An explicit one-dimensional consolidation solution with arbitrary drainage boundary for unsaturated soil

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Abstract: Existing solutions for analyzing one-dimensional (1-D) consolidation of unsaturated soil are only derived to cater for two extreme drainage conditions (fully drained and undrained). This study presents a new explicit solution for 1-D consolidation of unsaturated soil with any arbitrary drainage boundary. Based on the assumptions of two independent stress variables and the governing equations proposed by Fredlund, the eigenfunction expansion method is adopted to develop an explicit analytical solution to calculate excess pore-water and pore-air pressures in an unsaturated soil when it is subjected to external loads. The developed general solutions are expressed in terms of depth, z, and time, t. For any arbitrary drainage boundary, eigenvalues and eigenfunctions in the space domain are developed. The technique of Laplace transform is used to solve the coupled ordinary differential equations in the time domain. The newly derived explicit solution is verified with the existing semi-analytical method in the literature, and an excellent agreement is obtained. Compared with the semi-analytical solution, the newly derived analytical solution is more straightforward and explicit so that this solution is relatively easier to be implemented into a computer program to carry out a preliminary assessment of 1-D consolidation of unsaturated soil.

Keywords: Explicit analytical solution, unsaturated soil, 1-D consolidation, arbitrary drainage boundary, eigenfunction expansion method.

1.Introduction

The consolidation of soil describes the dissipation of excess pore-water pressure resulting from external stress. Since Terzaghi proposed the classical theory for one-dimensional (1-D) consolidation, it has been widely used for research and engineering applications to calculate the consolidation of saturated soil. However, almost all the earth surface is unsaturated and saturated soil (i.e., degree of saturation equal to 100%) is just a very special case of unsaturated soil. Assuming that the volume change of unsaturated soil results from the net stress and matric suction, Fredlund [1] attempted to develop the 1-D consolidation theory of unsaturated soil. In this study, Fredlund proposed two coupled nonlinear partial differential equations (PDEs) describing independent flows of air and water in unsaturated soil deposits. For many years, it has been a great challenge to solve this set of nonlinear PDEs using analytical methods. Thus, many studies chose some alternative numerical approaches to solve governing equations[1-6].

Most recently, researchers have made significant contributions to the analytical solutions for the 1-D consolidation of unsaturated soil. Qin *et al.*[7, 8] used the Laplace transform technique to solve the governing equations proposed by Fredlund. Shan *et al.*[9-11] introduced two parameters to rearrange the governing equations and used the method of separation of variables to obtain the closed-form solution to calculate excess pore-water and pore-air pressures. Following their pioneer works, more straightforward and explicit solution was developed by Zhou *et al.*[12] and Ho *et al.*[13]. This series of studies have inspired an increasing number of recent research works, including the analytical models to account for two-dimensional plain strain consolidation[14-16], axisymmetric consolidation[17-19], and consolidation of multi-layered soil[20].

These mathematical works generated valuable knowledge for the consolidation of unsaturated soil. However, all the above studies are only derived to cater for two extreme drainage conditions (fully drained and undrained). It has been reported that the boundaries of the soil stratum are actually partially-drained in most practical consolidation problems[21-23], which can be more complicated than those assumed fully permeable or impermeable boundaries.

For instance, in the preloading project of soft clay ground, when the drainage capability of the sand cushion is not very effective, there will become a partially-drained boundary on the top of the adjacent soft clay layer just beneath it [21]. Wang *et al.* [24-26] first incorporated arbitrary boundary conditions into the consolidation theory of unsaturated soil. In these studies, Wang *et al.*[24-26] transformed PDEs into an equivalent set of partial differential equations by introducing two new variables. Then, the solutions in the frequency domain could be easily derived using the method of Laplace transform. However, it was quite challenging to transform these solutions from the frequency domain to the time domain analytically. Wang *et al.*[24-26] used the numerical inversion of Laplace transform and obtained the semi-analytical solutions for excess pore-water and pore-air pressures. Although this set of semi-analytical solutions was correct, these solutions were not explicit in the time domain and derived from a complicated mathematical process, which made these formulas hard to use.

This study presents a new explicit analytical solution for 1-D consolidation of unsaturated soil with arbitrary drainage boundary. Based on the assumptions of two independent stress variables and the governing equations proposed by Fredlund, the eigenfunction expansion method is adopted to develop an explicit analytical solution to calculate excess pore-water and pore-air pressures in an unsaturated soil when it is subjected to external loads. The accuracy of the newly derived analytical solution is verified with the existing semi-analytical method in the literature[24], with various kinds of drainage boundaries[24]. Compared with semi-analytical solution[24], the newly derived analytical solution is more straightforward and explicit so that this solution is relatively easier to be implemented into a computer program to carry out a preliminary assessment of 1-D consolidation of unsaturated soil.

2.Mathematical model

2.1 Governing equations

Figure 1 illustrates a schematic diagram of the unsaturated soil layer when it is subjected to external loads. The soil stratum has an initial height, H, and the deformation only occurs in the vertical direction during the consolidation process.

In this paper, we adopt the same assumptions as those in previous studies[7-12, 14-18, 24-26].

(1) Solid grains and water phases are incompressible.

(2) The flows of air and water phases are assumed to be continuous.

(3) The effects of air diffusing through water, air dissolving in the water, and the movement of water vapor are neglected.

(4) The volume change of unsaturated soil results from the net stress and matric suction.

(5) The coefficients of permeability concerning air and water phases and volume change for the soil remain constant throughout the consolidation process.

Noted that the above assumptions may not account for all the cases, especially Assumption (5). For example, consolidation involves a series of complicated nonlinear problems in practical engineering projects. However, these assumptions are essential and widely used in the analytical analysis of unsaturated soil to deliver a prediction of dissipation for the excess pore pressures[7-18, 24-26].

