C}$ and $\hat{p}_{12} = \overline{C} - \underline{C}$, then the corresponding feasible region for $(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-)$ is

$$\hat{A} = \Big\{(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-) \in \mathbb{R}_+^{10} : \hat{p}_{11} = \overline{V} - \underline{C}, \tag{EC.14a}$$

$$\hat{r}_{11}^+ \leq \overline{C} - \overline{V}, \tag{EC.14b}$$

$$\hat{p}_{12} = \overline{C} - \underline{C}, \tag{EC.14c}$$

$$\hat{r}_{12}^+ = 0, \tag{EC.14d}$$

$$\hat{p}_{21} + \hat{r}_{21}^+ \leq \overline{V} - \underline{C}, \tag{EC.14e}$$

$$\hat{p}_{21} \geq \hat{r}_{21}^-, \tag{EC.14f}$$

$$\hat{p}_{22} + \hat{r}_{22}^+ \leq \overline{C} - \underline{C}, \tag{EC.14g}$$

$$\hat{r}_{21}^+ + \hat{r}_{21}^- \leq \overline{V} - \underline{C}, \tag{EC.14h}$$

$$\hat{r}_{22}^+ + \hat{r}_{22}^- \leq 2V, \tag{EC.14i}$$

$$\hat{p}_{22} + \hat{r}_{22}^+ \leq \overline{C} - \underline{C} + V, \tag{EC.14j}$$

$$\hat{r}_{22}^- - \hat{p}_{22} \leq \underline{C} + V - \overline{C}\Big\}. \tag{EC.14k}$$

Therefore, we can pick $\hat{r}_{12}^+ = 0$, $\hat{r}_{11}^+ = \frac{\overline{r}_1^+}{\overline{y}_1 - \overline{y}_2 + \overline{u}_2}$ if $\overline{y}_1 - \overline{y}_2 + \overline{u}_2 > 0$ and $\hat{r}_{11}^+$ free otherwise. We let $\hat{p}_{21} = \overline{V} - \underline{C}$, $\hat{p}_{22} = \frac{\overline{p}_2 - (\overline{V} - \underline{C})\overline{u}_2}{\overline{y}_2 - \overline{u}_2}$ if $\overline{y}_2 - \overline{u}_2 > 0$ and $\hat{p}_{22}$ free otherwise; $\hat{r}_{21}^+ = 0$, $\hat{r}_{22}^+ = \frac{\overline{r}_2^+}{\overline{y}_2 - \overline{u}_2}$ if $\overline{y}_2 - \overline{u}_2 > 0$ and $\hat{r}_{22}^+$ free otherwise; $\hat{r}_{21}^- = \overline{V} - \underline{C}$, $\hat{r}_{22}^- = \frac{\overline{r}_2^- - (\overline{V} - \underline{C})\overline{u}_2}{\overline{y}_2 - \overline{u}_2}$ if $\overline{y}_2 - \overline{u}_2 > 0$ and $\hat{r}_{22}^-$ free otherwise. It is easy to check that $(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-) \in A$ and $F(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-) = (\overline{p}_1, \overline{p}_2, \overline{r}_1^+, \overline{r}_2^+, \overline{r}_2^-)$.

Satisfy (EC.12b) **at equality.** In this case, we have the corresponding set $C_{(\text{EC.12b})}$ as follows, where $C_{(\text{EC.12b})} \subseteq C$, by substituting (EC.12b) at equality into other inequalities in C.

$$C_{(\text{EC.12b})} = \Big\{ \ \overline{p}_1 + \overline{r}_1^+ = (\overline{C} - \underline{C})(\overline{y}_1 - \overline{y}_2 + \overline{u}_2) + (\overline{C} - \underline{C})(\overline{y}_2 - \overline{u}_2), \tag{EC.15a}$$

$$\overline{p}_1 \leq (\overline{V} - \underline{C})(\overline{y}_1 - \overline{y}_2 + \overline{u}_2) + (\overline{C} - \underline{C})(\overline{y}_2 - \overline{u}_2), \tag{EC.15b}$$

$$\overline{p}_2 \geq \overline{r}_2^-, \tag{EC.15c}$$

$$\overline{p}_2 + \overline{r}_2^+ \leq (\overline{C} - \underline{C})\overline{y}_2 - (\overline{C} - \overline{V})\overline{u}_2, \tag{EC.15d}$$

$$\overline{p}_2+\overline{r}_2^+-\overline{p}_1\leq V\overline{y}_2-(\overline{C}+V-\overline{V})\overline{u}_2, \tag{EC.15e}$$

$$\overline{p}_1-\overline{p}_2+\overline{r}_2^-\leq(\overline{V}-\underline{C})\overline{y}_1+(\underline{C}+V-\overline{V})(\overline{y}_2-\overline{u}_2), \tag{EC.15f}$$

$$\left.\overline{r}_2^++\overline{r}_2^-\leq 2V\overline{y}_2-(\underline{C}+2V-\overline{V})\overline{u}_2\right\}. \tag{EC.15g}$$

We pick $\hat{p}_{12}=\overline{C}-\underline{C}$, $\hat{p}_{11}=\frac{\overline{p}_1-(\overline{C}-\underline{C})(\overline{y}_2-\overline{u}_2)}{\overline{y}_1-\overline{y}_2+\overline{u}_2}$ if $\overline{y}_1-\overline{y}_2+\overline{u}_2>0$ and $\hat{p}_{11}$ free otherwise; $\hat{r}_{12}^+=0$, $\hat{r}_{11}^+=\frac{\overline{r}_1^+}{\overline{y}_1-\overline{y}_2+\overline{u}_2}$ if $\overline{y}_1-\overline{y}_2+\overline{u}_2>0$ and $\hat{r}_{11}^+$ free otherwise. We let $\hat{p}_{21}=\overline{V}-\underline{C}$, $\hat{p}_{22}=\frac{\overline{p}_2-(\overline{V}-\underline{C})\overline{u}_2}{\overline{y}_2-\overline{u}_2}$ if $\overline{y}_2-\overline{u}_2>0$ and $\hat{p}_{22}$ free otherwise; $\hat{r}_{21}^+=0$, $\hat{r}_{22}^+=\frac{\overline{r}_2^+}{\overline{y}_2-\overline{u}_2}$ if $\overline{y}_2-\overline{u}_2>0$ and $\hat{r}_{22}^+$ free otherwise; $\hat{r}_{21}^-=\overline{V}-\underline{C}$, $\hat{r}_{22}^-=\frac{\overline{r}_2^--(\overline{V}-\underline{C})\overline{u}_2}{\overline{y}_2-\overline{u}_2}$ if $\overline{y}_2-\overline{u}_2>0$ and $\hat{r}_{22}^-$ free otherwise. By substituting the above equalities in A and utilizing inequalities in $C_{(\text{EC.12b})}$, we can check that $(\hat{p}_{11},\hat{p}_{12},\hat{p}_{21},\hat{p}_{22},\hat{r}_{11}^+,\hat{r}_{12}^+,\hat{r}_{21}^+,\hat{r}_{22}^+,\hat{r}_{21}^-,\hat{r}_{22}^-)\in A$ and $F(\hat{p}_{11},\hat{p}_{12},\hat{p}_{21},\hat{p}_{22},\hat{r}_{11}^+,\hat{r}_{12}^+,\hat{r}_{21}^+,\hat{r}_{22}^+,\hat{r}_{21}^-,\hat{r}_{22}^-)=(\overline{p}_1,\overline{p}_2,\overline{r}_1^+,\overline{r}_2^+,\overline{r}_2^-)$.

Satisfy (EC.12c) **at equality.** In this case, we have the corresponding set $C_{(\text{EC.12c})}$ as follows, where $C_{(\text{EC.12c})}\subseteq C$, by substituting (EC.12c) at equality into other inequalities in C.

$$C_{(\text{EC.12c})}=\Big\{\ \overline{p}_1\leq(\overline{V}-\underline{C})\overline{y}_1+(\overline{C}-\overline{V})(\overline{y}_2-\overline{u}_2), \tag{EC.16a}$$

$$\overline{p}_1\leq(\overline{V}-\underline{C})\overline{y}_1+(\underline{C}+V-\overline{V})(\overline{y}_2-\overline{u}_2), \tag{EC.16b}$$

$$\overline{p}_1+\overline{r}_1^+\leq(\overline{C}-\underline{C})\overline{y}_1, \tag{EC.16c}$$

$$\overline{p}_2=\overline{r}_2^-, \tag{EC.16d}$$

$$\overline{p}_2+\overline{r}_2^+\leq(\overline{C}-\underline{C})(\overline{y}_2-\overline{u}_2)+(\overline{V}-\underline{C})\overline{u}_2, \tag{EC.16e}$$

$$\overline{r}_2^++\overline{p}_2\leq 2V(\overline{y}_2-\overline{u}_2)+(\overline{V}-\underline{C})\overline{u}_2, \tag{EC.16f}$$

$$\left.\overline{p}_2+\overline{r}_2^+-\overline{p}_1\leq V\overline{y}_2-(\underline{C}+V-\overline{V})\overline{u}_2\right\}. \tag{EC.16g}$$

According to our assumption $\overline{C}>\underline{C}+V$, inequality (EC.16a) is dominated by (EC.16b). We let $\hat{p}_{11}=\overline{V}-\underline{C}$, $\hat{p}_{12}=\frac{\overline{p}_1-(\overline{V}-\underline{C})(\overline{y}_1-\overline{y}_2+\overline{u}_2)}{\overline{y}_2-\overline{u}_2}$ if $\overline{y}_2-\overline{u}_2>0$ and $\hat{p}_{12}$ free otherwise; $\hat{r}_{11}^+=\overline{C}-\overline{V}$, $\hat{r}_{12}^+=\frac{\overline{r}_1^+-(\overline{C}-\overline{V})(\overline{y}_1-\overline{y}_2+\overline{u}_2)}{\overline{y}_2-\overline{u}_2}$ if $\overline{y}_2-\overline{u}_2>0$ and $\hat{r}_{12}^+$ free otherwise. We pick $\hat{p}_{21}=\hat{r}_{21}^-=\overline{V}-\underline{C}$, $\hat{p}_{22}=\hat{r}_{22}^-=\frac{\overline{p}_2-(\overline{V}-\underline{C})\overline{u}_2}{\overline{y}_2-\overline{u}_2}$ if $\overline{y}_2-\overline{u}_2>0$, and $\hat{p}_{22}=\hat{r}_{22}^-$ but free otherwise; $\hat{r}_{21}^+=0$, $\hat{r}_{22}^+=\frac{\overline{r}_2^+}{\overline{y}_2-\overline{u}_2}$ if $\overline{y}_2-\overline{u}_2>0$, and $\hat{r}_{22}^+$ free otherwise. By substituting the above equalities in A and utilizing inequalities in $C_{(\text{EC.12c})}$, we can easily verify that $(\hat{p}_{11},\hat{p}_{12},\hat{p}_{21},\hat{p}_{22},\hat{r}_{11}^+,\hat{r}_{12}^+,\hat{r}_{21}^+,\hat{r}_{22}^+,\hat{r}_{21}^-,\hat{r}_{22}^-)\in A$ and $F(\hat{p}_{11},\hat{p}_{12},\hat{p}_{21},\hat{p}_{22},\hat{r}_{11}^+,\hat{r}_{12}^+,\hat{r}_{21}^+,\hat{r}_{22}^+,\hat{r}_{21}^-,\hat{r}_{22}^-)=(\overline{p}_1,\overline{p}_2,\overline{r}_1^+,\overline{r}_2^+,\overline{r}_2^-)$.

Satisfy (EC.12d) **at equality.** In this case, we have the corresponding set $C_{(\text{EC.12d})}$ as follows, where $C_{(\text{EC.12d})}\subseteq C$, by substituting (EC.12d) at equality into other inequalities in C.

$$C_{(\text{EC.12d})}=\Big\{\ \overline{p}_1\leq(\overline{V}-\underline{C})\overline{y}_1+(\overline{C}-\overline{V})(\overline{y}_2-\overline{u}_2), \tag{EC.17a}$$

$$\overline{p}_1+\overline{r}_1^+\leq(\overline{C}-\underline{C})\overline{y}_1, \tag{EC.17b}$$

$$\overline{p}_2\geq\overline{r}_2^-, \tag{EC.17c}$$

$$\overline{p}_2+\overline{r}_2^+=(\overline{C}-\underline{C})(\overline{y}_2-\overline{u}_2)+(\overline{V}-\underline{C})\overline{u}_2, \tag{EC.17d}$$

$$\overline{r}_2^- - \overline{p}_2 \le (2V + \underline{C} - \overline{C})(\overline{y}_2 - \overline{u}_2), \tag{EC.17e}$$

$$\overline{p}_1 - \overline{p}_2 + \overline{r}_2^- \le (\overline{V} - \underline{C})\overline{y}_1 + (\underline{C} + V - \overline{V})(\overline{y}_2 - \overline{u}_2), \tag{EC.17f}$$

$$\overline{p}_1 \ge (\overline{C} - \underline{C} - V)(\overline{y}_2 - \overline{u}_2)\Big\}. \tag{EC.17g}$$

We let $\hat{p}_{12} = \overline{C} - \underline{C} - V$, $\hat{p}_{11} = \frac{\overline{p}_1 - (\overline{C} - \underline{C} - V)(\overline{y}_2 - \overline{u}_2)}{\overline{y}_1 - \overline{y}_2 + \overline{u}_2}$ if $\overline{y}_1 - \overline{y}_2 + \overline{u}_2 > 0$ and $\hat{p}_{11}$ free otherwise; $\hat{r}_{12}^+ = V$, $\hat{r}_{11}^+ = \frac{\overline{r}_1^+ - V(\overline{y}_2 - \overline{u}_2)}{\overline{y}_1 - \overline{y}_2 + \overline{u}_2}$ if $\overline{y}_1 - \overline{y}_2 + \overline{u}_2 > 0$ and $\hat{r}_{11}^+$ free otherwise. We pick $\hat{p}_{22} = \overline{C} - \underline{C}$, $\hat{p}_{21} = \frac{\overline{p}_2 - (\overline{C} - \underline{C})(\overline{y}_2 - \overline{u}_2)}{\overline{u}_2}$ if $\overline{y}_2 - \overline{u}_2 > 0$, and $\hat{p}_{21}$ free otherwise; $\hat{r}_{22}^+ = 0$, $\hat{r}_{21}^+ = \frac{\overline{r}_2^+}{\overline{u}_2}$ if $\overline{u}_2 > 0$, and $\hat{r}_{21}^+$ free otherwise; $\hat{r}_{22}^- = 2V$, $\hat{r}_{21}^- = \frac{\overline{r}_2^- - 2V(\overline{y}_2 - \overline{u}_2)}{\overline{u}_2}$ if $\overline{u}_2 > 0$, and $\hat{r}_{21}^-$ free otherwise. By substituting the above equalities in A and utilizing inequalities in $C_{(\text{EC.12d})}$, we can easily verify that $(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-) \in A$ and $F(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-) = (\overline{p}_1, \overline{p}_2, \overline{r}_1^+, \overline{r}_2^+, \overline{r}_2^-)$.

Satisfy (EC.12e) **at equality.** In this case, we have the corresponding set $C_{(\text{EC.12e})}$ as follows, where $C_{(\text{EC.12e})} \subseteq C$, by substituting (EC.12e) at equality into other inequalities in C.

$$C_{(\text{EC.12e})} = \Big\{ \; \overline{p}_1 \le (\overline{V} - \underline{C})\overline{y}_1 + (\overline{C} - \overline{V})(\overline{y}_2 - \overline{u}_2), \tag{EC.18a}$$

$$\overline{p}_1 + \overline{r}_1^+ \le (\overline{C} - \underline{C})\overline{y}_1, \tag{EC.18b}$$

$$\overline{p}_2 \ge \overline{r}_2^-, \tag{EC.18c}$$

$$\overline{p}_2 - \overline{r}_2^- \le (\overline{C} - \underline{C} - 2V)(\overline{y}_2 - \overline{u}_2), \tag{EC.18d}$$

$$\overline{r}_2^+ + \overline{r}_2^- = 2V\overline{y}_2 - (\underline{C} + 2V - \overline{V})\overline{u}_2, \tag{EC.18e}$$

$$\overline{p}_1 - \overline{p}_2 + \overline{r}_2^- \le (\overline{V} - \underline{C})\overline{y}_1 + (\underline{C} + V - \overline{V})(\overline{y}_2 - \overline{u}_2), \tag{EC.18f}$$

$$\overline{p}_1 - \overline{p}_2 + \overline{r}_2^- \ge V(\overline{y}_2 - \overline{u}_2)\Big\}. \tag{EC.18g}$$

Note that inequality (EC.18a) is dominated by (EC.18d) and (EC.18f) since they imply that $\overline{p}_1 \le (\overline{V} - \underline{C})\overline{y}_1 + (\overline{C} - \underline{C} - V)(\overline{y}_2 - \overline{u}_2) \le (\overline{V} - \underline{C})\overline{y}_1 + (\overline{C} - \overline{V})(\overline{y}_2 - \overline{u}_2)$.

We let $\hat{p}_{12} = \overline{C} - \underline{C} - V$, $\hat{p}_{11} = \frac{\overline{p}_1 - (\overline{C} - \underline{C} - V)(\overline{y}_2 - \overline{u}_2)}{\overline{y}_1 - \overline{y}_2 + \overline{u}_2}$ if $\overline{y}_1 - \overline{y}_2 + \overline{u}_2 > 0$ and $\hat{p}_{11}$ free otherwise; $\hat{r}_{12}^+ = V$, $\hat{r}_{11}^+ = \frac{\overline{r}_1^+ - V(\overline{y}_2 - \overline{u}_2)}{\overline{y}_1 - \overline{y}_2 + \overline{u}_2}$ if $\overline{y}_1 - \overline{y}_2 + \overline{u}_2 > 0$ and $\hat{r}_{11}^+$ free otherwise. We pick $\hat{p}_{22} = \overline{C} - \underline{C} - V$, $\hat{p}_{21} = \frac{\overline{p}_2 - (\overline{C} - \underline{C} - V)(\overline{y}_2 - \overline{u}_2)}{\overline{u}_2}$ if $\overline{y}_2 - \overline{u}_2 > 0$, and $\hat{p}_{21}$ free otherwise; $\hat{r}_{22}^+ = V$, $\hat{r}_{21}^+ = \frac{\overline{r}_2^+ - V(\overline{y}_2 - \overline{u}_2)}{\overline{u}_2}$ if $\overline{u}_2 > 0$, and $\hat{r}_{21}^+$ free otherwise; $\hat{r}_{22}^- = V$, $\hat{r}_{21}^- = \frac{\overline{r}_2^- - V(\overline{y}_2 - \overline{u}_2)}{\overline{u}_2}$ if $\overline{u}_2 > 0$, and $\hat{r}_{21}^-$ free otherwise. By substituting the above equalities in A and utilizing inequalities in $C_{(\text{EC.12e})}$, we can easily verify that $(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-) \in A$ and $F(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-) = (\overline{p}_1, \overline{p}_2, \overline{r}_1^+, \overline{r}_2^+, \overline{r}_2^-)$.

Satisfy (EC.12f) **at equality.** In this case, we have the corresponding set $C_{(\text{EC.12f})}$ as follows, where $C_{(\text{EC.12f})} \subseteq C$, by substituting (EC.12f) at equality into other inequalities in C.

$$C_{(\text{EC.12f})} = \Big\{ \; \overline{p}_1 \le (\overline{V} - \underline{C})\overline{y}_1 + (\overline{C} - \overline{V})(\overline{y}_2 - \overline{u}_2), \tag{EC.19a}$$

$$\overline{p}_1 + \overline{r}_1^+ \le (\overline{C} - \underline{C})\overline{y}_1, \tag{EC.19b}$$

$$\overline{p}_1 \ge (\overline{V} - \underline{C})(\overline{y}_1 - \overline{y}_2 + \overline{u}_2) + V(\overline{y}_2 - \overline{u}_2), \tag{EC.19c}$$

$$\overline{p}_2 + \overline{r}_2^+ \le (\overline{C} - \underline{C})\overline{y}_2 - (\overline{C} - \overline{V})\overline{u}_2, \tag{EC.19d}$$

$$\overline{r}_2^+ + \overline{p}_2 - \overline{p}_1 \le V(\overline{y}_2 - \overline{u}_2) + (\overline{V} - \underline{C})\overline{u}_2 - (\overline{V} - \underline{C})(\overline{y}_1 - \overline{y}_2 + \overline{u}_2), \tag{EC.19e}$$

$$\overline{p}_1 - \overline{p}_2 + \overline{r}_2^- = (\overline{V} - \underline{C})\overline{y}_1 + (\underline{C} + V - \overline{V})(\overline{y}_2 - \overline{u}_2)\Big\}. \tag{EC.19f}$$

We let $\hat{p}_{11} = \overline{V} - \underline{C}$, $\hat{p}_{12} = \frac{\overline{p}_1 - (\overline{V} - \underline{C})(\overline{y}_1 - \overline{y}_2 + \overline{u}_2)}{\overline{y}_2 - \overline{u}_2}$ if $\overline{y}_2 - \overline{u}_2 > 0$ and $\hat{p}_{12}$ free otherwise; $\hat{r}_{11}^+ = \overline{C} - \overline{V}$, $\hat{r}_{12}^+ = \frac{\overline{r}_1^+ - (\overline{C} - \overline{V})(\overline{y}_1 - \overline{y}_2 + \overline{u}_2)}{\overline{y}_2 - \overline{u}_2}$ if $\overline{y}_2 - \overline{u}_2 > 0$ and $\hat{r}_{12}^+$ free otherwise. We pick $\hat{p}_{21} = \hat{r}_{21}^- = \overline{V} - \underline{C}$, $\hat{p}_{22} = \hat{r}_{22}^- = \frac{\overline{p}_2 - (\overline{V} - \underline{C})\overline{u}_2}{\overline{y}_2 - \overline{u}_2}$ if $\overline{y}_2 - \overline{u}_2 > 0$, and $\hat{p}_{22}, \hat{r}_{22}^-$ free otherwise; $\hat{r}_{21}^+ = 0$, $\hat{r}_{22}^+ = \frac{\overline{r}_2^+}{\overline{y}_2 - \overline{u}_2}$ if $\overline{y}_2 - \overline{u}_2 > 0$, and $\hat{r}_{22}^+$ free otherwise. By substituting the above equalities in A and utilizing inequalities in $C_{(\text{EC.12f})}$, we can easily verify that $(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-) \in A$ and $F(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-) = (\overline{p}_1, \overline{p}_2, \overline{r}_1^+, \overline{r}_2^+, \overline{r}_2^-)$.

