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Pre-positioning and Deployment of Reserved Inventories in a Supply Network: Structural Properties

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We study a two-stage decision problem, namely, the allocation and deployment of reserved inventories (RIs) in a supply network with random demand surges. The demand surge follows a time-dependent stochastic process and our objective is to minimize the expected total unmet demand in the presence of positive transshipment lead times. We first solve the optimal deployment problem given that the demand surges have occurred at some locations. We show that the optimal deployment policy is a 'nested' policy with respect to the shadow price at each location, where a shadow price represents the marginal reduction of the expected total unmet demand due to a marginal increase of RIs. Specifically, locations with higher shadow prices have higher priority in inventory allocation. We then consider the optimal allocation problem in the pre-positioning stage. We show that under certain conditions the optimal allocation is increasing in the total amount of RIs. We introduce a new stochastic order for distributions defined on sets called the *first order stochastic dominance* and use it to show that the expected total unmet demand is higher when one of the following is true: the demand surges tend to occur simultaneously at more locations, the post-surge delivery takes a longer time, more demand arrives earlier, or the demand has a higher volatility.

Keywords: Supply Network; Demand Surge; Inventory Planning; Stochastic Optimization

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1 Introduction

Demand surges –the significant demand increments in addition to the regular demand in a supply network– arise from various sources such as natural (e.g., a hurricane or tsunami) and man-made disasters (e.g., an oil spill or fire), structural demand such as acquisition and winning large contracts, new product introductions and viral products (Huang et al. 2016). In order to mitigate the disruptive impact of sudden demand surges on firms' inventory flow and production processes, firms usually either build up reactive capacities (Huang et al. 2016) or keep a certain amount of reserved inventories as a reactive buffer at warehouses in different geographical locations (Liu et al. 2016). In the latter case, the reserved inventories are often managed separately from the regular inventories that are used to satisfy the daily demand. For example, Johnson and Johnson (J&J) reserves inventories at their regional warehouses for possible demand surges from sudden disruptions (Sheffi 2005a, b). In particular, they set a red line on the inventory level at each warehouse so that the amount of inventory stored at each warehouse cannot go below the red line (Sheffi 2005a pp. 173-174). Similarly, the International Federation of Red Cross and Red Crescent Societies (IFRC) requires their suppliers to reserve and store an agreed amount of emergency relief commodities, either at their premises or in regional warehouses in Dubai, Kuala Lumpur, and Panama. The IFRC therefore has a constant guaranteed level of pre-positioned reserved inventories (RIs) at dedicated locations (www.ifrc.org).

When demand surge occurs at some specific location(s), the pre-positioned RIs at other non-surge locations need to be deployed to the surge-occurring location(s). Such deployment process creates a time lag between the demand and the arrival of the RIs, which results in an immediate mismatch between supply and demand and leads to a significant loss of demand. Consequently, firms, governments and non-for-profit organizations often target at minimizing the expected total unmet demand, i.e., the immediate mismatch between the supply of the RIs and the demand surge (see Huang et al. 2016). This is especially relevant for a supply network where the RIs are kept at multiple geographically different locations.

The pre-positioning of RIs in a supply network is challenging because of the uncertainties from both supply and demand sides. On the one hand, the deployment process is uncertain, which requires positive lead times that may vary with respect to the geographical locations of the allocated RIs and the demand surge locations. Consequently, the arrival sequence of RIs to a demand surge location may be uncertain. Also, the allocated RIs may be damaged in various degrees depending on the impact of the demand surge or the deployment process. On the other hand, the demand surge process itself may also be uncertain. In particular, the geographical locations of demand surges are usually highly uncertain (see detailed examples given in Huang et al. 2016) with its probability of occurrence at the different locations estimated through the historical data. Besides, once a demand surge occurs, its process may exhibit different time-evolving patterns. As a result, its attributes such as magnitude, duration, and variability of the surge may also be highly uncertain.

In this paper, we investigate how a central planner would strategically allocate and deploy RIs among different locations so as to mitigate the impact of demand surges. In particular, we study how uncertainties involved in both *pre-positioning* and *deployment* stages affect the firm's decisions. We formulate the firm's allocation and deployment problem as a two-stage stochastic optimization problem with supply and demand uncertainties.

In the deployment stage, we first derive the expected total unmet demand given the demand occurrence locations in closed form and examine its properties by using a novel ranking function that characterizes the arrival sequence of reserved inventories. Then, we show that the structure of the optimal deployment policy, given a set of affected locations, is similar to a 'nested' policy with respect to the shadow price at each location, where a shadow price represents the marginal reduction of the expected total unmet demand due to one unit increase of the RI. In particular, the set of locations with the highest shadow price is a subset of the affected locations, and they retain all their own RIs under optimality; a deployment can only happen from one location with a smaller shadow price to another location with a larger shadow price; and each affected location sequentially receives deployments from other locations (including itself) according to their shadow prices from the highest to the lowest. We also show that when the delivery lead time from a location $\mathbf{m}$ to an affected location $\mathbf{n}$ increases, the central planner should either increase the deployment amount from the location(s) that is (are) closer to $\mathbf{n}$ (compared to $\mathbf{m}$) to satisfy the additional demand arising from the increased lead time, or decrease the deployment amount from the location(s) that is (are) at or farther away from $\mathbf{n}$ (again compared to $\mathbf{m}$) because of the reduction of the future demand. We then compare the expected total unmet demand under the optimal deployment policy under different stochastic demand processes.

In the pre-positioning stage, the optimal allocation quantity at a location is positive only if the expected shadow price at that location is equal to a certain constant value. In particular, we show that when the Hessian matrix of the expected total unmet demand with respect to the allocation quantities is monotone, the optimal allocation quantity is increasing in the total amount of RIs. We then introduce the first order stochastic dominance on probability distributions that are defined on sets and characterize its equivalent representation using probability aggregation and disaggregation.

Using the properties of the first order stochastic dominance, we show that the expected total unmet demand is larger when one of the following is true: the demand surges tend to occur simultaneously at more locations, the probability distribution of the demand surge locations is more dispersed, the post-surge delivery takes a longer time, more demand arrives earlier, or the demand has a higher volatility. If more demand arrives earlier, then the amount of RIs that can be used to satisfy the demand is less due to the positive delivery lead time and thus, results in a larger expected total unmet demand. If the arrival demand is more volatile, then the probability that a larger demand arrives within a shorter time period is higher and hence, the expected total unmet demand is larger.

The remainder of this paper is organized as follows. We review the related literature in Section 2. The model formulation is discussed in Section 3. In Section 4, we derive the expected total unmet demand and analyze the optimal post-surge deployment policy. We characterize the properties of the optimal pre-surge allocation policy in Section 5 and discuss the impact of various factors on the expected total unmet demand. Section 6 concludes the paper. All the proofs are relegated to the online appendix.

2 Literature Review

Our work is closely related to the emerging studies of inventory planning for the random demand surge. Liu et al. (2016) consider pre-positioning reserved inventories with the objective to minimize operational cost while Huang et al. (2016) consider joint inventory stocking and capacity reserving problems for sudden demand surges. Our paper complements Liu et al. (2016) and Huang et al. (2016) by considering demand location uncertainty and delivery lead times. Also our focuses are different: we consider minimizing supply-demand mismatch while they consider cost minimization. Thus, our model is more suitable for the inventory products critical for human lives such as medicine and vaccine. Another related study is Wang et al. (2015), in which the authors investigate the relief inventory dispatch after the surge occurrence to minimize the unsatisfied demand. They decompose the dynamic deployment problem as a two multi-period multi-commodity network flow problems. Different from their work, we focus on finding the optimal RI pre-positioning and deployment policies by taking into account dynamics and uncertainties involved in the deployment process. Ni et al. (2018) use a min-max robust model to solve similar research questions for disaster response operations. However, in their paper, the demand surges are treated as random variables instead of time-evolving stochastic processes. Hence, they do not consider the transshipment lead time issue. In contrast, our model considers the transshipment lead times and captures the key features of the pre-positioning and deployment problem.

The benefit of reserving RIs is similar to that of physical/virtual inventory transshipment (see, e.g., Robinson 1990; Archibald and Thomas 1997; Rudi et al. 2001; Axsäter 2003; Dong and Rudi 2004; Zhang 2005; Herer and Tzur 2006; Sošić 2006; Hu et al. 2008; Zhao 2008; Huang and Sošić 2010; Liu et al. 2016). Under inventory transshipment, pooling benefits are achieved by sometimes serving demand at one location using inventory at another (see Paterson et al. 2011 for a review). Similarly, our paper benefits from pooling because the inventory reserve at one location can be sent to meet demand at other locations.

Studies on the lost-sales inventory system are also related. See Zipkin (2008a, 2008b), the review paper of Bijvank and Vis (2011), and the references therein for the related studies. Although results from the existing inventory management literature can be generally applied to pre-positioning of the reserved inventory, these studies usually base the optimization decision on performance measures such as cost, service level, and profit. In demand surge responses, particularly those related to humanitarian relief, the inventory management decision is primarily concerned with minimizing the unmet demand. This is different from the objectives found in the traditional inventory literature.

We note that our study can be applied to the inventory planning for disaster preparedness. The inventory pre-positioning has been a topic of interest in humanitarian organizations. Perhaps the most related work in this literature are Salmerón and Apte (2010) and Rawls and Turnquist (2010), where the former considers a stochastic model by incorporating uncertainties on disaster magnitudes while the latter addresses the pre-positioning problem by considering the uncertainty about if or where a natural disaster will occur. Neither, however, considers the time-evolving dynamics of supply and demand processes. Natarajan and Swaminathan(2014) study the optimal inventory procurement and replenishment in the presence of funding constraints under the context of humanitarian operations. The objective of our paper is also consistent with the central mission of humanitarian organizations to minimize the loss of life (Thomas 2003).

3 Model Setup

A central planner, such as a firm, a humanitarian organization, or a government, pre-positions M units of RIs to N stocking locations such as warehouses, denoted by $\mathcal{N} = \{\mathbf{1}, \mathbf{2}, ..., \mathbf{N}\}$, to anticipate sudden demand surges. Let $S_{\mathbf{n}}$ be the amount of RIs that the central planner allocates to $\mathbf{n}$. Then, $S_{\mathbf{1}} + S_{\mathbf{2}} + \cdots + S_{\mathbf{N}} = M$. At time zero, demand surge(s) occur(s) at a subset of locations $\eta \subseteq \mathcal{N}$ and then demands may arrive sequentially at these locations over time. The central planner uses the pre-positioned RIs to respond to the demand surge(s) within a certain time interval $[0, T]$. For example, in the disaster response stage, the first 72 hours are crucial for the humanitarian organizations/governments to plan and carry out the relief process (Duran et al., 2011); according to Centers for Disease Control and Prevention and American Water Works Association (2012), after the maximum respond time, large-scale international rescue teams arrive and meet all victims' needs. Suppose one unit of demand requires one unit of RI, and the demand is lost if he/she does not immediately receive the RI. The central planner needs to decide the allocation and deployment of the RIs before and after the surge(s) occur(s).

We assume that the amount of RIs available for deployment at a location is uncertain and depends on where the demand surge(s) occur(s). In general, the loss of RIs are mainly caused by two factors: bad management (e.g., mis-storage or misplacement) during the regular time, and the demand surge itself. At location $\mathbf{n}$, let $\nu_{\mathbf{n}} \in [0,1]$ denote the fraction of RIs available in the regular time due to the first factor and $\alpha_{\mathbf{n}} \in [0,1]$ be the fraction of RIs that would survive after the occurrence of demand surge. Then, the fraction of RIs available for deployment at location $\mathbf{n}$ given the set of demand surge locations η, denoted by $\nu_{\mathbf{n}}^{\eta} \in [0,1]$, can be expressed as $\nu_{\mathbf{n}}^{\eta} = \nu_{\mathbf{n}} - \nu_{\mathbf{n}}(1-\alpha_{\mathbf{n}})\mathbf{1}(\mathbf{n} \in \eta)$, where $\mathbf{1}(\cdot)$ is the indicator function. When $\nu_{\mathbf{n}} = 1$, $\nu_{\mathbf{n}}^{\eta} = 1-(1-\alpha_{\mathbf{n}})\mathbf{1}(\mathbf{n} \in \eta)$ and the loss of RIs only depends on whether the demand surge has occurred at location $\mathbf{n}$. When $\alpha_{\mathbf{n}} = 1$, $\nu_{\mathbf{n}}^{\eta} = \nu_{\mathbf{n}}$ and the loss of RIs is only due to the inventory mis-management in the regular time. Given the RI allocation in the pre-positioning stage, $\mathbf{S}$, and the set of affected locations, η, denote $s_{\mathbf{mn}}^{\eta}$ as the amount of RI deployed from the stocking location $\mathbf{m}$ to affected location $\mathbf{n}$. Then $s_{\mathbf{mn}}^{\eta} = 0$ for $\mathbf{n} \notin \eta$[1] and $\sum_{\mathbf{n}\in\eta} s_{\mathbf{mn}}^{\eta} = \nu_{\mathbf{m}}^{\eta} S_{\mathbf{m}}$, $\mathbf{m} \in \mathcal{N}$. Denote $\mathbf{s} = (s_{\mathbf{mn}}^{\eta})_{N\times N}$ as the deployment policy given η and the pre-positioning policy $\mathbf{S}$. Figure 1 illustrates the sequence of events regarding the dynamics of RI pre-positioning and supply before and during the demand surge.

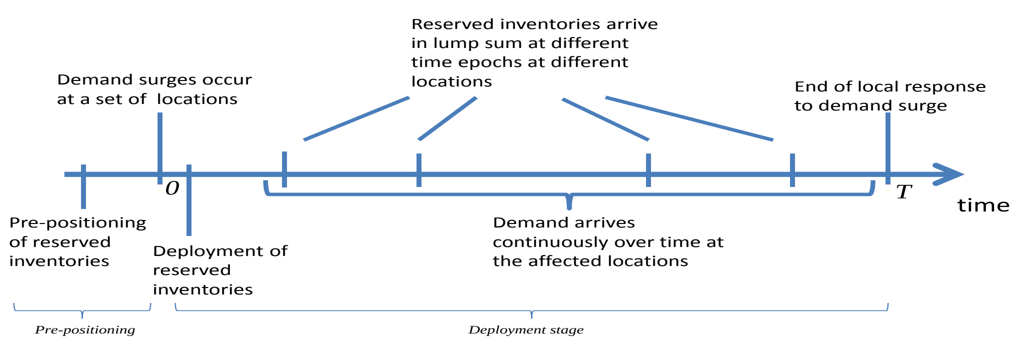

Figure 1: The Dynamics of RIs Before and After the Demand Surge

Our objective is to minimize the expected total unmet demand within the maximum response time T. Huang et al. (2016) consider a similar objective that maximizes the total demand that can

[1]It is obvious that deploying RIs to a location with no demand surge is suboptimal, therefore, $s_{\mathbf{mn}}^{\eta} = 0$ for $\mathbf{n} \notin \eta$.

be met immediately. Given deployment policy $\mathbf{s}$, denote the expected total unmet demand up to time T as $\mathcal{L}_d^{\eta}(\mathbf{s}|\mathbf{S}) \equiv \mathcal{L}_d^{\eta}(\mathbf{s})$. Then, at the deployment stage, the central planner solves the following optimization problem for the optimal deployment policy $\mathbf{s}^*$,

$$
\begin{aligned}
(\mathcal{D}) \qquad & \underset{\mathbf{s}}{\text{Min}} \;\; \mathcal{L}_d^{\eta}(\mathbf{s}) \\
& s.t. \quad \sum_{\mathbf{n}\in\eta} s_{\mathbf{mn}}^{\eta} = \nu_{\mathbf{m}}^{\eta} S_{\mathbf{m}}, \forall \eta \subseteq \mathcal{N}, \mathbf{m} \in \mathcal{N}, \\
& \qquad\;\; s_{\mathbf{mn}}^{\eta} \geq 0, \forall \eta \subseteq \mathcal{N}, \mathbf{m}, \mathbf{n} \in \mathcal{N}.
\end{aligned} \tag{1}
$$

Let $\mathcal{L}_p(\mathbf{S}) = \mathbb{E}_{\eta}\left[\mathcal{L}_d^{\eta}(\mathbf{s}^*)\right]$ be the expected total unmet demand up to time T under the optimal deployment policy. At the pre-positioning stage, the central planner solves for the optimal pre-positioning vector $\mathbf{S}^*$,

$$
\begin{aligned}
(\mathcal{P}) \qquad & \mathcal{L}^*(M) = \underset{\mathbf{S}}{\text{Min}} \;\; \mathcal{L}_p(\mathbf{S}), \\
& s.t. \quad \sum_{\mathbf{n}=\mathbf{1}}^{\mathbf{N}} S_{\mathbf{n}} = M, S_{\mathbf{n}} \geq 0, \forall \mathbf{n} \in \mathcal{N},
\end{aligned}
$$

where $\mathcal{L}^*(M)$ is the minimum expected total unmet demand given the total number of RIs.