Based on these assumptions, the governing equations can be formulated as follows[1]

$$\frac{\partial u_w}{\partial t} = -C_w \frac{\partial u_a}{\partial t} - c_v^w \frac{\partial u_w}{\partial z^2} \tag{1}$$

$$\frac{\partial u_a}{\partial t} = -C_a \frac{\partial u_w}{\partial t} - c_v^a \frac{\partial u_a}{\partial z^2} \tag{2}$$

in which u_w and u_a are the excess pore-water and pore-air pressure, respectively. C_w, $c^w{}_v$, C_a, and c^a_v are all constant parameters and can be expressed as

$$C_w = (\frac{m_1^w}{m_2^w} - 1), \quad c_v^w = \frac{k_w}{m_2^w \gamma_w},$$
$$C_a = \frac{1}{[\frac{m_1^a}{m_2^a} - 1 - \frac{(1-S_r)n}{\bar{u}_a m_2^a}]}, \quad c_v^a = \frac{k_a RT^\circ}{g[(m_1^a - m_2^a)\bar{u}_a - (1-S_r)n]\omega_a} \tag{3}$$

in which m_1^w and m_2^w denote the coefficients of water volume change with respect to a change in the net normal stress and matric suction, respectively; m_1^a and m_2^a represent the coefficients of air volume change in a soil element with respect to a change in net normal stress and matric suction, respectively; k_w and k_a denote the permeability coefficients for the water and air phase, respectively; γ_w (= 9.8kN/m^3) is the unit weight of water; g (= 9.8 m/s^2) is the acceleration of gravity; S_r and n are the degree of saturation and porosity, respectively. ω_a (= 0.029 kg/mol) is the molecular mass of air; $\bar{u}_a = u_a^0 + u_{atm}$, in which u_{atm} is the absolute pore-air pressure and u_a^0 is an initial excess pore-air pressure; R (= 8.314 *J/mol/K*) is the universal gas constant; $T1°$ is the absolute temperature.

2.2 Initial conditions and boundary conditions

Following is the distribution of initial excess pore-water and pore-air pressures

$$u_w(z,0) = u_w^0 \tag{4}$$

$$u_a(z,0) = u_a^0 \tag{5}$$

According to Wang *et al.*[24], the arbitrary drained boundary conditions are written as follows

$$\text{Water phase:}\begin{cases} \dfrac{\partial u_w(0,t)}{\partial z} - \dfrac{R_t}{H} u_w(0,t) = 0 \\ \dfrac{\partial u_w(H,t)}{\partial z} + \dfrac{R_b}{H} u_w(H,t) = 0 \end{cases} \tag{6}$$

$$\text{Air phase:}\begin{cases}\dfrac{\partial u_a(0,t)}{\partial z}-\dfrac{R_t}{H}u_a(0,t)=0\\ \dfrac{\partial u_a(H,t)}{\partial z}+\dfrac{R_b}{H}u_a(H,t)=0\end{cases} \tag{7}$$

where R_t and R_b are the parameters at the top and bottom boundaries, respectively, as shown in Fig.1. Obviously, for the fully permeable boundary, we can find that R_t=R_b=∞. As for the impermeable boundary, we can see that R_t=R_b=0.

3. Analytical solutions

3.1 Solution for excess pore pressures using eigenfunction expansion method

The general solution for u_w and u_a can be written as

$$u_w(z,t)=\sum_{i=1}^{\infty}Z^i(z)f_w^i(t) \tag{8}$$

$$u_a(z,t)=\sum_{i=1}^{\infty}Z^i(z)f_a^i(t) \tag{9}$$

where $Z^i(z)$ is the eigenfunction expressed in term of distance z; $f_w^i(t)$ and $f_a^i(t)$ are generalized Fourier coefficients varying with time t. Based on the boundary conditions shown in Eqs.(6) and (7), $Z^i(z)$ is written as

$$Z^i(z)=\frac{H\lambda_i}{R_t}\cos(\lambda_i z)+\sin(\lambda_i z) \tag{10}$$

in which λ_i are the eigenvalues that can be calculated as the positive roots of the following equation.

$$\lambda_i[-\frac{H\lambda_i}{R_t}\sin(\lambda_i H)+\cos(\lambda_i H)]+\frac{R_b}{H}[\frac{H\lambda_i}{R_t}\cos(\lambda_i H)+\sin(\lambda_i H)]=0 \tag{11}$$

Therefore, substituting Eqs.(8) and (9) into Eqs.(1) and (2) yields

$$\sum_{i=1}^{\infty}\left\{[\frac{\partial f_w^i(t)}{\partial t}+C_w\frac{\partial f_a^i(t)}{\partial t}-\lambda_i^2 c_v^w f_w^i(t)]Z^i(z)\right\}=0 \quad i=1,2,... \tag{12}$$

$$\sum_{i=1}^{\infty}\left\{[\frac{\partial f_a^i(t)}{\partial t}+C_a\frac{\partial f_w^i(t)}{\partial t}-\lambda_i^2 c_v^a f_a^i(t)]Z^i(z)\right\}=0 \quad i=1,2,... \tag{13}$$

Above a set of ordinary differential equations (ODEs) can be easily solved as shown in Appendix A. Thus, the solutions for $f_w^i(t)$ and $f_a^i(t)$ can be obtained as follows

$$f_w^i(t)=\exp[\frac{(c_v^a+c_v^w)\lambda_i t}{2(-C_aC_w+1)}]\times$$
$$[f_w^i(0)\cosh(\frac{\eta\lambda_i t}{2(C_aC_w-1)})+\frac{2C_wc_v^a f_a^i(0)+(c_v^a-c_v^w)f_w^i(0)}{\eta}\sinh(\frac{\eta\lambda_i t}{2(C_aC_w-1)})] \tag{14}$$

$$f_a^i(t)=\exp[\frac{(c_v^a+c_v^w)\lambda_i t}{2(-C_aC_w+1)}]\times$$
$$[f_a^i(0)\cosh(\frac{\eta\lambda_i t}{2(C_aC_w-1)})+\frac{2C_ac_v^w f_w^i(0)+(c_v^w-c_v^a)f_a^i(0)}{\eta}\sinh(\frac{\eta\lambda_i t}{2(C_aC_w-1)})] \tag{15}$$

where

$$\eta=\sqrt{\left(c_v^w-c_v^a\right)^2+4c_v^wc_v^aC_wC_a}; \tag{16}$$

and $f_w^i(0)$ and $f_a^i(0)$ are the initial value of $f_w^i(t)$ and $f_a^i(t)$, respectively.

Based on the orthogonality of eigenfunction, we can find

$$\int_0^H Z^i(z)Z^j(z)dz=0 \quad (i\neq j) \tag{17}$$

Thus, the values of $f_w^i(0)$ and $f_a^i(0)$ can be obtained

$$f_w^i(0)=\frac{\int_0^H[u_w(z,0)Z^i(z)]dz}{\int_0^H[Z^i(z)]^2dz}$$
$$=\frac{2u_w^0R_t[H\lambda_i\sin(H\lambda_i)-R_t\cos(H\lambda_i)+R_t]}{H^3\lambda_i^3+H^2\lambda_i^2\sin(H\lambda_i)\cos(H\lambda_i)-2H\lambda_iR_t\cos^2(H\lambda_i)+H\lambda_iR_t^2-R_t^2\sin(H\lambda_i)\cos(H\lambda_i)+2H\lambda_iR_t} \tag{18}$$

$$f_a^i(0) = \frac{\int_0^H [u_a(z,0)Z^i(z)]dz}{\int_0^H [Z^i(z)]^2 dz}$$

$$= \frac{2u_a^0 R_t[H\lambda_i \sin(H\lambda_i) - R_t \cos(H\lambda_i) + R_t]}{H^3\lambda_i^3 + H^2\lambda_i^2 \sin(H\lambda_i)\cos(H\lambda_i) - 2H\lambda_i R_t \cos^2(H\lambda_i) + H\lambda_i R_t^2 - R_t^2 \sin(H\lambda_i)\cos(H\lambda_i) + 2H\lambda_i R_t}$$

(19)

The detailed derivation for Eq.(17)is presented in Appendix B.