Satisfy (EC.12g) **at equality.** In this case, we have the corresponding set $C_{(\text{EC.12g})}$ as follows, where $C_{(\text{EC.12g})} \subseteq C$, by substituting (EC.12g) at equality into other inequalities in C.

$$C_{(\text{EC.12g})} = \Big\{ \ \overline{p}_1 \le (\overline{V} - \underline{C})\overline{y}_1 + (\overline{C} - \overline{V})(\overline{y}_2 - \overline{u}_2), \tag{EC.20a}$$

$$\overline{p}_1 + \overline{r}_1^+ \le (\overline{C} - \underline{C})\overline{y}_1, \tag{EC.20b}$$

$$\overline{p}_2 \ge \overline{r}_2^-, \tag{EC.20c}$$

$$\overline{p}_1 \le (\overline{C} - \underline{C} - V)(\overline{y}_2 - \overline{u}_2), \tag{EC.20d}$$

$$\overline{r}_2^+ + \overline{r}_2^- \le 2V\overline{y}_2 - (\underline{C} + 2V - \overline{V})\overline{u}_2, \tag{EC.20e}$$

$$\overline{p}_2 + \overline{r}_2^+ - \overline{p}_1 = V\overline{y}_2 - (\underline{C} + V - \overline{V})\overline{u}_2\Big\}. \tag{EC.20f}$$

We let $\hat{p}_{11} = 0$, $\hat{p}_{12} = \frac{\overline{p}_1}{\overline{y}_2 - \overline{u}_2}$ if $\overline{y}_2 - \overline{u}_2 > 0$ and $\hat{p}_{12}$ free otherwise; $\hat{r}_{11}^+ = \overline{C} - \underline{C}$, $\hat{r}_{12}^+ = \frac{\overline{r}_1^+ - (\overline{C} - \underline{C})(\overline{y}_1 - \overline{y}_2 + \overline{u}_2)}{\overline{y}_2 - \overline{u}_2}$ if $\overline{y}_2 - \overline{u}_2 > 0$ and $\hat{r}_{12}^+$ free otherwise. We pick $\hat{p}_{21} = \hat{r}_{21}^- = \overline{V} - \underline{C}$, $\hat{p}_{22} = \hat{r}_{22}^- = \frac{\overline{p}_2 - (\overline{V} - \underline{C})\overline{u}_2}{\overline{y}_2 - \overline{u}_2}$ if $\overline{y}_2 - \overline{u}_2 > 0$, and $\hat{p}_{22}, \hat{r}_{22}^-$ free otherwise; $\hat{r}_{21}^+ = 0$, $\hat{r}_{22}^+ = \frac{\overline{r}_2^+}{\overline{y}_2 - \overline{u}_2}$ if $\overline{y}_2 - \overline{u}_2 > 0$, and $\hat{r}_{22}^+$ free otherwise. By substituting the above equalities in A and utilizing inequalities in $C_{(\text{EC.12g})}$, we can easily verify that $(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-) \in A$ and $F(\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}, \hat{r}_{11}^+, \hat{r}_{12}^+, \hat{r}_{21}^+, \hat{r}_{22}^+, \hat{r}_{21}^-, \hat{r}_{22}^-) = (\overline{p}_1, \overline{p}_2, \overline{r}_1^+, \overline{r}_2^+, \overline{r}_2^-)$.

In summary, we have proved that for any point w in the faces of C, there exists a point $p \in A$ such that $Fp = w$. Thus, Proposition 5 holds. □

EC.2. Three-Period Convex Hulls

In this section, we present the complete three-period convex hull description under condition $\overline{C} - \overline{V} - 2V \ge 0$. Three-period convex hulls under other conditions are presented in Online Supplement EC.3.

Proposition EC.1. *If $\overline{C}-\overline{V}-2V\geq 0$, then the inequalities*

$$
\begin{aligned}
&(15)-(16),\\
&p_1\leq(\overline{V}-\underline{C})y_1+V(y_2-u_2)+(\overline{C}-\overline{V}-V)(y_3-u_3-u_2), && \text{(EC.21)}\\
&p_2\leq(\overline{V}-\underline{C})y_2+(\overline{C}-\overline{V})(y_3-u_3-u_2), && \text{(EC.22)}\\
&r_2^-\leq(\overline{V}-\underline{C})y_2+(\underline{C}+2V-\overline{V})(y_3-u_3-u_2), && \text{(EC.23)}\\
&r_2^++r_2^-\leq 2Vy_2-(\underline{C}+2V-\overline{V})u_2, && \text{(EC.24)}\\
&r_3^++r_3^-\leq 2Vy_3-(\underline{C}+V-\overline{V})u_2-(\underline{C}+2V-\overline{V})u_3, && \text{(EC.25)}\\
&p_1+r_1^+\leq(\overline{C}-\underline{C})y_1, && \text{(EC.26)}\\
&p_2+r_2^+\leq(\overline{V}+2V-\underline{C})y_2-2Vu_2+(\overline{C}-\overline{V}-2V)(y_3-u_3-u_2), && \text{(EC.27)}\\
&p_3+r_3^+\leq(\overline{C}-\underline{C})y_3-(\overline{C}-\overline{V})u_3-(\overline{C}-\overline{V}-V)u_2, && \text{(EC.28)}\\
&p_2-p_1\leq(\overline{V}-\underline{C})y_2+(\underline{C}+V-\overline{V})(y_3-u_3-u_2), && \text{(EC.29)}\\
&p_1-p_2+r_2^-\leq(\overline{V}-\underline{C})y_1+(\underline{C}+V-\overline{V})(y_2-u_2), && \text{(EC.30)}\\
&p_2-p_3+r_3^-\leq(\overline{V}-\underline{C})y_2+(\underline{C}+V-\overline{V})(y_3-u_3-u_2), && \text{(EC.31)}\\
&p_1-p_3+r_3^-\leq(\overline{V}-\underline{C})y_1+V(y_2-u_2)+(\underline{C}+V-\overline{V})(y_3-u_3-u_2), && \text{(EC.32)}\\
&p_3+r_3^+-p_1\leq 2Vy_3-(\underline{C}+V-\overline{V})u_2-(\underline{C}+2V-\overline{V})u_3, && \text{(EC.33)}\\
&p_2+r_2^+-p_1\leq Vy_2-(\underline{C}+V-\overline{V})u_2, && \text{(EC.34)}\\
&p_3+r_3^+-p_2\leq Vy_3-(\underline{C}+V-\overline{V})u_3, && \text{(EC.35)}\\
&p_3+r_3^++r_2^+\leq 2Vy_2+(\overline{C}-\underline{C}-V)y_3-(\overline{C}-\overline{V})u_2-(\overline{C}-\overline{V}-V)u_3, && \text{(EC.36)}\\
&p_3+r_3^+-p_2+r_2^-\leq 3Vy_3-(\underline{C}+2V-\overline{V})u_2-(\underline{C}+3V-\overline{V})u_3, && \text{(EC.37)}\\
&p_1-p_2+p_3+r_3^+\leq(\overline{V}-\underline{C})y_1+(\underline{C}+V-\overline{V})y_2+(\overline{C}-\underline{C})y_3-(\overline{C}-\overline{V})(u_2+u_3), && \text{(EC.38)}\\
&p_1-p_2+r_3^++r_3^-\leq(\overline{V}-\underline{C})y_1+(\underline{C}+V-\overline{V})y_2+2Vy_3-(\underline{C}+2V-\overline{V})(u_2+u_3), && \text{(EC.39)}
\end{aligned}
$$

are valid for conv(P_3^2).

Proof. See Online Supplement EC.2.1 for the detailed proof. □

Based on inequalities (15), (16), and (EC.21) - (EC.39), we introduce the linear programming description of conv(P_3^2) by adding minimum-up/-down time and nonnegative restrictions as follows:

$$
\begin{aligned}
Q_3^2:=\Big\{\ &(p,r^+,r^-,y,u)\in\mathbb{R}^{14}:(14a)-(14e),(15),(16),(\text{EC.21})-(\text{EC.39}),\\
&u_2\geq 0,\ u_3\geq 0\Big\}. && \text{(EC.40)}
\end{aligned}
$$

In the following, we show $Q_3^2=\text{conv}(P_3^2)$. Note that $\text{conv}(P_3^2)\subseteq Q_3^2$ because inequalities (15), (16), and (EC.21) - (EC.39) are valid for conv(P_3^2) by Proposition EC.1. So we only need to prove

$Q_3^2 \subseteq \text{conv}(P_3^2)$ by showing that all inequalities of Q_3^2 are facet-defining for $\text{conv}(P_3^2)$, all inequalities of P_3^2 are dominated by Q_3^2, and every extreme point of Q_3^2 is integral in variables y and u.

PROPOSITION EC.2. *The polytope Q_3^2 is full-dimensional.*

Proof. See Online Supplement EC.2.2 for the detailed proof. □

PROPOSITION EC.3. *Inequalities* (15), (16), *and* (EC.21) - (EC.39) *are facet-defining for conv*(P_3^2) *when* $\overline{C} - \overline{V} - 2V \geq 0$.

Proof. The proof is similar to that of Proposition 2 and thus is omitted here. □

PROPOSITION EC.4. *All inequalities of P_3^2 are dominated by inequalities of Q_3^2.*

Proof. The proof is similar to that of Proposition 4 and thus is omitted here. □

PROPOSITION EC.5. *Every extreme point in Q_3^2 is integral in y and u.*

Proof. See Online Supplement EC.2.3 for the detailed proof. □

According to Propositions EC.1 - EC.5, we have the following theorem.

THEOREM EC.1. *When* $\overline{C} - \overline{V} - 2V \geq 0$, *we have* $Q_3^2 = conv(P_3^2)$.

Proof. The proof is similar to that of Theorem 1 and thus is omitted here. □

EC.2.1. Proof of Proposition EC.1

Proof. To prove the validity of inequality (EC.21), we consider the shut-down time of the generator. It is clear that if the generator is offline in period $t = 1$, i.e., $y_1 = 0$, then $p_1 = 0$ due to (2e). Inequality (EC.21) becomes $0 \leq V(y_2 - u_2) + (\overline{C} - \overline{V} - V)(y_3 - u_3 - u_2)$, which is valid due to (2a). Now if the generator is online in period $t = 1$, then we discuss the cases when the generator shuts down. (i) If the generator shuts down in period $t = 2$, i.e., $y_1 = 1$ and $y_2 = 0$, then inequality (EC.21) converts to $p_1 \leq \overline{V} - \underline{C}$ which is valid due to the ramping-down constraints (2g); (ii) If the generator shuts down in period $t = 3$, i.e., $y_1 = y_2 = 1$ and $y_3 = 0$, then inequality (EC.21) converts to $p_1 \leq \overline{V} + V - \underline{C}$, which is valid due to (2g); (iii) If the generator shuts down after period $t = 3$, i.e., $y_1 = y_2 = y_3 = 1$, then inequality (EC.21) converts to $p_1 \leq \overline{C} - \underline{C}$, which is valid due to (2e).

Similarly by considering the shut-down time, we can prove the validity of inequality (EC.22). Thus, we omit the details here.

Inequality (EC.23) tightens the upper bound of regulation-down reserve r_2^-. Note that r_2^- is only bounded by p_2 in (2d). Due to inequality (EC.22), we have an inherent upper bound $r_2^- \leq p_2 \leq (\overline{V} - \underline{C})y_2 + (\overline{C} - \overline{V})(y_3 - u_3 - u_2)$, which can be further tightened by (EC.23) since $\overline{C} - \underline{C} - 2V > 0$, i.e., $\underline{C} + 2V - \overline{V} < \overline{C} - \overline{V}$. To show the validity of inequality (EC.23), it is clear that when $y_2 = 0$, inequality (EC.23) converts to $(\underline{C} + 2V - \overline{V})(y_3 - u_3 - u_2) \geq 0$, which is valid due to (2a). Next, we consider the cases with $y_2 = 1$. If the generator starts up in period $t = 2$, i.e., $u_2 = 1$, which leads to $y_2 = y_3 = 1$ due to (2a), then inequality (EC.23) becomes $r_2^- \leq \overline{V} - \underline{C}$, which is valid because $r_2^- \leq p_2$ due to inequality (2d) and $p_2 + r_2^+ \leq \overline{V} - \underline{C}$ due to (2e). If the generator starts up before $t = 2$, i.e., $u_2 = 0$, then we need to consider two possible shut-down times. First, if the generator shuts down in period $t = 3$, i.e., $y_2 = 1$ and $y_3 = u_2 = u_3 = 0$, then inequality (EC.23) becomes $r_2^- \leq \overline{V} - \underline{C}$, which is valid due to (2g). Second, if the generator shuts down after period $t = 3$, i.e., $y_2 = y_3 = 1$, then inequality (EC.23) becomes $r_2^- \leq 2V$, which is valid because $r_2^- \leq r_2^+ + r_2^- = (p_2 + r_2^+ - p_1) + (p_1 - p_2 + r_2^-) \leq V + V = 2V$, where the last inequality holds due to constraints (2f) and (2g).

We can show the validity of inequalities (EC.24) and (EC.25) with similar ideas as above, and thus we omit details here for brevity. Inequality (EC.26) is exactly from P, so it is obviously valid.

Now we show the validity of inequality (EC.27), which strengthens inequality (2e) by considering the start-up and shut-down decisions. If $y_2 = 0$, then inequality (EC.27) becomes $(\overline{C} - \overline{V} - 2V)(y_3 - u_3) \geq 0$, which is valid due to constraints (2a). If $y_2 = 1$, i.e., the generator is online in period $t = 2$, then we discuss when the generator starts up and shuts down.

1. If the generator starts up before period $t = 2$ and shuts down in period $t = 3$, i.e., $u_2 = y_3 = 0$, then (EC.27) becomes $p_2 + r_2^+ \leq \overline{V} + 2V - \underline{C}$, which is valid because $p_2 \leq \overline{V} - \underline{C}$ due to (2g), and $r_2^+ \leq V + p_1 - p_2 < 2V$ as $p_2 + r_2^+ - p_1 \leq V$ by constraints (2f) and (2g).
2. If the generator starts up before period $t = 2$ and shuts down after period $t = 3$, i.e., $u_2 = 0, y_3 = 1$, then (EC.27) becomes $p_2 + r_2^+ \leq \overline{C} - \underline{C}$, which is clearly valid.
3. If the generator starts up in period $t = 2$, then $u_2 = y_3 = 1$, and (EC.27) becomes $p_2 + r_2^+ \leq \overline{V} - \underline{C}$, which is valid due to ramping-up constraints (2f).

With similar proof ideas, we can also show the validity of inequalities (EC.28) and (EC.29) and thus omit the details.

Inequalities (EC.30) - (EC.32) are derived to strengthen the ramping-down constraints (2g), and inequalities (EC.33) - (EC.35) are derived to strengthen the ramping-up constraints (2f). To clarify how inequality (EC.30) strengthens the ramping-down constraint, we first derive an alternative representation of (2g) with $t = 2$, i.e.,

$$p_1 - p_2 + r_2^- \leq (\overline{V} - \underline{C})y_1 - (\underline{C} + V - \overline{V})(y_2 - u_2) + \overline{V}(1 - y_1) + (\underline{C} + V - \overline{V})u_2, \tag{EC.41}$$

whose RHS is larger than that of inequality (EC.30). That is, inequality (EC.30) strengthens the RHS of ramping-down constraint (EC.41) by the value of $\overline{V}(1 - y_1) + (\underline{C} + V - \overline{V})u_2$, which is non-negative because $\overline{V} < \underline{C} + V$, $y_1 \leq 1$, and $u_2 \geq 0$. Next, we show the validity of inequality (EC.30). If $y_1 = 1$ and $u_2 = 0$, then inequality (EC.30) is clearly valid since it is the same as (EC.41). If

$y_1 = u_2 = 0$, then $y_2 = 0$, and thus inequality (EC.30) is also valid. If $u_2 = 1$, then $y_1 = 0$ and $y_2 = 1$ due to constraints (2a) - (2b), and inequality (EC.30) converts to $p_2 \geq r_2^-$, which is valid due to (2d). Thus, we have shown the validity of inequality (EC.30). Following the same idea, we can also prove the validity of inequalities (EC.31) - (EC.35). We omit the proofs here for brevity.

For the validity of inequality (EC.36), we discuss the online/offline status of the generator.

1. If the generator is offline in period $t = 2$ and $t = 3$, then inequality (EC.36) is trivially valid.
2. If the generator is online in period $t = 2$ and shuts down in period $t = 3$, i.e., $y_2 = 1$, $y_3 = 0$, then $u_2 = u_3 = 0$ due to (2a) and (2c). Inequality (EC.36) becomes $r_2^+ \leq 2V$, which is valid based on inequality (EC.24).
3. If the generator starts up in period $t = 3$, then $y_3 = u_3 = 1$ and $y_2 = u_2 = 0$. Inequality (EC.36) converts to $p_3 + r_3^+ \leq \overline{V} - \underline{C}$, which is valid due to ramping-up constraints (2f).
4. If the generator is online in period $t = 2$ and $t = 3$, i.e., $y_2 = y_3 = 1$, then $u_3 = 0$ and inequality (EC.36) converts to $p_3 + r_3^+ + r_2^+ \leq \overline{C} - \underline{C} + V - (\overline{C} - \overline{V})u_2$. Note that we only need to prove the case where $u_2 = 1$ with a smaller RHS. When $u_2 = 1$, inequality (EC.36) becomes $p_3 + r_3^+ + r_2^+ \leq \overline{V} - \underline{C} + V$, whose validity can be shown by summing up $p_3 + r_3^+ - p_2 \leq V$ and $p_2 + r_2^+ \leq \overline{V} - \underline{C}$, where the second one is due to ramping-up constraints (2f).

The validity proof for inequality (16) also follows the similar technique of inequality (EC.36), thus we omit the proof here.

Inequality (15) tightens the ramping-down constraints (2g) by introducing variable r_2^+. Meanwhile, inequality (EC.37) strengthens the ramping-up constraints (2f) by introducing variable r_2^-. The validity proofs of inequalities (15) and (EC.37) are similar and we only show (15) is valid. To that end, we discuss the following four possible cases.

1. If $y_2 = y_3 = 0$, the validity trivially holds.
2. If $y_2 = 1$ and $y_3 = 0$, then $u_2 = u_3 = 0$ and inequality (15) becomes $p_2 + r_2^+ \leq \overline{V} + 2V - \underline{C}$ (which strengthens constraints (2e) since $\overline{V} + 2V < \overline{C}$). It is valid because $p_2 \leq \overline{V} - \underline{C}$ due to ramping-down constraint (2g), and $r_2^+ \leq 2V$ since $r_2^+ \leq r_2^+ + r_2^- = (p_2 + r_2^+ - p_1) + (p_1 - p_2 + r_2^-) \leq V + V = 2V$, where the last inequality holds due to constraints (2f) and (2g).
3. If $y_2 = 0$ and $y_3 = 1$, then $u_2 = 0$ and $u_3 = 1$. Inequality (15) converts to $p_3 \geq r_3^-$, which is valid due to (2d).
4. If $y_2 = y_3 = 1$, then $u_3 = 0$ and inequality (15) becomes $p_2 + r_2^+ - p_3 + r_3^- \leq 3V - (\underline{C} + 3V - \overline{V})u_2$. We further discuss two possible cases in terms of the value of u_2.
   (a) If $u_2 = 1$, then inequality (15) becomes $p_2 + r_2^+ - p_3 + r_3^- \leq \overline{V} - \underline{C}$, which is valid because $p_2 + r_2^+ \leq \overline{V} - \underline{C}$ due to ramping-up constraints (2f) and $p_3 \geq r_3^-$ due to constraints (2d).
   (b) If $u_2 = 0$, then inequality (15) becomes $p_2 + r_2^+ - p_3 + r_3^- \leq 3V$, which is valid because $p_2 + r_2^+ - p_3 \leq 2V$ due to ramping-down constraints (2g) and $r_3^- \leq 2V$ since $r_2^- \leq r_2^+ + r_2^- =$

$(p_2 + r_2^+ - p_1) + (p_1 - p_2 + r_2^-) \leq V + V = 2V$, where the last inequality holds due to constraints (2f) and (2g).

For inequality (EC.38), i.e., $p_1 - p_2 + p_3 + r_3^+ \leq (\overline{V} - \underline{C})y_1 + (\underline{C} + V - \overline{V})y_2 + (\overline{C} - \underline{C})y_3 - (\overline{C} - \overline{V})(u_2 + u_3)$, it tightens the summation of inequalities (EC.21) and (EC.35), with this summation represented as $p_1 - p_2 + p_3 + r_3^+ \leq (\overline{V} - \underline{C})y_1 + (\underline{C} + V - \overline{V})y_2 + (\overline{C} - \underline{C})y_3 - (\overline{C} - \overline{V})(u_2 + u_3) + (\overline{V} - \underline{C})(y_2 - y_3 + u_3)$. Note that $y_2 - y_3 + u_3 \geq 0$ due to constraints (2c). Therefore, we only need to discuss the case where $y_2 - y_3 + u_3 = 1$, otherwise inequality (EC.38) is the same as the summation of inequalities (EC.21) and (EC.35), leading to a redundant and clearly valid inequality. Due to constraints (2a) and (2b), $y_2 - y_3 + u_3 = 1$ leads to $y_2 = 1$ and $y_3 = u_3 = 0$, which further lead to $u_2 = 0$ and $y_1 = 1$ because the minimum-up time constraints (2a) with $L = 2$. It such case, inequality (EC.38) converts to $p_1 - p_2 \leq V$, which is valid due to ramping-down constraints (2g).

Finally, inequality (EC.39) can strengthen the summation of inequalities (EC.32) and (EC.35), i.e., $p_1 - p_2 + r_3^+ + r_3^- \leq (\overline{V} - \underline{C})y_1 + (\underline{C} + V - \overline{V})y_2 + 2Vy_3 - (\underline{C} + 2V - \overline{V})(u_2 + u_3) + (\overline{V} - \underline{C})(y_2 - y_3 + u_3)$, by the value of $(\overline{V} - \underline{C})(y_2 - y_3 + u_3)$. The validity proof for inequality (EC.39) is similar with that for inequality (EC.38). We omit the details for brevity. □

EC.2.2. Proof of Proposition EC.2

Proof. We prove that $dim(Q_3^2) = 14$ because there are 14 decision variables. We create 15 affinely independent points in Q_3^2. Since $\vec{0} \in Q_3^2$, we only need to create the other 14 linearly independent points in Q_3^2, as shown in Table EC.7. It is easy to verify that these points are linearly independent by Gaussian elimination, leading to a lower triangular matrix. □

Table EC.7 Linearly Independent Points in Q_3^2

p_1	p_2	p_3	r_1^+	r_2^+	r_3^+	r_1^-	r_2^-	r_3^-	y_1	y_2	y_3	u_2	u_3
0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\overline{V} - \underline{C}$	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	$\overline{C} - \underline{C}$	0	0	0	0	0	1	0	0	0	0
$\overline{V} - \underline{C}$	0	0	0	0	0	$\overline{V} - \underline{C}$	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	0	0	0
0	$\overline{V} - \underline{C}$	0	0	0	0	0	0	0	1	1	0	0	0
0	0	0	0	V	0	0	0	0	1	1	0	0	0
0	$\overline{V} - \underline{C}$	0	0	0	0	0	$\overline{V} - \underline{C}$	0	1	1	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0
0	0	0	0	0	0	0	0	0	0	1	1	1	0
0	0	0	0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	$\overline{V} - \underline{C}$	0	0	0	0	0	1	0	1
0	0	$\overline{V} - \underline{C}$	0	0	0	0	0	0	0	0	1	0	1
0	0	$\overline{V} - \underline{C}$	0	0	0	0	0	$\overline{V} - \underline{C}$	0	0	1	0	1

EC.2.3. Proof of Proposition EC.5

Proof. It is sufficient to show that every point $z \in Q_3^2$ can be written as $z = \sum_{s\in S} \lambda_s z^s$ for some $\lambda_s \geq 0$ and $\sum_{s\in S}\lambda_s = 1$, where $z^s \in P_3^2, s \in S$ with $y(z^s)$ and $u(z^s)$ binary.