Demand Surge and Time-evolving Process

Suppose conditioning on that some demand surges occur at time zero, the probability that they simultaneously occur at a set of locations η is P_{η}, $\eta \subseteq \mathcal{N}$. Denote

$$
\Pi = \left\{ \mathbf{P} \,\middle|\, \sum_{\eta \subseteq \mathcal{N}} P_{\eta} = 1; P_{\eta} \geq 0 \right\}
$$

as the set of all possible demand surge distribution $\mathbf{P}$s.[2] Clearly, Π is a convex and compact subset of $R^{2^{\mathcal{N}}}$. Here, we abuse the use of notation and let η also be the random variable defined on $(\mathcal{N}, 2^{\mathcal{N}}, \mathbf{P})$ representing the set of affected locations given that some demand surge has actually occurred. Once a demand surge has occurred at a location, it may result in different scenarios. Without loss of generality, we assume a single scenario setting where the magnitude of demand surge is fixed. The results also hold under the multiple scenarios where the demand-surge magnitude has multiple levels. We refer the readers to an earlier version of this paper for the detail of the multiple scenario setting. Let $(\mathcal{X}, \mathcal{F}, \mathbb{P})$ be a measurable space with probability measure $\mathbb{P}$. Denote $D_{\mathbf{n}} : [0, T] \times \mathcal{X} \to [0, \infty)$ as a measurable mapping representing the *cumulative* demand process at location $\mathbf{n}$ and $\mathcal{F}_t^{\mathbf{n}} = \sigma\{D_{\mathbf{n}}(s), 0 \leq s \leq t\}$ as the filtration generated by $D_{\mathbf{n}}(t)$. Let

[2] Here, we assume the demand surge has occurred at at least one location, that is, $\mathbf{P}(\emptyset) = 0$. This is because both pre-positioning and deployment decisions are made to reduce the demand loss during the surge, when there is no demand surge, i.e., $\eta = \emptyset$, the pre-positioning and deployment decisions become irrelevant.

$\mathcal{F} = \sigma\{\cup_{\mathbf{n}\in\mathcal{N}}\mathcal{F}_T^{\mathbf{n}}\}$. Then, $D_{\mathbf{n}}(t)$ is a random variable representing the cumulative number of demand arrivals up to time t if the surge occurs at $\mathbf{n}$, and $\{D_{\mathbf{n}}(t), t \in [0,T]\}$ is a continuous-time stochastic process defined on the common measurable space $(\mathcal{X}, \mathcal{F}, \mathbb{P})$ with *positive increments*. We further assume that $D_{\mathbf{n}}(t)$ has a density function on a continuous support. Examples of such stochastic processes include Gamma processes and log-normal processes, etc. Similar to Huang et al. (2016), we do not assume any parametric forms of the demand process in our analysis. For any time epoch $t \in [0,T]$, $D_{\mathbf{n}}(t)$ has a finite mean $\mu_{\mathbf{n}}(t)$ and a finite standard deviation $\sigma_{\mathbf{n}}(t)$. We also assume the boundary condition $D_{\mathbf{n}}(0) = 0$ almost everywhere (a.e.). Thus, $\mu_{\mathbf{n}}(0) = 0$ and $\sigma_{\mathbf{n}}(0) = 0$. Table 1 summarizes the frequently used notations.

4 Deployment Stage

In this section, we first characterize the expected total unmet demand and discuss its properties using a ranking function that is specific to the set of affected locations. Next, we discuss the impact of delivery lead times on the expected total unmet demand and the optimal deployment policy. Last, we discuss the impact of demand process on the expected total unmet demand.

4.1 Properties of Expected Total Unmet Demand and Optimal Deployment Policy

At time zero, when a demand surge occurs at location $\mathbf{n}$, $\mathbf{n} \in \eta$, we assume that other locations immediately deploy their RIs according to the deployment policy $\mathbf{s}$.[3] Denote $l_{\mathbf{mn}}$ as the shortest delivery lead time from location $\mathbf{m}$ to $\mathbf{n}$. We assume that $l_{\mathbf{mn}}$ is deterministic, $l_{\mathbf{nn}} = 0$ and $l_{\mathbf{mn}} > 0$ for all $\mathbf{n} \neq \mathbf{m}$, and the delivery lead times are symmetric: $l_{\mathbf{nm}} = l_{\mathbf{mn}}$. Without loss of generality, we also assume the delivery lead time is the shortest time among all possible delivery routes; that is, $l_{\mathbf{mn}} \leq l_{\mathbf{mk}} + l_{\mathbf{kn}}$, for all $\mathbf{m}, \mathbf{n}, \mathbf{k} \in \mathcal{N}$. Let $l \equiv (l_{\mathbf{nm}})_{N\times N}$ be the delivery lead time matrix for all locations. We define the following ranking functions to characterize the deployment process.

Location ranking function. Given $\mathbf{n} \in \mathcal{N}$, let $\mathbf{n}(k)$ be the location that has the kth shortest delivery lead time to $\mathbf{n}$, $k \in (1, 2, \cdots, N)$ (namely, the *location ranking function*). If there is a tie among the lead times, then the ranking of the tied locations can be arbitrary.[4] Naturally, $\mathbf{n}(1) = \mathbf{n}$; the location with the shortest delivery time to $\mathbf{n}$ is $\mathbf{n}$ itself, with a delivery lead time of $l_{\mathbf{nn}} = 0$. To facilitate our analysis, we define an artificial location ranked as the $(N+1)$th fastest to any location

[3] This strategy minimizes the total unmet demand with respect to any realized sample path of the demand arrival process so long as the delivered items at each location arrive in the same sequence as they are deployed. For the sake of space saving, we do not include the model and proofs of this result in the paper.

[4] The arbitrary assignment of tied locations will allow us to obtain one expression of the unmet demand given a deployment policy. However, it will not affect the value of the number of unmet demand.

Table 1: Summary of Frequently Used Notations

N	number of locations stockpiling the reserved inventory (RI)
M	total amount of reserved inventories
T	maximum response time
$\mathcal{N}$	inventory stockpiling location set $\{\mathbf{1}, \mathbf{2}, \dots, \mathbf{N}\}$
$\mathbf{m}, \mathbf{n}$	location indices, $\mathbf{m}, \mathbf{n} \in \mathcal{N}$
η	a subset of $\mathcal{N}$ representing the set of the affected locations
$\nu_{\mathbf{n}}^{\eta}$	the fraction of RIs available for deployment at location $\mathbf{n}$
$\nu_{\mathbf{n}}$	the fraction of RIs available in the regular periods at location $\mathbf{n}$
$\alpha_{\mathbf{n}}$	the fraction of RIs that would survive if demand surge occurs at location $\mathbf{n}$
$l_{\mathbf{mn}} = l_{\mathbf{nm}}$	shortest delivery lead time from $\mathbf{m}/\mathbf{n}$ to $\mathbf{n}/\mathbf{m}$, $l_{\mathbf{nn}} = 0$
l	delivery lead time matrix $(l_{\mathbf{mn}})_{N \times N}$
$\mathbf{n}(k)$	ranking function, the kth fastest delivery location to $\mathbf{n}$ $\mathbf{n}(1) = \mathbf{n}$ and $0 = l_{\mathbf{n}(1)\mathbf{n}} \leq l_{\mathbf{n}(2)\mathbf{n}} \leq \cdots < l_{\mathbf{n}(N)\mathbf{n}} \leq l_{\mathbf{n}(N+1)\mathbf{n}} \equiv T$. (when there is a tie among the delivery lead times, the ranking of tied locations can be arbitrary)
$\mathbf{n}^{-1}(\mathbf{m})$	the inverse function of $\mathbf{n}(k)$; if $\mathbf{n}(k) = \mathbf{m}$, then $\mathbf{n}^{-1}(\mathbf{m}) = k$
$\mathbb{R}^{N+}$	N-dimension non-negative real-value vector space $[0, \infty)^N$
$S_{\mathbf{m}}$	reserved inventories at location $\mathbf{m}$
$\mathbf{S}$	the pre-positioning vector of reserved inventories $(S_{\mathbf{1}}, S_{\mathbf{2}}, ...S_{\mathbf{N}})$
$\mathbf{S}^*$	optimal pre-positioning vector of reserved inventories $(S_{\mathbf{1}}^*, S_{\mathbf{2}}^*, ...S_{\mathbf{N}}^*)$
$s_{\mathbf{mn}}^{\eta}$	the amount of RIs deployed from $\mathbf{m}$ to $\mathbf{n}$ when demand surges occur at the set of locations η
$\mathbf{s}$	the deployment matrix given the set of affected locations η, $\mathbf{s} = (s_{\mathbf{mn}}^{\eta})_{N \times N}$
$\mathbf{s}^*$	the optimal deployment matrix given the set of affected locations η, $\mathbf{s} = (s_{\mathbf{mn}}^{\eta *})_{N \times N}$
P_{η}	probability of demand surge occurring at location set η conditioning on that surges have actually occurred, $0 < P_{\eta} < 1$, $\sum_{\eta \subseteq \mathcal{N}} P_{\eta} = 1$
$\mathbf{P}$	conditional demand surge probability distribution (P_{η})
$D_{\mathbf{n}}(t)$	cumulative demand up to time t given that a surge has occurred at location $\mathbf{n}$; $D_{\mathbf{n}}(0) = 0$
$\mu_{\mathbf{n}}(t), \sigma_{\mathbf{n}}(t)$	mean and standard deviation of $D_{\mathbf{n}}(t)$; $\mu_{\mathbf{n}}(0) = 0$, $\sigma_{\mathbf{n}}(0) = 0$
$D(\mathbf{n}, k)$	cumulative demand up to the end of the kth time bucket at location $\mathbf{n}$
$D_{\mathbf{n}k}$	number of demand arrived during the kth time bucket $[l_{\mathbf{n}(k)\mathbf{n}}, l_{\mathbf{n}(k+1)\mathbf{n}})$ at location $\mathbf{n}$
$\mathcal{L}_k^{\mathbf{n}}$	unmet demand up to time $l_{\mathbf{n}(k+1)\mathbf{n}}$ at location $\mathbf{n}$; $\mathcal{L}_0^{\mathbf{n}} = 0$
$\mathcal{L}_d^{\eta}(\mathbf{s})$	expected total unmet demand up to time T given the set of demand surge locations η and deployment policy $\mathbf{s}$
$\mathcal{L}_p(\mathbf{S})$	expected total unmet demand up to time T under the optimal deployment given RI allocation $\mathbf{S}$
$\mathcal{L}^*(M)$	expected total unmet demand up to time T under the optimal allocation $\mathbf{S}^*$

$\mathbf{n} \in \mathcal{N}$, and let $l_{\mathbf{n}(N+1)\mathbf{n}} \equiv T$.[5] For any location $\mathbf{n}$, the ranking function determines the sequence of supply arrivals, $0 = l_{\mathbf{n}(1)\mathbf{n}} < l_{\mathbf{n}(2)\mathbf{n}} \leq l_{\mathbf{n}(3)\mathbf{n}} \leq \cdots \leq l_{\mathbf{n}(N)\mathbf{n}} \leq l_{\mathbf{n}(N+1)\mathbf{n}} \equiv T$. For example, consider the ranking function for location $\mathbf{1}$, by definition $\mathbf{1}(1) = \mathbf{1}$. If $\mathbf{1}(2) = \mathbf{4}$, then the location with the 2nd shortest delivery lead time to $\mathbf{1}$ is $\mathbf{4}$ and it has a delivery lead time $l_{\mathbf{41}}$.

Ranking index function. Define $\mathbf{n}^{-1}(\mathbf{m})$, $\mathbf{n}, \mathbf{m} \in \mathcal{N}$ as the inverse function of $\mathbf{n}(k)$ (namely, the *ranking index function*). This generates a ranking index of $\mathbf{m}$ with respect to $\mathbf{n}$, according to the ordering of the delivery lead times required to deliver RIs to $\mathbf{n}$ from all locations. Following the aforementioned example, we have $\mathbf{1}^{-1}(\mathbf{1}) = 1$ and $\mathbf{1}^{-1}(\mathbf{4}) = 2$.

Given deployment policy $\mathbf{s}$, when a demand surge occurs at location $\mathbf{n} \in \eta$, RIs that are pre-positioned at $\mathbf{n}$, $s^{\eta}_{\mathbf{nn}}$, are immediately used to satisfy the arriving demands. In the meantime, other locations deliver their pre-positioned RIs to $\mathbf{n}$. When a delivery arrives, the total amount of RIs available at $\mathbf{n}$ would increase by a lump sum that equals the total amount of RIs that has just arrived. Thus, the supply-demand matching process at a specific surge location $\mathbf{n}$ can be divided into N time buckets with the kth bucket defined as $[l_{\mathbf{n}(k)\mathbf{n}}, l_{\mathbf{n}(k+1)\mathbf{n}})$, $k = 1, 2, \cdots, N$, which is the time interval between the arrival of the kth and $(k+1)$th delivery to $\mathbf{n}$. In such way, the total amount of RIs that can be used to satisfy the arrival demands at $\mathbf{n}$ within a given time bucket only depends on the inventory level at the beginning of that time bucket after the delivery has arrived. Denote $D_{\mathbf{n}k}$ as the number of demand arrivals at $\mathbf{n}$ that need the immediate RI supply within the kth time bucket, and let $D_{\mathbf{n}0} = 0$; denote $D(\mathbf{n}, k)$ as the cumulative demand up to the end of the kth time bucket. Then $D(\mathbf{n}, k) = D_{\mathbf{n}}(l_{\mathbf{n}(k+1)\mathbf{n}})$ and

$$D_{\mathbf{n}k} = D(\mathbf{n}, k) - D(\mathbf{n}, k-1) \geq 0 \ a.e..$$

If the available RIs at the beginning of a time bucket cannot meet all the demand arrivals within that time bucket, those who do not receive the RIs will be lost. Note that the terms *unmet demand* and *loss* are sometimes used interchangeably. Thus, the total unmet demand can be characterized by discretizing the supply-demand matching process into each time bucket.

Denote $\mathbf{s}_{(\mathbf{n})} = \left(s_{\mathbf{n}(1)\mathbf{n}}, s_{\mathbf{n}(2)\mathbf{n}}, \cdots, s_{\mathbf{n}(N)\mathbf{n}}\right)$ as the RI deployment vector with the elements sorted according to the ranking of the delivery lead times to $\mathbf{n}$ in *ascending order*. In this section, we drop the superscript η on $s^{\eta}_{\mathbf{mn}}$ for simplicity. Within the first time bucket $[l_{\mathbf{n}(1)\mathbf{n}}, l_{\mathbf{n}(2)\mathbf{n}}) = [0, l_{\mathbf{n}(2)\mathbf{n}})$, only the RIs at $\mathbf{n}$, $s_{\mathbf{n}(1)\mathbf{n}} = s_{\mathbf{nn}}$ units is available. Similarly, within the second time bucket $[l_{\mathbf{n}(2)\mathbf{n}}, l_{\mathbf{n}(3)\mathbf{n}})$, only the RIs at $\mathbf{n}$ (if any are left from the previous period) plus the RIs delivered from $\mathbf{n}(2)$ are available; in this case, $\left[s_{\mathbf{n}(1)\mathbf{n}} - D(\mathbf{n}, 1)\right]^{+} + s_{\mathbf{n}(2)\mathbf{n}}$ units are available, where $x^{+} = \max(x, 0)$.

[5] If the transportation infrastructure from $\mathbf{m}$ to $\mathbf{n}$ is totally destroyed, then $\mathbf{n}$ is inaccessible to $\mathbf{m}$ and thus, we let $l_{\mathbf{mn}} \equiv T$ (recall that T is the maximal response time).