As a result, the analytical solution to the excess pore-water pressure and pore-air pressure can be written as follows

$$u_w(z,t) = \sum_{i=1}^{\infty}[\frac{H\lambda_i}{R_t}\cos(\lambda_i z) + \sin(\lambda_i z)]f_w^i(t) \tag{20}$$

$$u_a(z,t) = \sum_{i=1}^{\infty}[\frac{H\lambda_i}{R_t}\cos(\lambda_i z) + \sin(\lambda_i z)]f_a^i(t) \tag{21}$$

3.3 Average degree of consolidation

According to Fredlund [1], the settlement, S_t, can be expressed as

$$S_t = |\int_0^H \varepsilon_v dz| \tag{22}$$

where ε_v is the volumetric strain

$$\varepsilon_v(z,t) = \left(m_2^s - m_1^s\right)\left[u_a(z,t) - u_a^0\right] - m_2^s\left[u_w(z,t) - u_w^0\right] \tag{23}$$

with $m_1^s = m_1^a + m_1^w, m_2^s = m_2^a + m_2^w$.

Thus, the average degree of consolidation, U_{avg}, can be obtained

$$U_{avg} = \frac{S_t}{S_\infty} \tag{24}$$

in which S_∞ is the final settlement.

4. Verification

In this section, the new-developed solution is verified with the semi-analytical method introduced by Wang *et al.*[24]. Basic parameters for the unsaturated soil layer are also taken from the literature [24]:

$$\begin{aligned}
&H = 10\text{m}, n_0 = 50\%; \quad S_{r0} = 80\%; \quad k_a = 10^{-9}\text{m/s}; \quad k_w = 10^{-10}\text{m/s}; \\
&m_1^s = -2.5\times10^{-4}\text{kPa}^{-1}; \quad m_1^w = -0.5\times10^{-4}\text{kPa}^{-1}; \quad m_2^s = -1.0\times10^{-4}\text{kPa}^{-1}; \\
&m_2^w = -2\times10^{-4}\text{kPa}^{-1}; \quad u_a^0 = 5\text{kPa}; \quad u_w^0 = 40\text{kPa}; \quad u_{atm} = 101.3\text{kPa}; \\
&R = 8.314\text{J.mol}^{-1}\text{K}^{-1}; \quad \omega_a = 0.029\text{kg.mol}^{-1}; \quad T^\circ = \left(t^\circ + 273.16\right)\text{K}; \quad t^\circ = 20^\circ\text{C}
\end{aligned} \tag{25}$$

The investigation point is located at z=8 m. Figure 2 and Figure 3 illustrate the dissipation of excess pore-water pressure and excess pore-air pressure, respectively, and Figure 4 shows the profiles for the average degree of consolidation. All these values are obtained from the new-derived analytical solution and the semi-analytical solution. As observed, an excellent agreement is obtained, which powerfully demonstrates the correctness of the proposed solution.

5. Conclusion

This study presents a new explicit analytical solution for the 1-D consolidation of unsaturated soil with arbitrary drainage boundary. The eigenfunction expansion method is adopted to obtain the analytical solution to calculate excess pore-water and pore-air pressures. The newly derived explicit solution is verified with the existing semi-analytical method in the literature [24], and an excellent agreement is obtained. Compared with the semi-analytical solution introduced in [24], the new-developed analytical solution is more simple and explicit so that this solution is relatively easier to be implemented into a computer program to carry out a preliminary assessment of 1-D consolidation of unsaturated soil.

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Appendix A

Following is the derivation of the solutions to $f_w^i(t)$ and $f_a^i(t)$.

Applying Laplace transform to Eq.(12) and (13) yields:

$$[p\tilde{f}_w^i(p) - f_w^i(0)] + C_w[p\tilde{f}_a^i(p) - f_a^i(0)] - \lambda_i^2 c_v^w \tilde{f}_w^i(p) = 0 \tag{A-1}$$

$$[p\tilde{f}_a^i(p) - f_a^i(0)] + C_a[p\tilde{f}_w^i(p) - f_w^i(0)] - \lambda_i^2 c_v^a \tilde{f}_a^i(p) = 0 \tag{A-2}$$

where $\tilde{f}_w^i(p) = \int_0^{+\infty} f_w^i(t)e^{-pt}dt$ and $\tilde{f}_a^i(p) = \int_0^{+\infty} f_a^i(t)e^{-pt}dt$ are the image functions of $f_w^i(t)$ and $f_a^i(t)$. Solving the above equations gives.

$$\tilde{f}_w^i(p) = \frac{f_w^i(0)(C_w C_a - 1)p + \lambda_i^2 c_v^a [f_w^i(0) + C_w f_a^i(0)]}{(C_a C_w - 1)p^2 + (c_v^w + c_v^a)\lambda_i^2 p - \lambda_i^4 c_v^w c_v^a} \tag{A-3}$$

$$\tilde{f}_a^i(p) = \frac{f_a^i(0)(C_w C_a - 1)p + \lambda_i^2 c_v^w [f_a^i(0) + C_a f_w^i(0)]}{(C_a C_w - 1)p^2 + (c_v^w + c_v^a)\lambda_i^2 p - \lambda_i^4 c_v^w c_v^a} \tag{A-4}$$

Thus, the solutions for $f_w^i(t)$ and $f_a^i(t)$ can be obtained using the inverse Laplace transform

$$\begin{aligned} f_w^i(t) &= \exp[\frac{(c_v^a + c_v^w)\lambda_i t}{2(-C_a C_w + 1)}]\times \\ &[f_w^i(0)\cosh(\frac{\eta\lambda_i t}{2(C_a C_w - 1)}) + \frac{2C_w c_v^a f_a^i(0) + (c_v^a - c_v^w) f_w^i(0)}{\eta}\sinh(\frac{\eta\lambda_i t}{2(C_a C_w - 1)})] \end{aligned} \tag{A-5}$$

$$\begin{aligned} f_a^i(t) &= \exp[\frac{(c_v^a + c_v^w)\lambda_i t}{2(-C_a C_w + 1)}]\times \\ &[f_a^i(0)\cosh(\frac{\eta\lambda_i t}{2(C_a C_w - 1)}) + \frac{2C_a c_v^w f_w^i(0) + (c_v^w - c_v^a) f_a^i(0)}{\eta}\sinh(\frac{\eta\lambda_i t}{2(C_a C_w - 1)})] \end{aligned} \tag{A-6}$$

Appendix B

Following is the derivation of the orthogonality of eigenfunction.