For any given $\bar{z} = (\bar{p}_1,\bar{p}_2,\bar{p}_3,\bar{r}_1^+,\bar{r}_2^+,\bar{r}_3^+,\bar{r}_1^-,\bar{r}_2^-,\bar{r}_3^-,\bar{y}_1,\bar{y}_2,\bar{y}_3,\bar{u}_2,\bar{u}_3) \in Q_3^2$, we can pick $z^1,\dots,z^6 \in P_3^2$ such that

$$
\begin{aligned}
z^1 &= (\hat{p}_{11},0,0,\hat{r}_{11}^+,0,0,\hat{r}_{11}^-,0,0,1,0,0,0,0),\\
z^2 &= (\hat{p}_{12},\hat{p}_{22},0,\hat{r}_{12}^+,\hat{r}_{22}^+,0,\hat{r}_{12}^-,\hat{r}_{22}^-,0,1,1,0,0,0),\\
z^3 &= (\hat{p}_{13},\hat{p}_{23},\hat{p}_{33},\hat{r}_{13}^+,\hat{r}_{23}^+,\hat{r}_{33}^+,\hat{r}_{13}^-,\hat{r}_{23}^-,\hat{r}_{33}^-,1,1,1,0,0),\\
z^4 &= (0,\hat{p}_{24},\hat{p}_{34},0,\hat{r}_{24}^+,\hat{r}_{34}^+,0,\hat{r}_{24}^-,\hat{r}_{34}^-,0,0,1,1,1,0),\\
z^5 &= (0,0,\hat{p}_{35},0,0,\hat{r}_{35}^+,0,0,\hat{r}_{35}^-,0,0,1,0,1),\\
z^6 &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0).
\end{aligned}
$$

In addition, we define $\lambda_1 = \bar{y}_1 - \bar{y}_2 + \bar{u}_2$, $\lambda_2 = \bar{y}_2 - \bar{y}_3 + \bar{u}_3$, $\lambda_3 = \bar{y}_3 - \bar{u}_2 - \bar{u}_3$, $\lambda_4 = \bar{u}_2$, $\lambda_5 = \bar{u}_3$, and $\lambda_6 = 1 - \bar{y}_1 - \bar{u}_2 - \bar{u}_3$. Note that $\sum_{s=1}^6 \lambda_s = 1$. Due to (2a) - (2c), we have $\lambda_s \geq 0$ for each $s \in [1,6]_{\mathbb{Z}}$.

It is clear that $\bar{y}_i = y_i(\bar{z}) = \sum_{s=1}^6 \lambda_s y_i(z^s)$ for $i = 1,2$ and $\bar{u}_2 = u_2(\bar{z}) = \sum_{s=1}^4 \lambda_s u_2(z^s)$. So we only need to determine the values of $\hat{\omega} \doteq \{\hat{p}_{11},\hat{p}_{12},\hat{p}_{13},\hat{p}_{22},\hat{p}_{23},\hat{p}_{24},\hat{p}_{33},\hat{p}_{34},\hat{p}_{35},\hat{r}_{11}^+,\hat{r}_{12}^+,\hat{r}_{13}^+,\hat{r}_{22}^+,\hat{r}_{23}^+,\hat{r}_{24}^+,\hat{r}_{33}^+,\hat{r}_{34}^+,\hat{r}_{35}^+,\hat{r}_{11}^-,\hat{r}_{12}^-,\hat{r}_{13}^-,\hat{r}_{22}^-,\hat{r}_{23}^-,\hat{r}_{24}^-,\hat{r}_{33}^-,\hat{r}_{34}^-,\hat{r}_{35}^-\}$ and show that $\bar{p}_1 = \sum_{i=1}^3 \lambda_i \hat{p}_{1i}$, $\bar{p}_2 = \sum_{i=2}^4 \lambda_i \hat{p}_{2i}$, $\bar{p}_3 = \sum_{i=3}^5 \lambda_i \hat{p}_{3i}$, $\bar{r}_1^+ = \sum_{i=1}^3 \lambda_i \hat{r}_{1i}^+$, $\bar{r}_2^+ = \sum_{i=2}^4 \lambda_i \hat{r}_{2i}^+$, $\bar{r}_3^+ = \sum_{i=3}^5 \lambda_i \hat{r}_{3i}^+$, $\bar{r}_1^- = \sum_{i=1}^3 \lambda_i \hat{r}_{1i}^-$, $\bar{r}_2^- = \sum_{i=2}^4 \lambda_i \hat{r}_{2i}^-$, and $\bar{r}_3^- = \sum_{i=3}^5 \lambda_i \hat{r}_{3i}^-$. The corresponding feasible region for $\hat{\omega}$ is described as $A = A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$, where $A_1 = \{(\hat{p}_{11},\hat{r}_{11}^+,\hat{r}_{11}^-) \in \mathbb{R}_+^3 : \hat{p}_{11} \geq \hat{r}_{11}^-,\ \hat{p}_{11} \leq \overline{V} - \underline{C},\ \hat{p}_{11} + \hat{r}_{11}^+ \leq \overline{C} - \underline{C}\}$, $A_2 = \{(\hat{p}_{12},\hat{r}_{12}^+,\hat{r}_{12}^-,\hat{p}_{22},\hat{r}_{22}^+,\hat{r}_{22}^-) \in \mathbb{R}_+^6 : \hat{p}_{12} \geq \hat{r}_{12}^-,\ \hat{p}_{22} \geq \hat{r}_{22}^-,\ \hat{p}_{22} \leq \overline{V} - \underline{C},\ \hat{p}_{12} + \hat{r}_{12}^+ \leq \overline{C} - \underline{C},\ \hat{p}_{22} + \hat{r}_{22}^+ \leq \overline{C} - \underline{C},\ \hat{p}_{12} - \hat{p}_{22} + \hat{r}_{22}^- \leq V,\ \hat{p}_{22} + \hat{r}_{22}^+ - \hat{p}_{12} \leq V\}$, $A_3 = \{(\hat{p}_{13},\hat{r}_{13}^+,\hat{r}_{13}^-,\hat{p}_{23},\hat{r}_{23}^+,\hat{r}_{23}^-,\hat{p}_{33},\hat{r}_{33}^+,\hat{r}_{33}^-) \in \mathbb{R}_+^9 : \hat{p}_{13} \geq \hat{r}_{13}^-,\ \hat{p}_{23} \geq \hat{r}_{23}^-,\ \hat{p}_{33} \geq \hat{r}_{33}^-,\ \hat{p}_{13} + \hat{r}_{13}^+ \leq \overline{C} - \underline{C},\ \hat{p}_{23} + \hat{r}_{23}^+ \leq \overline{C} - \underline{C},\ \hat{p}_{33} + \hat{r}_{33}^+ \leq \overline{C} - \underline{C},\ \hat{p}_{13} - \hat{p}_{23} + \hat{r}_{23}^- \leq V,\ \hat{p}_{23} + \hat{r}_{23}^+ - \hat{p}_{13} \leq V,\ \hat{p}_{23} - \hat{p}_{33} + \hat{r}_{33}^- \leq V,\ \hat{p}_{33} + \hat{r}_{33}^+ - \hat{p}_{23} \leq V\}$, $A_4 = \{(\hat{p}_{24},\hat{r}_{24}^+,\hat{r}_{24}^-,\hat{p}_{34},\hat{r}_{34}^+,\hat{r}_{34}^-) \in \mathbb{R}_+^6 : \hat{p}_{24} \geq \hat{r}_{24}^-,\ \hat{p}_{34} \geq \hat{r}_{34}^-,\ \hat{p}_{24} + \hat{r}_{24}^+ \leq \overline{V} - \underline{C},\ \hat{p}_{34} + \hat{r}_{34}^+ \leq \overline{C} - \underline{C},\ \hat{p}_{24} - \hat{p}_{34} + \hat{r}_{34}^- \leq V,\ \hat{p}_{34} + \hat{r}_{34}^+ - \hat{p}_{24} \leq V\}$, and $A_5 = \{(\hat{p}_{35},\hat{r}_{35}^+,\hat{r}_{35}^-) \in \mathbb{R}_+^3 : \hat{p}_{35} \geq \hat{r}_{35}^-,\ \hat{p}_{35} + \hat{r}_{35}^+ \leq \overline{V} - \underline{C}\}$.

To show that $\bar{p}_i = \sum_{s=1}^6 \lambda_s p_i(z^s)$, $\bar{r}_i^+ = \sum_{s=1}^6 \lambda_s r_i^+(z^s)$, and $\bar{r}_i^- = \sum_{s=1}^6 \lambda_s r_2^-(z^s)$, for $i = 1,2,3$, it suffices to prove that given a fixed $(\bar{y}_1,\bar{y}_2,\bar{y}_3,\bar{u}_2,\bar{u}_3) \in B = \{(y_1,y_2,y_3,u_2,u_3) \in [0,1]^5 : (2a) - (2c)\}$, i.e., fixing λ_i for any $i \in [1,6]_{\mathbb{Z}}$, for any point $(\bar{p}_1,\bar{p}_2,\bar{p}_3,\bar{r}_1^+,\bar{r}_2^+,\bar{r}_3^+,\bar{r}_1^-,\bar{r}_2^-,\bar{r}_3^-)$ in the set

$$
C = \Big\{ \quad (p_1,p_2,p_3,r_1^+,r_2^+,r_3^+,r_1^-,r_2^-,r_3^-) \in \mathbb{R}_+^9 :
$$

$$
p_1 \leq (\overline{V} - \underline{C})\lambda_1 + (\overline{V} + V - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3, \tag{EC.42a}
$$

$$
p_2 \leq (\overline{V} - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3 + (\overline{V} - \underline{C})\lambda_4, \tag{EC.42b}
$$

$$r_2^- \le (\overline{V} - \underline{C})\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, \tag{EC.42c}$$

$$r_2^+ + r_2^- \le 2V\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, \tag{EC.42d}$$

$$r_3^+ + r_3^- \le 2V\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5, \tag{EC.42e}$$

$$p_1 + r_1^+ \le (\overline{C} - \underline{C})\lambda_1 + (\overline{C} - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3, \tag{EC.42f}$$

$$p_2 + r_2^+ \le (\overline{V} + 2V - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3 + (\overline{V} - \underline{C})\lambda_4, \tag{EC.42g}$$

$$p_3 + r_3^+ \le (\overline{C} - \underline{C})\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5, \tag{EC.42h}$$

$$p_i \ge r_i^-, \forall i = 1, 2, 3, \tag{EC.42i}$$

$$p_2 - p_1 \le (\overline{V} - \underline{C})\lambda_2 + V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, \tag{EC.42j}$$

$$p_1 - p_2 + r_2^- \le (\overline{V} - \underline{C})\lambda_1 + V\lambda_2 + V\lambda_3, \tag{EC.42k}$$

$$p_2 - p_3 + r_3^- \le (\overline{V} - \underline{C})\lambda_2 + V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, \tag{EC.42l}$$

$$p_1 - p_3 + r_3^- \le (\overline{V} - \underline{C})\lambda_1 + (\overline{V} + V - \underline{C})\lambda_2 + 2V\lambda_3 + V\lambda_4, \tag{EC.42m}$$

$$p_3 + r_3^+ - p_1 \le 2V\lambda_3 + V\lambda_4 + (\overline{V} - \underline{C})\lambda_5, \tag{EC.42n}$$

$$p_2 + r_2^+ - p_1 \le V\lambda_2 + V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, \tag{EC.42o}$$

$$p_3 + r_3^+ - p_2 \le V\lambda_3 + V\lambda_4 + (\overline{V} - \underline{C})\lambda_5, \tag{EC.42p}$$

$$p_3 + r_3^+ + r_2^+ \le 2V\lambda_2 + (\overline{C} - \underline{C} + V)\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5, \tag{EC.42q}$$

$$r_2^+ + r_3^+ + r_3^- \le 2V\lambda_2 + 4V\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5, \tag{EC.42r}$$

$$p_2 + r_2^+ - p_3 + r_3^- \le (\overline{V} + 2V - \underline{C})\lambda_2 + 3V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, \tag{EC.42s}$$

$$p_3 + r_3^+ - p_2 + r_2^- \le 3V\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5, \tag{EC.42t}$$

$$p_1 - p_2 + p_3 + r_3^+ \le (\overline{V} - \underline{C})\lambda_1 + V\lambda_2 + (\overline{C} - \underline{C} + V)\lambda_3 + V\lambda_4 + (\overline{V} - \underline{C})\lambda_5, \tag{EC.42u}$$

$$p_1 - p_2 + r_3^+ + r_3^- \le (\overline{V} - \underline{C})\lambda_1 + V\lambda_2 + 3V\lambda_3 + V\lambda_4 + (\overline{V} - \underline{C})\lambda_5 \Big\}, \tag{EC.42v}$$

there exists $\hat{\omega} \in A$ such that

$$\overline{p}_1 = \textstyle\sum_{i=1}^3 \lambda_i \hat{p}_{1i}, \ \overline{r}_1^+ = \sum_{i=1}^3 \lambda_i \hat{r}_{1i}^+, \ \overline{r}_1^- = \sum_{i=1}^3 \lambda_i \hat{r}_{1i}^-, \tag{EC.43a}$$

$$\overline{p}_2 = \textstyle\sum_{i=2}^4 \lambda_i \hat{p}_{2i}, \ \overline{r}_2^+ = \sum_{i=2}^4 \lambda_i \hat{r}_{2i}^+, \ \overline{r}_2^- = \sum_{i=2}^4 \lambda_i \hat{r}_{2i}^-, \tag{EC.43b}$$

$$\overline{p}_3 = \textstyle\sum_{i=3}^5 \lambda_i \hat{p}_{3i}, \ \overline{r}_3^+ = \sum_{i=3}^5 \lambda_i \hat{r}_{3i}^+, \ \overline{r}_3^- = \sum_{i=3}^5 \lambda_i \hat{r}_{3i}^-. \tag{EC.43c}$$

That is, the linear transformation $F: A \to C$ is surjective, where the matrix F is defined by setting $F_{i,(3i-2)} = \lambda_1, i = 1,4,7;$ $F_{i,(3i-1)} = \lambda_2, i = 1,4,7;$ $F_{i,(3i)} = \lambda_3, i = 1,4,7;$ $F_{i,(3i-2)} = \lambda_2, i = 2,5,8;$ $F_{i,(3i-1)} = \lambda_3, i = 2,5,8;$ $F_{i,(3i)} = \lambda_4, i = 2,5,8;$ $F_{i,(3i-2)} = \lambda_3, i = 3,6,9;$ $F_{i,(3i-1)} = \lambda_4, i = 3,6,9;$ $F_{i,(3i)} = \lambda_5, i = 3,6,9;$ $F_{i,j} = 0$ otherwise.

Note that any point in C can be represented as a convex combination of the extreme points in C since C is compact. Thus, it suffices to prove that for any extreme point $\omega \in C$, there exists a point $p \in A$ such that $Fp = \omega$. Furthermore, it is sufficient to show this conclusion holds for any point on the faces of C, i.e., satisfying one of (EC.42a) - (EC.42v) at equality, which implies this conclusion holds for any extreme point.

Satisfying (EC.42a) at equality. In this case, we have the corresponding set $C_{(\text{EC.42a})}$ as follows, where $C_{(\text{EC.42a})} \subseteq C$, by substituting (EC.42a) at equality into (EC.42b) - (EC.42v).

$$
\begin{aligned}
C_{(\text{EC.42a})} = \Big\{ & (p_2, p_3, r_1^+, r_2^+, r_3^+, r_1^-, r_2^-, r_3^-) \in \mathbb{R}_+^8 : \\
& p_2 \le (\overline{V} - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.44a)} \\
& r_2^- \le (\overline{V} - \underline{C})\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.44b)} \\
& r_2^+ + r_2^- \le 2V\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.44c)} \\
& r_3^+ + r_3^- \le 2V\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5, && \text{(EC.44d)} \\
& r_1^+ \le (\overline{C} - \overline{V})\lambda_1 + (\overline{C} - \overline{V} - V)\lambda_2, && \text{(EC.44e)} \\
& p_2 + r_2^+ \le (\overline{V} + 2V - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.44f)} \\
& p_3 + r_3^+ \le (\overline{C} - \underline{C})\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5, && \text{(EC.44g)} \\
& r_1^- \le (\overline{V} - \underline{C})\lambda_1 + (\overline{V} + V - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3, && \text{(EC.44h)} \\
& p_i \ge r_i^-, \forall i = 2, 3, && \text{(EC.44i)} \\
& p_2 \le (\overline{V} - \underline{C})\lambda_1 + (2\overline{V} + V - 2\underline{C})\lambda_2 + +(\overline{C} + V - \underline{C})\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.44j)} \\
& -p_2 + r_2^- \le -(\overline{V} - \underline{C})\lambda_2 - (\overline{C} - V - \underline{C})\lambda_3, && \text{(EC.44k)} \\
& p_2 - p_3 + r_3^- \le (\overline{V} - \underline{C})\lambda_2 + V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.44l)} \\
& -p_3 + r_3^- \le -(\overline{C} - 2V - \underline{C})\lambda_3 + V\lambda_4, && \text{(EC.44m)} \\
& p_3 + r_3^+ \le (\overline{V} - \underline{C})\lambda_1 + (\overline{V} + V - \underline{C})\lambda_2 + (\overline{C} + 2V - \underline{C})\lambda_3 + V\lambda_4 + (\overline{V} - \underline{C})\lambda_5, && \\
& && \text{(EC.44n)} \\
& p_2 + r_2^+ \le (\overline{V} - \underline{C})\lambda_1 + (\overline{V} + 2V - \underline{C})\lambda_2 + (\overline{C} + V - \underline{C})\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.44o)} \\
& p_3 + r_3^+ - p_2 \le V\lambda_3 + V\lambda_4 + (\overline{V} - \underline{C})\lambda_5, && \text{(EC.44p)} \\
& p_3 + r_3^+ + r_2^+ \le 2V\lambda_2 + (\overline{C} - \underline{C} + V)\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5, && \text{(EC.44q)} \\
& r_2^+ + r_3^+ + r_3^- \le 2V\lambda_2 + 4V\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5, && \text{(EC.44r)} \\
& p_2 + r_2^+ - p_3 + r_3^- \le (\overline{V} + 2V - \underline{C})\lambda_2 + 3V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.44s)} \\
& p_3 + r_3^+ - p_2 + r_2^- \le 3V\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5, && \text{(EC.44t)} \\
& -p_2 + p_3 + r_3^+ \le -(\overline{V} - \underline{C})\lambda_2 + V\lambda_3 + V\lambda_4 + (\overline{V} - \underline{C})\lambda_5, && \text{(EC.44u)} \\
& -p_2 + r_3^+ + r_3^- \le -(\overline{V} - \underline{C})\lambda_2 - (\overline{C} - \underline{C} - 3V)\lambda_3 + V\lambda_4 + (\overline{V} - \underline{C})\lambda_5 \Big\}. && \text{(EC.44v)}
\end{aligned}
$$

Note that in $C_{(\text{EC.42a})}$, some inequalities are dominated by others. Inequality (EC.44d) is dominated by the summation of inequalities (EC.44l) and (EC.44u). Inequality (EC.44j) is dominated by inequality (EC.44a). Inequality (EC.44k) and (EC.44m) are dominated by inequality (EC.44i). Inequality (EC.44n) is dominated by inequality (EC.44g). Inequality (EC.44o) is dominated by inequality (EC.44f). Inequality (EC.44p) is dominated by inequality (EC.44u). Inequality (EC.44q) is dominated by the summation of inequalities (EC.44f) and (EC.44u). Inequality (EC.44r) is dominated by the summation of inequalities (EC.44s) and (EC.44u). Inequality (EC.44t) is dominated by the summation of inequalities (EC.44b) and (EC.44u). After eliminating the dominated inequalities, $C_{(\text{EC.42a})}$ can be equivalently represented as

$$
\begin{aligned}
C'_{(\text{EC.42a})} = \Big\{ & (r_1^+, r_1^-, p_2, r_2^+, r_2^-, p_3, r_3^+, r_3^-) \in \mathbb{R}^8_+ : \\
& r_1^+ \le (\overline{C} - \overline{V})\lambda_1 + (\overline{C} - \overline{V} - V)\lambda_2, && \text{(EC.45a)} \\
& r_1^- \le (\overline{V} - \underline{C})\lambda_1 + (\overline{V} + V - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3, && \text{(EC.45b)} \\
& p_2 \le (\overline{V} - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.45c)} \\
& r_2^- \le (\overline{V} - \underline{C})\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.45d)} \\
& r_2^+ + r_2^- \le 2V\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.45e)} \\
& p_2 + r_2^+ \le (\overline{V} + 2V - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.45f)} \\
& p_3 + r_3^+ \le (\overline{C} - \underline{C})\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5, && \text{(EC.45g)} \\
& p_2 \ge r_2^-, && \text{(EC.45h)} \\
& p_3 \ge r_3^-, && \text{(EC.45i)} \\
& p_2 - p_3 + r_3^- \le (\overline{V} - \underline{C})\lambda_2 + V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.45j)} \\
& p_2 + r_2^+ - p_3 + r_3^- \le (\overline{V} + 2V - \underline{C})\lambda_2 + 3V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.45k)} \\
& -p_2 + p_3 + r_3^+ \le -(\overline{V} - \underline{C})\lambda_2 + V\lambda_3 + V\lambda_4 + (\overline{V} - \underline{C})\lambda_5, && \text{(EC.45l)} \\
& -p_2 + r_3^+ + r_3^- \le -(\overline{V} - \underline{C})\lambda_2 - (\overline{C} - \underline{C} - 3V)\lambda_3 + V\lambda_4 + (\overline{V} - \underline{C})\lambda_5 \Big\}. && \text{(EC.45m)}
\end{aligned}
$$

Since $\overline{p}_1 = (\overline{V} - \underline{C})\lambda_1 + (\overline{V} + V - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3$, it is clear that we can pick $\hat{p}_{11} = \overline{V} - \underline{C}$, $\hat{p}_{12} = \overline{V} + V - \underline{C}$, and $\hat{p}_{13} = \overline{C} - \underline{C}$. Then due to the inequalities of A_3, we choose $\hat{r}_{13}^+ = 0$. As we can observe that in $C'_{(\text{EC.42a})}$, the values of the variables defined in the first period, i.e., r_1^+ and r_1^-, do not affect the values of other variables. Thus, we can first show that $\overline{r}_1^+ = \sum_{i=1}^3 \lambda_i \hat{r}_{1i}^+$ and $\overline{r}_1^- = \sum_{i=1}^3 \lambda_i \hat{r}_{1i}^-$ hold with $\hat{r}_{1i}^+, \hat{r}_{1i}^- \in A^1$, by picking $\hat{r}_{11}^+ = \overline{C} - \overline{V}, \hat{r}_{12}^+ = \overline{C} - \overline{V} - V$ if $r_1^+ = (\overline{C} - \overline{V})\lambda_1 + (\overline{C} - \overline{V} - V)\lambda_2$; $\hat{r}_{11}^+ = \hat{r}_{12}^+ = 0$ if $r_1^+ = 0$; $\hat{r}_{11}^- = \overline{V} - \underline{C}, \hat{r}_{12}^- = \overline{V} + V - \underline{C}, \hat{r}_{13}^- = \overline{C} - \underline{C}$ if $r_1^- = (\overline{V} - \underline{C})\lambda_1 + (\overline{V} + V - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3$; and $\hat{r}_{11}^- = \hat{r}_{12}^- = \hat{r}_{13}^- = 0$ if $r_1^- = 0$.