Let $\mathcal{L}_k^{\mathbf{n}}$, $k = 0, 1, \cdots, N$ be the total unmet demand up to time $l_{\mathbf{n}(k+1)\mathbf{n}}$ at $\mathbf{n}$, and $\mathcal{L}_0^{\mathbf{n}} = 0$. Then, it can be computed recursively,

$$\mathcal{L}_0^{\mathbf{n}} = 0 \text{ and } \mathcal{L}_k^{\mathbf{n}} = \left[D(\mathbf{n}, k) - \mathcal{L}_{k-1}^{\mathbf{n}} - \sum_{i=1}^{k} s_{\mathbf{n}(i)\mathbf{n}} \right]^+ + \mathcal{L}_{k-1}^{\mathbf{n}}, \quad \forall k \in \{1, 2, \cdots, N\}, \mathbf{n} \in \eta$$

and expressed in the following closed form:

$$\mathcal{L}_k^{\mathbf{n}} = \max_{0 \le i \le k} \left\{ D(\mathbf{n}, i) - \sum_{v=1}^{i} s_{\mathbf{n}(v)\mathbf{n}} \right\}, \ \forall k \in \{1, 2, \cdots, N\}, \mathbf{n} \in \eta.$$

Moreover, $\mathcal{L}_k^{\mathbf{n}}$ is jointly convex and supermodular in $\mathbf{s}_{(\mathbf{n})}$ on $\mathbb{R}^{N+}$ and coordinatewise decreasing in $s_{\mathbf{mn}}$, $\mathbf{m} \in \mathcal{N}$. The closed form expression of $\mathcal{L}_k^{\mathbf{n}}$ enables us to derive $\mathcal{L}_d^{\eta}(\mathbf{s})$, the expected total unmet demand given the deployment policy $\mathbf{s}$, as stated in (1):

$$\mathcal{L}_d^{\eta}(\mathbf{s}) = \sum_{\mathbf{n} \in \eta} \mathbb{E}_{\mathbb{P}} \left[\max_{0 \le k \le N} \left\{ D(\mathbf{n}, k) - \sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}} \right\} \right].$$

To facilitate the further analysis, we define

$$\sum_{u=1}^{0} a_u = 0, \quad \sum_{u=i+1}^{i} a_u = 0, \quad \text{and} \quad \sum_{u=r+1}^{i} a_u = -\sum_{u=i+1}^{r} a_u \text{ if } r > i,$$

for a given sequence of real numbers $\{a_u\}_{u=1}^{\infty}$. Define $\pi^{\eta}(i)$, $i = 1, 2, ...$ as a set of locations with the ith largest shadow price under the optimal deployment policy, and $(\pi^{\eta})^{-1}(\mathbf{m})$ as the rank of the shadow price at location $\mathbf{m}$. $(\pi^{\eta})^{-1}(\mathbf{m}) = i$, $\mathbf{m} \in \mathcal{N}$, if and only if $\mathbf{m} \in \pi^{\eta}(i)$. The following proposition summarizes the structure of the optimal deployment policy.

Proposition 1. *The optimal deployment policy satisfies (1) $\pi^{\eta}(1) \subseteq \eta$; (2) for any $\mathbf{n} \in \pi^{\eta}(1)$, we have $s_{\mathbf{nn}}^* = S_{\mathbf{n}}$; (3) for any $\mathbf{m} \in \pi^{\eta}(i)$, $s_{\mathbf{mn}}^* > 0$ only if $(\pi^{\eta})^{-1}(\mathbf{n}) \le i$; and (4) given any affected location $\mathbf{n} \in \eta$, for all k such that $s_{\mathbf{n}(k)\mathbf{n}}^* > 0$, $(\pi^{\eta})^{-1}(\mathbf{n}(k))$ is increasing in k.*

Proposition 1 provides the central planner some information about the priority level (or importance level) of RIs at different locations. The first statement of Proposition 1 shows that the set of locations with the largest shadow price (highest priority) is a subset of the affected locations. The second statement implies that those affected locations with the largest shadow price keep all their own RIs and do not deploy anything to other affected locations. Note that the affected locations that are not in this subset may still deploy some RIs to other affected locations under optimality. The third statement indicates that a deployment can only happen from one location with a smaller shadow price to another location with a larger shadow price. The last statement shows that when RIs must be deployed from multiple source locations to one affected location, those deployments must be arranged in a descending order according to the source locations' shadow prices.

4.2 Impact of Delivery Lead Time on Optimal Deployment Policy

For any demand surge location $\mathbf{n} \in \eta$, if the delivery lead time $l_{\mathbf{mn}}$ increases, the cumulative demand up to the $(\mathbf{n}^{-1}(\mathbf{m}) - 1)$th time bucket, $D(\mathbf{n}, \mathbf{n}^{-1}(\mathbf{m}) - 1)$ increases. Thus, one would expect that the expected unmet demand would increase. As a result, one would also expect that it is optimal to send more RIs to $\mathbf{n}$ for deployments that arrive earlier than $l_{\mathbf{mn}}$ to cover the demand up to the $(\mathbf{n}^{-1}(\mathbf{m}) - 1)$th time bucket and less for those that arrive at $l_{\mathbf{mn}}$ or later. However, this statement is not always true. To see this, we present the impact of the delivery lead time on the expected total unmet demand and its first-order derivatives with respect to any deployment policy in the following Proposition 2. We then show the impact of the delivery lead time on both the optimal expected total unmet demand and the optimal deployment policy in Proposition 3.

Proposition 2. *When demand surge occurs at the location set η, given $\mathbf{n} \in \eta$, (1) the expected total unmet demand $\mathcal{L}_d^\eta(\mathbf{s})$ is increasing in $l_{\mathbf{mn}}$, (2) the marginal expected total unmet demand, $\frac{\partial \mathcal{L}_d^\eta(\mathbf{s})}{\partial s_{\mathbf{n}(i)\mathbf{n}}}$, is decreasing in $l_{\mathbf{mn}}$ for all $i = 1, ..., \mathbf{n}^{-1}(\mathbf{m}) - 1$ but increasing in $l_{\mathbf{mn}}$ for all $i = \mathbf{n}^{-1}(\mathbf{m}), ..., N$, and (3) when $l_{\mathbf{mn}}$ increases, the change in $\frac{\partial \mathcal{L}_d^\eta(\mathbf{s})}{\partial s_{\mathbf{n}(i)\mathbf{n}}}$ is decreasing in both $i = 1, ..., \mathbf{n}^{-1}(\mathbf{m}) - 1$ and $i = \mathbf{n}^{-1}(\mathbf{m}), ..., N$.*

The first statement of Proposition 2 shows that when the delivery lead time $l_{\mathbf{mn}}$ increases, the expected loss would increase because the cumulative demand $D(\mathbf{n}, \mathbf{n}^{-1}(\mathbf{m}) - 1)$ increases for any sample path. The second statement implies that when the delivery from a location $\mathbf{m}$ to an affected location $\mathbf{n}$ is lengthened, those RIs delivered earlier from other locations become more important (because they need to cover demands in a longer time span) while those RIs delivered later from other locations become less important (because they only need to cover demands in a shorter time span). The third statement indicates that lengthening leadtime $l_{\mathbf{mn}}$ has a larger impact on those deployments happening nearby the deployment from $\mathbf{m}$ to $\mathbf{n}$ than those farther away from it. This is because the deployments from a closer location are more likely to be used as $l_{\mathbf{mn}}$ increases, whereas those from a farther location are less likely to be used. In other words, as $l_{\mathbf{mn}}$ increases, the deployments arriving before $l_{\mathbf{mn}}$ become more important and those arriving at or later than $l_{\mathbf{mn}}$ become less important. This impact, however, decreases as the locations are ranked away from $\mathbf{m}$.

The monotonic impact of the delivery lead times on the expected total unmet demand under the optimal deployment policy can be characterized by adopting the stochastic comparison arguments. Intuitively, the longer the delivery lead times, the larger and more uncertain the cumulative demand up to the arrival of the RIs, and hence the larger the expected total unmet demand. However, it is notoriously difficult to examine the impact of the delivery lead times on the optimal deployment policy because, despite the expected total unmet demand is convex that ensures the optimal solution can be found by local search, we still need to evaluate and compare the change of the expected total unmet demand in every feasible direction with respect to the change of the deployment policy. The comparison of the change of the expected total unmet demand in different directions requires additional mathematical properties besides joint convexity. However, the expected total unmet demand, given an allocation policy $S_{\mathbf{m}}$, $\mathbf{m} \in \mathcal{N}$, is neither supermodular nor submodular in the deployment policy (the feasible domain of the deployment policy is not a sub-lattice and it cannot be converted to a sub-lattice using simple linear transformations), and it is also not multimodular in the deployment matrix and delivery lead times (see Altman et al. (2003) for the definition of multimodularity) and cannot be converted to a multimodular function through simple linear transformations. Consequently, we are only able to partially characterize the impact of the delivery lead times on the optimal deployment policy by applying the second-order properties given in Proposition 2.

Below, we provide the monotonicity result of the expected total unmet demand and the optimal deployment policy with respect to the delivery lead times.

Proposition 3. *Assume that demand surges occur at the location set η and for any location $n \in \eta$, if the delivery lead time from $\mathbf{m}$ to $\mathbf{n}$, $l_{\mathbf{mn}}$ increases, (1) the optimal expected total unmet demand $\mathcal{L}_d^{\eta}(\mathbf{s}^*)$ increases. Moreover, if the optimal deployment policy is unique, then we have (2) either for any location $\mathbf{m}'$ that satisfies $\mathbf{n}^{-1}(\mathbf{m}') < \mathbf{n}^{-1}(\mathbf{m})$, at least one of the optimal deployment quantities $s^*_{\mathbf{m}'\mathbf{n}}$ increases, or for any location $\mathbf{m}'$ that satisfies $\mathbf{n}^{-1}(\mathbf{m}') \geq \mathbf{n}^{-1}(\mathbf{m})$, at least one of the optimal deployment quantities $s^*_{\mathbf{m}'\mathbf{n}}$ decreases, and (3) there exists a $k < \mathbf{n}^{-1}(\mathbf{m})$ such that the cumulative optimal deployment quantities $\sum_{v=1}^{k} s^*_{\mathbf{n}(v)\mathbf{n}}$ increases, and/or there exists a $k \geq \mathbf{n}^{-1}(\mathbf{m})$ such that the cumulative optimal deployment quantities $\sum_{v=1}^{k} s^*_{\mathbf{n}(v)\mathbf{n}}$ decreases.*

The first statement of Proposition 3 implies that the optimal expected total unmet demand increases in the delivery lead times. This is because for any given deployment policy, when the delivery lead time from location $\mathbf{m}$ to $\mathbf{n}$ increases, the demand in the $(\mathbf{n}^{-1}(\mathbf{m}) - 1)$th bucket is larger and hence the expected loss is higher. Moreover, the second statement suggests that if we increase $l_{\mathbf{mn}}$, two cases may occur under optimality. One possibility is that when the delivery lead time from location $\mathbf{m}$ to $\mathbf{n}$ increases, then the delivery quantity from some location $\mathbf{m}'$ that is closer to location $\mathbf{n}$ than $\mathbf{m}$ would increase. The increment is then used to cover the additional demand in the $(\mathbf{n}^{-1}(\mathbf{m}) - 1)$th bucket at $\mathbf{n}$ due to the increase of the delivery lead time $l_{\mathbf{mn}}$.

Another possibility is that when the delivery lead time from location $\mathbf{m}$ to $\mathbf{n}$ increases, the delivery quantity from some location $\mathbf{m}'$ that is farther away from location $\mathbf{n}$ than $\mathbf{m}$ would decrease. This is because the delivery quantities that arrive later than $l_{\mathbf{mn}}$ has less value since the demand in the $\mathbf{n}^{-1}(\mathbf{m})$th bucket decreases. Consequently, we can obtain the similar result with respect to the cumulative deployment quantities at the surge location $\mathbf{n}$ when $l_{\mathbf{mn}}$ increases, which is stated in the last statement of Proposition 3.

Recall that the expected total unmet demand $\mathcal{L}_d^{\eta}$ can be fully characterized by the cumulative demand up to the kth time bucket $D(\mathbf{n}, k)$, $k = 1, ..., T+1$, and the location ranking functions of the affected locations. As a result, analyzing the impact of delivery lead times is equivalent to analyzing the demand processes and the location ranking functions. This is because one can normalize all lead times such that the time length between successive RI arrivals is one and rescale the demand process so that the demand in each time bucket is equivalent to the demand in the corresponding original unnormalized time unit. Thus, below we focus on analyzing the impact of demand processes.

4.3 Impact of Demand Processes on the Expected Total Unmet Demand

For any demand surge location $\mathbf{n} \in \eta$, the expected total unmet demand $\mathcal{L}_d^{\mathbf{n}}$ is an increasing and convex function of $D(\mathbf{n}, i)$, $i = 1, 2...N$, the cumulative demand up to the end of the ith time bucket. Thus, the expected total unmet demand increases if more demands arrive earlier or the demand arrival process is more uncertain. To see this, we first provide the following definition.

Definition 1. *For any two continuous-time stochastic processes $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$*[6]*,*

1. *the stochastic process $\{X(t), t \geq 0\}$ first-order stochastically dominates the stochastic process $\{Y(t), t \geq 0\}$, i.e., $X(\cdot) \succeq_{st} Y(\cdot)$, if and only if given any $t = \xi$, random variables $X(\xi)$ and $Y(\xi)$ satisfy $X(\xi) \geq_{st} Y(\xi)$;*

2. *the stochastic process $\{X(t), t \geq 0\}$ second-convex-order stochastically dominates the stochastic process $\{Y(t), t \geq 0\}$, i.e., $X(\cdot) \succeq_{cx} Y(\cdot)$, if and only if given any $t = \xi$, random variables $X(\xi)$ and $Y(\xi)$ satisfy $X(\xi) \geq_{cx} Y(\xi)$;*

3. *the stochastic process $\{X(t), t \geq 0\}$ increasingly second-convex-order stochastically dominates the stochastic process $\{Y(t), t \geq 0\}$, i.e., $X(\cdot) \succeq_{icx} Y(\cdot)$, if and only if given any $t = \xi$, random variables $X(\xi)$ and $Y(\xi)$ satisfy $X(\xi) \geq_{icx} Y(\xi)$.*[7]

[6] See Shaked and Shanthikumar (2006) for the definition of $\geq_{st}$, $\geq_{cx}$ and $\geq_{icx}$.

[7] Alternatively, parts (1) - (3) can be defined using sample path arguments. Take part 1 for example: $X(\cdot) \succeq_{st} Y(\cdot)$,

Suppose the demand surge occurs at $\mathbf{n} \in \eta$. It can be easily shown that if two demand processes $D^1_{\mathbf{n}}(\cdot)$ and $D^2_{\mathbf{n}}(\cdot)$ satisfy $D^1_{\mathbf{n}}(\cdot) \succeq_{st} D^2_{\mathbf{n}}(\cdot)$, $D^1_{\mathbf{n}}(\cdot) \succeq_{cx} D^2_{\mathbf{n}}(\cdot)$, or $D^1_{\mathbf{n}}(\cdot) \succeq_{icx} D^2_{\mathbf{n}}(\cdot)$, $\mathcal{L}^{\eta}_d(\mathbf{s}^*)$ is larger under $D^1_{\mathbf{n}}(\cdot)$ than under $D^2_{\mathbf{n}}(\cdot)$. In particular, when demand arrives earlier, the demand process is stochastically larger and hence, $\mathcal{L}^{\eta}_d(\mathbf{s}^*)$ is larger. Let $\kappa_{\mathbf{n}} \geq 0$ be the largest possible cumulative demand up to time T, that is,

$$\kappa_{\mathbf{n}} = \inf\{\kappa | D_{\mathbf{n}}(T) \leq \kappa \ a.e.\}.$$

Then, the expected total unmet demand $\mathcal{L}^{\eta}_d(\mathbf{s}^*)$ takes the largest value $\sum_{\mathbf{n}\in\eta}[\kappa_{\mathbf{n}} - s^*_{\mathbf{nn}}]^+$ when the demands at the affected locations occur only at time zero and equals $\kappa_{\mathbf{n}}$. This provides an upper bound for $\mathcal{L}^{\eta}_d(\mathbf{s}^*)$ when the support of the demand process is known.

$\mathcal{L}^{\eta}_d(\mathbf{s}^*)$ is also larger when the demand process is more uncertain. By applying Lemma 1 from Feng and Shanthikumar (2017), we can construct the following demand process to generate the largest uncertainty while fixing $\mu_{\mathbf{n}}(t)$, the mean of the cumulative demand process: $D_{\mathbf{n}}(t)$ is a demand process where customers arrive in a lump sum of size $\kappa_{\mathbf{n}}$ (the largest possible demand size) with the probability that the demand arrives before t is $\mu_{\mathbf{n}}(t)/\kappa_{\mathbf{n}}$. By constructing this process, $\mathcal{L}^{\eta}_d(\mathbf{s}^*)$ takes the largest value $\sum_{\mathbf{n}\in\eta}\int_0^T \left(\kappa_{\mathbf{n}} - \sum_{v=1}^{k(t)} s^*_{\mathbf{n}(v)\mathbf{n}}\right)^+ d\mu_{\mathbf{n}}(t)/\kappa_{\mathbf{n}}$, where $k(t) \in \{1, 2, 3...\}$ and satisfies $l_{\mathbf{n}(k(t))\mathbf{n}} \leq t < l_{\mathbf{n}(k(t)+1)\mathbf{n}}$. This provides an upper bound for $\mathcal{L}^{\eta}_d(\mathbf{s}^*)$ when the mean of the demand process is known.