According to Eq.(10), we can find

$$\frac{d^2}{dz^2}Z^i(z) = -\lambda_i^2 Z^i(z) \tag{B-1}$$

As for $i \neq j$, we can find

$$\frac{d}{dz}Z^{i,z}(z) = -\lambda_i^2 Z^i(z) \tag{B-2}$$

$$\frac{d}{dz}Z^{j,z}(z) = -\lambda_j^2 Z^j(z) \tag{B-3}$$

where $Z^{i,z}(z)$ and $Z^{j,z}(z)$ are the first order of ODEs of $Z^i(z)$ and $Z^j(z)$ with respect to depth, respectively. Rearranging Eqs.(B-2) and (B-3) yields

$$Z^j(z)\frac{d}{dz}Z^{i,z}(z) = -\lambda_i^2 Z^j(z)Z^i(z) \tag{B-4}$$

$$Z^i(z)\frac{d}{dz}Z^{j,z}(z) = -\lambda_j^2 Z^i(z)Z^j(z) \tag{B-5}$$

Thus,

$$\int_0^H [\lambda_i^2 Z^j(z)Z^i(z) - \lambda_j^2 Z^i(z)Z^j(z)]dz = \int_0^H [Z^i(z)\frac{d}{dz}Z^{j,z}(z) - Z^j(z)\frac{d}{dz}Z^{i,z}(z)]dz \tag{B-6}$$

On the right hand of Eq.(B-6), we find

$$\int_0^H Z^i(z)d[Z^{j,z}(z)] = [Z^i(z)Z^{j,z}(z)]_0^H - \int_0^H Z^{j,z}(z)Z^{i,z}(z)dz \tag{B-7}$$

$$\int_0^H Z^j(z)d[Z^{i,z}(z)] = [Z^j(z)Z^{i,z}(z)]_0^H - \int_0^H Z^{j,z}(z)Z^{i,z}(z)dz \tag{B-8}$$

Substituting Eqs.(B-7) and (B-8) into Eq.(B-6) gives

$$\begin{aligned}&\int_0^H (\lambda_i^2 - \lambda_j^2)Z^i(z)Z^j(z)dz \\ &= [Z^i(H)Z^{j,z}(H) - Z^j(H)Z^{i,z}(H)] - [Z^i(0)Z^{j,z}(0) - Z^j(0)Z^{i,z}(0)]\end{aligned} \tag{B-9}$$

where

$$Z^i(0) = \frac{H\lambda_i}{R_t} \tag{B-10}$$

$$Z^j(0) = \frac{H\lambda_j}{R_t} \tag{B-11}$$

$$Z^{i,z}(0) = \lambda_i \tag{B-12}$$

$$Z^{j,z}(0) = \lambda_j \tag{B-13}$$

$$Z^{i}(H)=\frac{H\lambda_{i}}{R_{t}}\cos(\lambda_{i}H)+\sin(\lambda_{i}H) \tag{B-14}$$

$$Z^{j}(H)=\frac{H\lambda_{j}}{R_{t}}\cos(\lambda_{j}H)+\sin(\lambda_{j}H) \tag{B-15}$$

$$Z^{i,z}(H)=-\frac{H\lambda_{i}^{2}}{R_{t}}\sin(\lambda_{i}H)+\lambda_{i}\cos(\lambda_{i}H) \tag{B-16}$$

$$Z^{j,z}(H)=-\frac{H\lambda_{j}^{2}}{R_{t}}\sin(\lambda_{j}H)+\lambda_{j}\cos(\lambda_{j}H) \tag{B-17}$$

Thus, substituting Eqs.(B-10)-(B-17) into Eq.(B-9) results in

$$\begin{aligned}&(\lambda_{i}^{2}-\lambda_{j}^{2})\int_{0}^{H}Z^{i}(z)Z^{j}(z)dz\\&=\lambda_{j}[\frac{H\lambda_{i}}{R_{t}}\cos(\lambda_{i}H)+\sin(\lambda_{i}H)][-\frac{H\lambda_{j}}{R_{t}}\sin(\lambda_{j}H)+\cos(\lambda_{j}H)]\\&-\lambda_{i}[\frac{H\lambda_{j}}{R_{t}}\cos(\lambda_{j}H)+\sin(\lambda_{j}H)][-\frac{H\lambda_{i}}{R_{t}}\sin(\lambda_{i}H)+\cos(\lambda_{i}H)]\end{aligned} \tag{B-18}$$

On the other hand, it is noted that the eigenvalues, λ_i and λ_j ,are obtained from Eq.(11).

$$\lambda_{i}[-\frac{H\lambda_{i}}{R_{t}}\sin(\lambda_{i}H)+\cos(\lambda_{i}H)]=\frac{R_{b}}{H}[\frac{H\lambda_{i}}{R_{t}}\cos(\lambda_{i}H)+\sin(\lambda_{i}H)] \tag{B-19}$$

$$\lambda_{j}[-\frac{H\lambda_{j}}{R_{t}}\sin(\lambda_{j}H)+\cos(\lambda_{j}H)]=\frac{R_{b}}{H}[\frac{H\lambda_{j}}{R_{t}}\cos(\lambda_{j}H)+\sin(\lambda_{j}H)] \tag{B-20}$$

Substituting Eqs.(B-19) and (B-20) into Eq.(B-18) gives

$$\begin{aligned}&(\lambda_{i}^{2}-\lambda_{j}^{2})\int_{0}^{H}Z^{i}(z)Z^{j}(z)dz\\&=\frac{R_{b}}{H}[\frac{H\lambda_{i}}{R_{t}}\cos(\lambda_{i}H)+\sin(\lambda_{i}H)][\frac{H\lambda_{j}}{R_{t}}\cos(\lambda_{j}H)+\sin(\lambda_{j}H)]\\&-\frac{R_{b}}{H}[\frac{H\lambda_{j}}{R_{t}}\cos(\lambda_{j}H)+\sin(\lambda_{j}H)][\frac{H\lambda_{i}}{R_{t}}\cos(\lambda_{i}H)+\sin(\lambda_{i}H)]\end{aligned} \tag{B-21}$$

It can be seen that the value of the right hand of Eq.(B-21) is equal to zero. Since the values of λ_i and λ_j are different, we can obtain the orthogonal relationship as follow

$$\int_{0}^{H}Z^{i}(z)Z^{j}(z)dz=0 \quad (i\neq j) \tag{B-22}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