Therefore, we can from now on only discuss the variables defined in the second and third periods. The corresponding feasible region for $\hat{\psi} \doteq \{\hat{p}_{22}, \hat{p}_{23}, \hat{p}_{24}, \hat{p}_{33}, \hat{p}_{34}, \hat{p}_{35}, \hat{r}_{22}^+, \hat{r}_{23}^+, \hat{r}_{24}^+, \hat{r}_{33}^+, \hat{r}_{34}^+, \hat{r}_{35}^+, \hat{r}_{22}^-, \hat{r}_{23}^-,$

$\hat{r}_{24}^-, \hat{r}_{33}^-, \hat{r}_{34}^-, \hat{r}_{35}^-\}$ is $A^1 = \cap_{i=2}^5 A_i^1$, where $A_2^1 = \{(\hat{p}_{22}, \hat{r}_{22}^+, \hat{r}_{22}^-) \in \mathbb{R}_+^3 : \hat{p}_{22} \geq \hat{r}_{22}^-,\ \hat{p}_{22} \leq \overline{V} - \underline{C},\ \hat{p}_{22} + \hat{r}_{22}^+ \leq \overline{V} + 2V - \underline{C}\}$, $A_3^1 = \{(\hat{p}_{23}, \hat{r}_{23}^+, \hat{r}_{23}^-, \hat{p}_{33}, \hat{r}_{33}^+, \hat{r}_{33}^-) \in \mathbb{R}_+^6 : \hat{p}_{23} \geq \hat{r}_{23}^-,\ \hat{p}_{33} \geq \hat{r}_{33}^-,\ \hat{p}_{23} + \hat{r}_{23}^+ \leq \overline{C} - \underline{C},\ \hat{p}_{33} + \hat{r}_{33}^+ \leq \overline{C} - \underline{C},\ \hat{p}_{23} - \hat{p}_{33} + \hat{r}_{33}^- \leq V,\ \hat{p}_{33} + \hat{r}_{33}^+ - \hat{p}_{23} \leq V\}$, $A_4^1 = \{(\hat{p}_{24}, \hat{r}_{24}^+, \hat{r}_{24}^-, \hat{p}_{34}, \hat{r}_{34}^+, \hat{r}_{34}^-) \in \mathbb{R}_+^6 : \hat{p}_{24} \geq \hat{r}_{24}^-,\ \hat{p}_{34} \geq \hat{r}_{34}^-,\ \hat{p}_{24} + \hat{r}_{24}^+ \leq \overline{V} - \underline{C},\ \hat{p}_{34} + \hat{r}_{34}^+ \leq \overline{C} - \underline{C},\ \hat{p}_{24} - \hat{p}_{34} + \hat{r}_{34}^- \leq V,\ \hat{p}_{34} + \hat{r}_{34}^+ - \hat{p}_{24} \leq V\}$, and $A_5^1 = \{(\hat{p}_{35}, \hat{r}_{35}^+, \hat{r}_{35}^-) \in \mathbb{R}_+^3 : \hat{p}_{35} \geq \hat{r}_{35}^-,\ \hat{p}_{35} + \hat{r}_{35}^+ \leq \overline{V} - \underline{C}\}$. Meanwhile, the constraints with the variables defined in the second and third periods in $C'_{\text{(EC.42a)}}$ can be described as follows.

$$
\begin{aligned}
C^1_{\text{(EC.42a)}} = \Big\{ & (p_2, r_2^+, r_2^-, p_3, r_3^+, r_3^-) \in \mathbb{R}_+^6 : \\
& p_2 \leq (\overline{V} - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.46a)} \\
& r_2^- \leq (\overline{V} - \underline{C})\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.46b)} \\
& r_2^+ + r_2^- \leq 2V\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.46c)} \\
& p_2 + r_2^+ \leq (\overline{V} + 2V - \underline{C})\lambda_2 + (\overline{C} - \underline{C})\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.46d)} \\
& p_3 + r_3^+ \leq (\overline{C} - \underline{C})\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5, && \text{(EC.46e)} \\
& p_2 \geq r_2^-, && \text{(EC.46f)} \\
& p_3 \geq r_3^-, && \text{(EC.46g)} \\
& p_2 - p_3 + r_3^- \leq (\overline{V} - \underline{C})\lambda_2 + V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.46h)} \\
& p_2 + r_2^+ - p_3 + r_3^- \leq (\overline{V} + 2V - \underline{C})\lambda_2 + 3V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.46i)} \\
& -p_2 + p_3 + r_3^+ \leq -(\overline{V} - \underline{C})\lambda_2 + V\lambda_3 + V\lambda_4 + (\overline{V} - \underline{C})\lambda_5, && \text{(EC.46j)} \\
& -p_2 + r_3^+ + r_3^- \leq -(\overline{V} - \underline{C})\lambda_2 - (\overline{C} - \underline{C} - 3V)\lambda_3 + V\lambda_4 + (\overline{V} - \underline{C})\lambda_5 \Big\}. && \text{(EC.46k)}
\end{aligned}
$$

In the following, we will keep fixing one of the inequalities in $C^1_{\text{(EC.42a)}}$ at equality, i.e., leading to a point $\{\overline{p}_2, \overline{p}_3, \overline{r}_2^+, \overline{r}_3^+, \overline{r}_2^-, \overline{r}_3^-\}$ satisfying this inequality at equality, and show that this point, say $\bar{\omega}$, can be represented using a point $\hat{p} \in A^1$ such that $F\hat{p} = \bar{\omega}$.

1) Satisfying (EC.46a) at equality. We can pick $\hat{p}_{22} = \overline{V} - \underline{C}$, $\hat{p}_{23} = \overline{C} - \underline{C}$ and $\hat{p}_{24} = \overline{V} - \underline{C}$, which satisfy the constraints in set A^1. Due to inequalities of A^1, we can fix $\hat{r}_{23}^+ = \hat{r}_{24}^+ = 0$, and the corresponding feasible region for $\hat{\psi}$ is $A^2 = \cap_{i=2}^5 A_i^2$, where $A_2^2 = \{(\hat{r}_{22}^+, \hat{r}_{22}^-) \in \mathbb{R}_+^2 : \hat{r}_{22}^- \leq \overline{V} - \underline{C},\ \hat{r}_{22}^+ \leq 2V\}$, $A_3^2 = \{(\hat{r}_{23}^+, \hat{r}_{23}^-, \hat{p}_{33}, \hat{r}_{33}^+, \hat{r}_{33}^-) \in \mathbb{R}_+^5 : \hat{r}_{23}^- \leq \overline{C} - \underline{C},\ \hat{p}_{33} \geq \hat{r}_{33}^-,\ \hat{p}_{33} + \hat{r}_{33}^+ \leq \overline{C} - \underline{C}\}$, $A_4^2 = \{(\hat{r}_{24}^-, \hat{p}_{34}, \hat{r}_{34}^+, \hat{r}_{34}^-) \in \mathbb{R}_+^4 : \hat{r}_{24}^- \leq \overline{V} - \underline{C},\ \hat{p}_{34} \geq \hat{r}_{34}^-,\ \hat{p}_{34} + \hat{r}_{34}^+ \leq \overline{V} + V - \underline{C}\}$, and $A_5^2 = \{(\hat{p}_{35}, \hat{r}_{35}^+, \hat{r}_{35}^-) \in \mathbb{R}_+^3 : \hat{p}_{35} \geq \hat{r}_{35}^-,\ \hat{p}_{35} + \hat{r}_{35}^+ \leq \overline{V} - \underline{C}\}$. Meanwhile, $C^1_{\text{(EC.42a)}}$ reduces to the following set:

$$
\begin{aligned}
C^2_{\text{(EC.42a)}} = \Big\{ & (p_3, r_2^+, r_3^+, r_2^-, r_3^-) \in \mathbb{R}_+^5 : \\
& r_2^+ \leq 2V\lambda_2, && \text{(EC.47a)} \\
& r_2^- \leq (\overline{V} - \underline{C})\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.47b)} \\
& r_2^+ + r_2^- \leq 2V\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4, && \text{(EC.47c)}
\end{aligned}
$$

$$p_3 + r_3^+ \le (\overline{C} - \underline{C})\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5, \tag{EC.47d}$$

$$p_3 \ge r_3^-, \tag{EC.47e}$$

$$r_3^+ + r_3^- \le 3V\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5 \Big\}. \tag{EC.47f}$$

Note that all the inequalities both in $C^2_{(\text{EC.42a})}$ and in each $A_i^2, i \in [2,5]_{\mathbb{Z}}$ contain decision variables that are defined in the same periods. Thus, it suffices to show that for any extreme point $\bar{\omega} \in C^2_{(\text{EC.42a})}$, there exists a point $\hat{p} \in A^2$ such that $F\hat{p} = \bar{\omega}$. Note that the inequalities in $C^2_{(\text{EC.42a})}$ are separable with respect to the variables in the second and third periods. Thus, we define sets $C^{2,t}_{(\text{EC.42a})}, t = 2,3$ by letting $C^{2,2}_{(\text{EC.42a})} = \{(r_2^+, r_2^-) \in \mathbb{R}^2_+ : r_2^+ \le 2V\lambda_2,\ r_2^- \le (\overline{V} - \underline{C})\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4,\ r_2^+ + r_2^- \le 2V\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4\}$ and $C^{2,3}_{(\text{EC.42a})} = \{(p_3, r_3^+, r_3^-) \in \mathbb{R}^3_+ : p_3 + r_3^+ \le (\overline{C} - \underline{C})\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5,\ p_3 \ge r_3^-,\ r_3^+ + r_3^- \le 3V\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5\}$. That is, $C^2_{(\text{EC.42a})} = \cap_{t=2}^3 C^{2,t}_{(\text{EC.42a})}$. We consider two possible cases.

1.1) Consider $t = 2$. We further consider one of the inequalities in $C^{2,2}_{(\text{EC.42a})}$ is satisfied at equality as follows.

   1.1.1) When $r_2^+ = 2V\lambda_2$. We can pick $\hat{r}_{22}^+ = 2V$. Note that we have fixed $\hat{r}_{23}^+ = \hat{r}_{24}^+ = 0$. It is easy to verify that $\hat{r}_{2i}^+ \in A^2, i = 2,3,4$, and $r_2^+ = \sum_{i=2}^4 \lambda_i \hat{r}_{2i}^+$ holds. Now the feasible region of r_2^- is $0 \le r_2^- \le 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4$. If $r_2^- = 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4$, we can pick $\hat{r}_{23}^- = 2V$ and $\hat{r}_{24}^- = \overline{V} - \underline{C}$. Otherwise, we can let $\hat{r}_{23}^- = \hat{r}_{24}^- = 0$. Note that we have previously set $\hat{r}_{22}^- = 0$. Now we can easily verify that $\hat{r}_{2i}^- \in A^2, i = 2,3,4$, and $r_2^- = \sum_{i=2}^4 \lambda_i \hat{r}_{2i}^-$ holds.

   1.1.2) When $r_2^- = (\overline{V} - \underline{C})\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4$. We can pick $\hat{r}_{23}^- = 2V$ and $\hat{r}_{24}^- = \overline{V} - \underline{C}$. Note that we have previously set $\hat{r}_{22}^- = 0$. Now we can easily verify that $\hat{r}_{2i}^- \in A^2, i = 2,3,4$, and $r_2^- = \sum_{i=2}^4 \lambda_i \hat{r}_{2i}^-$ holds. Furthermore, now $0 \le r_2^+ \le (\underline{C} + 2V - \overline{V})\lambda_2$. If $r_2^+ = (\underline{C} + 2V - \overline{V})\lambda_2$, then we can let $\hat{r}_{22}^+ = \underline{C} + 2V - \overline{V}$ or 0 otherwise. Note that we have fixed $\hat{r}_{23}^+ = \hat{r}_{24}^+ = 0$ before. Now it is easy to verify that $\hat{r}_{2i}^+ \in A^2, i = 2,3,4$, and $r_2^+ = \sum_{i=2}^4 \lambda_i \hat{r}_{2i}^+$ holds.

   1.1.3) When $r_2^+ + r_2^- = 2V\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4$. Then $r_2^- = 2V\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4 - r_2^+$ and $\underline{C} + 2V - \overline{C} \le r_2^+ \le 2V\lambda_2$. The extreme points of (r_1^+, r_1^-) are $(2V\lambda_2, 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4)$ and $((\underline{C} + 2V - \overline{C})\lambda_2, (\overline{V} - \underline{C})\lambda_2 + 2V\lambda_3 + (\overline{V} - \underline{C})\lambda_4)$, which have been proved by previous discussions.

1.2) For $t = 3$. We further consider one of the inequalities in $C^{2,3}_{(\text{EC.42a})}$ is satisfied at equality as follows.

   1.2.1) When $p_3 + r_3^+ = (\overline{C} - \underline{C})\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5$. Substituting this equality into $C^{2,3}_{(\text{EC.42a})}$, we have $C^{2,3'}_{(\text{EC.42a})} = \{(p_3, r_3^+, r_3^-) \in \mathbb{R}^3_+ : p_3 + r_3^+ = (\overline{C} - \underline{C})\lambda_3 + (\overline{V} + V - \underline{C})\lambda_4 + (\overline{V} - \underline{C})\lambda_5,\ p_3 \ge r_3^-,\ p_3 \ge r_3^- + (\overline{C} - 3V - \underline{C})\lambda_3\}$.

A. If $\overline{C}-3V-\underline{C}>0$, then we can set $p_3=r_3^-+(\overline{C}-3V-\underline{C})\lambda_3$. We can let $\hat{p}_{33}+\hat{r}_{33}^+=\overline{C}-\underline{C}$, $\hat{r}_{33}^-=\hat{p}_{33}-(\overline{C}-3V-\underline{C})$, $\hat{r}_{34}^-=\hat{p}_{34}$, $\hat{p}_{34}+\hat{r}_{34}^+=\overline{V}+V-\underline{C}$, $\hat{r}_{35}^-=\hat{p}_{35}$, and $\hat{p}_{35}+\hat{r}_{35}^+=\overline{V}-\underline{C}$, which satisfy the constraints in A^2.

B. If $\overline{C}-3V-\underline{C}\le 0$, then we can set $p_3=r_3^-$. We can let $\hat{p}_{33}+\hat{r}_{33}^+=\overline{C}-\underline{C}$, $\hat{r}_{33}^-=\hat{p}_{33}$, $\hat{r}_{34}^-=\hat{p}_{34}$, $\hat{p}_{34}+\hat{r}_{34}^+=\overline{V}+V-\underline{C}$, $\hat{r}_{35}^-=\hat{p}_{35}$, $\hat{p}_{35}+\hat{r}_{35}^+=\overline{V}-\underline{C}$, which satisfy the constraints in A^2.

1.2.2) When $p_3=r_3^-$. Substituting this equality into $C^{2,3}_{(\text{EC.42a})}$, we have $C^{2,3'}_{(\text{EC.42a})}=\{(p_3,r_3^+,r_3^-)\in\mathbb{R}^3_+: p_3+r_3^+\le(\overline{C}-\underline{C})\lambda_3+(\overline{V}+V-\underline{C})\lambda_4+(\overline{V}-\underline{C})\lambda_5,\ p_3=r_3^-,\ r_3^++p_3\le 3V\lambda_3+(\overline{V}+V-\underline{C})\lambda_4+(\overline{V}-\underline{C})\lambda_5\}$.

A. If $\overline{C}-3V-\underline{C}>0$, then we can set $p_3+r_3^+=3V\lambda_3+(\overline{V}+V-\underline{C})\lambda_4+(\overline{V}-\underline{C})\lambda_5$. We can let $\hat{p}_{33}+\hat{r}_{33}^+=3V$, $\hat{r}_{33}^-=\hat{p}_{33}$, $\hat{r}_{34}^-=\hat{p}_{34}$, $\hat{p}_{34}+\hat{r}_{34}^+=\overline{V}+V-\underline{C}$, $\hat{r}_{35}^-=\hat{p}_{35}$, and $\hat{p}_{35}+\hat{r}_{35}^+=\overline{V}-\underline{C}$, which satisfy the constraints in A^2.

B. If $\overline{C}-3V-\underline{C}\le 0$, then we can set $p_3+r_3^+=(\overline{C}-\underline{C})\lambda_3+(\overline{V}+V-\underline{C})\lambda_4+(\overline{V}-\underline{C})\lambda_5$. We can let $\hat{p}_{33}+\hat{r}_{33}^+=\overline{C}-\underline{C}$, $\hat{r}_{33}^-=\hat{p}_{33}$, $\hat{r}_{34}^-=\hat{p}_{34}$, $\hat{p}_{34}+\hat{r}_{34}^+=\overline{V}+V-\underline{C}$, $\hat{r}_{35}^-=\hat{p}_{35}$, and $\hat{p}_{35}+\hat{r}_{35}^+=\overline{V}-\underline{C}$, which satisfy the constraints in A^2.

1.2.3) When $r_3^++r_3^-=3V\lambda_3+(\overline{V}+V-\underline{C})\lambda_4+(\overline{V}-\underline{C})\lambda_5$, which implies $\overline{C}-3V-\underline{C}>0$. Substituting this equality into $C^{2,3}_{(\text{EC.42a})}$, we have $C^{2,3'}_{(\text{EC.42a})}=\{(p_3,r_3^+,r_3^-)\in\mathbb{R}^3_+: p_3+r_3^+\le(\overline{C}-\underline{C})\lambda_3+(\overline{V}+V-\underline{C})\lambda_4+(\overline{V}-\underline{C})\lambda_5,\ p_3+r_3^+\ge 3V\lambda_3+(\overline{V}+V-\underline{C})\lambda_4+(\overline{V}-\underline{C})\lambda_5,\ r_3^++r_3^-=3V\lambda_3+(\overline{V}+V-\underline{C})\lambda_4+(\overline{V}-\underline{C})\lambda_5\}$.

A. If $p_3+r_3^+=3V\lambda_3+(\overline{V}+V-\underline{C})\lambda_4+(\overline{V}-\underline{C})\lambda_5$, then we can pick $\hat{p}_{33}+\hat{r}_{33}^+=3V$, $\hat{r}_{33}^-=\hat{p}_{33}$, $\hat{r}_{34}^-=\hat{p}_{34}$, $\hat{p}_{34}+\hat{r}_{34}^+=\overline{V}+V-\underline{C}$, $\hat{r}_{35}^-=\hat{p}_{35}$, and $\hat{p}_{35}+\hat{r}_{35}^+=\overline{V}-\underline{C}$, which satisfy the constraints in A^2.

B. If $p_3+r_3^+=(\overline{C}-\underline{C})\lambda_3+(\overline{V}+V-\underline{C})\lambda_4+(\overline{V}-\underline{C})\lambda_5$, then we can pick $\hat{p}_{33}+\hat{r}_{33}^+=\overline{C}-\underline{C}$, $\hat{r}_{33}^-=\hat{p}_{33}$, $\hat{r}_{34}^-=\hat{p}_{34}$, $\hat{p}_{34}+\hat{r}_{34}^+=\overline{V}+V-\underline{C}$, $\hat{r}_{35}^-=\hat{p}_{35}$, and $\hat{p}_{35}+\hat{r}_{35}^+=\overline{V}-\underline{C}$, which satisfy the constraints in A^2.

Now we have shown that any point $\bar{\omega}$ satisfying (EC.46a) at equality can be represented by a point $\hat{p}\in A$ such that $F\hat{p}=\bar{\omega}$. We can also similarly show that this statement holds when the other inequalities in C are satisfied at equality following the same idea. Due to the similarity, we omit the details for brevity.

In summary, we have proved that any point ω on the faces of C, there exists a point $p\in A$ such that $Fp=\omega$. This completes the proof. □

EC.3. Three-Period Convex Hulls under Other Conditions

In this section, we first consider the case where $L=\ell=1$ and $\overline{C}-\overline{V}-2V\ge 0$. The original constraint set, denoted by P_3^1, can be described as follows:

$$P_3^1:=\Big\{\ (p,r^+,r^-,y,u)\in\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{B}^3\times\mathbb{B}^2:$$

$$
\begin{aligned}
&(14c)-(14h),\\
&y_t \geq u_t,\ t=2,3, && \text{(EC.48a)}\\
&y_{t-1}+u_t \leq 1,\ t=2,3\Big\}. && \text{(EC.48b)}
\end{aligned}
$$

Similar to the previous case, we can derive the convex hull representation of P_3^1 as follows:

THEOREM EC.2. *When $L=\ell=1$ and $\overline{C}-\overline{V}-2V\geq 0$, the convex hull representation for the three-period set P_3^1 is*

$$
\begin{aligned}
Q_3^1 = conv(P_3^1) = \Big\{&(p,r^+,r^-,y,u)\in\mathbb{R}^{14}:\ (14c)-(14e),(\text{EC.26}),(\text{EC.40}),(\text{EC.48a})-(\text{EC.48b}),\\
&p_1 \leq (\overline{V}-\underline{C})y_1 + V(y_2-u_2)+(\overline{C}-\overline{V}-V)(y_3-u_3), && \text{(EC.49a)}\\
&p_1 \leq (\overline{V}-\underline{C})y_1 + (\overline{C}-\overline{V})(y_2-u_2), && \text{(EC.49b)}\\
&p_2 \leq (\overline{V}-\underline{C})y_2 + (\overline{C}-\overline{V})(y_3-u_3), && \text{(EC.49c)}\\
&r_2^- \leq (\overline{V}-\underline{C})y_2 + (\underline{C}+2V-\overline{V})(y_3-u_3), && \text{(EC.49d)}\\
&r_2^+ + r_2^- \leq 2Vy_2 - (\underline{C}+2V-\overline{V})u_2, && \text{(EC.49e)}\\
&r_3^+ + r_3^- \leq 2Vy_3 - (\underline{C}+2V-\overline{V})u_3, && \text{(EC.49f)}\\
&r_3^+ + r_3^- \leq (\underline{C}+V-\overline{V})(y_2-u_2)+(\overline{V}+V-\underline{C})y_3 - Vu_3, && \text{(EC.49g)}\\
&p_2 + r_2^+ \leq (\overline{C}-\underline{C})y_2 - (\overline{C}-\overline{V})u_2, && \text{(EC.49h)}\\
&p_2 + r_2^+ \leq (\overline{V}-\underline{C})y_2 + 2V(y_2-u_2)+(\overline{C}-\overline{V}-2V)(y_3-u_3), && \text{(EC.49i)}\\
&p_3 + r_3^+ \leq (\overline{C}-\underline{C})y_3 - (\overline{C}-\overline{V})u_3, && \text{(EC.49j)}\\
&p_3 + r_3^+ \leq (\overline{C}-\overline{V}-V)(y_2-u_2)+(\overline{V}+V-\underline{C})y_3 - Vu_3, && \text{(EC.49k)}\\
&p_2 - p_1 \leq (\overline{V}-\underline{C})y_2 + (\underline{C}+V-\overline{V})(y_3-u_3), && \text{(EC.49l)}\\
&p_1 - p_2 + r_2^- \leq (\overline{V}-\underline{C})y_1 + (\underline{C}+V-\overline{V})(y_2-u_2), && \text{(EC.49m)}\\
&p_2 - p_3 + r_3^- \leq (\overline{V}-\underline{C})y_2 + (\underline{C}+V-\overline{V})(y_3-u_3), && \text{(EC.49n)}\\
&p_2 - p_3 + r_3^- \leq Vy_2 + (\underline{C}+V-\overline{V})u_2, && \text{(EC.49o)}\\
&p_1 - p_3 + r_3^- \leq (\overline{V}-\underline{C})y_1 + V(y_2-u_2)+(\underline{C}+V-\overline{V})(y_3-u_3), && \text{(EC.49p)}\\
&p_1 - p_3 + r_3^- \leq (\overline{V}-\underline{C})y_1 + (\underline{C}+2V-\overline{V})(y_2-u_2), && \text{(EC.49q)}\\
&p_3 + r_3^+ - p_1 \leq 2Vy_3 - (\underline{C}+2V-\overline{V})u_3, && \text{(EC.49r)}\\
&p_3 + r_3^+ - p_1 \leq (\underline{C}+V-\overline{V})(y_2-u_2)+(\overline{V}+V-\underline{C})y_3 - Vu_3, && \text{(EC.49s)}\\
&p_2 + r_2^+ - p_1 \leq Vy_2 - (\underline{C}+V-\overline{V})u_2, && \text{(EC.49t)}\\
&p_3 + r_3^+ - p_2 \leq Vy_3 - (\underline{C}+V-\overline{V})u_3, && \text{(EC.49u)}\\
&p_3 + r_3^+ + r_2^+ \leq 2Vy_2 + (\overline{C}-\underline{C}-V)(y_3-u_3) - (\underline{C}+2V-\overline{V})u_2, && \text{(EC.49v)}
\end{aligned}
$$

$$p_2 + r_2^+ - p_3 + r_3^- \le 3Vy_2 - (\underline{C} + 3V - \overline{V})u_2, \quad \text{(EC.49w)}$$

$$p_2 + r_2^+ - p_3 + r_3^- \le (\overline{V} + 2V - \underline{C})y_2 - 2Vu_2 + (\underline{C} + V - \overline{V})(y_3 - u_3), \quad \text{(EC.49x)}$$

$$p_3 + r_3^+ - p_2 + r_2^- \le 3Vy_3 - (\underline{C} + 3V - \overline{V})u_3, \quad \text{(EC.49y)}$$

$$p_3 + r_3^+ - p_2 + r_2^- \le (\overline{V} + V - \underline{C})y_3 - Vu_3 + (\underline{C} + 2V - \overline{V})(y_2 - u_2)\Big\}. \quad \text{(EC.49z)}$$

Theorem EC.3. *For the case $L = 1$, $\ell = 2$ and $\overline{C} - \overline{V} - 2V \ge 0$, the convex hull representation of the original set is the same as Q_3^1 except that (EC.48b) is replaced by (14b). For the case $L = 2$, $\ell = 1$ and $\overline{C} - \overline{V} - 2V \ge 0$, the convex hull representation of the original set is the same as Q_3^2 except that (14b) is replaced by (EC.48b).*

In the following, we provide the convex hull representations for other cases with $L = \ell = 2$ or $L = \ell = 1$. Each of them contains only a subset of inequalities in Q_3^1 or Q_3^2.