For the rest of the paper, we assume that the demand process $\{D_{\mathbf{n}}(t), t \geq 0\}$ at location $n \in \eta$ is independent of η, the set of demand surge locations. We have the following result.

Proposition 4. *For any two sets of affected locations $\eta \subseteq \eta' \subseteq \mathcal{N}$, we must have $\mathcal{L}^{\eta}_d(\mathbf{s}^*) \leq \mathcal{L}^{\eta'}_d(\mathbf{s}^*)$.*

The following corollary provides comparisons of the expected total unmet demand when the demand processes can be transformed from one to another by rescaling the time t.

Corollary 1. *For any two demand processes $\bar{D}^1_{\mathbf{n}}(t)$ and $\bar{D}^2_{\mathbf{n}}(t)$, if there exist two increasing functions $g_1(t)$ and $g_2(t)$, $g_1(t) \geq g_2(t) \geq 0$ for all $t \geq 0$ such that for some demand process $\bar{D}_{\mathbf{n}}(t)$, $\bar{D}^1_{\mathbf{n}}(t)$ has the same distribution as $\bar{D}_{\mathbf{n}}(g_1(t))$ and $\bar{D}^2_{\mathbf{n}}(t)$ has the same distribution as $\bar{D}_{\mathbf{n}}(g_2(t))$ for all $t \geq 0$, then $\mathcal{L}^{\eta}_d(\mathbf{s}^*)$ is larger under $\bar{D}^1_{\mathbf{n}}(t)$ than under $\bar{D}^2_{\mathbf{n}}(t)$.*

5 Pre-positioning Stage

In this section, we discuss the allocation of RIs before demand surge occurs. We identify the conditions under which the optimal allocation is coordinate-wise increasing in the total amount of

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if and only if given any $t = \xi$, there exist random variable X_ξ that has the same distribution as $X(\xi)$ and random variable Y_ξ that has the same distribution as $Y(\xi)$ such that $X_\xi \geq Y_\xi$ in sample path.

RIs. We also introduce the first order stochastic dominance (FOSD) on the probability distribution of the surge locations and show that the expected total unmet demand increases in the order of FOSD on the demand surge occurring probabilities.

5.1 The Optimal RI Allocation and Its Monotonicity Property

According to the constrained deployment problem ($\mathcal{D}$) (see §3 and §4), $\mathcal{L}_d^\eta(\mathbf{s}^*)$ is jointly convex in $\mathcal{S}$ and coordinatewise decreasing in $S_\mathbf{n}$, $\mathbf{n} \in \mathcal{N}$. Note that at the pre-positioning stage, before the demand surge occurs, the expected total unmet demand, $\mathcal{L}_p(\mathbf{S}) = \mathbb{E}_\eta\left[\mathcal{L}_d^\eta(\mathbf{s}^*)\right]$, is a linear combination of $\mathcal{L}_d^\eta(\mathbf{s}^*)$. Thus, $\mathcal{L}_p(\mathbf{S})$ is also jointly convex in $\mathcal{S}$ and coordinatewise decreasing in $S_\mathbf{n}$, $\mathbf{n} \in \mathcal{N}$. Given an allocation policy, let $\lambda = \mathbb{E}_\eta\left[\nu_\mathbf{n}^\eta \pi_\mathbf{n}\right]$ be the expected shadow price of location $\mathbf{n}$ under the optimal deployment policy. Because the shadow prices are non-negative and $\mathcal{L}_d^\eta(\mathbf{s}^*)$ is jointly convex in $\mathbf{S}$, it can be shown that $\mathcal{L}^*(M) = \mathcal{L}_p(\mathbf{S}^*)$ is always decreasing and convex in M according to the constrained optimization problem ($\mathcal{P}$). Thus, under optimality, the expected total unmet demand is decreasing in the total amount of RIs. However, the marginal reduction of the expected total unmet demand decreases as the number of RI increases.

Because $\mathcal{L}_p(\mathbf{S})$ is jointly convex in $\mathbf{S}$, for the pre-positioning problem ($\mathcal{P}$), the KKT conditions are also sufficient conditions. Therefore, the shadow price under optimality λ^* is a constant independent of location n. The KKT conditions for the optimal allocation policy $\mathbf{S}^*$ are

$$\frac{\partial \mathcal{L}_p(\mathbf{S}^*)}{\partial S_\mathbf{n}} = -\lambda^* \text{ if } S_\mathbf{n}^* > 0; \quad \text{and} \quad \frac{\partial \mathcal{L}_p(\mathbf{S}^*)}{\partial S_\mathbf{n}} \geq -\lambda^* \text{ if } S_\mathbf{n}^* = 0.$$

Thus, under optimality, the locations that store positive amounts of RIs have identical partial derivative values that equal the negative shadow price, i.e., $-\lambda^*$.

Below we provide sufficient conditions under which the optimal allocation is increasing in M, the total amount of RIs.

Proposition 5. *The optimal allocation is coordinate-wise increasing in M if the following matrix*

$$\begin{pmatrix} \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_\mathbf{1} \partial S_\mathbf{1}} & \cdots & \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_\mathbf{1} \partial S_\mathbf{N}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_\mathbf{N} \partial S_\mathbf{1}} & \cdots & \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_\mathbf{N} \partial S_\mathbf{N}} \end{pmatrix}^{-1} \begin{pmatrix} \epsilon \\ \vdots \\ \epsilon \end{pmatrix} \tag{2}$$

is element-wise non-negative for some $\epsilon > 0$ and all $\mathbf{S} \geq \mathbf{0}$. In particular, the Hessian matrix of $\mathcal{L}_p(\mathbf{S})$ satisfies the above sufficient condition if it is monotone[8] *for all $\mathbf{S} \geq \mathbf{0}$. Suppose demand surge only occurs at one location; then there exists an optimal allocation policy that is coordinate-wise*

[8] A matrix is *monotone* if its inverse is element-wise positive (see Mangasarian 1968 for equivalent definitions of monotone matrices).

increasing in M if there exists a set of vertical $\{0,1\}$ vectors u_k, $k=1,...,K$ such that the Hessian matrix can be represented as

$$\left(\frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_{\mathbf{n}}^2}\right)_{N\times N} = \sum_{k=1}^{K} c_k u_k u_k^{'},$$

where $u_k^{'}$ is the transpose of u_k and $c_k > 0$ for all $k = 1,...,K$. Also, if $\mathbf{S}$ is a two dimensional vector, there always exists an optimal allocation policy that is coordinate-wise increasing in M.

Proposition 5 provides sufficient conditions under which the optimal allocation policy is coordinate-wise increasing in M. Note that Proposition 5 does not impose any (local) modularity conditions on the objective function. Instead, the sufficient condition implies that all of the row vectors of the inverse Hessian matrix lay in the half-space of the N-dimensional vector space, in which the inner product of the row vectors to $(1,...,1)_{1\times N}$ is non-negative. This condition is weaker than the objective function being $M^{\natural}$-convex, under which the inverse Hessian matrix is a diagonally dominant M-matrix.

If demand surge occurs only at a single location, then it is optimal to deploy all units to the demand surge location. Let $P_{\mathbf{n}}$ denote the probability of demand surge occurring at $\mathbf{n}$. Then, $\frac{\partial \mathcal{L}_p(\mathbf{S})}{\partial S_{\mathbf{n}}}$ can be explicitly expressed as

$$\frac{\partial \mathcal{L}_p(\mathbf{S})}{\partial S_{\mathbf{n}}} = -\sum_{\mathbf{m}\in\mathcal{N}} P_{\mathbf{m}} \nu_{\mathbf{n}}^{\mathbf{m}} \sum_{k=\mathbf{m}^{-1}(\mathbf{n})}^{N} \text{Prob}\left(\sum_{u=r+1}^{k} D_{\mathbf{m}u} \geq \sum_{u=r+1}^{k} \nu_{\mathbf{m}(u)}^{\mathbf{m}} S_{\mathbf{m}(u)}, \forall r = 0,1,\cdots,N\right).$$

The sufficient condition becomes the row diagonally dominant property on the Hessian matrix. To see this, consider $\{0,1\}$ column vectors u_{ij} $(i \neq j)$ with the ith and jth elements equal to 1 and the rest equal to 0, and u_{ii} with the ith element equal to 1 and the rest equal to zero. Then,

$$\left(\frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_{\mathbf{n}}^2}\right)_{N\times N} = \sum_{i,j=1}^{N} c_{ij} u_{ij} u_{ij}^{'},$$

where $c_{ij} = \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_{\mathbf{i}} \partial S_{\mathbf{j}}}$ for $\mathbf{i} \neq \mathbf{j}$ and $c_{ii} = \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_{\mathbf{i}}^2} - \sum_{\mathbf{j}\neq\mathbf{i}} \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_{\mathbf{i}} \partial S_{\mathbf{j}}}$.

5.2 Demand Surge Occurring Probability

We start this subsection with a two-location hurricane strike example ($\mathcal{N} = \{\mathbf{1},\mathbf{2}\}$). There exist various scenarios for the striking events. For the illustration purpose, we consider the following three typical ones. In the first scenario (Scenario 1), the demand surge only occurs at either one location with equal probability. That is, the demand surge probability distribution $\mathbf{P}^1$ is $P_{\{\mathbf{1}\}}^1 = P_{\{\mathbf{2}\}}^1 = 1/2$ and the hurricane only strikes either $\mathbf{1}$ or $\mathbf{2}$ but never both simultaneously. In the second scenario

(Scenario 2), the demand surge may simultaneously occur at both locations with the demand surge probability distribution $\mathbf{P}^2$ to be $P^2_{\{\mathbf{1},\mathbf{2}\}} = P^2_{\{\mathbf{1}\}} = P^2_{\{\mathbf{2}\}} = 1/3$. That is, hurricane strikes either **1**, or **2**, or both simultaneously, with an equal chance. In the third scenario (Scenario 3), the demand surges always simultaneously occur at both locations and the demand surge probability distribution $\mathbf{P}^3$ is $P^3_{\{\mathbf{1},\mathbf{2}\}} = 1$. That is, hurricane always simultaneously strikes locations **1** and **2**. It can be shown that, in this two-location example, as the demand surges occur more likely at more locations, the minimum expected total unmet demand becomes larger.

In general, is it always true that the expected total unmet demand is higher if demand surges are more likely to simultaneously occur at more locations? In order to answer this question, we need to compare the expected total unmet demand under different demand surge probability distributions, where the demand surge probability distributions are probability measures defined on the class of subsets of locations. As such, in this section, we develop stochastic comparison tools to compare probability distributions defined on sets. Specifically, we introduce the *first order stochastic dominance* ($\succeq_{FOSD}$) for distributions defined on sets. We will show that this stochastic order is parallel to that defined on real numbers in Shaked and Shanthilkumar (2006).

Below, we first provide the formal definition of the first order stochastic dominance for distributions defined on sets. We then show some of its properties and apply them to the conditional demand surge probability distributions to discuss their impact on the expected total unmet demand.

5.2.1 Definition of the First Order Stochastic Dominance for Distributions Defined on Sets

Define a binary random variable for location $i \in \mathcal{N}$, denoted by X_i, and let

$$X_i = \begin{cases} 1, & \text{when demand surge occurs at } i, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for any subset of locations $\eta \in 2^{\mathcal{N}}$, the conditional probability distribution $\mathbf{P} = (P_\eta) \in \Pi$ corresponds to an N dimensional 0-1 multivariate distribution $(X_1, ..., X_N)$, i.e.,

$$\mathbf{P}_\eta = \text{Prob}(X_i = 1, \mathbf{i} \in \eta; X_j = 0, \mathbf{j} \notin \eta).$$

Consequently, the distribution on set $\mathbf{P}$ captures the *joint distribution* of demand surges at all locations. As a result, defining a stochastic order for distributions defined on sets is equivalent to defining a multivariate order for random variables $(X_1, ..., X_N)$ defined on $\{0,1\}^N$. To better present our insights, below we directly define the first order stochastic dominance on the conditional probability distributions $\mathbf{P} \in \Pi$. However, we may refer to $\mathbf{P}$ and $(X_1, ..., X_N)$ interchangeably in the later discussion.

We say that a real valued set function f defined on $2^{\mathcal{N}}$ is increasing if and only if for any subsets $S \subseteq T \in 2^{\mathcal{N}}$, we have $f(S) \leq f(T)$. The definition of the first order stochastic dominance for distributions defined on sets is as follows.

Definition 2. *For any conditional probability distributions* $\mathbf{P}^1$ *and* $\mathbf{P}^2$ *defined on* $2^{\mathcal{N}}$, $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$ *if and only if for any increasing set function* f *defined on* $2^{\mathcal{N}}$, *we have* $E\left[f|\mathbf{P}^1\right] \geq E\left[f|\mathbf{P}^2\right]$.

Analogue to the first order stochastic dominance defined on real valued random variables in Shaked and Shanthikumar (2006), Definition 2 provides an equivalent definition of the first order stochastic dominance for random variables defined on sets.

Definition 2 also allows us to compare our FOSD order with the majorization orders defined in Xu and Li (2000). In particular, our FOSD order implies the majorization order from the roots ($\geq_{\mathcal{T}_r}$) defined in Xu and Li (2000) (see Definition 2.1) since for any set $K \subseteq \mathcal{N}$,

$$f_K(S) = \begin{cases} 1, & K \subseteq S, \\ 0, & \text{otherwise}, \end{cases}$$

is an increasing function defined on $2^{\mathcal{N}}$. However, the reverse is not necessarily true. Consider a three-location example with location set $\mathcal{N} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ and the following two conditional distributions: $\mathbf{P}^1(\mathbf{1}) = \mathbf{P}^1(\mathbf{2}) = \mathbf{P}^1(\mathbf{1}, \mathbf{2}) = \mathbf{P}^1(\mathbf{1}, \mathbf{3}) = \mathbf{P}^1(\mathbf{2}, \mathbf{3}) = 0.1$, $\mathbf{P}^1(\mathbf{3}) = 0.09$, and $\mathbf{P}^1(\mathbf{1}, \mathbf{2}, \mathbf{3}) = 0.41$; $\mathbf{P}^2(\mathbf{1}) = \mathbf{P}^2(\mathbf{2}) = \mathbf{P}^1(\mathbf{1}, \mathbf{3}) = 0.2$, $\mathbf{P}^1(\mathbf{1}, \mathbf{2}) = \mathbf{P}^1(\mathbf{3}) = 0$, $\mathbf{P}^1(\mathbf{2}, \mathbf{3}) = 0.1$, and $\mathbf{P}^1(\mathbf{1}, \mathbf{2}, \mathbf{3}) = 0.3$. Then, we have $\mathbf{P}^1 \geq_{\mathcal{T}_r} \mathbf{P}^2$. However, consider the increasing function

$$f(S) = \begin{cases} 1, & S \neq \{\mathbf{3}\}, \\ 0, & S = \{\mathbf{3}\}, \end{cases}$$

under which $E[f|\mathbf{P}^1] = 0.91 < E[f|\mathbf{P}^2] = 1$. Thus, $\mathbf{P}^1$ is not FOSD larger than $\mathbf{P}^2$.

For any conditional probability distribution $\mathbf{P} \in \Pi$, the majorization order from the roots in Xu and Li (2000) is equivalent to the upper orthant order in Shaked and Shanthilkumar (2006) (see equation 6.G.1) for the corresponding N dimensional 0-1 multivariate random variables. From the above discussion, our FOSD order also implies the upper orthant order. Moreover, the marginal distribution of $(X_1, ..., X_N)$ can be determined by $\mathbf{P}$ as $\text{Prob}(X_i = 1) = \sum_{\mathbf{i} \in S, S \subseteq \mathcal{N}} \mathbf{P}(S)$ for all $\mathbf{i} \in \mathcal{N}$. For the multivariate distribution $(X_1, ..., X_N)$ with the same marginal distribution, our FOSD order implies the positive quadrant order (PQD), see equations 9.A.1 and 9.A.15 in Shaked and Shanthilkumar (2006).

To better understand whether one distribution FOSD dominates the other distribution defined on sets, we first introduce two concepts, *disaggregation* and *aggregation*, for a given conditional demand surge probability distribution.