1. Fredlund DG, Hasan JU. One-dimensional consolidation theory: unsaturated soils. *Canadian Geotechnical Journal*. 1979;16 (3):521-31.
2. Wong TT, Fredlund DG, Krahn JJCGJ. A numerical study of coupled consolidation in unsaturated soils. *Can Geotech J*. 1998;35 (6):926-37.
3. Conte E. Consolidation analysis for unsaturated soils. *Can Geotech J*. 2004;41 (4):599-612.
4. Zhou W-H, Zhao L-S. One-Dimensional Consolidation of Unsaturated Soil Subjected to Time-Dependent Loading with Various Initial and Boundary Conditions. *Int J Geomech*. 2014;14 (2):291-301.
5. Yuan S, Zhong H. A weak form quadrature element formulation for coupled analysis of unsaturated soils. *Computers and Geotechnics*. 2016;76:1-11.
6. Moradi M, Keshavarz A, Fazeli A. One dimensional consolidation of multi-layered unsaturated soil under partially permeable boundary conditions and time-dependent loading. *Comput Geotech*. 2019;107:45-54.
7. Qin A-f, Chen G-j, Tan Y-w, Sun D-a. Analytical solution to one-dimensional consolidation in unsaturated soils. *Applied Mathematics and Mechanics*. 2008;29 (10):1329-40.
8. Qin A, Sun Da, Tan Y. Analytical solution to one-dimensional consolidation in unsaturated soils under loading varying exponentially with time. *Comput Geotech*. 2010;37 (1-2):233-8.
9. Shan Z, Ling D, Ding H. Exact solutions for one-dimensional consolidation of single-layer unsaturated soil. *Int J Numer Anal Met*. 2012;36 (6):708-22.
10. Shan Z, Ling D, Ding H. Exact solutions for the one-dimensional transient response of unsaturated single-layer porous media. *Comput Geotech*. 2013;48:127-33.
11. Shan Z-d, Ling D-s, Ding H-j. Analytical solution for 1D consolidation of unsaturated soil with mixed boundary condition. *Journal of Zhejiang University SCIENCE A*. 2013;14 (1):61-70.
12. Zhou W-H, Zhao L-S, Li X-B. A simple analytical solution to one-dimensional consolidation for unsaturated soils. *Int J Numer Anal Met*. 2014;38 (8):794-810.
13. Ho L, Fatahi B, Khabbaz H. Analytical solution for one-dimensional consolidation of unsaturated soils using eigenfunction expansion method. *Int J Numer Anal Met*. 2014;38 (10):1058-77.
14. Huang M, Li J. Generalized analytical solution for 2D plane strain consolidation of unsaturated soil with time-dependent drainage boundaries. *Comput Geotech*. 2018;103:218-28.
15. Wang L, Xu Y, Xia X, Sun Da. Semi-analytical solutions to two-dimensional plane strain consolidation for unsaturated soil. *Comput Geotech*. 2018;101:100-13.
16. Wang L, Xu Y, Xia X, He Y, Li T. Semi-analytical solutions to two-dimensional plane strain

consolidation for unsaturated soils under time-dependent loading. *Comput Geotech*. 2019;109:144-65.
17. Qin A, Sun Da, Yang L, Weng Y. A semi-analytical solution to consolidation of unsaturated soils with the free drainage well. *Comput Geotech*. 2010;37 (7-8):867-75.
18. Zhou W-H, Zhao L-S, Lok TM-H, Mei G-x, Li X-B. Analytical Solutions to the Axisymmetrical Consolidation of Unsaturated Soils. *Journal of Engineering Mechanics*. 2018;144 (1).
19. Ho L, Fatahi B, Khabbaz H. Analytical solution to axisymmetric consolidation in unsaturated soils with linearly depth-dependent initial conditions. *Computers and Geotechnics*. 2016;74:102-21.
20. Shan Z, Ling D, Ding H. Analytical solution for the 1D consolidation of unsaturated multi-layered soil. *Comput Geotech*. 2014;57:17-23.
21. Xie KH, Xie XY, Gao X. Theory of one dimensional consolidation of two-layered soil with partially drained boundaries. *Computers and Geotechnics*. 1999;24 (4):265-78.
22. Mesri GJWRR. One-dimensional consolidation of a clay layer with impeded drainage boundaries. *Water Resour Res*. 1973;9 (4):1090-3.
23. Liu J-C, Lei GH. One-dimensional consolidation of layered soils with exponentially time-growing drainage boundaries. *Comput Geotech*. 2013;54:202-9.
24. Wang L, Sun Da, Li L, Li P, Xu Y. Semi-analytical solutions to one-dimensional consolidation for unsaturated soils with symmetric semi-permeable drainage boundary. *Comput Geotech*. 2017;89:71-80.
25. Wang L, Sun DA, Qin AF, Xu YF. Semi-analytical solution to one-dimensional consolidation for unsaturated soils with semi-permeable drainage boundary under time-dependent loading. *Int J Numer Anal Met*. 2017;41 (16):1636-55.
26. Wang L, Xu Y, Xia X, He Y. Semi-analytical solutions of two-dimensional plane strain consolidation in unsaturated soils subjected to the lateral semipermeable drainage boundary. *Int J Numer Anal Met*. 2019;43 (17):2628-51.