Theorem EC.4. *If $\underline{C} \le \overline{V} < \underline{C} + V$, $\overline{C} - \overline{V} - 2V < 0$, and $\overline{C} - \underline{C} - 2V \ge 0$, then when $L = \ell = 2$, the corresponding convex hull representation for the three-period problem is*

$$\Big\{(p, r^+, r^-, y, u) \in \mathbb{R}^{14} : (14a) - (14e), (EC.21) - (EC.36), (EC.38) - (EC.40)\Big\}.$$

When $L = \ell = 1$, the convex hull representation for the three-period problem is

$$\Big\{(p, r^+, r^-, y, u) \in \mathbb{R}^{14} : (14c) - (14e), (EC.26), (EC.40), (EC.48a) - (EC.48b), (EC.49) - (EC.49g), (EC.49i) - (EC.49t)\Big\}.$$

Theorem EC.5. *If $\underline{C} \le \overline{V} < \underline{C} + V$, $\overline{C} - \underline{C} - 2V < 0$, and $\overline{C} - \underline{C} - V \ge 0$, then when $L = \ell = 2$, the corresponding convex hull representation for the three-period problem is*

$$\Big\{(p, r^+, r^-, y, u) \in \mathbb{R}^{14} : (14a) - (14e), (EC.21) - (EC.22), (EC.26) - (EC.31), (EC.34) - (EC.35), (EC.38), (EC.40)\Big\}.$$

When $L = \ell = 1$, the convex hull representation for the three-period problem is

$$\Big\{(p, r^+, r^-, y, u) \in \mathbb{R}^{14} : (14c) - (14e), (EC.26), (EC.40), (EC.48a) - (EC.48b), (EC.49) - (EC.49b), (EC.49g), (EC.49i) - (EC.49n), (EC.49s) - (EC.49t)\Big\}.$$

Theorem EC.6. *If $\underline{C} \le \overline{V} < \underline{C} + V$ and $\overline{C} - \underline{C} - V < 0$, then when $L = \ell = 2$, the corresponding convex hull representation for the three-period problem is*

$$\Big\{(p, r^+, r^-, y, u) \in \mathbb{R}^{14} : (14a) - (14e), (EC.21) - (EC.22), (EC.26) - (EC.28), (EC.40)\Big\}.$$

When $L = \ell = 1$, the convex hull representation for the three-period problem is

$$\Big\{(p, r^+, r^-, y, u) \in \mathbb{R}_+^{14} : (14c) - (14e), (EC.26), (EC.40), (EC.49a) - (EC.49b), (EC.49g), (EC.49i)\Big\}.$$

EC.4. Proofs of Multi-Period Formulations

EC.4.1. Proof of Proposition 6

Proof. (**Validity**) For the validity of (17), we discuss the following two possible cases.

1. If $y_{t-k}=0$, then $p_{t-k}=0$ due to constraints (2e). It follows that (17) is valid since $y_{t-s}-\sum_{i=s}^{\min\{k,s+L-1\}}u_{t-i}\geq 0$ for all $s\in[1,k-1]_{\mathbb{Z}}$ and $y_t-\sum_{i=0}^{\min\{k,L-1\}}u_{t-i}\geq 0$ due to minimum-up time constraints (2a).
2. If $y_{t-k}=1$, then we discuss the following two possible cases in terms of the value of u_{t-k}.
   (a) If $u_{t-k}=1$, then we have $p_{t-k}\leq \overline{V}-\underline{C}$ due to ramping-up constraints (2f). It follows that inequality (17) is valid since $y_{t-s}-\sum_{i=s}^{\min\{k,s+L-1\}}u_{t-i}\geq 0$ for all $s\in[1,k-1]_{\mathbb{Z}}$ and $y_t-\sum_{i=0}^{\min\{k,L-1\}}u_{t-i}\geq 0$ due to minimum-up time constraints (2a).
   (b) If $u_{t-k}=0$, then it means that the generator starts up in a period prior to period $t-k$. We further discuss the following two possible cases based on when the generator shuts down right after period $t-k$.
      i. If the generator shuts down in period $t-\overline{s}$ for some $\overline{s}\in[1,k-1]_{\mathbb{Z}}$, i.e., $y_{t-\overline{s}}=0$, then $u_{t-s}=0$ for all $s\in[\overline{s},\min\{k,k+L-2\}]_{\mathbb{Z}}$. It follows that (17) becomes $p_{t-k}\leq(\overline{V}-\underline{C})+(k-\overline{s}-1)V+V\sum_{s=1}^{\overline{s}-1}(y_{t-s}-\sum_{i=s}^{\min\{k,s+L-1\}}u_{t-i})+(\overline{C}-\overline{V}-(k-1)V)(y_t-\sum_{i=0}^{\min\{k,L-1\}}u_{t-i})$, which is valid because $p_{t-k}\leq\overline{V}-\underline{C}+(k-\overline{s}-1)V$ due to ramping-down constraints (2g), $y_{t-s}-\sum_{i=s}^{\min\{k,s+L-1\}}u_{t-i}\geq 0$ for all $s\in[1,\overline{s}-1]_{\mathbb{Z}}$, and $y_t-\sum_{i=0}^{\min\{k,L-1\}}u_{t-i}\geq 0$ due to minimum-up time constraints (2a).
      ii. If the generator shuts down in period $\overline{t}$ for some $\overline{t}\geq t$, then (17) becomes $p_{t-k}\leq\overline{C}-\underline{C}$, which is valid due to (2e).

(**Facet-defining**) We create $5T-1$ affinely independent points in conv(P) that satisfy inequality (17) at equality. Since $\vec{0}\in$ conv(P), we create the other $5T-2$ linearly independent points in conv(P) in the following groups. Here we only consider condition (1), i.e., $k=\lfloor\frac{\overline{C}-\overline{V}}{V}\rfloor+1$, as the proof under condition (2) is similar. It follows that $\overline{C}-\overline{V}-(k-1)V<V$.

First, we create T linearly independent points $(\overline{p}_s^{\alpha},\overline{r}_s^{+,\alpha},\overline{r}_s^{-,\alpha},\overline{y}_s^{\alpha},\overline{u}_s^{\alpha})_{\alpha=1}^T$ as follows.

1) For each $\alpha\in[1,t-k-1]_{\mathbb{Z}}$ (totally $t-k-1$ points), we let

$$\overline{p}_s^{\alpha}=\begin{cases}\overline{V}-\underline{C}, & s\in[1,\alpha]_{\mathbb{Z}}\\ 0, & s\in[\alpha+1,T]_{\mathbb{Z}}\end{cases},\quad \overline{y}_s^{\alpha}=\begin{cases}1, & s\in[1,\alpha]_{\mathbb{Z}}\\ 0, & s\in[\alpha+1,T]_{\mathbb{Z}}\end{cases},$$
$$\overline{r}_s^{+,\alpha}=\overline{r}_s^{-,\alpha}=0,\ \forall s\in[1,T]_{\mathbb{Z}},\text{ and }\overline{u}_s^{\alpha}=0,\ \forall s\in[1,T]_{\mathbb{Z}}.$$

2) For each $\alpha\in[t-k,t-1]_{\mathbb{Z}}$ (totally k points), we let

$$\overline{p}_s^{\alpha}=\begin{cases}\overline{V}+(\alpha-t+k)V-\underline{C}, & s\in[1,t-k-1]_{\mathbb{Z}}\\ \overline{V}+(\alpha-s)V-\underline{C}, & s\in[t-k,\alpha]_{\mathbb{Z}}\\ 0, & s\in[\alpha+1,T]_{\mathbb{Z}}\end{cases},\quad \overline{y}_s^{\alpha}=\begin{cases}1, & s\in[1,\alpha]_{\mathbb{Z}}\\ 0, & s\in[\alpha+1,T]_{\mathbb{Z}}\end{cases},$$
$$\overline{r}_s^{+,\alpha}=\overline{r}_s^{-,\alpha}=0,\ \forall s\in[1,T]_{\mathbb{Z}},\text{ and }\overline{u}_s^{\alpha}=0,\ \forall s\in[1,T]_{\mathbb{Z}}.$$

3) For each $\alpha\in[t,T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we let

$$\overline{p}_s^{\alpha} = \begin{cases} \overline{C} - \underline{C}, & s \in [1, t-k]_{\mathbb{Z}} \\ \overline{V} + (t-s)V - \underline{C}, & s \in [t-k+1, t-1]_{\mathbb{Z}} \\ \overline{V} - \underline{C}, & s \in [t, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases}, \quad \overline{y}_s^{\alpha} = \begin{cases} 1, & s \in [1, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases},$$
$$\overline{r}_s^{+,\alpha} = \overline{r}_s^{-,\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}, \text{ and } \overline{u}_s^{\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}.$$

Second, we create another T linearly independent points $(\grave{p}_s^{\alpha}, \grave{r}_s^{+,\alpha}, \grave{r}_s^{-,\alpha}, \grave{y}_s^{\alpha}, \grave{u}_s^{\alpha})_{\alpha=1}^{T}$ as follows.

4) For each $\alpha \in [1, t-k-1]_{\mathbb{Z}}$ (totally $t-k-1$ points), we let

$$\grave{p}_s^{\alpha} = \grave{r}_s^{-,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [1, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases}, \quad \grave{y}_s^{\alpha} = \begin{cases} 1, & s \in [1, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases},$$
$$\grave{r}_s^{+,\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}, \text{ and } \grave{u}_s^{\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}.$$

5) For each $\alpha \in [t-k, t-1]_{\mathbb{Z}}$ (totally k points), we let

$$\grave{p}_s^{\alpha} = \begin{cases} \overline{V} + (\alpha - t + k)V - \underline{C}, & s \in [1, t-k-1]_{\mathbb{Z}} \\ \overline{V} + (\alpha - s)V - \underline{C}, & s \in [t-k, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases}, \quad \grave{r}_s^{-,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [1, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases},$$
$$\grave{y}_s^{\alpha} = \begin{cases} 1, & s \in [1, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases}, \quad \grave{r}_s^{+,\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}, \text{ and } \grave{u}_s^{\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}.$$

6) For each $\alpha \in [t, T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we let

$$\grave{p}_s^{\alpha} = \begin{cases} \overline{C} - \underline{C}, & s \in [1, t-k]_{\mathbb{Z}} \\ \overline{V} + (t-s)V - \underline{C}, & s \in [t-k+1, t-1]_{\mathbb{Z}} \\ \overline{V} - \underline{C}, & s \in [t, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases}, \quad \grave{r}_s^{-,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [1, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases},$$
$$\grave{y}_s^{\alpha} = \begin{cases} 1, & s \in [1, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases}, \quad \grave{r}_s^{+,\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}, \text{ and } \grave{u}_s^{\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}.$$

Next, we create another T linearly independent points $(\acute{p}_s^{\alpha}, \acute{r}_s^{+,\alpha}, \acute{r}_s^{-,\alpha}, \acute{y}_s^{\alpha}, \acute{u}_s^{\alpha})_{\alpha=1}^{T}$ as follows.

7) For each $\alpha \in [1, t-k-1]_{\mathbb{Z}}$ (totally $t-k-1$ points), we let

$$\acute{p}_s^{\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [1, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases}, \quad \acute{r}_s^{+,\alpha} = \begin{cases} \min\{V, \overline{C} - \overline{V}\}, & s \in [1, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases},$$
$$\acute{y}_s^{\alpha} = \begin{cases} 1, & s \in [1, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases}, \quad \acute{r}_s^{-,\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}, \text{ and } \acute{u}_s^{\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}.$$

8) For each $\alpha \in [t-k, t-1]_{\mathbb{Z}}$ (totally k points), we let

$$\acute{p}_s^{\alpha} = \begin{cases} \overline{V} + (\alpha - t + k)V - \underline{C}, & s \in [1, t-k-1]_{\mathbb{Z}} \\ \overline{V} + (\alpha - s)V - \underline{C}, & s \in [t-k, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases},$$
$$\acute{r}_s^{+,\alpha} = \begin{cases} \min\{V, \overline{C} - \overline{V} - (\alpha - s)V\}, & s \in [t-k, \alpha]_{\mathbb{Z}} \\ 0, & o.w. \end{cases}, \quad \acute{y}_s^{\alpha} = \begin{cases} 1, & s \in [1, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases},$$
$$\acute{r}_s^{-,\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}, \text{ and } \acute{u}_s^{\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}.$$

9) For each $\alpha \in [t, T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we let

$$\acute{p}_s^{\alpha} = \begin{cases} \overline{C} - \underline{C}, & s \in [1, t-k]_{\mathbb{Z}} \\ \overline{V} + (t-s)V - \underline{C}, & s \in [t-k+1, t-1]_{\mathbb{Z}} \\ \overline{V} - \underline{C}, & s \in [t, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases},$$

$$\acute{r}_s^{+,\alpha} = \begin{cases} 0, & s \in [1, t-k]_{\mathbb{Z}} \\ \min\{V, \overline{C} - \overline{V} - (t-s)V\}, & s \in [t-k+1, t-1]_{\mathbb{Z}} \\ \min\{V, \overline{C} - \overline{V}\}, & s \in [t, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^{\alpha} = \begin{cases} 1, & s \in [1, \alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1, T]_{\mathbb{Z}} \end{cases},$$

$$\acute{r}_s^{-,\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}, \text{ and } \acute{u}_s^{\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}}.$$

In addition, we create $T-1$ linearly independent points $(\dot{p}_s^{\alpha}, \dot{r}_s^{+,\alpha}, \dot{r}_s^{-,\alpha}, \dot{y}_s^{\alpha}, \dot{u}_s^{\alpha})_{\alpha=2}^{T}$ as follows.

10) For each $\alpha \in [2, t-k-1]_{\mathbb{Z}}$ (totally $t-k-2$ points), we let

$$\dot{p}_s^{\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [\alpha, \alpha+L-1]_{\mathbb{Z}} \setminus \{t-k\} \\ 0, & o.w. \end{cases}, \quad \dot{r}_s^{+,\alpha} = \dot{r}_s^{-,\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}},$$

$$\dot{y}_s^{\alpha} = \begin{cases} 1, & s \in [\alpha, \alpha+L-1]_{\mathbb{Z}} \\ 0, & s \in o.w. \end{cases}, \text{ and } \dot{u}_s^{\alpha} = \begin{cases} 1, & s = \alpha \\ 0, & o.w. \end{cases}.$$

Here, if $\dot{y}_{t-k}^{\alpha} = \dot{y}_{t-k+1}^{\alpha} = 1$, then we let $\dot{p}_{t-k}^{\alpha} = \overline{V} + V - \underline{C}$. If $\dot{y}_{t-k}^{\alpha} = 1$ and $\dot{y}_{t-k+1}^{\alpha} = 0$, then $\dot{p}_{t-k}^{\alpha} = \overline{V} - \underline{C}$. If $\dot{y}_{t-k}^{\alpha} = 0$, then $\dot{p}_{t-k}^{\alpha} = 0$.

11) For each $\alpha \in [t-k, T]_{\mathbb{Z}}$ (totally $T-t+k+1$ points), we let

$$\dot{p}_s^{\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [\alpha, \min\{\alpha+L-1, T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases}, \quad \dot{r}_s^{+,\alpha} = \dot{r}_s^{-,\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}},$$

$$\dot{y}_s^{\alpha} = \begin{cases} 1, & s \in [\alpha, \min\{\alpha+L-1, T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases}, \text{ and } \dot{u}_s^{\alpha} = \begin{cases} 1, & s = \alpha \\ 0, & o.w. \end{cases}.$$

Finally, we create another $T-1$ linearly independent points $(\hat{p}_s^{\alpha}, \hat{r}_s^{+,\alpha}, \hat{r}_s^{-,\alpha}, \hat{y}_s^{\alpha}, \hat{u}_s^{\alpha})_{\alpha=2}^{T}$ as follows.

12) For each $\alpha \in [2, t-k-1]_{\mathbb{Z}}$ (totally $t-k-2$ points), we let

$$\hat{p}_s^{\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [\alpha, \alpha+L-1]_{\mathbb{Z}} \setminus \{t-k\} \\ 0, & o.w. \end{cases}, \quad \hat{r}_s^{-,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [\alpha, \alpha+L-1]_{\mathbb{Z}} \\ 0, & o.w. \end{cases},$$

$$\hat{r}_s^{+,\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}},$$

$$\hat{y}_s^{\alpha} = \begin{cases} 1, & s \in [\alpha, \alpha+L-1]_{\mathbb{Z}} \\ 0, & s \in o.w. \end{cases}, \text{ and } \hat{u}_s^{\alpha} = \begin{cases} 1, & s = \alpha \\ 0, & o.w. \end{cases}.$$

Here, if $\hat{y}_{t-k}^{\alpha} = \hat{y}_{t-k+1}^{\alpha} = 1$, then we let $\hat{p}_{t-k}^{\alpha} = \overline{V} + V - \underline{C}$. If $\hat{y}_{t-k}^{\alpha} = 1$ and $\hat{y}_{t-k+1}^{\alpha} = 0$, then $\hat{p}_{t-k}^{\alpha} = \overline{V} - \underline{C}$. If $\hat{y}_{t-k}^{\alpha} = 0$, then $\hat{p}_{t-k}^{\alpha} = 0$.

13) For each $\alpha \in [t-k, T]_{\mathbb{Z}}$ (totally $T-t+k+1$ points), we let

$$\hat{p}_s^{\alpha} = \hat{r}_s^{-,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [\alpha, \min\{\alpha+L-1, T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases}, \quad \hat{r}_s^{+,\alpha} = 0, \ \forall s \in [1, T]_{\mathbb{Z}},$$

$$\hat{y}_s^{\alpha} = \begin{cases} 1, & s \in [\alpha, \min\{\alpha+L-1, T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases}, \text{ and } \hat{u}_s^{\alpha} = \begin{cases} 1, & s = \alpha \\ 0, & o.w. \end{cases}.$$

In summary, we have created $5T-2$ points. The points $(\overline{p}_s^{\alpha}, \overline{r}_s^{+,\alpha}, \overline{r}_s^{-,\alpha}, \overline{y}_s^{\alpha}, \overline{u}_s^{\alpha})_{\alpha=1}^{T}$ are linearly independent because they form a lower triangular matrix by the construction. Similarly, $(\grave{p}_s^{\alpha}, \grave{r}_s^{+,\alpha}, \grave{r}_s^{-,\alpha}, \grave{y}_s^{\alpha}, \grave{u}_s^{\alpha})_{\alpha=1}^{T}$, $(\acute{p}_s^{\alpha}, \acute{r}_s^{+,\alpha}, \acute{r}_s^{-,\alpha}, \acute{y}_s^{\alpha}, \acute{u}_s^{\alpha})_{\alpha=1}^{T}$, $(\dot{p}_s^{\alpha}, \dot{r}_s^{+,\alpha}, \dot{r}_s^{-,\alpha}, \dot{y}_s^{\alpha}, \dot{u}_s^{\alpha})_{\alpha=2}^{T}$, and $(\hat{p}_s^{\alpha}, \hat{r}_s^{+,\alpha}, \hat{r}_s^{-,\alpha}, \hat{y}_s^{\alpha}, \hat{u}_s^{\alpha})_{\alpha=2}^{T}$ are linearly independent. By performing Gaussian elimination among $(\overline{p}_s^\alpha, \overline{r}_s^{+,\alpha}, \overline{r}_s^{-,\alpha}, \overline{y}_s^\alpha, \overline{u}_s^\alpha)_{\alpha=1}^T$, $(\grave{p}_s^\alpha, \grave{r}_s^{+,\alpha}, \grave{r}_s^{-,\alpha}, \grave{y}_s^\alpha, \grave{u}_s^\alpha)_{\alpha=1}^T$, $(\acute{p}_s^\alpha, \acute{r}_s^{+,\alpha}, \acute{r}_s^{-,\alpha}, \acute{y}_s^\alpha, \acute{u}_s^\alpha)_{\alpha=1}^T$, $(\dot{p}_s^\alpha, \dot{r}_s^{+,\alpha}, \dot{r}_s^{-,\alpha}, \dot{y}_s^\alpha, \dot{u}_s^\alpha)_{\alpha=2}^T$ and $(\hat{p}_s^\alpha, \hat{r}_s^{+,\alpha}, \hat{r}_s^{-,\alpha}, \hat{y}_s^\alpha, \hat{u}_s^\alpha)_{\alpha=2}^T$, we conclude that all of them are linearly independent. □

EC.4.2. Proof of Proposition 7

Proof. (**Validity**) It is clear that both (18) and (19) are valid when $y_t = 0$ because in this case, we have $r_t^- = 0$ due to (2d) and (2e) and every term on the RHS of (18) and (19) is nonnegative. We only need to consider $y_t = 1$.