Definition 3. (Disaggregation) *For any conditional demand surge probability distribution* $\mathbf{P} \in \Pi$, *a probability distribution* $\bar{\mathbf{P}}$ *is a disaggregation of* $\mathbf{P}$ *on the multiset of* $2^{\mathcal{N}}$ *if it satisfies* $P_\eta = \sum_i \bar{P}_{\eta^i}$ *for any* $\eta \subseteq \mathcal{N}$, *where* η^i *is the ith duplicate of* η *and* $\bar{P}_{\eta^i}$ *is its probability.*

Disaggregation of a probability distribution basically keeps some event set unchanged but divide the event's probability into smaller ones. To illustrate, consider a disaggregation of $\mathbf{P}^1$ in the aforementioned two-location hurricane strike example. Suppose the hurricane strike can be classified into two event features: without flooding (denoted with superscript 1 on location set) and with flooding (denoted with superscript 2 on location set). Then, the following distribution is an example of disaggregation of $\mathbf{P}^1$: $\bar{P}^1_{\{\mathbf{1}\}^1} = 1/3$, $\bar{P}^1_{\{\mathbf{1}\}^2} = 1/6$, $\bar{P}^1_{\{\mathbf{2}\}^1} = 1/3$ and $\bar{P}^1_{\{\mathbf{2}\}^2} = 1/6$. In this way, 1/2, the overall probability of striking location **1** or **2** is divided into 1/3 for the event without flooding and 1/6 for the event with flooding.

We next introduce the concept *aggregation*.

Definition 4. (Aggregation) *For any disaggregation* $\mathbf{Q}$, *a conditional demand surge probability distribution* $\hat{\mathbf{Q}}$ *is an aggregation of* $\mathbf{Q}$ *if for some partition* $\{\Sigma_j, j = 1, 2, ...\}$ *of the multiset of* $2^{\mathcal{N}}$ *we have* $\hat{Q}_{\cup_{\eta^i \in \Sigma_j}} = \sum_{\eta^i \in \Sigma_j} Q_{\eta^i}$ *for all* $j = 1, 2, ...$

Let us continue with the disaggregation result of the above two-location hurricane example. Suppose people observe that hurricane with a flooding feature happens if and only if hurricane strikes locations **1** and **2** simultaneously. Hence, there is no need to separate these two locations for this scenario. One can use the union set $\{\mathbf{1}, \mathbf{2}\}$ to record the affected locations for the events with flooding feature. An aggregation of the disaggregation result in the foregoing two-location hurricane example is $\hat{\bar{P}}^1_{\{\mathbf{1},\mathbf{2}\}} = \bar{P}^1_{\{\mathbf{1}\}^2} + \bar{P}^1_{\{\mathbf{2}\}^2} = 1/3$, $\hat{\bar{P}}^1_{\{\mathbf{1}\}} = \bar{P}^1_{\{\mathbf{1}\}^1} = 1/3$ and $\hat{\bar{P}}^1_{\{\mathbf{2}\}} = \bar{P}^1_{\{\mathbf{2}\}^1} = 1/3$. This aggregation result is exactly $\mathbf{P}^2$. Hence, we can say that $\mathbf{P}^2$ can be generated from $\mathbf{P}^1$, by first disaggregation and then aggregation. Note that the outcome of conducting disaggregation or aggregation operations may not be unique. That is, multiple distributions can be generated through applying disaggregation-aggregation operations. The following proposition demonstrates that the generated distribution is FOSD larger than the original distribution.

Proposition 6. *For any* $\mathbf{P}^1, \mathbf{P}^2 \in \Pi$, $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$ *if and only if* $\mathbf{P}^1$ *can be generated from* $\mathbf{P}^2$ *by first applying disaggregation to* $\mathbf{P}^2$ *and then conducting aggregation over the disaggregation result. That is, there exists some disaggregation and aggregation of* $\mathbf{P}^2$ *such that* $\mathbf{P}^1 = \hat{\bar{\mathbf{P}}}^2$.

Proposition 6 implies that the conditional distribution that demand surges would occur at every location is FOSD larger than any other distributions. If one conditional demand surge probability distribution is FOSD larger than another conditional demand surge probability distribution, then *the demand surges under the former distribution would occur simultaneously at more locations with higher probabilities.* This is the intuitive understanding of the stochastic comparison of two distributions defined on sets. In the foregoing two-location hurricane example, $\mathbf{P}^3 \succeq_{FOSD} \mathbf{P}^2 \succeq_{FOSD} \mathbf{P}^1$.

Moreover, the proof of Proposition 6 implies that the FOSD order can be represented by the set of all simple transformations. Mathematically speaking, denote $\mathcal{E}_K$ as a set function that takes the value of one at K and zero at all other sets, i.e.,

$$\mathcal{E}_K(S) = \begin{cases} 1, & \text{if } S = K, \\ 0, & \text{otherwise.} \end{cases}$$

$\mathcal{T}_{\sigma,\rho,\delta_*}(\mathbf{P})$ is a simple transformation of $\mathbf{P}$ if $\mathcal{T}_{\sigma,\rho,\delta_*}(\mathbf{P}) = \mathbf{P} + \sum_{S\in\sigma,T\in\rho,S\subset T} \delta_{ST}(\mathcal{E}_T - \mathcal{E}_S)$ is a probability distribution on $2^{\mathcal{N}}$, where $\sigma, \rho \subseteq 2^{\mathcal{N}}, \sigma \cap \rho = \emptyset$, and $\delta_* = \{\delta_{ST} | \delta_{ST} \geq 0, S \in \sigma, T \in \rho\}$. Define the set of all simple transformations as $\mathcal{I} = \left\{\mathcal{T}_{\sigma,\rho,\delta_*} | \sigma, \rho \in 2^{\mathcal{N}}, \sigma \cap \rho = \emptyset, \delta_* \geq 0\right\}$. Then, $\mathbf{P}^2 \geq_{FOSD} \mathbf{P}^1$ if and only if there exists a $\mathcal{T}_{\sigma,\rho,\delta_*}$ such that $\mathcal{T}_{\sigma,\rho,\delta_*}(\mathbf{P}^1) = \mathbf{P}^2$. Below, we illustrate this simple transformation by referring back to the aforementioned two-location hurricane example. let $\rho = \{\{\mathbf{1}, \mathbf{2}\}\}$ be the singleton set that contains the set of two locations and $\sigma = \{\{\mathbf{1}\}, \{\mathbf{2}\}\}$ the set that contains two singleton sets of one location. Then, we have $T = \{\mathbf{1}, \mathbf{2}\}$ and $S = \{\mathbf{1}\}$ or $\{\mathbf{2}\}$. The simple transformation between the demand surge probability distributions $\mathbf{P}^1$ and $\mathbf{P}^2$ can be written as $\mathbf{P}^2 = \mathbf{P}^1 + \frac{1}{6}\left(\mathcal{E}_{\{\mathbf{1},\mathbf{2}\}} - \mathcal{E}_{\{\mathbf{1}\}}\right) + \frac{1}{6}\left(\mathcal{E}_{\{\mathbf{1},\mathbf{2}\}} - \mathcal{E}_{\{\mathbf{2}\}}\right)$. For example,

$$\mathbf{P}^2_{\{\mathbf{1}\}} = \mathbf{P}^1_{\{\mathbf{1}\}} + \frac{\left(\mathcal{E}_{\{\mathbf{1},\mathbf{2}\}}(\{\mathbf{1}\}) - \mathcal{E}_{\{\mathbf{1}\}}(\{\mathbf{1}\})\right)}{6} + \frac{\left(\mathcal{E}_{\{\mathbf{1},\mathbf{2}\}}(\{\mathbf{1}\}) - \mathcal{E}_{\{\mathbf{2}\}}(\{\mathbf{1}\})\right)}{6} = \frac{1}{2} + \frac{0-1}{6} + \frac{0-0}{6} = \frac{1}{3}.$$

Similarly, we can get $\mathbf{P}^3 = \mathbf{P}^2 + \frac{1}{3}\left(\mathcal{E}_{\{\mathbf{1},\mathbf{2}\}} - \mathcal{E}_{\{\mathbf{1}\}}\right) + \frac{1}{3}\left(\mathcal{E}_{\{\mathbf{1},\mathbf{2}\}} - \mathcal{E}_{\{\mathbf{2}\}}\right)$.

5.2.2 The Impact of Demand Surge Probability Distributions

We now apply the FOSD order defined by us for the distributions defined on sets to investigate the impact of demand surge probability distributions on the expected total unmet demand. First, our FOSD order also implies positive correlations of the 0-1 random variables $X_1, ..., X_N$.

Proposition 7. $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$ *implies that for any* $i_1, i_2, ..., i_m \in \mathcal{N}$,

1. $E[\Pi_{k=1}^m X_{i_k}]$ *is larger under* $\mathbf{P}^1$ *than under* $\mathbf{P}^2$.

2. $\sum_{k=1}^m X_{i_k}$ *is stochastically larger (in the sense of the univariate first order stochastic dominance defined in Shaked and Shanthilkumar 2006) under* $\mathbf{P}^1$ *than that under* $\mathbf{P}^2$ *for* $m = 1, ..., N$.

The proof is straightforward from the aggregation of the probability distributions and hence is omitted here. Note that the functions $h(x_{i_1}, ..., x_{i_m}) = x_{i_1}...x_{i_m},\ i_1, i_2, ..., i_m \in \mathcal{N}$ is a class of coordinate-wise increasing and supermodular functions defined on $\{0,1\}^N$. The first statement of Proposition 7, $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$ implies $E[h(X_{i_1}, ..., X_{i_m})|\mathbf{P}^1] \geq E[h(X_{i_1}, ..., X_{i_m})|\mathbf{P}^2]$ for all $i_1, i_2, ..., i_m \in \mathcal{N}$. Therefore, the FOSD order $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$ implies $E[g(X_1, ..., X_N)|\mathbf{P}^1] \geq E[g(X_1, ..., X_N)|\mathbf{P}^2]$ for all gs defined on $\{0,1\}^N$ that are linear combinations of the functions $h(x_{i_1}, ..., x_{i_m})$, $i_1, i_2, ..., i_m \in \mathcal{N}$ with non-negative coefficients.

While the conventional correlation of random variables emphasizes on X_is simultaneously taking a smaller value 0 and a larger value 1, our FOSD order emphasizes only on the event that the random variables X_is simultaneously taking the value of 1. This difference allows us to capture the increasing of the expected loss when demand surge occurs at more locations. The second statement of Proposition 7 implies that if the conditional demand surge probability distributions satisfy $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$, the probability that demand surge occurs at multiple locations is larger under $\mathbf{P}^1$.

When $\mathbf{P}$ is the conditional demand surge probability distribution, the probability of no disaster occurring is zero, that is, $\mathbf{P}(X_1 = 0, ..., X_N = 0) = 0$. Therefore, if $\mathbf{P}^1 \succ_{FOSD} \mathbf{P}^2$, then there exists a pair X_i and X_j such that the covariance $Cov(X_i, X_j)$, the correlation $Corr(X_i, X_j)$, and the conditional probability $P(X_i = 1|X_j = 1)$ are larger under $\mathbf{P}^1$ than those under $\mathbf{P}^2$. When $N = 2$, we have the following properties regarding the FOSD order.

Proposition 8. *If $N = 2$ and $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$ are two conditional demand surge probability distributions, $Cov(X_1, X_2)$, $Corr(X_1, X_2)$, and $P(X_1 = 1|X_2 = 1)$ are larger under $\mathbf{P}^1$ than those under $\mathbf{P}^2$.*

The proof is straightforward through the computation of each term and thus is omitted here. Propositions 4 and 6 jointly lead to the following result.

Proposition 9. *For any two demand surge occurring probabilities $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$, we have $\mathcal{L}^*(M)$ is larger under $\mathbf{P}^1$ than under $\mathbf{P}^2$.*

Proposition 9 implies that if demand surges tend to simultaneously occur at more locations, it is more difficult to satisfy demands. This result can be easily extended to the distributions which are mixtures of $\mathbf{P}^1$ and $\mathbf{P}^2$. Consider a mixture distribution $\mathbf{P}_\alpha = \alpha\mathbf{P}^1 + (1-\alpha)\mathbf{P}^2$ with a scalar $\alpha \in [0,1]$. Then, for any two distributions $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$, $\mathbf{P}_\alpha$ is stochastically larger in α in the FOSD order, i.e., $\mathbf{P}_{\alpha_1} \succeq_{FOSD} \mathbf{P}_{\alpha_2}$ for all $\alpha_1 > \alpha_2$. This coupled with Proposition 9 leads to the following Corollary 2, where we explicitly write $\mathcal{L}^*(M, \mathbf{P})$ to replace $\mathcal{L}^*(M)$ for the purpose of clarity.

Corollary 2. *For any conditional demand surge probability distributions* $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$*,* $\mathbf{P}^1, \mathbf{P}^2 \in \Pi$*,* $\mathcal{L}^*(M, \mathbf{P}_\alpha)$ *is increasing in* α*.*

6 Conclusion

In this study, we examine both the optimal pre-positioning and the optimal deployment decisions of the reserved inventories within a supply network. Our aim is to minimize the expected total unmet demand during a demand surge. We incorporate the dynamics of both the supply and demand processes by considering the demand processes as general stochastic processes and the supply processes as sequences of the arrival epochs of the RIs transhipped from the unaffected locations to affected locations within the supply network. We then derive the expected total unmet demand and describe its properties.

For the deployment problem (after the occurrence of the demand surge), we show that the optimal deployment policy is a 'nested'-type policy with respect to the shadow price of each location. A subset of the affected locations have the highest shadow prices. Under optimality, they keep their inventory and only deploy to themselves. Affected locations receive the RIs from the non-affected locations in the decreasing order of their shadow prices.

For the pre-positioning decision problem, the optimal allocation policy balances the expected shadow price of each location. By stochastic comparison, we show that the expected total unmet demand is larger if any of the following is true: the demand surges occur simultaneously at more locations, the probability distribution of the demand surge location is more dispersed, the post-surge delivery takes a longer time, more demand arrives at the early times, or the demand has a higher volatility. These findings, we believe, have deepened our understanding of the operations of RI pre-positioning and deployment.

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Online Appendix

"Pre-positioning and Deployment of Reserved Inventories in a Supply Network: Structural Properties "

In the proofs, for notational simplicity, we some times use the non-boldface letters to represent the location index of the corresponding boldface letters.

Proof of Proposition 1: We need the following two lemmas to prove Proposition 1.

Lemma 1. *$\partial\mathcal{L}_d^{\eta}(\mathbf{s})/\partial s_{\mathbf{mn}}$ exists if and only if*

$$Prob\left(\sum_{u=r}^{k} D_{\mathbf{n}u} = \sum_{u=r}^{k} s_{\mathbf{n}(u)\mathbf{n}}\right) = 0,$$

for all $r = 1, ..., N$ and $k = 0, ..., N$. If $\partial\mathcal{L}_d^{\eta}(\mathbf{s})/\partial s_{\mathbf{mn}}$ exists, then it has the following expression:

$$\frac{\partial\mathcal{L}_d^{\eta}(\mathbf{s})}{\partial s_{\mathbf{mn}}} = -\sum_{k=\mathbf{n}^{-1}(\mathbf{m})}^{N} Prob\left(\sum_{u=r+1}^{k} D_{\mathbf{n}u} > \sum_{u=r+1}^{k} s_{\mathbf{n}(u)\mathbf{n}}, \forall r = 0, 1, \cdots, N\right) \text{ for all } \mathbf{n} \in \eta \qquad (3)$$

and $\partial\mathcal{L}_d^{\eta}(\mathbf{s})/\partial s_{\mathbf{mn}} = 0$ for all $\mathbf{n} \in \mathcal{N}\backslash\eta$. In addition, if $D_{\mathbf{n}}(t)$ has a positive density function, then

$$\frac{\partial^2\mathcal{L}_d^{\eta}(\mathbf{s})}{\partial s_{\mathbf{mn}}^2} \geq \frac{\partial^2\mathcal{L}_d^{\eta}(\mathbf{s})}{\partial s_{\mathbf{mn}}\partial s_{\mathbf{m'n}}}, \forall \mathbf{m}, \mathbf{m'} \in \mathcal{N}, \mathbf{m} \neq \mathbf{m'}.$$

Lemma 1 illustrates how the expected total unmet demand would change with respect to the deployment quantity from $\mathbf{m}$ to $\mathbf{n}$. When the deployment quantity increases, the expected total unmet demand is reduced by the probability that this quantity is used in some time bucket after its arrival at location $\mathbf{n}$, which can be decomposed into the sum of probabilities that it is used in the kth time bucket for $k \geq \mathbf{n}^{-1}(\mathbf{m})$. Moreover, the second order property implies that the marginal effect of using RIs at location $\mathbf{m}$ to relieve demand at location $\mathbf{n}$ is more sensitive to the change of the inventory amount at $\mathbf{m}$ itself than that at another location $\mathbf{m}'$. This second order property can be utilized to show the monotonicity results for the optimal allocation decision in the latter part of the paper.