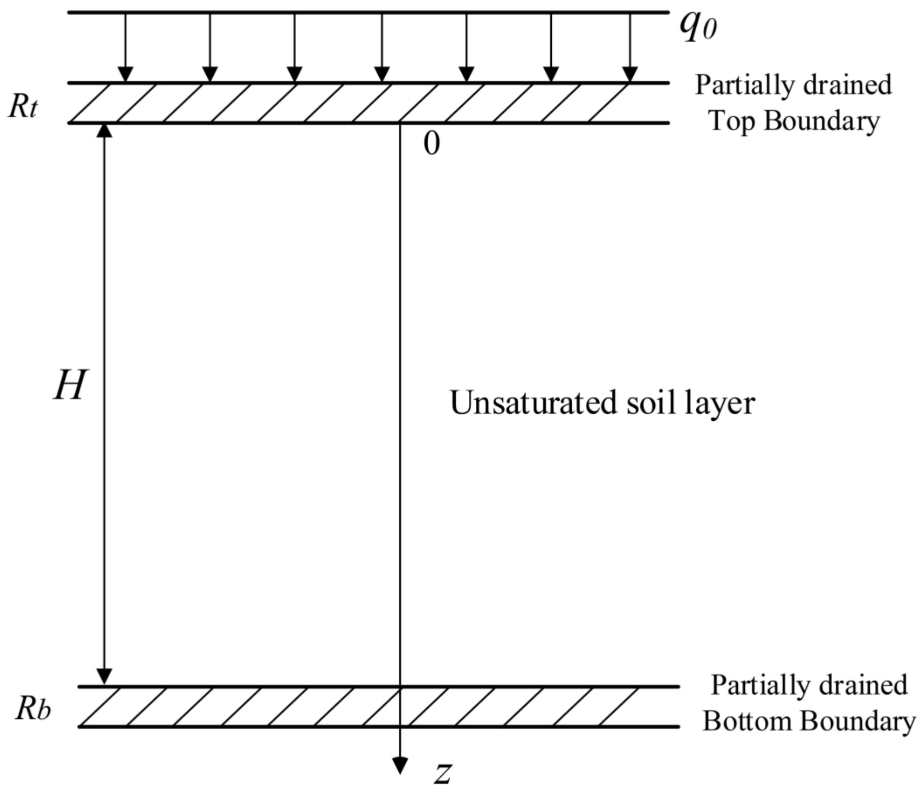

Fig.1. Geometry for analysis of 1-D consolidation of unsaturated soil

141x117mm (600 x 600 DPI)

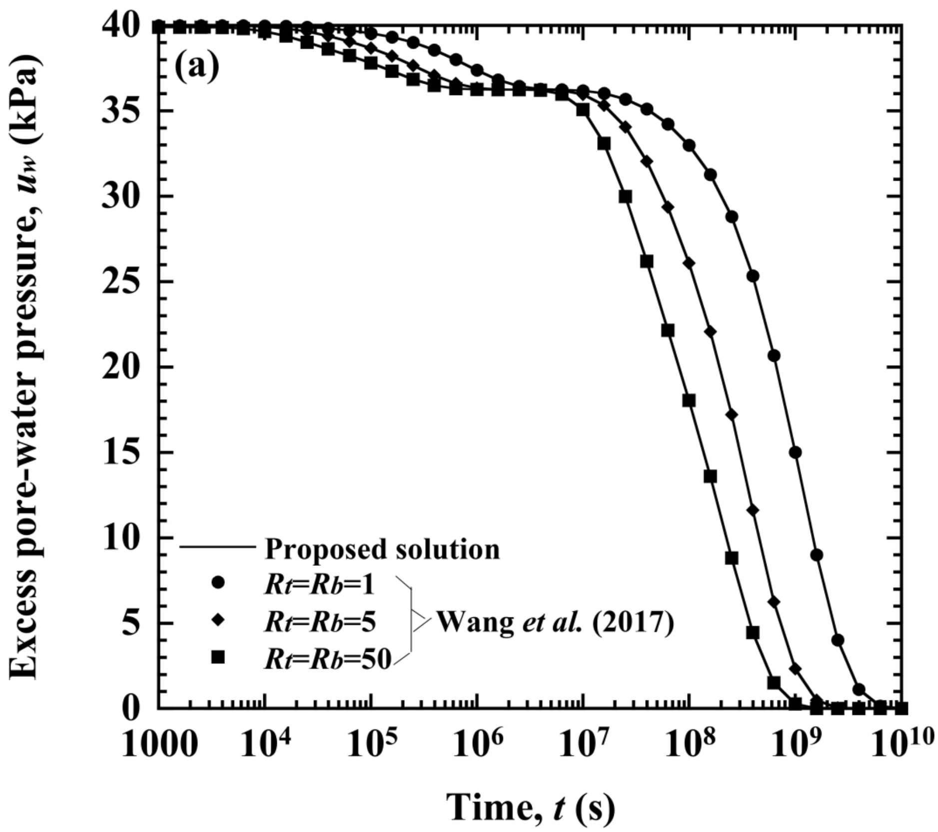

Fig.2. Verification profiles for the excess pore-water pressure: (a) Rt=Rb=1,5, and 50;

160x139mm (600 x 600 DPI)

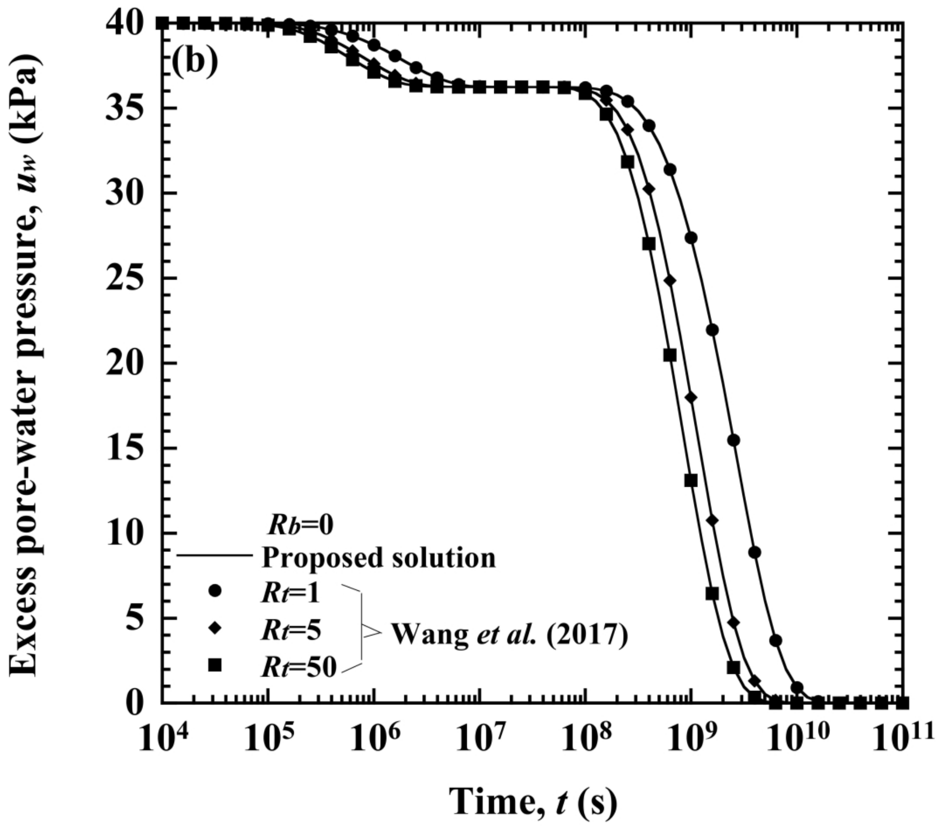

Fig.2. Verification profiles for the excess pore-water pressure: (b) Rb=0, Rt=1,5, and 50;

161x138mm (600 x 600 DPI)