First, for (18), we further discuss the following two possible cases.

1. If $y_{t+k} = 0$, then inequality (18) becomes $r_t^- \le (\overline{V} - \underline{C})$, which is valid due to constraints (2g).
2. If $y_{t+k} = 1$, then we discuss the following three possible cases in terms of the value of u_{t+k-s} for $s \in [0, \min\{L-1, t+k-2\}]_{\mathbb{Z}}$.
   (a) If $u_{t+k-s} = 0$ for all $s \in [0, \min\{L-1, t+k-2\}]_{\mathbb{Z}}$, then inequality (18) converts to $r_t^- \le (k+1)V = 2V$, which is valid because $r_t^- \le r_t^+ + r_t^- = (p_t + r_t^+ - p_{t-1}) + (p_{t-1} - p_t + r_t^-) \le V + V = 2V$.
   (b) If $u_{t+k-\overline{s}} = 1$ for some $\overline{s} \in [0, \min\{L-1, k, t+k-2\}]_{\mathbb{Z}} = [0, \min\{L-1, k\}]_{\mathbb{Z}}$, then we have $u_t = 1$. Hence, inequality (18) becomes $r_t^- \le (\overline{V} - \underline{C})$, which is valid due to (2f).
   (c) If $u_{t+k-\overline{s}} = 1$ for some $\overline{s} \in [\min\{L-1, k\} + 1, \min\{L-1, t+k-2\}]_{\mathbb{Z}} = [\min\{L, k+1\}, \min\{L-1, t+k-2\}]_{\mathbb{Z}}$, then inequality (18) converts to $r_t^- \le \overline{V} + (t-\overline{s})V - \underline{C}$, which is valid because $r_t^- \le p_t \le p_t + r_t^+ \le \overline{V} + (t-\overline{s})V - \underline{C}$ due to ramping-up constraints (2f).

Next, for (19), we prove the case $L = 1$ while the case $L \ge 2$ can be proved similarly. Then we have $q = \max\{a \in S\}$. We further discuss the following two possible cases in terms of the values of t_1 and t_2, where $t_1 \le t$ is the start-up time of the generator right before t and $t_2 > \max\{t, t_1 + L - 1\}$ is the time when the generator shuts down right after t.

1. If $\max\{2, t+q-L+1\} \le t_1 \le t$, i.e., $u_{t_1} = 1$ for some $t_1 \in [\max\{2, t+q-L+1\}, t]_{\mathbb{Z}}$. In this case, we have

$$
\begin{aligned}
r_t^- &\le p_t \le p_t + r_t^+ \\
&\le \overline{V} + (t-t_1)V - \underline{C} \\
&\le \overline{V} - \underline{C} + \sum_{i \in S \cup \{1\}} (d_i - i)V \left(y_{t+i} - \sum_{s=\max\{2, t+i-L+1\}}^{t+i} u_s \right) \\
&\quad + \left(\underline{C} + V - \overline{V}\right) \left(y_{t+k} - \sum_{s=\max\{2, s'\}}^{t+k} u_s \right) + (t-t_1)V \\
&= \text{RHS of (19)}.
\end{aligned}
$$

The first inequality holds due to (2d) and the third inequality holds due to ramping-up constraints (2f). It follows that (19) is valid.

2. If $t_1 < \max\{2, t+q-L+1\}$, then we further discuss the following two possible cases.
   (a) If $\max\{2, t+q-L+1\} = 2$, then we have $t_1 < 2$, i.e., $u_s = 0$ for all $s \in [2,t]_{\mathbb{Z}}$, then we discuss the following two possible cases.
      i. If $t_2 > t+k$, i.e., $y_{t+i} = 1$ for all $i \in [1,k]_{\mathbb{Z}}$, then inequality (19) converts to $r_t^- \le V + \sum_{i \in S \cup \{1\}} (d_i - i)V = kV$, which is valid because $r_t^- \le 2V$ due to constraints (2g) and (2f).
      ii. If $t < t_2 \le t+k$, then $y_s = 1$ for all $s \in [t+1, t_2-1]_{\mathbb{Z}}$ and $y_s = 0$ for all $s \in [t_2, t+k]_{\mathbb{Z}}$. We let $p = \min\{a \in S \cup \{k\} : a \ge t_2 - t\}$. In this case, we have

$$
\begin{aligned}
r_t^- &\le p_t \le \overline{V} - \underline{C} + (t_2 - t - 1)V \\
&\le \overline{V} - \underline{C} + (p-1)V \\
&= \overline{V} - \underline{C} + \sum_{i \in \{S \cup \{1\}\} \cap [1, t_2-t-1]_{\mathbb{Z}}} (d_i - i)V \\
&\le \overline{V} - \underline{C} + \sum_{i \in \{S \cup \{1\}\} \cap [1, t_2-t-1]_{\mathbb{Z}}} (d_i - i)V \\
&\quad + \sum_{i \in S \cup \{1\} \cap [t_2-t, k-1]_{\mathbb{Z}}} (d_i - i)V(y_{t+i} - \sum_{s=\max\{2, t+i-L+1\}}^{t+i} u_s) \\
&\quad + (\underline{C} + V - \overline{V})(y_{t+k} - \sum_{s=\max\{2, t+k-L+1\}}^{t+k} u_s) \\
&= \text{RHS of (19)}.
\end{aligned}
$$

The second inequality is due to constraints (2g), the third inequality is due to the definition of p, and the third equation is due to the definition of d_i for any $i \in S \cup \{1\}$. It follows that (19) is valid.

   (b) If $\max\{2, t+q-L+1\} = t+q-L+1$, then we have $t+q-L+1 > t_1$, i.e., $t_1 + L - 1 < t+q$. We discuss the following two possible cases in terms of t_2.
      i. If $t_2 > t+k$, i.e., $y_{t+i} = 1$ for all $i \in [1,k]_{\mathbb{Z}}$, then inequality (19) converts to

$$
r_t^- \le V + \sum_{i \in \{S \cup \{1\}\} \cap [t_1+L-t, k-1]_{\mathbb{Z}}} (d_i - i)V. \tag{EC.50}
$$

Note that $\sum_{i \in \{S \cup \{1\}\} \cap [t_1+L-t, k-1]_{\mathbb{Z}}} (d_i - i)V \ge (k-q)V \ge V$. It follows that (EC.50) is valid because in this case we have $r_t^- \le 2V \le V + \sum_{i \in \{S \cup \{1\}\} \cap [t_1+L-t, k-1]_{\mathbb{Z}}} (d_i - i)V$.
      ii. If $\max\{t, t_1+L-1\} < t_2 \le t+k$, then $y_s = 1$ for all $s \in [t+1, t_2-1]_{\mathbb{Z}}$ and $y_s = 0$ for all $s \in [t_2, t+k]_{\mathbb{Z}}$. Note that if $t_2 = t+1$, then we have $r_t^- \le \overline{V} - \underline{C} \le$ RHS of (19), indicating the validity of (19). If $t_2 \ge t+2$, then $t_1 + L - 1 = t_1 \le t < t+1 < t_2$, where the first equation is due to the condition $L = 1$.

A. If $t_2 > t+q$, then inequality (19) becomes $r_t^- \le \overline{V} + (k-1)V - \underline{C} + \sum_{i\in\{S\cup\{1\}\}\cap[t_2-t,k-1]_{\mathbb{Z}}}(d_i-i)V(y_{t+i} - \sum_{s=\max\{2,t+i-L+1\}}^{t+i} u_s) + (\underline{C}+V-\overline{V})(y_{t+k} - \sum_{s=\max\{2,t+k-L+1\}}^{t+k} u_s)$, which is valid because in this case $r_t^- \le \overline{V} + (t_2-t-1)V - \underline{C} \le \overline{V} + (k-1)V - \underline{C}$ due to ramping-down constraints (2g).

B. If $t_2 \le t+q$, then in this case, we have

$$\begin{aligned}
r_t^- &\le \overline{V} + (t_2-t-1)V - \underline{C} \\
&\le \overline{V} + (q-1)V - \underline{C} \\
&= \overline{V} + \sum_{i\in\{S\cup\{1\}\}\cap[1,t_2-t-1]_{\mathbb{Z}}} (d_i-i)V - \underline{C} \\
&\le \overline{V} + \sum_{i\in\{S\cup\{1\}\}\cap[1,t_2-t-1]_{\mathbb{Z}}} (d_i-i)V - \underline{C} \\
&+ \sum_{i\in\{S\cup\{1\}\}\cap[t_2-t,k-1]_{\mathbb{Z}}} (d_i-i)V(y_{t+i} - \sum_{s=\max\{2,t+i-L+1\}}^{t+i} u_s) \\
&+ (\underline{C}+V-\overline{V})(y_{t+k} - \sum_{s=\max\{2,t+k-L+1\}}^{t+k} u_s) \\
&= \text{RHS of (19)}.
\end{aligned}$$

It follows that (19) is valid.

(Facet-defining) Under the condition $k=1$ and $t+k=T$, inequality (18) can be rewritten as

$$r_{T-1}^- \le \left(\overline{V} - \underline{C}\right) y_{T-1} + \left(\underline{C} + 2V - \overline{V}\right) \left(y_T - \sum_{s=\max\{2,T-L+1\}}^{T} u_s \right) + \sum_{s=\max\{2,T-L+1\}}^{T-2} (T-s-1)Vu_s. \tag{EC.51}$$

We create $5T-1$ affinely independent points in conv(P) that satisfy inequality (EC.51) at equality. Since $\vec{0} \in$ conv(P), we create the other $5T-2$ linearly independent points in conv(P) in the following groups.

First, we create T linearly independent points $(\bar{p}_s^\alpha, \bar{r}_s^{+,\alpha}, \bar{r}_s^{-,\alpha}, \bar{y}_s^\alpha, \bar{u}_s^\alpha)_{\alpha=1}^T$ as follows.

1) For each $\alpha \in [1, T-2]_{\mathbb{Z}}$ (totally $T-2$ points), we create $(p^\alpha, r^{+,\alpha}, r^{-,\alpha}) \in$ conv(P) such that $y_s^\alpha = 1$ for all $s \in [1,\alpha]_{\mathbb{Z}}$ and $y_s^\alpha = 0$ otherwise; $u_s^\alpha = 0$ for all $s \in [2,T]_{\mathbb{Z}}$; $p_s^\alpha = r_s^{+,\alpha} = r_s^{-,\alpha} = 0$ for all $s \in [1,T]_{\mathbb{Z}}$.
2) For $\alpha = T-1$ (totally one point), we create $(p^\alpha, r^{+,\alpha}, r^{-,\alpha}) \in$ conv(P) such that $y_s^\alpha = 1$ for all $s \in [1,\alpha]_{\mathbb{Z}}$ and $y_s^\alpha = 0$ otherwise; $u_s^\alpha = 0$ for all $s \in [2,T]_{\mathbb{Z}}$; $p_s^\alpha = r_s^{-,\alpha} = \overline{V} - \underline{C}$ for all $s \in [1,\alpha]_{\mathbb{Z}}$ and $p_s^\alpha = r_s^{-,\alpha} = 0$ otherwise; $r_s^{+,\alpha} = 0$ for all $s \in [1,T]_{\mathbb{Z}}$.
3) For $\alpha = T$ (totally one point), we create $(p^\alpha, r^{+,\alpha}, r^{-,\alpha}) \in$ conv(P) such that $y_s^\alpha = 1$ for all $s \in [1,\alpha]_{\mathbb{Z}}$ and $y_s^\alpha = 0$ otherwise; $u_s^\alpha = 0$ for all $s \in [2,T]_{\mathbb{Z}}$; $p_s^\alpha = r_s^{-,\alpha} = V$ for all $s \in [1,T-2]_{\mathbb{Z}} \cup \{T\}$ and $p_s^\alpha = r_s^{-,\alpha} = 2V$ otherwise; $r_s^{+,\alpha} = 0$ for all $s \in [1,T]_{\mathbb{Z}}$.

Second, we create another $T-1$ linearly independent points $(\tilde{p}^\alpha, \tilde{r}^{+,\alpha}, \tilde{r}^{-,\alpha}) \in \text{conv}(P)$ as follows.

4) For each $\alpha \in [1, T-2]_{\mathbb{Z}}$ (totally $T-2$ points), we create $(\tilde{p}^\alpha, \tilde{r}^{+,\alpha}, \tilde{r}^{-,\alpha}) \in \text{conv}(P)$ such that $\tilde{y}_s^\alpha = 1$ for each $s \in [1, \alpha]_{\mathbb{Z}}$ and $\tilde{y}_s^\alpha = 0$ otherwise; $\tilde{u}_s^\alpha = 0$ for all $s \in [2, T]_{\mathbb{Z}}$; $\tilde{p}_s^\alpha = \overline{V} - \underline{C}$ for all $s \in [1, \alpha]_{\mathbb{Z}}$ and $\tilde{p}_s^\alpha = 0$ otherwise; $\tilde{r}_s^{+,\alpha} = \tilde{r}_s^{-,\alpha} = 0$ for all $s \in [1, T]_{\mathbb{Z}}$.
5) For $\alpha = T-1$ (totally one point), we let $(\tilde{p}^\alpha, \tilde{r}^{+,\alpha}, \tilde{r}^{-,\alpha}) \in \text{conv}(P)$ such that $\tilde{y}_s^\alpha = 1$ for $s = T$ and $\tilde{y}_s^\alpha = 0$ otherwise; $\tilde{u}_s^\alpha = 1$ for $s = T$ and $\tilde{u}_s^\alpha = 0$ otherwise; $\tilde{p}_s^\alpha = \overline{V} - \underline{C}$ for $s = T$ and $\tilde{p}_s^\alpha = 0$ otherwise; $\tilde{r}_s^{+,\alpha} = \tilde{r}_s^{-,\alpha} = 0$ for all $s \in [1, T]_{\mathbb{Z}}$.

Next, we create another T linearly independent points $(\dot{p}^\alpha, \dot{r}^{+,\alpha}, \dot{r}^{-,\alpha}) \in \text{conv}(P)$ as follows.

6) For each $\alpha \in [1, T-2]_{\mathbb{Z}}$ (totally $T-2$ points), we create $(\dot{p}^\alpha, \dot{r}^{+,\alpha}, \dot{r}^{-,\alpha}) \in \text{conv}(P)$ such that $\dot{y}_s^\alpha = 1$ for each $s \in [1, \alpha]_{\mathbb{Z}}$ and $\dot{y}_s^\alpha = 0$ otherwise; $\dot{u}_s^\alpha = 0$ for all $s \in [2, T]_{\mathbb{Z}}$; $\dot{r}_s^{+,\alpha} = V$ for all $s \in [1, \alpha]_{\mathbb{Z}}$ and $\dot{r}_s^{+,\alpha} = 0$ otherwise; $\dot{p}_s^\alpha = \dot{r}_s^{-,\alpha} = 0$ for all $s \in [1, T]_{\mathbb{Z}}$.
7) For each $\alpha = T-1$ (totally one point), we create $(\dot{p}^\alpha, \dot{r}^{+,\alpha}, \dot{r}^{-,\alpha}) \in \text{conv}(P)$ such that $\dot{y}_s^\alpha = 1$ for each $s \in [1, \alpha]_{\mathbb{Z}}$ and $\dot{y}_s^\alpha = 0$ otherwise; $\dot{u}_s^\alpha = 0$ for all $s \in [2, T]_{\mathbb{Z}}$; $\dot{r}_s^{+,\alpha} = V$ for all $s \in [1, \alpha-1]_{\mathbb{Z}}$, $\dot{r}_s^{+,\alpha} = \underline{C} + V - \overline{V}$ for $s = \alpha$ and $\dot{r}_s^{+,\alpha} = 0$ otherwise; $\dot{p}_s^\alpha = \dot{r}_s^{-,\alpha} = \overline{V} - \underline{C}$ for $s = \alpha$ and $\dot{p}_s^\alpha = \dot{r}_s^{-,\alpha} = 0$ otherwise.
8) For each $\alpha = T$ (totally one point), we create $(\dot{p}^\alpha, \dot{r}^{+,\alpha}, \dot{r}^{-,\alpha}) \in \text{conv}(P)$ such that $\dot{y}_s^\alpha = 1$ for each $s \in [1, \alpha]_{\mathbb{Z}}$ and $\dot{y}_s^\alpha = 0$ otherwise; $\dot{u}_s^\alpha = 0$ for all $s \in [2, T]_{\mathbb{Z}}$; $\dot{p}_s^\alpha = \overline{C} - \underline{C}$ for $s = T-1$ and $\dot{p}_s^\alpha = \overline{C} - \underline{C} - V$ otherwise; $\dot{r}_s^{+,\alpha} = V$ for $s = T$ and $\dot{r}_s^{+,\alpha} = 0$ otherwise; $\dot{r}_s^{-,\alpha} = 2V$ for $s = T-1$ and $\dot{r}_s^{-,\alpha} = 0$ otherwise.

Next, we generate another $T-1$ linearly independent points $(\hat{p}^\alpha, \hat{r}^{+,\alpha}, \hat{r}^{-,\alpha}) \in \text{conv}(P)$ as follows.

9) For each $\alpha \in [1, T-2]_{\mathbb{Z}}$ (totally $T-2$ points), we create $(\hat{p}^\alpha, \hat{r}^{+,\alpha}, \hat{r}^{-,\alpha}) \in \text{conv}(P)$ such that $\hat{y}_s^\alpha = 1$ for each $s \in [1, \alpha]_{\mathbb{Z}}$ and $\hat{y}_s^\alpha = 0$ otherwise; $\hat{u}_s^\alpha = 0$ for all $s \in [2, T]_{\mathbb{Z}}$; $\hat{p}_s^\alpha = \hat{r}_s^{-,\alpha} = \overline{V} - \underline{C}$ for all $s \in [1, \alpha]_{\mathbb{Z}}$ and $\hat{p}_s^\alpha = \hat{r}_s^{-,\alpha} = 0$ otherwise; $\hat{r}_s^{+,\alpha} = 0$ for all $s \in [1, T]_{\mathbb{Z}}$.
10) For $\alpha = T-1$ (totally one point), we create $(\hat{p}^\alpha, \hat{r}^{+,\alpha}, \hat{r}^{-,\alpha}) \in \text{conv}(P)$ such that $\hat{y}_s^\alpha = 1$ for $s = T$ and $\hat{y}_s^\alpha = 0$ otherwise; $\hat{u}_s^\alpha = 1$ for $s = T$ and $\hat{u}_s^\alpha = 0$ otherwise; $\hat{p}_s^\alpha = \hat{r}_s^{-,\alpha} = \overline{V} - \underline{C}$ for $s = T$ and $\hat{p}_s^\alpha = \hat{r}_s^{-,\alpha} = 0$ otherwise; $\hat{r}_s^{+,\alpha} = 0$ for all $s \in [1, T]_{\mathbb{Z}}$.

Next, we create $T-1$ linearly independent points $(\check{p}^\alpha, \check{r}^{+,\alpha}, \check{r}^{-,\alpha})_{\alpha=2}^T \in \text{conv}(P)$ as follows.

11) For each $\alpha \in [2, \max\{2, T-L+1\} - 2]_{\mathbb{Z}}$ (totally $\max\{0, T-L-2\} \equiv [T-L-2]^+$ points), we let $(\check{p}^\alpha, \check{r}^{+,\alpha}, \check{r}^{-,\alpha}) \in \text{conv}(P)$ such that $\check{y}_s^\alpha = 1$ for all $s \in [\alpha, \alpha+L-1]_{\mathbb{Z}}$ and $\check{y}_s^\alpha = 0$ otherwise; $\check{u}_s^\alpha = 1$ for $s = \alpha$ and $\check{u}_s^\alpha = 0$ otherwise; $\check{p}_s^\alpha = \check{r}_s^{+,\alpha} = \check{r}_s^{-,\alpha} = 0$ for all $s \in [1, T]_{\mathbb{Z}}$.
12) If $T-L-1 > 0$, then for $\alpha = \max\{2, T-L+1\} - 1$ (totally 1 point if $T-L-1 > 0$), we let $(\check{p}^\alpha, \check{r}^{+,\alpha}, \check{r}^{-,\alpha}) \in \text{conv}(P)$ such that $\check{y}_s^\alpha = 1$ for all $s \in [\alpha, \alpha+L-1]_{\mathbb{Z}}$ and $\check{y}_s^\alpha = 0$ otherwise; $\check{u}_s^\alpha = 1$ for $s = \alpha$ and $\check{u}_s^\alpha = 0$ otherwise; $\check{p}_s^\alpha = \check{r}_s^{-,\alpha} = \overline{V} - \underline{C}$ for all $s \in [\alpha, \alpha+L-1]_{\mathbb{Z}}$, and $\check{p}_s^\alpha = \check{r}_s^{-,\alpha} = 0$ otherwise; $\check{r}_s^{+,\alpha} = 0$ for all $s \in [1, T]_{\mathbb{Z}}$.

Note that if $T-L-1 > 0$, then we have created $T-L-1$ points. Otherwise, we did not create any point.

13) For each $\alpha \in [\max\{2, T-L+1\}, T-1]_{\mathbb{Z}}$ (totally $\min\{T-2, L-1\}$ points), we let $(\check{p}^{\alpha}, \check{r}^{+,\alpha}, \check{r}^{-,\alpha}) \in \operatorname{conv}(P)$ such that $\check{y}_s^{\alpha} = 1$ for all $s \in [\alpha, T]_{\mathbb{Z}}$ and $\check{y}_s^{\alpha} = 0$ otherwise; $\check{u}_s^{\alpha} = 1$ for $s = \alpha$ and $\check{u}_s^{\alpha} = 0$ otherwise; $\check{p}_s^{\alpha} = \check{r}_s^{-,\alpha} = \overline{V} - \underline{C} + (s-\alpha)V$ for all $s \in [\alpha, T-2]_{\mathbb{Z}}$, $\check{p}_s^{\alpha} = \check{r}_s^{-,\alpha} = \overline{V} - \underline{C} + (T-1-\alpha)V$ for all $s \in [T-1, T]_{\mathbb{Z}}$ and $\check{p}_s^{\alpha} = \check{r}_s^{-,\alpha} = 0$ otherwise; $\check{r}_s^{+,\alpha} = 0$ for all $s \in [1, T]_{\mathbb{Z}}$.
14) For each $\alpha = T$ (totally one point), we let $(\check{p}^{\alpha}, \check{r}^{+,\alpha}, \check{r}^{-,\alpha}) \in \operatorname{conv}(P)$ such that $\check{y}_s^{\alpha} = 1$ for all $s \in [\alpha, T]_{\mathbb{Z}}$ and $\check{y}_s^{\alpha} = 0$ otherwise; $\check{u}_s^{\alpha} = 1$ for $s = \alpha$ and $\check{u}_s^{\alpha} = 0$ otherwise; $\check{p}_s^{\alpha} = \check{r}_s^{+,\alpha} = \check{r}_s^{-,\alpha} = 0$ for all $s \in [1, T]_{\mathbb{Z}}$.

Finally, we create another point as follows.