Proof of Lemma 1: Because $\mathcal{L}_d^{\eta}(\mathbf{s})$ is the sum of the expected total unmet demand at each affected location $\mathbf{n}$, by definition, the left and right derivative of $\mathcal{L}_d^{\eta}(\mathbf{s})$ with respect to $s_{\mathbf{mn}}$ are

$$\frac{\partial\mathcal{L}_d^{\eta-}(\mathbf{s})}{\partial s_{\mathbf{mn}}} = -\sum_{k=\mathbf{n}^{-1}(\mathbf{m})}^{N} \text{Prob}\left(D(\mathbf{n}, k) - \sum_{u=1}^{k} s_{\mathbf{n}(u)\mathbf{n}} \geq D(\mathbf{n}, r) - \sum_{u=1}^{r} s_{\mathbf{n}(u)\mathbf{n}},\ \forall\ r = 0, 1, 2, ...N\right)$$

and

$$\frac{\partial\mathcal{L}_d^{\eta+}(\mathbf{s})}{\partial s_{\mathbf{mn}}} = -\sum_{k=\mathbf{n}^{-1}(\mathbf{m})}^{N} \text{Prob}\left(D(\mathbf{n}, k) - \sum_{u=1}^{k} s_{\mathbf{n}(u)\mathbf{n}} > D(\mathbf{n}, r) - \sum_{u=1}^{r} s_{\mathbf{n}(u)\mathbf{n}},\ \forall\ r = 0, 1, 2, ...N\right),$$

respectively. Thus, $\frac{\partial \mathcal{L}_d^{\eta -}(\mathbf{s})}{\partial s_{\mathbf{mn}}} = \frac{\partial \mathcal{L}_d^{\eta +}(\mathbf{s})}{\partial s_{\mathbf{mn}}}$ if and only if

$$Prob\left(\sum_{u=r}^{k} D_{\mathbf{n}u} = \sum_{u=r}^{k} s_{\mathbf{n}(u)\mathbf{n}}\right) = 0,$$

for all $r = 1, ..., N$ and $k = 0, ..., N$. If $\frac{\partial \mathcal{L}_d^{\eta}(\mathbf{s})}{\partial s_{\mathbf{mn}}}$ exists,

$$\begin{aligned}
\frac{\partial \mathcal{L}_d^{\eta}(\mathbf{s})}{\partial s_{\mathbf{mn}}} &= -\sum_{k=\mathbf{n}^{-1}(\mathbf{m})}^{N} \text{Prob}\left(D(\mathbf{n},k) - \sum_{u=1}^{k} s_{\mathbf{n}(u)\mathbf{n}} \geq D(\mathbf{n},r) - \sum_{u=1}^{r} s_{\mathbf{n}(u)\mathbf{n}},\ \forall\ r = 0,1,2,...N\right) \\
&= -\sum_{k=\mathbf{n}^{-1}(\mathbf{m})}^{N} \text{Prob}\left(D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=1}^{k} s_{\mathbf{n}(u)\mathbf{n}} - \sum_{u=1}^{r} s_{\mathbf{n}(u)\mathbf{n}},\ \forall\ r = 0,1,2,...N\right) \\
&= -\sum_{k=\mathbf{n}^{-1}(\mathbf{m})}^{N} \text{Prob}\left(D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s_{\mathbf{n}(u)\mathbf{n}}, \forall\ r = 0,1,2,...N\right).
\end{aligned}$$

We now show how to prove the second-order derivative relationship. For any $\mathbf{s}_{(\mathbf{n})} \geq 0$, a scalar $\delta \geq 0$, and a sample path of $\{D_{\mathbf{n}}(t), 0 \leq t \leq T\}$, $\mathbf{n} \in \eta$, consider locations $\mathbf{m}, \mathbf{m}' \in \mathcal{N}$ and $\mathbf{m}' \neq \mathbf{m}$. Define $\mathbf{s}^{\mathbf{m}}_{(\mathbf{n})} = \left(s^{\mathbf{m}}_{\mathbf{n}(1)\mathbf{n}}, s^{\mathbf{m}}_{\mathbf{n}(2)\mathbf{n}}, \ldots, s^{\mathbf{m}}_{\mathbf{n}(N)\mathbf{n}}\right)$, where $s^{\mathbf{m}}_{\mathbf{n}(v)\mathbf{n}} = s_{\mathbf{n}(v)\mathbf{n}} + \delta$ if $\mathbf{n}(v) = \mathbf{m}$ and $s^{\mathbf{m}}_{\mathbf{n}(v)\mathbf{n}} = s_{\mathbf{n}(v)\mathbf{n}}$ if $\mathbf{n}(v) \neq \mathbf{m}$; $\mathbf{s}^{\mathbf{m}'}_{(\mathbf{n})} = \left(s^{\mathbf{m}'}_{\mathbf{n}(1)\mathbf{n}}, s^{\mathbf{m}'}_{\mathbf{n}(2)\mathbf{n}}, \ldots, s^{\mathbf{m}'}_{\mathbf{n}(N)\mathbf{n}}\right)$, where $s^{\mathbf{m}'}_{\mathbf{n}(v')\mathbf{n}} = s_{\mathbf{n}(v')\mathbf{n}} + \delta$ if $\mathbf{n}(v') = \mathbf{m}'$ and $s^{\mathbf{m}'}_{\mathbf{n}(v')\mathbf{n}} = s_{\mathbf{n}(v')\mathbf{n}}$ if $\mathbf{n}(v') \neq \mathbf{m}'$.

With respect to location $\mathbf{n}$, the rankings of locations $\mathbf{m}$ and $\mathbf{m}'$ are either (1) $\mathbf{n}^{-1}(\mathbf{m}') < \mathbf{n}^{-1}(\mathbf{m})$, or (2) $\mathbf{n}^{-1}(\mathbf{m}') > \mathbf{n}^{-1}(\mathbf{m})$. Now, first consider the case that $\mathbf{n}^{-1}(\mathbf{m}') < \mathbf{n}^{-1}(\mathbf{m})$. For any integer $r \in [0, N]$ and $k \geq \mathbf{n}^{-1}(\mathbf{m})$, we have the following three scenarios.

(i). $r \leq \mathbf{n}^{-1}(\mathbf{m}')$. Then $\sum_{u=r+1}^{k} s^{\mathbf{m}}_{\mathbf{n}(u)\mathbf{n}} = \sum_{u=r+1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(u)\mathbf{n}}$. Thus, $D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s^{\mathbf{m}}_{\mathbf{n}(u)\mathbf{n}}$ implies $D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(u)\mathbf{n}}$.

(ii). $\mathbf{n}^{-1}(\mathbf{m}') < r \leq \mathbf{n}^{-1}(\mathbf{m})$. Then $\sum_{u=r+1}^{k} s^{\mathbf{m}}_{\mathbf{n}(u)\mathbf{n}} = \sum_{u=r+1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(u)\mathbf{n}} + \delta$. Thus, $D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s^{\mathbf{m}}_{\mathbf{n}(u)\mathbf{n}}$ implies $D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(u)\mathbf{n}}$.

(iii). $r > \mathbf{n}^{-1}(\mathbf{m})$. Then $\sum_{u=r+1}^{k} s^{\mathbf{m}}_{\mathbf{n}(u)\mathbf{n}} = \sum_{u=r+1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(u)\mathbf{n}}$. Thus, $D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s^{\mathbf{m}}_{\mathbf{n}(u)\mathbf{n}}$ implies $D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(u)\mathbf{n}}$.

Therefore, the event $\left\{D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s^{\mathbf{m}}_{\mathbf{n}(u)\mathbf{n}}, \forall r = 0,1,...,N\right\}$ is a subset of the event $\left\{D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(u)\mathbf{n}}, \forall r = 0,1,...,N\right\}$. As this holds in general for any given sample path, for any $k \geq \mathbf{n}^{-1}(\mathbf{m})$,

$$\begin{aligned}
&\text{Prob}\left(D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s^{\mathbf{m}}_{\mathbf{n}(u)\mathbf{n}}, \forall r = 0,1,...,N\right) \\
\leq\ &\text{Prob}\left(D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(u)\mathbf{n}}, \forall r = 0,1,...,N\right).
\end{aligned}$$

Next, consider the case that $\mathbf{n}^{-1}(\mathbf{m}') > \mathbf{n}^{-1}(\mathbf{m})$. Note that

$$\begin{aligned}
&\sum_{k=\mathbf{n}^{-1}(\mathbf{m})}^{N} \text{Prob}\left(D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s_{\mathbf{n}(u)\mathbf{n}}, \forall r = 0,1,...,N \right) \\
&= \sum_{k=\mathbf{n}^{-1}(\mathbf{m})}^{N} \text{Prob}\left(D(\mathbf{n},k) - \sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}} \geq D(\mathbf{n},r) - \sum_{v=1}^{r} s_{\mathbf{n}(v)\mathbf{n}}, \forall r = 0,1,...,N \right) \\
&= \text{Prob}\left(\arg\max_{0\leq k\leq N} \left\{ D(\mathbf{n},k) - \sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}} \right\} \subseteq \left\{ \mathbf{n}^{-1}(\mathbf{m}), \cdots, N \right\} \right)
\end{aligned}$$

and the fact that $\sum_{v=1}^{k} s^{\mathbf{m}}_{\mathbf{n}(v)\mathbf{n}} \geq \sum_{v=1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(v)\mathbf{n}}$ for all $k \geq \mathbf{n}^{-1}(\mathbf{m})$, and $\sum_{v=1}^{k} s^{\mathbf{m}}_{\mathbf{n}(v)\mathbf{n}} = \sum_{v=1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(v)\mathbf{n}}$ for all $k < \mathbf{n}^{-1}(\mathbf{m})$. Thus, $D(\mathbf{n},k) - \sum_{v=1}^{k} s^{\mathbf{m}}_{\mathbf{n}(v)\mathbf{n}} \leq D(\mathbf{n},k) - \sum_{v=1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(v)\mathbf{n}}$ for all $k \geq \mathbf{n}^{-1}(\mathbf{m})$, and $D(\mathbf{n},k) - \sum_{v=1}^{k} s^{\mathbf{m}}_{\mathbf{n}(v)\mathbf{n}} = D(\mathbf{n},k) - \sum_{v=1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(v)\mathbf{n}}$ for all $k < \mathbf{n}^{-1}(\mathbf{m})$. Therefore, if

$$\arg\max_{0\leq k\leq N} \left\{ D(\mathbf{n},k) - \sum_{v=1}^{k} s^{\mathbf{m}}_{\mathbf{n}(v)\mathbf{n}} \right\} \subseteq \left\{ \mathbf{n}^{-1}(\mathbf{m}), \cdots, N \right\},$$

we can have $\arg\max_{0\leq k\leq N} \left\{ D(\mathbf{n},k) - \sum_{v=1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(v)\mathbf{n}} \right\} \subseteq \left\{ \mathbf{n}^{-1}(\mathbf{m}), \cdots, N \right\}$. Hence

$$\begin{aligned}
&\sum_{k=\mathbf{n}^{-1}(\mathbf{m})}^{N} \text{Prob}\left(D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s^{\mathbf{m}}_{\mathbf{n}(u)\mathbf{n}}, \forall r = 0,1,...,N \right) \\
\leq \quad &\sum_{k=\mathbf{n}^{-1}(\mathbf{m})}^{N} \text{Prob}\left(D(\mathbf{n},k) - D(\mathbf{n},r) \geq \sum_{u=r+1}^{k} s^{\mathbf{m}'}_{\mathbf{n}(u)\mathbf{n}}, \forall r = 0,1,...,N \right).
\end{aligned}$$

Then, define $\mathbf{s}^{\mathbf{m}}$ and $\mathbf{s}^{\mathbf{m}'}$ as the deployment policy with (m,n)th and (m',n)th entry equal to $s_{\mathbf{mn}} + \delta$ and $s_{\mathbf{m'n}} + \delta$, respectively. Combining the above two cases and from (3), we have

$$\frac{\partial \mathcal{L}^{\eta}_{d}(\mathbf{s}^{\mathbf{m}})}{\partial s_{\mathbf{mn}}} \geq \frac{\partial \mathcal{L}^{\eta}_{d}(\mathbf{s}^{\mathbf{m}'})}{\partial s_{\mathbf{mn}}}.$$

Consequently,

$$\frac{\partial^2 \mathcal{L}^{\eta}_{d}(\mathbf{s})}{\partial s^2_{\mathbf{mn}}} = \lim_{\delta\to 0} \frac{\frac{\partial \mathcal{L}^{\eta}_{d}(\mathbf{s}^{\mathbf{m}})}{\partial s_{\mathbf{mn}}} - \frac{\partial \mathcal{L}^{\eta}_{d}(\mathbf{s})}{\partial s_{\mathbf{mn}}}}{\delta} \geq \lim_{\delta\to 0} \frac{\frac{\partial \mathcal{L}^{\eta}_{d}(\mathbf{s}^{\mathbf{m}'})}{\partial s_{\mathbf{mn}}} - \frac{\partial \mathcal{L}^{\eta}_{d}(\mathbf{s})}{\partial s_{\mathbf{mn}}}}{\delta} = \frac{\partial^2 \mathcal{L}^{\eta}_{d}(\mathbf{s})}{\partial s_{\mathbf{mn}} \partial s_{\mathbf{m'n}}}.$$

□

The following lemma can be directly obtained from (3).

Lemma 2. *The partial derivatives* $\frac{\partial \mathcal{L}^{\eta}_{d}(\mathbf{s})}{\partial s_{\mathbf{n}(i)\mathbf{n}}}$ *are increasing in* i *for all* $\mathbf{n} \in \eta$.

Lemma 2 implies that the sooner the RIs arrive at an affected location, the more they can reduce the expected total unmet demand. Thus, there exists an optimal deployment policy $\mathbf{s}^*$ such that

(1) there is no *circular deployment*, i.e., $s^*_{\mathbf{n}_1\mathbf{n}_2}s^*_{\mathbf{n}_2\mathbf{n}_3}\cdots s^*_{\mathbf{n}_k\mathbf{n}_1}=0$, for all $\mathbf{n}_1,\mathbf{n}_2,...,\mathbf{n}_k\in\mathcal{N}$ because otherwise, we can reduce the expected total unmet demand by reducing $s_{\mathbf{n}_1\mathbf{n}_2}, s_{\mathbf{n}_2\mathbf{n}_3}, ...,$ and $s_{\mathbf{n}_m\mathbf{n}_1}$ and increasing $s_{\mathbf{n}_1\mathbf{n}_1}, s_{\mathbf{n}_2\mathbf{n}_2}, ...,$ and $s_{\mathbf{n}_k\mathbf{n}_k}$; and (2) there is no *mutual deployment*, i.e., if $\mathbf{n}_1,\mathbf{n}_2\in\eta$, $\mathbf{n}_1^{-1}(\mathbf{m}_2)<\mathbf{n}_1^{-1}(\mathbf{m}_1)$ and $\mathbf{n}_2^{-1}(\mathbf{m}_1)<\mathbf{n}_2^{-1}(\mathbf{m}_2)$, then $s^*_{\mathbf{m}_1\mathbf{n}_1}s^*_{\mathbf{m}_2\mathbf{n}_2}=0$, because otherwise, we can improve the deployment policy by reducing $s_{\mathbf{m}_1\mathbf{n}_1}$ and $s_{\mathbf{m}_2\mathbf{n}_2}$. For example, locations **1** and **2** are affected locations. Location **3** is closer to location **1** while location **4** is closer to location **2**. Then, simultaneously sending RIs from location **3** to location **2** and from location **4** to location **1** is suboptimal.

Now we are ready to proof Proposition 1. (1) We proof by contradiction. If there exists a location m with rank 1 and $\mathbf{m}\notin\eta$, then there exists a location $\mathbf{n}\in\eta$ such that $s^*_{\mathbf{mn}}>0$. From Lemma 2, we know that $-\pi_{\mathbf{m}}=\frac{\partial\mathcal{L}}{\partial s^*_{\mathbf{mn}}}>\frac{\partial\mathcal{L}}{\partial s^*_{\mathbf{nn}}}\geq -p\pi_{\mathbf{n}}$. This contradicts $\mathbf{m}\in\pi^\eta(1)$. (2) Similar to (1), if there exists $\mathbf{n}\in\pi^\eta(1)$ such that $s^*_{\mathbf{nn}}<S_{\mathbf{n}}$, then there exists a location $\mathbf{n}'\in\eta$ such that $s^*_{\mathbf{n}'\mathbf{n}}>0$. Following the same argument in (1), this contradicts $\mathbf{n}\in\pi^\eta(1)$. (3) and (4) can also be obtained following the similar arguments in (1). □

Proof of Proposition 2: (1) For a given $\mathbf{n}\in\eta$, and a sample path of $D_{\mathbf{n}}(t)$, we have $D(\mathbf{n},\mathbf{n}^{-1}(\mathbf{m})-1)$ is increasing in $l_{\mathbf{mn}}$ while $D(\mathbf{n},k)$, $k\neq\mathbf{n}^{-1}(\mathbf{m})-1$ stays unchanged. As a result,

$$\max_{0\leq k\leq N}\left\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\right\}$$

increases in $l_{\mathbf{mn}}$. Thus, taking expectations over $D_{\mathbf{n}}(t)$ for all $\mathbf{n}\in\eta$, we have $\mathcal{L}^\eta_d(\mathbf{s})$ is increasing in $l_{\mathbf{mn}}$.