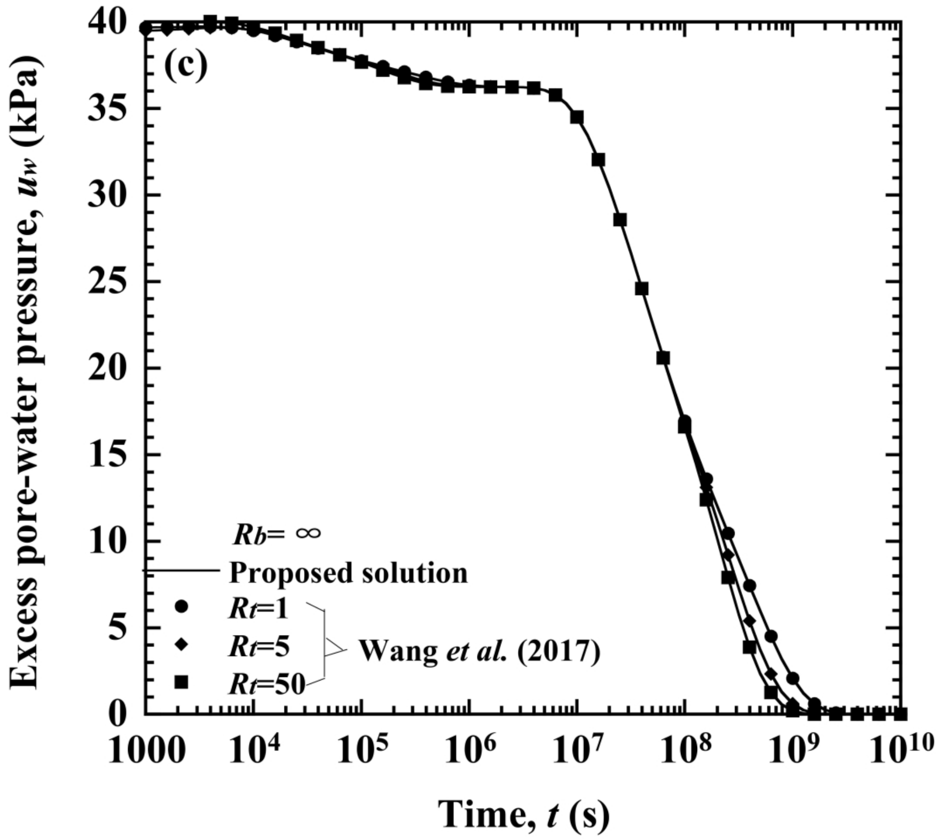

Fig.2. Verification profiles for the excess pore-water pressure: (c) Rb=∞, Rt=1,5, and 50.

157x138mm (600 x 600 DPI)

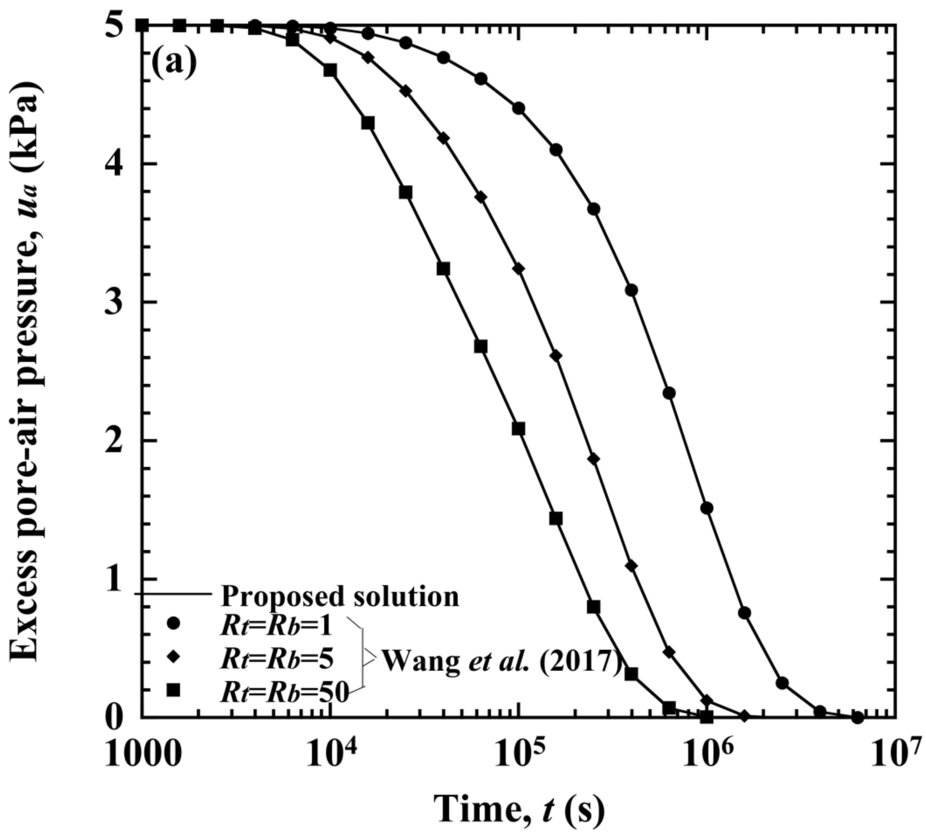

Fig.3. Verification profiles for the excess pore-air pressure:(a)Rt=Rb=1,5, and 50;

158x135mm (600 x 600 DPI)