15) For $\alpha = T$ (totally one point), we create $(\hat{p}^{\alpha}, \hat{r}^{+,\alpha}, \hat{r}^{-,\alpha}) \in \operatorname{conv}(P)$ such that $\hat{y}_s^{\alpha} = 1$ for $s = T$ and $\hat{y}_s^{\alpha} = 0$ otherwise; $\hat{u}_s^{\alpha} = 1$ for $s = T$ and $\hat{u}_s^{\alpha} = 0$ otherwise; $\hat{r}_s^{+,\alpha} = \overline{V} - \underline{C}$ for $s = T$ and $\hat{r}_s^{+,\alpha} = 0$ otherwise; $\hat{p}_s^{\alpha} = \hat{r}_s^{-,\alpha} = 0$ for all $s \in [1, T]_{\mathbb{Z}}$.

In summary, we have created $5T-2$ points. The points of $(\overline{p}_s^{\alpha}, \overline{r}_s^{+,\alpha}, \overline{r}_s^{-,\alpha}, \overline{y}_s^{\alpha}, \overline{u}_s^{\alpha})_{\alpha=1}^T$ are linearly independent because they form a lower triangular matrix by the construction. Similarly, $(\grave{p}_s^{\alpha}, \grave{r}_s^{+,\alpha}, \grave{r}_s^{-,\alpha}, \grave{y}_s^{\alpha}, \grave{u}_s^{\alpha})_{\alpha=1}^T$, $(\acute{p}_s^{\alpha}, \acute{r}_s^{+,\alpha}, \acute{r}_s^{-,\alpha}, \acute{y}_s^{\alpha}, \acute{u}_s^{\alpha})_{\alpha=1}^T$, $(\dot{p}_s^{\alpha}, \dot{r}_s^{+,\alpha}, \dot{r}_s^{-,\alpha}, \dot{y}_s^{\alpha}, \dot{u}_s^{\alpha})_{\alpha=2}^T$ and $(\hat{p}_s^{\alpha}, \hat{r}_s^{+,\alpha}, \hat{r}_s^{-,\alpha}, \hat{y}_s^{\alpha}, \hat{u}_s^{\alpha})_{\alpha=2}^T$ are linearly independent. By performing Gaussian elimination among $(\overline{p}_s^{\alpha}, \overline{r}_s^{+,\alpha}, \overline{r}_s^{-,\alpha}, \overline{y}_s^{\alpha}, \overline{u}_s^{\alpha})_{\alpha=1}^T$, $(\grave{p}_s^{\alpha}, \grave{r}_s^{+,\alpha}, \grave{r}_s^{-,\alpha}, \grave{y}_s^{\alpha}, \grave{u}_s^{\alpha})_{\alpha=1}^T$, $(\acute{p}_s^{\alpha}, \acute{r}_s^{+,\alpha}, \acute{r}_s^{-,\alpha}, \acute{y}_s^{\alpha}, \acute{u}_s^{\alpha})_{\alpha=1}^T$, $(\dot{p}_s^{\alpha}, \dot{r}_s^{+,\alpha}, \dot{r}_s^{-,\alpha}, \dot{y}_s^{\alpha}, \dot{u}_s^{\alpha})_{\alpha=2}^T$, and $(\hat{p}_s^{\alpha}, \hat{r}_s^{+,\alpha}, \hat{r}_s^{-,\alpha}, \hat{y}_s^{\alpha}, \hat{u}_s^{\alpha})_{\alpha=2}^T$, we conclude that all of them are linearly independent. □

EC.4.3. Proof of Proposition 9

Proof. (**Validity**) For the validity of (20), we discuss the following four possible cases.

1. If $y_t = 0$, then $u_t = u_{t-1} = 0$ due to constraints (2a) and $L \in [2, T-1]_{\mathbb{Z}}$. According to (2d) and (2e), we have $r_t^- = r_t^+ = 0$ and inequality (20) is clearly valid.
2. If $y_t = 1$ and $u_t = 1$, then $y_{t-1} = u_{t-1} = 0$ due to minimum-up time constraints (2a) and $L \in [2, T-1]_{\mathbb{Z}}$. Inequality (20) becomes $r_t^+ + r_t^- \le \overline{V} - \underline{C}$ and its validity is proved by constraints (2d) and (2f).
3. If $y_t = 1$, $u_t = 0$, and $u_{t-1} = 1$, then (20) converts to $r_t^+ + r_t^- \le \overline{V} + V - \underline{C}$, which is valid according to the above case.
4. If $y_t = 1$ and $u_{t-1} = u_t = 0$, then $y_{t-1} = 1$ and inequality (20) becomes $r_t^+ + r_t^- \le 2V$, which is valid because $r_t^+ + r_t^- = (p_t + r_t^+ - p_{t-1}) + (p_{t-1} - p_t + r_t^-) \le V + V = 2V$, where the inequality holds due to (2f) and (2g).

(**Facet-defining**) We create $5T-1$ affinely independent points in $\operatorname{conv}(P)$ that satisfy inequality (20) at equality. Since $\vec{0} \in \operatorname{conv}(P)$, we create the other $5T-2$ linearly independent points in $\operatorname{conv}(P)$ in the following groups.

First, we create T linearly independent points $(\overline{p}_s^{\alpha}, \overline{r}_s^{+,\alpha}, \overline{r}_s^{-,\alpha}, \overline{y}_s^{\alpha}, \overline{u}_s^{\alpha})_{\alpha=1}^T$ as follows.

1) For each $\alpha \in [1, t-1]_{\mathbb{Z}}$ (totally $t-1$ points), we let

$$\overline{p}_s^\alpha = \begin{cases} \overline{V} - \underline{C}, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \quad \overline{y}_s^\alpha = \begin{cases} 1, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases},$$
$$\overline{r}_s^{+,\alpha} = \overline{r}_s^{-,\alpha} = 0, \ \forall s \in [1,T]_{\mathbb{Z}}, \text{ and } \overline{u}_s^\alpha = 0, \ \forall s \in [1,T]_{\mathbb{Z}}.$$

2) For $\alpha = t$ (totally one point), we let

$$\overline{p}_s^\alpha = \begin{cases} V, & s \in [1,\alpha-1]_{\mathbb{Z}} \\ \overline{V} - \underline{C}, & s = \alpha \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \quad \overline{r}_s^{+,\alpha} = \begin{cases} 2V - \overline{V} + \underline{C}, & s = t \\ 0, & o.w. \end{cases}, \quad \overline{r}_s^{-,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s = t \\ 0, & o.w. \end{cases},$$
$$\overline{y}_s^\alpha = \begin{cases} 1, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \overline{u}_s^\alpha = 0, \ \forall s \in [1,T]_{\mathbb{Z}}.$$

3) For each $\alpha \in [t+1,T]_{\mathbb{Z}}$ (totally $T-t$ points), we let

$$\overline{p}_s^\alpha = \begin{cases} V, & s \in [1,t]_{\mathbb{Z}} \\ \overline{V} - \underline{C}, & s \in [t+1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \quad \overline{r}_s^{+,\alpha} = \begin{cases} 2V - \overline{V} + \underline{C}, & s = t \\ 0, & o.w. \end{cases}, \quad \overline{r}_s^{-,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s = t \\ 0, & o.w. \end{cases},$$
$$\overline{y}_s^\alpha = \begin{cases} 1, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \overline{u}_s^\alpha = 0, \ \forall s \in [1,T]_{\mathbb{Z}}.$$

Second, we create another T linearly independent points $(\grave{p}_s^\alpha, \grave{r}_s^{+,\alpha}, \grave{r}_s^{-,\alpha}, \grave{y}_s^\alpha, \grave{u}_s^\alpha)_{\alpha=1}^T$ as follows.

4) For each $\alpha \in [1,t-1]_{\mathbb{Z}}$ (totally $t-1$ points), we let

$$\grave{p}_s^\alpha = \begin{cases} \overline{V} - \underline{C}, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \quad \grave{r}_s^{+,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \quad \grave{r}_s^{-,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases},$$
$$\grave{y}_s^\alpha = \begin{cases} 1, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \grave{u}_s^\alpha = 0, \ \forall s \in [1,T]_{\mathbb{Z}}.$$

5) For $\alpha = t$ (totally one point), we let

$$\grave{p}_s^\alpha = \begin{cases} V, & s \in [1,\alpha-1]_{\mathbb{Z}} \\ \overline{V} - \underline{C}, & s = \alpha \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \quad \grave{r}_s^{+,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [1,\alpha-1]_{\mathbb{Z}} \\ 2V - \overline{V} + \underline{C}, & s = \alpha \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases},$$
$$\grave{r}_s^{-,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \quad \grave{y}_s^\alpha = \begin{cases} 1, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \grave{u}_s^\alpha = 0, \ \forall s \in [1,T]_{\mathbb{Z}}.$$

6) For each $\alpha \in [t+1,T]_{\mathbb{Z}}$ (totally $T-t$ points), we let

$$\grave{p}_s^\alpha = \begin{cases} V, & s \in [1,t]_{\mathbb{Z}} \\ \overline{V} - \underline{C}, & s \in [t+1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \quad \grave{r}_s^{+,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [1,t-1]_{\mathbb{Z}} \cup [t+1,\alpha]_{\mathbb{Z}} \\ 2V - \overline{V} + \underline{C}, & s = t \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}$$
$$\grave{r}_s^{-,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \quad \grave{y}_s^\alpha = \begin{cases} 1, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \grave{u}_s^\alpha = 0, \ \forall s \in [1,T]_{\mathbb{Z}}.$$

Next, we create another T linearly independent points $(\acute{p}_s^\alpha, \acute{r}_s^{+,\alpha}, \acute{r}_s^{-,\alpha}, \acute{y}_s^\alpha, \acute{u}_s^\alpha)_{\alpha=1}^T$ as follows.

7) For each $\alpha \in [1,t-1]_{\mathbb{Z}}$ (totally $t-1$ points), we let

$$\acute{p}_s^\alpha = \begin{cases} \overline{V} - \underline{C}, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \quad \acute{r}_s^{+,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \quad \acute{r}_s^{-,\alpha} = 0, \ \forall s \in [1,T]_{\mathbb{Z}},$$
$$\acute{y}_s^\alpha = \begin{cases} 1, & s \in [1,\alpha]_{\mathbb{Z}} \\ 0, & s \in [\alpha+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \acute{u}_s^\alpha = 0, \ \forall s \in [1,T]_{\mathbb{Z}}.$$

8) For each $\alpha \in [t,T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we let

$$\acute{p}_s^\alpha = \begin{cases} V, & s\in[1,t-1]_{\mathbb{Z}} \\ \overline{V}-\underline{C}, & s\in[t,\alpha]_{\mathbb{Z}} \\ 0, & s\in[\alpha+1,T]_{\mathbb{Z}} \end{cases}, \quad \acute{r}_s^{+,\alpha} = \begin{cases} \overline{V}-\underline{C}, & s\in[1,t-1]_{\mathbb{Z}}\cup[t+1,\alpha]_{\mathbb{Z}} \\ 2V-\overline{V}+\underline{C}, & s=t \\ 0, & s\in[\alpha+1,T]_{\mathbb{Z}} \end{cases},$$

$$\acute{r}_s^{-,\alpha} = \begin{cases} \overline{V}-\underline{C}, & s=t \\ 0, & o.w. \end{cases}, \quad \acute{y}_s^\alpha = \begin{cases} 1, & s\in[1,\alpha]_{\mathbb{Z}} \\ 0, & s\in[\alpha+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \acute{u}_s^\alpha = 0,\ \forall s\in[1,T]_{\mathbb{Z}}.$$

In addition, we create $T-1$ linearly independent points $(\dot{p}_s^\alpha, \dot{r}_s^{+,\alpha}, \dot{r}_s^{-,\alpha}, \dot{y}_s^\alpha, \dot{u}_s^\alpha)_{\alpha=2}^T$ as follows.

9) For each $\alpha\in[2,t-2]_{\mathbb{Z}}$ (totally $t-3$ points), we let

$$\dot{p}_s^\alpha = \begin{cases} \overline{V}-\underline{C}, & s\in[\alpha,\alpha+L-1]_{\mathbb{Z}}\setminus\{t-1\} \\ V, & s=[\alpha,\alpha+L-1]_{\mathbb{Z}}\cap\{t-1\} \text{ and if } t\in[\alpha,\alpha+L-1]_{\mathbb{Z}} \\ 0, & o.w. \end{cases},$$

$$\dot{r}_s^{+,\alpha} = \begin{cases} 2V-\overline{V}+\underline{C}, & s\in[\alpha,\alpha+L-1]_{\mathbb{Z}}\cap\{t\} \\ 0, & s\in o.w. \end{cases}, \quad \dot{r}_s^{-,\alpha} = \begin{cases} \overline{V}-\underline{C}, & s\in[\alpha,\alpha+L-1]_{\mathbb{Z}}\cap\{t\} \\ 0, & s\in o.w. \end{cases},$$

$$\dot{y}_s^\alpha = \begin{cases} 1, & s\in[\alpha,\alpha+L-1]_{\mathbb{Z}} \\ 0, & s\in o.w. \end{cases}, \text{ and } \dot{u}_s^\alpha = \begin{cases} 1, & s=\alpha \\ 0, & o.w. \end{cases}.$$

10) For $\alpha=t-1$ (totally one points), we let

$$\dot{p}_s^\alpha = \begin{cases} \overline{V}-\underline{C}, & s\in\{t-1\}\cup[t+1,\min\{\alpha+L-1,T\}]_{\mathbb{Z}} \\ \overline{V}+V-\underline{C}, & s=t \\ 0, & o.w. \end{cases},$$

$$\dot{r}_s^{+,\alpha} = 0,\ \forall s\in[1,T]_{\mathbb{Z}}, \quad \dot{r}_s^{-,\alpha} = \begin{cases} \overline{V}+V-\underline{C}, & s\in\{t\}\cap[\alpha,\min\{\alpha+L-1,T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases},$$

$$\dot{y}_s^\alpha = \begin{cases} 1, & s\in[\alpha,\min\{\alpha+L-1,T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases}, \text{ and } \dot{u}_s^\alpha = \begin{cases} 1, & s=\alpha \\ 0, & o.w. \end{cases}.$$

11) For $\alpha=t$ (totally one points), we let

$$\dot{p}_s^\alpha = \begin{cases} \overline{V}-\underline{C}, & s\in[\alpha,\min\{\alpha+L-1,T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases}, \quad \dot{y}_s^\alpha = \begin{cases} 1, & s\in[\alpha,\min\{\alpha+L-1,T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases}, \text{ and}$$

$$\dot{u}_s^\alpha = \begin{cases} 1, & s=\alpha \\ 0, & o.w. \end{cases}, \quad \dot{r}_s^{+,\alpha} = 0,\ \forall s\in[1,T]_{\mathbb{Z}}, \quad \dot{r}_s^{-,\alpha} = \begin{cases} \overline{V}-\underline{C}, & s\in\{t\}\cap[\alpha,\min\{\alpha+L-1,T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases}.$$

12) For each $\alpha\in[t+1,T]_{\mathbb{Z}}$ (totally $T-t$ points), we let

$$\dot{p}_s^\alpha = \begin{cases} \overline{V}-\underline{C}, & s\in[\alpha,\min\{\alpha+L-1,T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases}, \quad \dot{r}_s^{+,\alpha} = \dot{r}_s^{-,\alpha} = 0,\ \forall s\in[1,T]_{\mathbb{Z}},$$

$$\dot{y}_s^\alpha = \begin{cases} 1, & s\in[\alpha,\min\{\alpha+L-1,T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases}, \text{ and } \dot{u}_s^\alpha = \begin{cases} 1, & s=\alpha \\ 0, & o.w. \end{cases}.$$

Finally, we create another $T-1$ linearly independent points $(\hat{p}_s^\alpha, \hat{r}_s^{+,\alpha}, \hat{r}_s^{-,\alpha}, \hat{y}_s^\alpha, \hat{u}_s^\alpha)_{\alpha=2}^T$ as follows.

13) For each $\alpha\in[2,t-2]_{\mathbb{Z}}$ (totally $t-3$ points), we let

$$\hat{p}_s^\alpha = \begin{cases} \overline{V}-\underline{C}, & s\in[\alpha,\alpha+L-1]_{\mathbb{Z}}\setminus\{t-1\} \\ V, & s=[\alpha,\alpha+L-1]_{\mathbb{Z}}\cap\{t-1\} \text{ and if } t\in[\alpha,\alpha+L-1]_{\mathbb{Z}} \\ 0, & o.w. \end{cases},$$

$$\hat{r}_s^{-,\alpha} = \begin{cases} \overline{V}-\underline{C}, & s\in[\alpha,\alpha+L-1]_{\mathbb{Z}} \\ 0, & s\in o.w. \end{cases}, \quad \hat{r}_s^{+,\alpha} = \begin{cases} 2V-\overline{V}+\underline{C}, & s\in[\alpha,\alpha+L-1]_{\mathbb{Z}}\cap\{t\} \\ 0, & s\in o.w. \end{cases},$$

$$\hat{y}_s^\alpha = \begin{cases} 1, & s\in[\alpha,\alpha+L-1]_{\mathbb{Z}} \\ 0, & s\in o.w. \end{cases}, \text{ and } \hat{u}_s^\alpha = \begin{cases} 1, & s=\alpha \\ 0, & o.w. \end{cases}.$$

14) For $\alpha = t-1$ (totally one point), we let

$$\hat{p}_s^\alpha = \begin{cases} \overline{V} - \underline{C}, & s \in [\alpha, \min\{\alpha+L-1, T\}]_{\mathbb{Z}} \setminus \{t\} \\ \overline{V} + V - \underline{C}, & s = t \\ 0, & o.w. \end{cases},$$

$$\hat{r}_s^{+,\alpha} = 0,\ \forall s \in [1,T]_{\mathbb{Z}},\ \ \hat{r}_s^{-,\alpha} = \begin{cases} \overline{V} + V - \underline{C}, & s \in \{t\} \cap [\alpha, \min\{\alpha+L-1, T\}]_{\mathbb{Z}} \\ \overline{V} - \underline{C}, & s \in [\alpha, \min\{\alpha+L-1, T\}]_{\mathbb{Z}} \setminus \{t\} \\ 0, & o.w. \end{cases},$$

$$\hat{y}_s^\alpha = \begin{cases} 1, & s \in [\alpha, \min\{\alpha+L-1, T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases}, \text{ and } \hat{u}_s^\alpha = \begin{cases} 1, & s = \alpha \\ 0, & o.w. \end{cases}.$$

15) For each $\alpha \in [t, T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we let

$$\hat{p}_s^\alpha = \hat{r}_s^{-,\alpha} = \begin{cases} \overline{V} - \underline{C}, & s \in [\alpha, \min\{\alpha+L-1, T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases},\ \ \hat{r}_s^{+,\alpha} = 0,\ \forall s \in [1,T]_{\mathbb{Z}},$$

$$\hat{y}_s^\alpha = \begin{cases} 1, & s \in [\alpha, \min\{\alpha+L-1, T\}]_{\mathbb{Z}} \\ 0, & o.w. \end{cases}, \text{ and } \hat{u}_s^\alpha = \begin{cases} 1, & s = \alpha \\ 0, & o.w. \end{cases}.$$

In summary, we have created $5T-2$ points. The points $(\overline{p}_s^\alpha, \overline{r}_s^{+,\alpha}, \overline{r}_s^{-,\alpha}, \overline{y}_s^\alpha, \overline{u}_s^\alpha)_{\alpha=1}^T$ are linearly independent because they form a lower triangular matrix by the construction. Similarly, $(\grave{p}_s^\alpha, \grave{r}_s^{+,\alpha}, \grave{r}_s^{-,\alpha}, \grave{y}_s^\alpha, \grave{u}_s^\alpha)_{\alpha=1}^T$, $(\acute{p}_s^\alpha, \acute{r}_s^{+,\alpha}, \acute{r}_s^{-,\alpha}, \acute{y}_s^\alpha, \acute{u}_s^\alpha)_{\alpha=1}^T$, $(\dot{p}_s^\alpha, \dot{r}_s^{+,\alpha}, \dot{r}_s^{-,\alpha}, \dot{y}_s^\alpha, \dot{u}_s^\alpha)_{\alpha=2}^T$ and $(\hat{p}_s^\alpha, \hat{r}_s^{+,\alpha}, \hat{r}_s^{-,\alpha}, \hat{y}_s^\alpha, \hat{u}_s^\alpha)_{\alpha=2}^T$ are linearly independent. By performing Gaussian elimination among $(\overline{p}_s^\alpha, \overline{r}_s^{+,\alpha}, \overline{r}_s^{-,\alpha}, \overline{y}_s^\alpha, \overline{u}_s^\alpha)_{\alpha=1}^T$, $(\grave{p}_s^\alpha, \grave{r}_s^{+,\alpha}, \grave{r}_s^{-,\alpha}, \grave{y}_s^\alpha, \grave{u}_s^\alpha)_{\alpha=1}^T$, $(\acute{p}_s^\alpha, \acute{r}_s^{+,\alpha}, \acute{r}_s^{-,\alpha}, \acute{y}_s^\alpha, \acute{u}_s^\alpha)_{\alpha=1}^T$, $(\dot{p}_s^\alpha, \dot{r}_s^{+,\alpha}, \dot{r}_s^{-,\alpha}, \dot{y}_s^\alpha, \dot{u}_s^\alpha)_{\alpha=2}^T$ and $(\hat{p}_s^\alpha, \hat{r}_s^{+,\alpha}, \hat{r}_s^{-,\alpha}, \hat{y}_s^\alpha, \hat{u}_s^\alpha)_{\alpha=2}^T$, we conclude that all of them are linearly independent. □

EC.4.4. Proof of Proposition 10

Proof. We show the validity of inequality (21) by discussing the following two possible cases in terms of the value of y_t.

1. If $y_t = 0$, then $u_{t-s} = 0$ for all $s \in [0, \min\{k, L-1\}]_{\mathbb{Z}}$ due to constraints (2a). Inequality (21) becomes $-p_{t-k} + r_{t-k}^- \leq 0$, which is valid due to constraints (2d).
2. If $y_t = 1$, then we consider the following two possible cases.
   (a) If $u_{t-s} = 0$ for all $s \in [0, \min\{k, L-1\}]_{\mathbb{Z}}$, then inequality (21) converts to $p_t + r_t^+ - p_{t-k} + r_{t-k}^- \leq (k+2)V$, which is valid because $p_t + r_t^+ - p_{t-k} \leq kV$ due to ramping-up constraints (2f), and $r_{t-k}^- \leq 2V$ due to constraints (2g) (i.e., $r_{t-k}^- \leq V + p_{t-k} - p_{t-k-1}$) and constraints (2f) (i.e., $p_{t-k} - p_{t-k-1} \leq p_{t-k} + r_{t-k}^+ - p_{t-k-1} \leq V$).
   (b) If $u_{t-\overline{s}} = 1$ for some $\overline{s} \in [0, \min\{k, L-1\}]_{\mathbb{Z}}$, then inequality (21) becomes $p_t + r_t^+ - p_{t-k} + r_{t-k}^- \leq \overline{V} + \overline{s}V - \underline{C}$, which is valid because $p_t + r_t^+ - p_{t-k} \leq \overline{V} + \overline{s}V - \underline{C}$ due to ramping-up constraints (2f) and $r_{t-k}^- \leq p_{t-k}$ due to constraints (2g) and (2f).