(2) Note that from Lemma 1, $\frac{\partial\mathcal{L}^\eta_d(\mathbf{s})}{\partial s_{\mathbf{mn}}}$ satisfies

$$\frac{\partial\mathcal{L}^\eta_d(\mathbf{s})}{\partial s_{\mathbf{mn}}}=-Prob\left(\max_{\mathbf{n}^{-1}(\mathbf{m})\leq k\leq N}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\}\geq\max_{0\leq k\leq\mathbf{n}^{-1}(\mathbf{m})-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\}\right).$$

When $l_{\mathbf{mn}}$ increases, $D(\mathbf{n},\mathbf{n}^{-1}(\mathbf{m})-1)$ increases, thus, for any $\mathbf{m}'$ such that $\mathbf{n}^{-1}(\mathbf{m}')<\mathbf{n}^{-1}(\mathbf{m})$, $\frac{\partial\mathcal{L}^\eta_d(\mathbf{s})}{\partial s_{\mathbf{m}'\mathbf{n}}}$ decreases in $l_{\mathbf{mn}}$; for any $\mathbf{m}'$ such that $\mathbf{n}^{-1}(\mathbf{m}')\geq\mathbf{n}^{-1}(\mathbf{m})$, $\frac{\partial\mathcal{L}^\eta_d(\mathbf{s})}{\partial s_{\mathbf{m}'\mathbf{n}}}$ increases in $l_{\mathbf{mn}}$.

(3) For any $\mathbf{m}'$ such that $\mathbf{n}^{-1}(\mathbf{m}') < \mathbf{n}^{-1}(\mathbf{m})$, consider increasing $l_{\mathbf{mn}}$ by a positive Δ_l, then

$$\left.\frac{\partial \mathcal{L}_d^\eta(\mathbf{s})}{\partial s_{\mathbf{m}'\mathbf{n}}}\right|_{l_{\mathbf{mn}}+\Delta_l} - \left.\frac{\partial \mathcal{L}_d^\eta(\mathbf{s})}{\partial s_{\mathbf{m}'\mathbf{n}}}\right|_{l_{\mathbf{mn}}} =$$

$$-Prob\Bigg(\max_{\mathbf{n}^{-1}(\mathbf{m}')\le k\le N}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\} < \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\},$$
$$\left.\left(D(\mathbf{n},\mathbf{n}^{-1}(\mathbf{m})-1)-\sum_{v=1}^{\mathbf{n}^{-1}(\mathbf{m})-1}s_{\mathbf{n}(v)\mathbf{n}}\right)\right|_{l_{\mathbf{mn}}} < \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\},$$
$$\left.\left(D(\mathbf{n},\mathbf{n}^{-1}(\mathbf{m})-1)-\sum_{v=1}^{\mathbf{n}^{-1}(\mathbf{m})-1}s_{\mathbf{n}(v)\mathbf{n}}\right)\right|_{l_{\mathbf{mn}}+\Delta_l} > \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\}\Bigg)$$

For any $\mathbf{m}''$ such that $\mathbf{n}^{-1}(\mathbf{m}'') < \mathbf{n}^{-1}(\mathbf{m}')$, if we have

$$\max_{\mathbf{n}^{-1}(\mathbf{m}'')\le k\le N}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\} < \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}'')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\},$$

then

$$\max_{\mathbf{n}^{-1}(\mathbf{m}')\le k\le N}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\} < \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\},$$

and

$$\max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}'')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\} = \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\}$$

hold.

As a result. we have that

$$\left.\left(D(\mathbf{n},\mathbf{n}^{-1}(\mathbf{m})-1)-\sum_{v=1}^{\mathbf{n}^{-1}(\mathbf{m})-1}s_{\mathbf{n}(v)\mathbf{n}}\right)\right|_{l_{\mathbf{mn}}} < \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}'')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\}$$

implies

$$\left.\left(D(\mathbf{n},\mathbf{n}^{-1}(\mathbf{m})-1)-\sum_{v=1}^{\mathbf{n}^{-1}(\mathbf{m})-1}s_{\mathbf{n}(v)\mathbf{n}}\right)\right|_{l_{\mathbf{mn}}} < \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\};$$

and

$$\left.\left(D(\mathbf{n},\mathbf{n}^{-1}(\mathbf{m})-1)-\sum_{v=1}^{\mathbf{n}^{-1}(\mathbf{m}-1)}s_{\mathbf{n}(v)\mathbf{n}}\right)\right|_{l_{\mathbf{mn}}+\Delta_l} > \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}'')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\}$$

implies

$$\left.\left(D(\mathbf{n},\mathbf{n}^{-1}(\mathbf{m})-1)-\sum_{v=1}^{\mathbf{n}^{-1}(\mathbf{m})-1}s_{\mathbf{n}(v)\mathbf{n}}\right)\right|_{l_{\mathbf{mn}}+\Delta_l} > \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k}s_{\mathbf{n}(v)\mathbf{n}}\}.$$

Thus,

$$\begin{aligned}
&\left.\frac{\partial \mathcal{L}_d^\eta(\mathbf{s})}{\partial s_{\mathbf{m}'\mathbf{n}}}\right|_{l_{\mathbf{mn}}+\Delta_l} - \left.\frac{\partial \mathcal{L}_d^\eta(\mathbf{s})}{\partial s_{\mathbf{m}'\mathbf{n}}}\right|_{l_{\mathbf{mn}}} = \\
&-Prob\Bigg(\max_{\mathbf{n}^{-1}(\mathbf{m}')\le k\le N}\{D(\mathbf{n},k)-\sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}}\} < \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}}\},\\
&\quad \left.\left(D(\mathbf{n},\mathbf{n}^{-1}(\mathbf{m})-1)-\sum_{v=1}^{\mathbf{n}^{-1}(\mathbf{m})-1} s_{\mathbf{n}(v)\mathbf{n}}\right)\right|_{l_{\mathbf{mn}}} < \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}}\},\\
&\quad \left.\left(D(\mathbf{n},\mathbf{n}^{-1}(\mathbf{m})-1)-\sum_{v=1}^{\mathbf{n}^{-1}(\mathbf{m})-1} s_{\mathbf{n}(v)\mathbf{n}}\right)\right|_{l_{\mathbf{mn}}+\Delta_l} > \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}}\}\Bigg)\\
\le\ &-Prob\Bigg(\max_{\mathbf{n}^{-1}(\mathbf{m}'')\le k\le N}\{D(\mathbf{n},k)-\sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}}\} < \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}'')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}}\},\\
&\quad \left.\left(D(\mathbf{n},\mathbf{n}^{-1}(\mathbf{m})-1)-\sum_{v=1}^{\mathbf{n}^{-1}(\mathbf{m})-1} s_{\mathbf{n}(v)\mathbf{n}}\right)\right|_{l_{\mathbf{mn}}} < \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}'')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}}\},\\
&\quad \left.\left(D(\mathbf{n},\mathbf{n}^{-1}(\mathbf{m})-1)-\sum_{v=1}^{\mathbf{n}^{-1}(\mathbf{m})-1} s_{\mathbf{n}(v)\mathbf{n}}\right)\right|_{l_{\mathbf{mn}}+\Delta_l} > \max_{0\le k\le \mathbf{n}^{-1}(\mathbf{m}'')-1}\{D(\mathbf{n},k)-\sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}}\}\Bigg)\\
=\ &\left.\frac{\partial \mathcal{L}_d^\eta(\mathbf{s})}{\partial s_{\mathbf{m}''\mathbf{n}}}\right|_{l_{\mathbf{mn}}+\Delta_l} - \left.\frac{\partial \mathcal{L}_d^\eta(\mathbf{s})}{\partial s_{\mathbf{m}''\mathbf{n}}}\right|_{l_{\mathbf{mn}}}.
\end{aligned}$$

That is, when $l_{\mathbf{mn}}$ increases, the change of $\frac{\partial \mathcal{L}_d^\eta(\mathbf{s})}{\partial s_{\mathbf{m}'\mathbf{n}}}$ decreases in $\mathbf{m}'$ for all $\mathbf{n}^{-1}(\mathbf{m}') < \mathbf{n}^{-1}(\mathbf{m})$. Similarly, we can show the same is true for all $\mathbf{n}^{-1}(\mathbf{m}') \ge \mathbf{n}^{-1}(\mathbf{m})$. Finally, the change is positive at $\mathbf{n}^{-1}(\mathbf{m}') = \mathbf{n}^{-1}(\mathbf{m})+1$ but negative at $\mathbf{m}'=\mathbf{m}$. Thus, part (3) holds. □

We now consider the deployment decision problem ($\mathcal{D}$) stated in §3. Since $\mathcal{L}_d^\eta(\mathbf{s})$ is convex in $\mathbf{s}$, the Karush-Kuhn-Tucker (KKT) conditions are also sufficient conditions for the optimal deployment decision. Under the KKT conditions, there exists a constant $\pi_{\mathbf{m}} \ge 0$, $\mathbf{m}\in\mathcal{N}$, such that the left and right derivatives of $\mathcal{L}_d^\eta(\mathbf{s})$ with respect to $s_{\mathbf{mn}}$ satisfy

$$\frac{\partial \mathcal{L}_d^{\eta-}(\mathbf{s}^*)}{\partial s_{\mathbf{mn}}} \le -\pi_{\mathbf{m}},\ \text{and}\ \frac{\partial \mathcal{L}_d^{\eta+}(\mathbf{s}^*)}{\partial s_{\mathbf{mn}}} \ge -\pi_{\mathbf{m}},\ \text{if}\ s^*_{\mathbf{mn}}>0;\ \text{and}\ \frac{\partial \mathcal{L}_d^{\eta+}(\mathbf{s}^*)}{\partial s_{\mathbf{mn}}} \ge -\pi_{\mathbf{m}},\ \text{if}\ s^*_{\mathbf{mn}}=0,$$

for all $\mathbf{m}\in\mathcal{N}$ and $\mathbf{n}\in\eta$. If $\partial\mathcal{L}_d^\eta(\mathbf{s}^*)/\partial s_{\mathbf{mn}}$ exists, then

$$\frac{\partial \mathcal{L}_d^{\eta}(\mathbf{s}^*)}{\partial s_{\mathbf{mn}}} = -\pi_{\mathbf{m}},\ \text{if}\ s^*_{\mathbf{mn}}>0;\ \text{and}\ \frac{\partial \mathcal{L}_d^{\eta}(\mathbf{s}^*)}{\partial s_{\mathbf{mn}}} \ge -\pi_{\mathbf{m}},\ \text{if}\ s^*_{\mathbf{mn}}=0,$$

for all $\mathbf{m}\in\mathcal{N}$ and $\mathbf{n}\in\eta$. Here, $\pi_{\mathbf{m}}$ can be explained as the *shadow price* at location $\mathbf{m}$, which can be used to measure the priority level of location m. A location with a higher shadow price has a higher priority. If $\partial\mathcal{L}_d^\eta(\mathbf{s}^*)/\partial s_{\mathbf{mn}}$ exists, the amount deployed from $\mathbf{m}$ to each affected location

$\mathbf{n}$ either equals zero or has a shadow price of $\pi_{\mathbf{m}}$ that is independent of the affected location. Lemma 2 indicates that the shadow prices of the locations delivering to the affected location $\mathbf{n}$ are decreasing in the ranking function of $\mathbf{n}$. Thus, under optimality, locations with higher shadow prices sequentially receive RIs from those with lower shadow prices in the descending order of their shadow prices. Moreover, because $\mathcal{L}_d^{\eta}(\mathbf{s}^*)$ is jointly convex in $\mathbf{S}$ and coordinately decreasing in $S_{\mathbf{m}}$, $\mathbf{m} \in \mathcal{N}$, the expected total unmet demand under the optimal deployment is decreasing in the amount of RIs allocated to (or available at) each location. As a result, from the Envelop theorem, the shadow price of each location $\mathbf{n}$ is decreasing in $S_{\mathbf{n}}$ (or $\nu_{\mathbf{n}}^{\eta} S_{\mathbf{n}}$). In addition, according to Lemma 1 and because of the supermodularity of $\mathcal{L}_d^{\eta}(\mathbf{s})$ in $\mathbf{s}$, the shadow price of each location is more sensitive to the change of RIs at its own location than that at other locations. In other words, if we increase the available amount of RIs at a location, then the shadow price at this location would decrease at a higher rate than that at other locations.

Proof of Proposition 3: (1) Given the deployment matrix $\mathbf{s}$, when $l_{\mathbf{mn}}$ increases, $D(\mathbf{n}, \mathbf{n}^{-1}(\mathbf{m})-1)$ stochastically increases. Because $\mathcal{L}_N^{\eta}$ is an increasing function of $D(\mathbf{n}, \mathbf{n}^{-1}(\mathbf{m})-1)$, by definition of the first-order stochastic dominance in Shaked and Shanthilkumar (2006), we have $\mathbb{E}_{\mathbb{P}}[\mathcal{L}_N^{\eta}]$ is increasing in $l_{\mathbf{mn}}$. Thus, $\mathcal{L}_d^{\eta}(\mathbf{s})$ is increasing in $l_{\mathbf{mn}}$. Taking the minimum over all possible deployment policies, $\mathcal{L}_d^{\eta}(\mathbf{s}^*)$ is also increasing in $l_{\mathbf{mn}}$.

(2) Given $l_{\mathbf{mn}}$ and the corresponding optimal deployment policy $\mathbf{s}^*$, consider any feasible perturbation of $\mathbf{s}^*$, ϵ. Denote $N \times |\eta|$ base matrices $e_{ji'}$ with (j, i')th entry equal to 1 and other entries zero. Then ϵ can be decomposed into $\epsilon = \sum_{\mathbf{i},\mathbf{i'} \in \eta, \mathbf{j} \in \mathcal{N}} a_{ii'}^j \gamma_{ii'}^j$, where $a_{ii'}^j \in \mathbb{R}$ and $\gamma_{ii'}^j = e_{ji} - e_{ji'}$.

Then,

$$
\begin{aligned}
0 \le \left[\mathcal{L}_d^{\eta}(\mathbf{s}^* + \epsilon) - \mathcal{L}_d^{\eta}(\mathbf{s}^*)\right]|_{l_{\mathbf{mn}}} \quad = \quad & \mathbb{E}_{\mathbb{P}}\left[\max_{0 \le k \le N}\left\{D(\mathbf{n}, k) - \sum_{v=1}^{k}\left(s_{\mathbf{n}(v)\mathbf{n}} + \sum_{u=1}^{N}\left(a_{nu}^{\mathbf{n}(v)} - a_{un}^{\mathbf{n}(v)}\right)\right)\right\}\right]_{l_{\mathbf{mn}}} \\
- \quad & \mathbb{E}_{\mathbb{P}}\left[\max_{0 \le k \le N}\left\{D(\mathbf{n}, k) - \sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}}\right\}\right]_{l_{\mathbf{mn}}}
\end{aligned}
$$

If for all $\mathbf{n}^{-1}(\mathbf{m'}) < \mathbf{n}^{-1}(\mathbf{m})$, $\sum_{u=1}^{N}\left(a_{nu}^{m'} - a_{un}^{m'}\right) < 0$, and for all $\mathbf{n}^{-1}(\mathbf{m'}) \ge \mathbf{n}^{-1}(\mathbf{m})$, $\sum_{u=1}^{N}\left(a_{nu}^{m'} - a_{un}^{m'}\right) > 0$, then because of the second part of Proposition 2, we have that for an arbitrary small $\delta > 0$,

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{P}}\left[\max_{0 \le k \le N}\left\{D(\mathbf{n}, k) - \sum_{v=1}^{k}\left(s_{\mathbf{n}(v)\mathbf{n}} + \sum_{u=1}^{N}\left(a_{nu}^{\mathbf{n}(v)} - a_{un}^{\mathbf{n}(v)}\right)\right)\right\} - \max_{0 \le k \le N}\left\{D(\mathbf{n}, k) - \sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}}\right\}\right]_{l_{\mathbf{mn}}} \\
\le \; & \mathbb{E}_{\mathbb{P}}\left[\max_{0 \le k \le N}\left\{D(\mathbf{n}, k) - \sum_{v=1}^{k}\left(s_{\mathbf{n}(v)\mathbf{n}} + \sum_{u=1}^{N}\left(a_{nu}^{\mathbf{n}(v)} - a_{un}^{\mathbf{n}(v)}\right)\right)\right\} - \max_{0 \le k \le N}\left\{D(\mathbf{n}, k) - \sum_{v=1}^{k} s_{\mathbf{n}(v)\mathbf{n}}\right\}\right]_{l_{\mathbf{mn}}+\delta} \\
= \; & \left[\mathcal{L}_d^{\eta}(\mathbf{s}^* + \epsilon) - \mathcal{L}_d^{\eta}(\mathbf{s}^*)\right]|_{l_{\mathbf{mn}}+\delta}.
\end{aligned}
$$

Because of the uniqueness of the optimal solution, $\mathbf{s}^* + \epsilon$ is not optimal.