The facet-defining proof is similar to that of Proposition 9 and thus is omitted here. □

EC.5. Proof of Theorem 3

Proof. We first derive a dynamic programming algorithm to solve problem (22) and then reformulate the dynamic program into a linear program, which can be further reformulated to be an integral formulation by duality theory.

First, to derive a dynamic program for problem (22), we define two value functions $V_{\uparrow}(t)$ and $V_{\downarrow}(t)$. We use $V_{\uparrow}(t)$ to denote the optimal value from period t to the end if the generator starts up in period t (i.e., $y_{t-1}=0$ and $y_t=1$), and $V_{\downarrow}(t)$ to denote the optimal value from period t to the end if the generator shuts down in period $t+1$ (i.e., $y_t=1$ and $y_{t+1}=0$). Meanwhile, we use $C(t,k)$ to denote the optimal value if the generator starts up in period t and shuts down in period $k+1$; that is, the optimal value of a economic dispatch problem considering the generator that stays online since periods t through period k. Therefore, we can derive the corresponding dynamic programming algorithm by setting the following bellman equations:

$$V_{\uparrow}(t) = \min_{\substack{k\in[\min\{\min\{t+L-1,\\ T-1\},1+T(t-1)\},T-1]_{\mathbb{Z}}}} \left\{ \mathrm{SD} + C(t,k) + V_{\downarrow}(k), C(t,T) + V_{\downarrow}(T) \right\},\ \forall t \in [1,T]_{\mathbb{Z}}, \tag{EC.52a}$$

$$V_{\downarrow}(t) = \min_{k\in[t+\ell+1,T]_{\mathbb{Z}}} \left\{ \mathrm{SU} + V_{\uparrow}(k), 0 \right\},\ \forall t \in [1, T-\ell-1]_{\mathbb{Z}}, \tag{EC.52b}$$

$$V_{\downarrow}(t) = 0,\ \forall t \in [T-\ell, T]_{\mathbb{Z}}. \tag{EC.52c}$$

Meanwhile, we use z to denote the optimal value of problem (22) and it follows that

$$z = \min_{t\in[2,T]_{\mathbb{Z}}} \left\{ \mathrm{SU} + V_{\uparrow}(t), V_{\uparrow}(1), 0 \right\}. \tag{EC.53}$$

Note that equations (EC.52a) indicate that when the generator stays online since period t, it either stays online until period k after satisfying the minimum-up time requirement or until the end; equations (EC.52b) indicate that when the generator shuts down in period $t+1$, it either stays offline until period k after satisfying the minimum-down time requirement or until the end; equations (EC.52c) indicate that once the generator shuts down in a period later than $T-\ell$, it cannot start up again due to the minimum-down time requirement.

Next, by converting (EC.52) and (EC.53) to constraints, we reformulate the above dynamic program with (EC.52) and (EC.53) as the following equivalent linear program:

$$\max\ z \tag{EC.54a}$$

$$(\alpha_t) \quad \text{s.t.}\ z \le \mathrm{SU} + V_{\uparrow}(t),\ \forall t \in [2,T]_{\mathbb{Z}}, \tag{EC.54b}$$

$$(\alpha_1) \quad z \le V_{\uparrow}(1),\ t = 1, \tag{EC.54c}$$

$$(\beta_{tk}) \quad V_{\uparrow}(t) \le \mathrm{SD} + C(t,k) + V_{\downarrow}(k),$$

$$\forall k \in [\min\{t+L-1, T-1\}, T-1]_{\mathbb{Z}}, \forall t \in [2,T]_{\mathbb{Z}}, \tag{EC.54d}$$

$$(\beta_{tk}) \qquad V_{\uparrow}(t) \leq \mathrm{SD} + C(t,k) + V_{\downarrow}(k), \forall k \in [1, T-1]_{\mathbb{Z}}, t = 1, \tag{EC.54e}$$

$$(\beta_{tk}) \qquad V_{\uparrow}(t) \leq C(t,T) + V_{\downarrow}(T),\ \forall t \in [1,T]_{\mathbb{Z}}, \tag{EC.54f}$$

$$(\gamma_{tk}) \qquad V_{\downarrow}(t) \leq \mathrm{SU} + V_{\uparrow}(k),\ \forall k \in [t+\ell+1, T]_{\mathbb{Z}}, \forall t \in [1, T-\ell-1]_{\mathbb{Z}}, \tag{EC.54g}$$

$$(\theta_t) \qquad V_{\downarrow}(t) = 0,\ \forall t \in [T-\ell, T]_{\mathbb{Z}}, \tag{EC.54h}$$

$$z \leq 0,\ V_{\downarrow}(t) \leq 0,\ \forall t \in [1, T-\ell-1]_{\mathbb{Z}}. \tag{EC.54i}$$

Note that this linear program cannot be solved directly because $C(t,k)$ represents an optimization problem by itself for each given feasible t and k. In the following, we replace $C(t,k)$ with its equivalent formulation due to duality theory and thereby provide an integral formulation to reformulate (EC.54). To that end, we use the notations in the brackets on the left side of problem (EC.54) to denote the dual variables of constraints (EC.54b) - (EC.54i).

First, given any feasible $(t,k) \in \mathcal{TK}$, we characterize $C(t,k)$ by considering three possible cases in terms of the values of t and k: (i) $t \geq 2$ and $k \leq T-1$; (ii) $t \geq 2$ and $k = T$; (iii) $t = 1$ and $k \leq T-1$; and (iv) $t = 1$ and $k = T$. For the first case, $C(t,k)$ can be calculated as follows:

$$C(t,k) = \min \sum_{s=t}^{k} \phi_s \tag{EC.55a}$$

$$(\lambda_s^-) \qquad \text{s.t. } -p_s + r_s^- \leq 0,\ \forall s \in [t,k]_{\mathbb{Z}}, \tag{EC.55b}$$

$$(\lambda_s^+) \qquad p_s + r_s^+ \leq \overline{C} - \underline{C},\ \forall s \in [t,k]_{\mathbb{Z}}, \tag{EC.55c}$$

$$(\mu_t) \qquad p_t + r_t^+ \leq \overline{V} - \underline{C}, \tag{EC.55d}$$

$$(\mu_k) \qquad p_k \leq \overline{V} - \underline{C}, \tag{EC.55e}$$

$$(\sigma_s^+) \qquad p_s + r_s^+ - p_{s-1} \leq V,\ \forall s \in [t+1,k]_{\mathbb{Z}}, \tag{EC.55f}$$

$$(\sigma_s^-) \qquad p_{s-1} - p_s + r_s^- \leq V,\ \forall s \in [t+1,k]_{\mathbb{Z}}, \tag{EC.55g}$$

$$(\delta_{sj}) \qquad \phi_s \geq \nu_j^s p_s + \nu_j^s \underline{C} + \varphi_j + \mathrm{RU} r_s^+ + \mathrm{RD} r_s^-,\ \forall s \in [t,k]_{\mathbb{Z}}, j \in [1,N]_{\mathbb{Z}}. \tag{EC.55h}$$

In the above model, we use the notations in the brackets on the left side of problem (EC.55) to denote the dual variables of constraints (EC.55b) - (EC.55h). For the second case, i.e., $t \geq 2$ and $k = T$, $C(t,T)$ can be calculated in the same way except that constraint (EC.55e) should be removed because the generator does not have to shut down in period $T+1$ if it stays online until time T. For the third case, i.e., $t = 1$ and $k \leq T-1$, $C(1,k)$ can be calculated in the same way to (EC.55) except that constraint (EC.55d) should be removed because the generator is allowed to stay in either online or offline status in the first period. For the fourth case, i.e., $t = 1$ and $k = T$, both constraints (EC.55d) and (EC.55e) should be removed.

Next, we replace $C(t,k)$ in (EC.54) with its equivalent optimization models for any feasible $(t,k) \in \mathcal{TK}$. To that end, we take the dual of model (EC.55) and include its dual formulation into

(EC.54). In particular, for the case where $t \geq 2$ and $k \leq T-1$, the corresponding dual formulation can be described as follows:

$$C(t,k) = \max \sum_{s=t}^{k} (\overline{C} - \underline{C})\lambda_s^+ + (\overline{V} - \underline{C})(\mu_t + \mu_k) + \sum_{s=t+1}^{k} V(\sigma_s^+ + \sigma_s^-) + \sum_{s=t}^{k}\sum_{j=1}^{N} (\nu_j^s \underline{C} + \varphi_j)\delta_{sj} \tag{EC.56a}$$

$$(q_{tk}^t) \qquad \text{s.t.} \quad \lambda_t^+ - \lambda_t^- + \mu_t - \sigma_{t+1}^+ + \sigma_{t+1}^- - \sum_{j=1}^{N} \nu_j^t \delta_{tj} = 0, \tag{EC.56b}$$

$$(q_{tk}^k) \qquad \lambda_k^+ - \lambda_k^- + \mu_k + \sigma_k^+ - \sigma_k^- - \sum_{j=1}^{N} \nu_j^k \delta_{kj} = 0, \tag{EC.56c}$$

$$(q_{tk}^s) \qquad \lambda_s^+ - \lambda_s^- + \sigma_s^+ - \sigma_{s+1}^+ - \sigma_s^- + \sigma_{s+1}^- - \sum_{j=1}^{N} \nu_j^s \delta_{sj} = 0, \ \forall s \in [t+1, k-1]_{\mathbb{Z}}, \tag{EC.56d}$$

$$(m_{tk}^t) \qquad \lambda_t^+ + \mu_t - \sum_{j=1}^{N} \mathrm{RU}\delta_{tj} \leq 0, \tag{EC.56e}$$

$$(m_{tk}^s) \qquad \lambda_s^+ + \sigma_s^+ - \sum_{j=1}^{N} \mathrm{RU}\delta_{sj} \leq 0, \ \forall s \in [t+1, k]_{\mathbb{Z}}, \tag{EC.56f}$$

$$(n_{tk}^t) \qquad \lambda_t^- - \sum_{j=1}^{N} \mathrm{RD}\delta_{tj} \leq 0, \tag{EC.56g}$$

$$(n_{tk}^s) \qquad \lambda_s^- + \sigma_s^- - \sum_{j=1}^{N} \mathrm{RD}\delta_{sj} \leq 0, \ \forall s \in [t+1, k]_{\mathbb{Z}}, \tag{EC.56h}$$

$$(w_{tk}^s) \qquad \sum_{j=1}^{N} \delta_{sj} = 1, \ \forall s \in [t, k]_{\mathbb{Z}}, \tag{EC.56i}$$

$$\lambda_s^{\pm} \leq 0, \ \forall s \in [t, k]_{\mathbb{Z}}, \ \mu_t \leq 0, \ \mu_k \leq 0, \ \sigma_s^{\pm} \leq 0, \ \forall s \in [t+1, k]_{\mathbb{Z}},$$
$$\delta_{sj} \geq 0, \ \forall j \in [1, N]_{\mathbb{Z}}, s \in [t, k]_{\mathbb{Z}}, \tag{EC.56j}$$

where we use the notations in the brackets on the left side of problem (EC.56) to denote the dual variables of constraints (EC.56b) - (EC.56j). When $k = T$, the same dual formulation can be obtained except that the dual variable μ_k should be removed from model (EC.56). Similarly, when $t = 1$, dual variable μ_t should be removed from model (EC.56). For simplicity, we refer to (EC.56) as the dual formulation for any feasible (t,k), where μ_k is removed from (EC.56) when $k = T$ and μ_t is removed when $t = 1$.

Therefore, by redefining $C(t,k)$ as a decision variable and replacing it with its dual formulation in (EC.54), an integrated linear program can be obtained as follows:

$$\max \ z \tag{EC.57a}$$

$$\text{s.t.} \quad \text{(EC.54b)} - \text{(EC.54i)}, \tag{EC.57b}$$

$$(p_{tk}) \quad C(t,k) \le \sum_{s=t}^{k}(\overline{C}-\underline{C})\lambda_s^+ + (\overline{V}-\underline{C})(\mu_t+\mu_k) + \sum_{s=t+1}^{k} V(\sigma_s^+ + \sigma_s^-) + \sum_{s=t}^{k}\sum_{j=1}^{N}(\nu_j^s \underline{C} + \varphi_j)\delta_{sj}, \quad \forall t \in [2,T]_{\mathbb{Z}}, k \in [\min\{t+L-1, T-1\}, T-1]_{\mathbb{Z}}, \tag{EC.57c}$$

$$(p_{tk}) \quad C(t,k) \le \sum_{s=t}^{T}(\overline{C}-\underline{C})\lambda_s^+ + (\overline{V}-\underline{C})\mu_t + \sum_{s=t+1}^{T} V(\sigma_s^+ + \sigma_s^-) + \sum_{s=t}^{T}\sum_{j=1}^{N}(\nu_j^s \underline{C} + \varphi_j)\delta_{sj}, \quad \forall t \in [2,T]_{\mathbb{Z}}, k = T, \tag{EC.57d}$$

$$(p_{tk}) \quad C(t,k) \le \sum_{s=t}^{k}(\overline{C}-\underline{C})\lambda_s^+ + (\overline{V}-\underline{C})\mu_k + \sum_{s=t+1}^{k} V(\sigma_s^+ + \sigma_s^-) + \sum_{s=t}^{k}\sum_{j=1}^{N}(\nu_j^s \underline{C} + \varphi_j)\delta_{sj}, \quad \forall t = 1, k \in [1, T-1]_{\mathbb{Z}}, \tag{EC.57e}$$

$$(p_{tk}) \quad C(t,k) \le \sum_{s=t}^{T}(\overline{C}-\underline{C})\lambda_s^+ + \sum_{s=t+1}^{T} V(\sigma_s^+ + \sigma_s^-) + \sum_{s=t}^{T}\sum_{j=1}^{N}(\nu_j^s \underline{C} + \varphi_j)\delta_{sj}, \ t = 1, k = T, \tag{EC.57f}$$

$$(\text{EC.56b}) - (\text{EC.56j}), \ \forall (t,k) \in \mathcal{TK}, \tag{EC.57g}$$

where (EC.57c), (EC.57d), (EC.57e), and (EC.57f) represent the objective function (EC.56a) under the case (1) $t \ge 2$ and $k \le T-1$, (2) $t \ge 2$ and $k = T$, (3) $t = 1$ and $k \le T-1$, and (4) $t = 1$ and $k = T$, respectively. We use the notations in the brackets on the left side of problem (EC.57) to denote the dual variables of constraints (EC.57c) - (EC.57f).

By strong duality, we take the dual of the above linear program (EC.57) and obtain the equivalent dual linear program as follows.

$$\min \sum_{t=2}^{T} \mathrm{SU}\alpha_t + \sum_{k=1}^{T-1} \mathrm{SD}\beta_{1k} + \sum_{t=2}^{T}\sum_{k=t+L-1}^{T-1} \mathrm{SD}\beta_{tk} + \sum_{t=1}^{T-\ell-1}\sum_{k=t+\ell+1}^{T} \mathrm{SU}\gamma_{tk} + \sum_{(t,k)\in\mathcal{TK}}\sum_{s=t}^{k} w_{tk}^s \tag{EC.58a}$$

$$\text{s.t.} \ \sum_{t=1}^{T} \alpha_t \le 1, \tag{EC.58b}$$

$$-\alpha_t + \sum_{k=1}^{T} \beta_{tk} = 0, \ t = 1, \tag{EC.58c}$$

$$-\alpha_t + \sum_{k=\min\{t+L-1,T\}}^{T} \beta_{tk} = 0, \ \forall t \in [2, \ell+1]_{\mathbb{Z}}, \tag{EC.58d}$$

$$-\alpha_t + \sum_{k=\min\{t+L-1,T\}}^{T} \beta_{tk} - \sum_{k=1}^{t-\ell-1} \gamma_{kt} = 0, \ \forall t \in [\ell+2, T]_{\mathbb{Z}}, \tag{EC.58e}$$

$$-\sum_{k=1}^{t-L+1} \beta_{kt} + \sum_{k=t+\ell+1}^{T} \gamma_{tk} \le 0, \ \forall t \in [1, T-\ell-1]_{\mathbb{Z}}, \tag{EC.58f}$$

$$\theta_t - \sum_{k=1}^{t-L+1} \beta_{kt} = 0, \ \forall t \in [T-\ell, T]_{\mathbb{Z}}, \tag{EC.58g}$$

$$p_{tk} - \beta_{tk} = 0, \ \forall (t,k) \in \mathcal{TK}, \tag{EC.58h}$$

$$q_{tk}^s + m_{tk}^s \le (\overline{C}-\underline{C}) p_{tk}, \ \forall (t,k) \in \mathcal{TK}, \forall s \in [t,k]_{\mathbb{Z}}, \tag{EC.58i}$$

$$-q_{tk}^s + n_{tk}^s \le 0,\ \forall (t,k) \in \mathcal{TK}, \forall s \in [t,k]_{\mathbb{Z}}, \tag{EC.58j}$$

$$q_{tk}^t + m_{tk}^t \le (\overline{V} - \underline{C}) p_{tk},\ \forall (t,k) \in \mathcal{TK} \text{ with } t \ge 2, \tag{EC.58k}$$

$$q_{tk}^k \le (\overline{V} - \underline{C}) p_{tk},\ \forall (t,k) \in \mathcal{TK} \text{ with } k \le T-1, \tag{EC.58l}$$

$$q_{tk}^{s-1} - q_{tk}^s + n_{tk}^s \le V p_{tk},\ \forall (t,k) \in \mathcal{TK}, \forall s \in [t+1,k]_{\mathbb{Z}}, \tag{EC.58m}$$

$$q_{tk}^s + m_{tk}^s - q_{tk}^{s-1} \le V p_{tk},\ \forall (t,k) \in \mathcal{TK}, \forall s \in [t+1,k]_{\mathbb{Z}}, \tag{EC.58n}$$

$$w_{tk}^s - \nu_j^s q_{tk}^s - \mathrm{RU} m_{tk}^s - \mathrm{RD} n_{tk}^s \ge (\nu_j^s \underline{C} + \varphi_j) p_{tk},\ \forall (t,k) \in \mathcal{TK}, \forall s \in [t,k]_{\mathbb{Z}}, \forall j \in [1,N]_{\mathbb{Z}}, \tag{EC.58o}$$

$$\alpha, \beta, \gamma, p, m, n \ge 0. \tag{EC.58p}$$

Due to (EC.58h), we replace p with β and obtain the following equivalent formulation:

$$\min \sum_{t=2}^{T} \mathrm{SU}\alpha_t + \sum_{k=1}^{T-1} \mathrm{SD}\beta_{1k} + \sum_{t=2}^{T} \sum_{k=t+L-1}^{T-1} \mathrm{SD}\beta_{tk} + \sum_{t=1}^{T-\ell-1} \sum_{k=t+\ell+1}^{T} \mathrm{SU}\gamma_{tk} + \sum_{(t,k)\in\mathcal{TK}} \sum_{s=t}^{k} w_{tk}^s \tag{EC.59a}$$

$$\text{s.t.} \ \sum_{t=1}^{T} \alpha_t \le 1, \tag{EC.59b}$$

$$-\alpha_t + \sum_{k=1}^{T} \beta_{tk} = 0,\ t = 1, \tag{EC.59c}$$

$$-\alpha_t + \sum_{k=\min\{t+L-1,T\}}^{T} \beta_{tk} = 0,\ \forall t \in [2, \ell+1]_{\mathbb{Z}}, \tag{EC.59d}$$

$$-\alpha_t + \sum_{k=\min\{t+L-1,T\}}^{T} \beta_{tk} - \sum_{k=1}^{t-\ell-1} \gamma_{kt} = 0,\ \forall t \in [\ell+2, T]_{\mathbb{Z}}, \tag{EC.59e}$$

$$-\sum_{k=1}^{t-L+1} \beta_{kt} + \sum_{k=t+\ell+1}^{T} \gamma_{tk} \le 0,\ \forall t \in [1, T-\ell-1]_{\mathbb{Z}}, \tag{EC.59f}$$

$$\theta_t - \sum_{k=1}^{t-L+1} \beta_{kt} = 0,\ \forall t \in [T-\ell, T]_{\mathbb{Z}}, \tag{EC.59g}$$

$$q_{tk}^s + m_{tk}^s \le (\overline{C} - \underline{C}) \beta_{tk},\ \forall (t,k) \in \mathcal{TK}, \forall s \in [t,k]_{\mathbb{Z}}, \tag{EC.59h}$$

$$-q_{tk}^s + n_{tk}^s \le 0,\ \forall (t,k) \in \mathcal{TK}, \forall s \in [t,k]_{\mathbb{Z}}, \tag{EC.59i}$$

$$q_{tk}^t + m_{tk}^t \le (\overline{V} - \underline{C}) \beta_{tk},\ \forall (t,k) \in \mathcal{TK} \text{ with } t \ge 2, \tag{EC.59j}$$

$$q_{tk}^k \le (\overline{V} - \underline{C}) \beta_{tk},\ \forall (t,k) \in \mathcal{TK} \text{ with } k \le T-1, \tag{EC.59k}$$

$$q_{tk}^{s-1} - q_{tk}^s + n_{tk}^s \le V \beta_{tk},\ \forall (t,k) \in \mathcal{TK}, \forall s \in [t+1,k]_{\mathbb{Z}}, \tag{EC.59l}$$

$$q_{tk}^s + m_{tk}^s - q_{tk}^{s-1} \le V \beta_{tk},\ \forall (t,k) \in \mathcal{TK}, \forall s \in [t+1,k]_{\mathbb{Z}}, \tag{EC.59m}$$

$$w_{tk}^s - \nu_j^s q_{tk}^s - \mathrm{RU} m_{tk}^s - \mathrm{RD} n_{tk}^s \ge (\nu_j^s \underline{C} + \varphi_j) \beta_{tk},\ \forall (t,k) \in \mathcal{TK}, \forall s \in [t,k]_{\mathbb{Z}}, \forall j \in [1,N]_{\mathbb{Z}}, \tag{EC.59n}$$

$$\alpha, \beta, \gamma, m, n \geq 0. \tag{EC.59o}$$

Following the same proof for Theorem 1 and Proposition 1 in Guan et al. (2018), we can show that the extreme points of the polytope (EC.59b) - (EC.59o) are binary with respect to α, β, γ, and θ, and accordingly there exists an optimal solution, which is binary with respect to α, β, γ, and θ, to the formulation (EC.59) due to the linear objective function. Meanwhile, similar to Proposition 2 in Guan et al. (2018), we can also show that if $(\alpha^*, \beta^*, \gamma^*, \theta^*, q^*, m^*, n^*)$ is an optimal solution to the formulation (EC.59), then we have the solution $(p^*, r^{+*}, r^{-*}, y^*, u^*)$ with

$$p_s^* = \sum_{(t,k)\in\mathcal{TK}, t\leq s\leq k} q_{tk}^{s*},\ r_s^{+*} = \sum_{(t,k)\in\mathcal{TK}, t\leq s\leq k} m_{tk}^{s*},\ r_s^{-*} = \sum_{(t,k)\in\mathcal{TK}, t\leq s\leq k} n_{tk}^{s*},$$

$$y_s^* = \sum_{(t,k)\in\mathcal{TK}, t\leq s\leq k} \beta_{tk}^*,\ \text{and}\ u_s^* = \alpha_s^* + \sum_{(t,k)\in\mathcal{TK}, k=s} \gamma_{tk}^*,\ \forall s \in [1,T]_{\mathbb{Z}}$$

is an optimal solution to the original problem (22). It follows that Theorem 3 holds. □