(3) can be shown directly following (2) by considering the cumulative deployment quantities. $\square$

Proof of Proposition 4: For any $\eta \subseteq \eta'$, let $\mathbf{s}^{*'}$ be the minimizer of the expected total unmet demand $\mathcal{L}_d^{\eta'}\left(\mathbf{s}^{*'}\right)$ and $\mathbf{s}^*$ be the minimizer of the expected total unmet demand $\mathcal{L}_d^{\eta}(\mathbf{s}^*)$. Because the demand processes $D(n,t)$ $n \in \eta$ is independent of η and η', provided that the demand surge has occurred at n, we must have

$$\begin{aligned}
\mathcal{L}_d^{\eta'}\left(\mathbf{s}^{*'}\right) &= \sum_{\mathbf{n}\in\eta'} \mathbb{E}_{\mathbb{P}}\left[\max_{0\le k\le N}\left\{D(\mathbf{n},k) - \sum_{v=1}^{k} s^{*'}_{\mathbf{n}(v)\mathbf{n}}\right\}\right] \\
&\ge \sum_{\mathbf{n}\in\eta} \mathbb{E}_{\mathbb{P}}\left[\max_{0\le k\le N}\left\{D(\mathbf{n},k) - \sum_{v=1}^{k} s^{*'}_{\mathbf{n}(v)\mathbf{n}}\right\}\right] \\
&\ge \mathcal{L}_d^{\eta}(\mathbf{s}^*).
\end{aligned}$$

Proof of Proposition 5: We prove the monotonicity result by showing that the greedy algorithm below converges to an optimal allocation policy. Liu (2017) introduces an algorithm that increases the decision variables in multiple coordinates. Here, we construct a greedy algorithm that also increases the decision variables in multiple coordinates. Assume the tolerance level $tol > 0$, consider the following greedy algorithm,

Greedy Algorithm:

Step 1. Set the initial values $\mathbf{S} = \mathbf{0}$.

Step 2. If $-tol \le \sum_{n=\mathbf{1}}^{\mathbf{N}} S_n - M \le tol$, *then set* $S_n^* = S_n$ *and stop; otherwise, go to the next step.*

Step 3. Compute $\frac{\partial \mathcal{L}_p(\mathbf{S})}{\partial S_n}$, $n = 1, ..., N$ *to find locations with the steepest descending direction and add a marginal amount of RIs at those locations. We use a vector* $\Delta\mathbf{S}$ *to record the marginal increased amount at all locations. Then we set its nth-element* $\Delta S_n \ge 0$ *for* $n \in \arg\min_m \left\{\frac{\partial \mathcal{L}_p(\mathbf{S})}{\partial S_m}\right\}$ *and set* $\Delta S_n = 0$ *for* $n \notin \arg\min_m \left\{\frac{\partial \mathcal{L}_p(\mathbf{S})}{\partial S_m}\right\}$, *under the requirement that* $\arg\min_m \left\{\frac{\partial \mathcal{L}_p(\mathbf{S})}{\partial S_m}\right\} \subseteq \arg\min_m \left\{\frac{\partial \mathcal{L}_p(\mathbf{S}+\Delta\mathbf{S})}{\partial S_m}\right\}$. *Set* $\mathbf{S} = \mathbf{S} + \Delta\mathbf{S}$ *and go back to Step 2.*

Below, we show that if the greedy algorithm terminates at a feasible solution, then it satisfies the KKT conditions and hence is optimal.

To do so, we only need to show that a nonzero $\Delta\mathbf{S}$ exists in every iteration. At Step 2 of any iteration, we must maintain the elements in $\arg\min_n\{\frac{\partial \mathcal{L}_p(\mathbf{S})}{\partial S_n}\}$ as $\Delta\mathbf{S}$ increases, which implies that the first order derivative in all directions of $\arg\min_n\{\frac{\partial \mathcal{L}_p(\mathbf{S})}{\partial S_n}\}$ have the same increment. Let

$i_1, i_2, ..., i_k$ be the directions in $\arg\min_n\{\frac{\partial \mathcal{L}_p(\mathbf{S})}{\partial S_n}\}$. When $\Delta\mathbf{S}$ is sufficiently small, the above statement is equivalent to that $\Delta\mathbf{S}$ is the solution to the following equation

$$\begin{pmatrix} \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_{i_1}\partial S_{i_1}} & \cdots & \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_{i_1}\partial S_{i_k}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_{i_k}\partial S_{i_1}} & \cdots & \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_{i_k}\partial S_{i_k}} \end{pmatrix} \begin{pmatrix} \Delta S_{i_1} \\ \vdots \\ \Delta S_{i_k} \end{pmatrix} = \begin{pmatrix} \epsilon \\ \vdots \\ \epsilon \end{pmatrix}$$

for some $\epsilon > 0$. Because the Hessian is monotone, the inverse of the sub-Hessian matrix is an element-wise non-negative matrix. Thus,

$$\begin{pmatrix} \Delta S_{i_1} \\ \vdots \\ \Delta S_{i_k} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_{i_1}\partial S_{i_1}} & \cdots & \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_{i_1}\partial S_{i_k}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_{i_k}\partial S_{i_1}} & \cdots & \frac{\partial^2 \mathcal{L}_p(\mathbf{S})}{\partial S_{i_k}\partial S_{i_k}} \end{pmatrix}^{-1} \begin{pmatrix} \epsilon \\ \vdots \\ \epsilon \end{pmatrix} \tag{4}$$

is element-wise non-negative.

If demand surge occurs at a single location $\mathbf{n}$, then the optimal deployment policy is to deploy all RIs to the demand surge location, i.e., $s^*_{\mathbf{mn}} = S_m$. Because $\mathcal{L}_d^{\mathbf{n}}$ is supermodar in $s_{\mathbf{mn}}$, the Hessian matrix is a non-negative symmetric matrix. Furthermore, if the Hessian matrix can be represented by the $\{0, 1\}$ vectors, then from Theorem 3.2 in Dellacherie et al. (2016), the Hessian matrix is a symmetric potential of order n. Thus, its inverse is a row diagonally dominant M-matrix. This implies that the solution in (4) is non-negative and nonezero. This hence proves Proposition 5 for the single location case.

If there are only two locations, the sub-Hessian matrix is at most a two by two matrix. Using the cross dominance property in Lemma 1, the diagonal elements of the inverse of the Hessian matrix is always larger in absolute value than the off diagonal elements. Thus, $\Delta\mathbf{S}$ can be found by multiplying the inverse of the sub-Hessian matrix with a vector of all ones. Thus, Proposition 5 holds for the two location case. □

Proof of Corollary 1: The first result is obvious. For the second result, we only need to show that $\bar{a} < \infty$. By definition of $f(t)$, there exits a $\tilde{t} > 0$ such that $f(t)$ is increasing and convex on $[0, \tilde{t}]$ and increasing and concave on $[\tilde{t}, \infty)$. Let $a_1 = f(\tilde{t})/\tilde{t}$. Then, because $f(t)$ is increasing and convex on $[0, \tilde{t}]$, we must have $a_1 t \geq f(t)$ for all $t \in [0, \tilde{t}]$. Note that $f(t)$ is increasing and concave on $[\tilde{t}, \infty)$ and it has no interception with $x = 0$ (another concave function). Therefore, by the separation theorem, there exists $0 < a_2 < \infty$ such that $a_2 t \geq f(t)$ for all $t \in [\tilde{t}, \infty)$. Let $a = \max\{a_1, a_2\}$, then we have the second result. Similarly, we can derive the last result.

Proof of Proposition 6: We prove Proposition 6 in two parts.

(a) The disaggregation-then-aggregation process of generating $\mathbf{P}^1$ from $\mathbf{P}^2$ implies that $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$.

By definition $\mathbf{P}^1 = \hat{\bar{\mathbf{P}}}^2$. For any increasing function f, $f(\cup_{\eta^i \in \Sigma_j} \eta^i) \geq f(\eta^i)$. Thus, $E[f|\mathbf{P}^1] = \sum_j \mathbf{P}^1(\cup_{\eta^i \in \Sigma_j} \eta^i) f(\cup_{\eta^i \in \Sigma_j} \eta^i) \geq \sum_j \sum_{\eta^i \in \Sigma_j} \bar{\mathbf{P}}^2(\eta^i) f(\eta^i) = E[f|\mathbf{P}^2]$.

(b) $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$ implies that there exists a disaggregation-then-aggregation process between $\mathbf{P}^1$ and $\mathbf{P}^2$.

We just need to show that if $\mathbf{P}^1$ cannot be generated by disaggregation-then-aggregation from $\mathbf{P}^2$, then $\mathbf{P}^1$ is not FOSD larger or equal to $\mathbf{P}^2$.

Note that for any multi-set η^i that is generated from a disaggregation of $\mathbf{P}^2$, $\bar{P}^2_{\eta^i}$ probability has been passed to some set ξ in the aggregation step to generate $\mathbf{P}^1$ such that $\eta \subseteq \xi \subseteq \mathcal{N}$. We denote $\eta \to \xi$ as the transformation that moves a positive probability from set η to ξ during the disaggregation and aggregation processes. Consider three sets $S_1 \subseteq S_2 \subseteq S_3$, and $S_1 \to S_2$ and $S_2 \to S_3$. Then there exists an equivalent transformation such that either $S_1 \to S_2$ and $S_1 \to S_3$, or $S_1 \to S_3$ and $S_2 \to S_3$, depending on the probability distribution $\mathbf{P}^2$ and the amount of probability mass moving from one set to another. As a result, for any (disaggregation-aggregation) transformation, there always exists a simple transformation $\mathcal{T} : \mathbf{P}^2 \to \mathbf{P}^1$ such that for every set $S \subseteq \mathcal{N}$ with $\mathbf{P}^1(S) \neq \mathbf{P}^2(S)$, it is either a pure recipient, i.e., $T \to S$ for some $T \subseteq \mathcal{N}$ but does not exist $T' \subseteq \mathcal{N}$ such that $S \to T'$, or a pure sender i.e., $S \to T$ for some $T \subseteq \mathcal{N}$ but does not exist $T' \subseteq \mathcal{N}$ such that $T' \to S$. As a result, if $\mathbf{P}^2$ can be converted to $\mathbf{P}^1$ by some simple transformation, then for $S \subseteq \mathcal{N}$, it is a pure recipient if and only if $\mathbf{P}^1(S) > \mathbf{P}^2(S)$ and a pure sender if and only if $\mathbf{P}^1(S) < \mathbf{P}^2(S)$. Thus, studying the disaggregation and aggregation of $\mathbf{P}$ is equivalent to studying the simple transformations $\mathcal{T}$.

Given two probability distributions $\mathbf{P}^1$ and $\mathbf{P}^2$, we divide $2^{\mathcal{N}}$ into three mutually exclusive classes: the senders $\sigma = \{S|\mathbf{P}^1(S) < \mathbf{P}^2(S), S \subseteq \mathcal{N}\}$, the receivers $\rho = \{S|\mathbf{P}^1(S) > \mathbf{P}^2(S), S \subseteq \mathcal{N}\}$, and the invariants $\iota = \{S|\mathbf{P}^1(S) = \mathbf{P}^2(S), S \subseteq \mathcal{N}\}$. Then, $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$ if and only if there exists a simple transformation $\mathcal{T} : \mathbf{P}^2 \to \mathbf{P}^1$ such that the probabilities can be transferred from the set of senders to the receivers. Equivalently, the following transportation problem has a feasible solution:

$$
\begin{aligned}
(PT) \quad & \max 1 \\
s.t. \quad & \sum_{T \in \rho, S \subset T} x_{ST} = \mathbf{P}^2(S) - \mathbf{P}^1(S), \text{ for all } S \in \sigma, \\
& \sum_{S \in \sigma, S \subset T} x_{ST} = \mathbf{P}^1(T) - \mathbf{P}^2(T), \text{ for all } T \in \rho, \\
& x_{ST} \geq 0, \text{ for all } S \in \sigma, T \in \rho, \text{ and } S \subset T.
\end{aligned}
$$

The transportation problem (PT) has a feasible solution if and only if for any $\sigma' \subseteq \sigma$ we have $\sum_{S\in\sigma'}\left(\mathbf{P}^2(S)-\mathbf{P}^1(S)\right) \le \sum_{\exists S\in\sigma', S\subset T, T\in\rho}\left(\mathbf{P}^1(T)-\mathbf{P}^2(T)\right)$. That is, the total amount of probabilities sent out from σ' should be not greater than the maximum capacity the adjacent receivers can receive.

Thus, for any $\mathbf{P}^1$ and $\mathbf{P}^2$ such that $\mathbf{P}^1$ is NOT FOSD larger than $\mathbf{P}^2$, there exists $\sigma' \subseteq \sigma$ such that $\sum_{S\in\sigma'}\left(\mathbf{P}^2(S)-\mathbf{P}^1(S)\right) > \sum_{\exists S\in\sigma', S\subset T, T\in\rho}\left(\mathbf{P}^1(T)-\mathbf{P}^2(T)\right)$. Rearranging the terms, we have $\sum_{S\in\sigma'}\mathbf{P}^2(S) + \sum_{\exists S\in\sigma', S\subset T, T\in\rho}\mathbf{P}^2(T) > \sum_{\exists S\in\sigma', S\subset T, T\in\rho}\mathbf{P}^1(T) + \sum_{S\in\sigma'}\mathbf{P}^1(S)$.

Consider the increasing set function that equals 1 for all sets that contain some set from σ' and equal 0 for all other sets:

$$f_{\sigma'}(K) = \begin{cases} 1, & \text{if there exists some set } S\in\sigma' \text{ such that } S\subseteq K, \\ 0, & \text{otherwise.} \end{cases}$$

We compare its expected value under $\mathbf{P}^1$ and $\mathbf{P}^2$ as follows:

$$\begin{aligned} E[f|\mathbf{P}^2] &= \sum_{\exists S\in\sigma', S\subseteq K}\mathbf{P}^2(K) \\ &= \sum_{S\in\sigma'}\mathbf{P}^2(S) + \sum_{\exists S\in\sigma', S\subset T, T\in\rho}\mathbf{P}^2(T) + \sum_{\exists S\in\sigma', S\subset I, I\in\iota}\mathbf{P}^2(I) \\ &= \sum_{S\in\sigma'}\mathbf{P}^2(S) + \sum_{\exists S\in\sigma', S\subset T, T\in\rho}\mathbf{P}^2(T) + \sum_{\exists S\in\sigma', S\subset I, I\in\iota}\mathbf{P}^1(I) \\ &> \sum_{S\in\sigma'}\mathbf{P}^1(S) + \sum_{\exists S\in\sigma', S\subset T, T\in\rho}\mathbf{P}^1(T) + \sum_{\exists S\in\sigma', S\subset I, I\in\iota}\mathbf{P}^1(I) \\ &= \sum_{\exists S\in\sigma', S\subseteq K}\mathbf{P}^1(K) \\ &= E[f|\mathbf{P}^1]. \end{aligned}$$

The inequality follows from the fact that $\sum_{S\in\sigma'}\mathbf{P}^2(S)+\sum_{\exists S\in\sigma', S\subset T, T\in\rho}\mathbf{P}^2(T) > \sum_{\exists S\in\sigma', S\subset T, T\in\rho}\mathbf{P}^1(T)+\sum_{S\in\sigma'}\mathbf{P}^1(S)$. Thus, (b) is true.

Proof of Proposition 9: Because $\mathbf{P}^1 \succeq_{FOSD} \mathbf{P}^2$ we have $\mathbf{P}^1 = \hat{\bar{\mathbf{P}}}^2$. Note that $\mathcal{L}^*(M)$ does not change when $\mathbf{P}^2$ is changed to $\bar{\mathbf{P}}^2$. However, according to Proposition 4, for a given allocation policy, the expected total unmet demand would increase under aggregation. Thus, the optimal expected total unmet demand would also increase under aggregation.

References

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