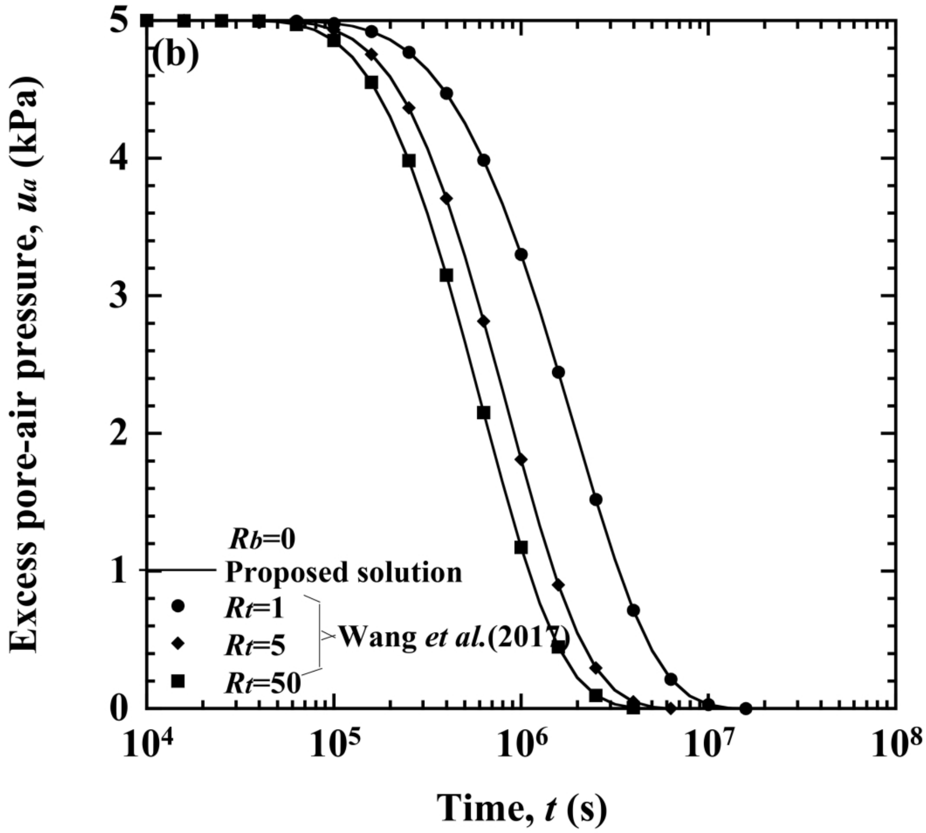

Fig.3. Verification profiles for the excess pore-air pressure:(b) Rb=0, Rt=1,5, and 50;

159x136mm (600 x 600 DPI)

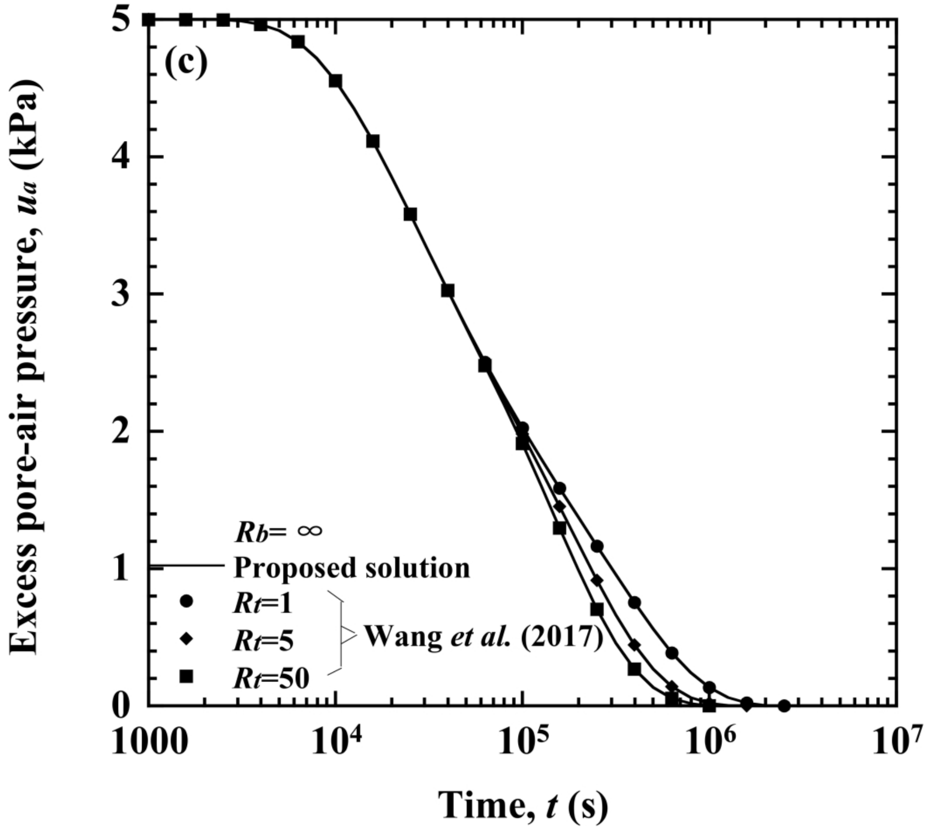

Fig.3. Verification profiles for the excess pore-air pressure:(c) Rb=∞, Rt=1,5, and 50.

159x136mm (600 x 600 DPI)

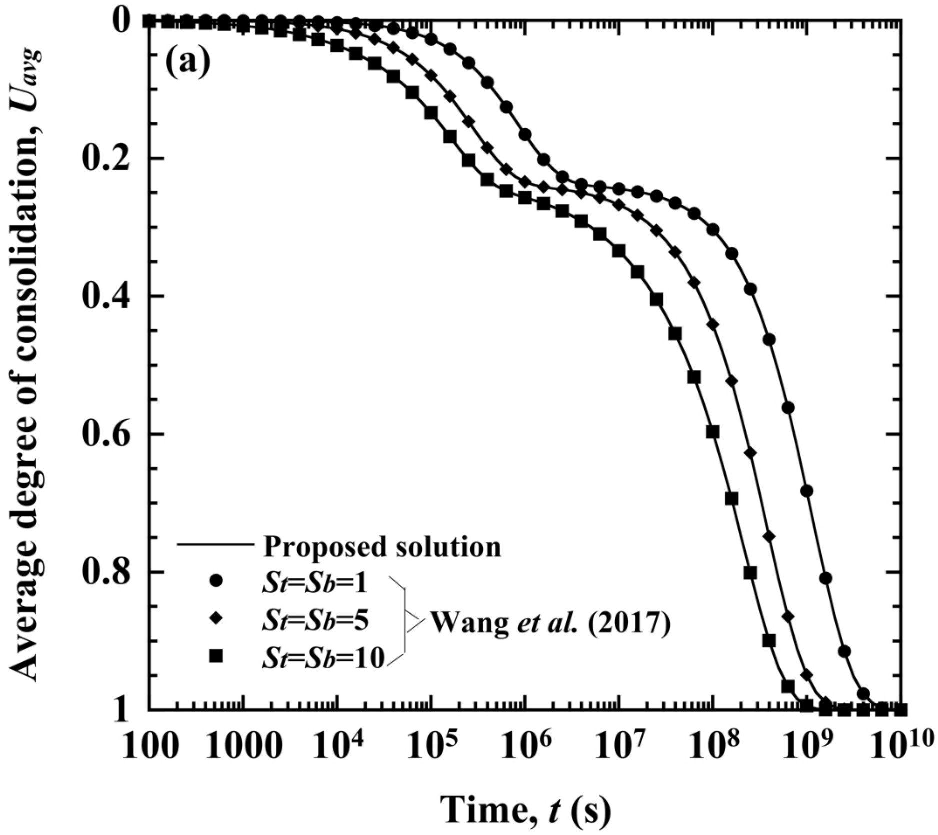

Fig.4. Verification profiles for the average degree of consolidation:(a) Rt=Rb=1,5, and 50;

158x136mm (600 x 600 DPI)

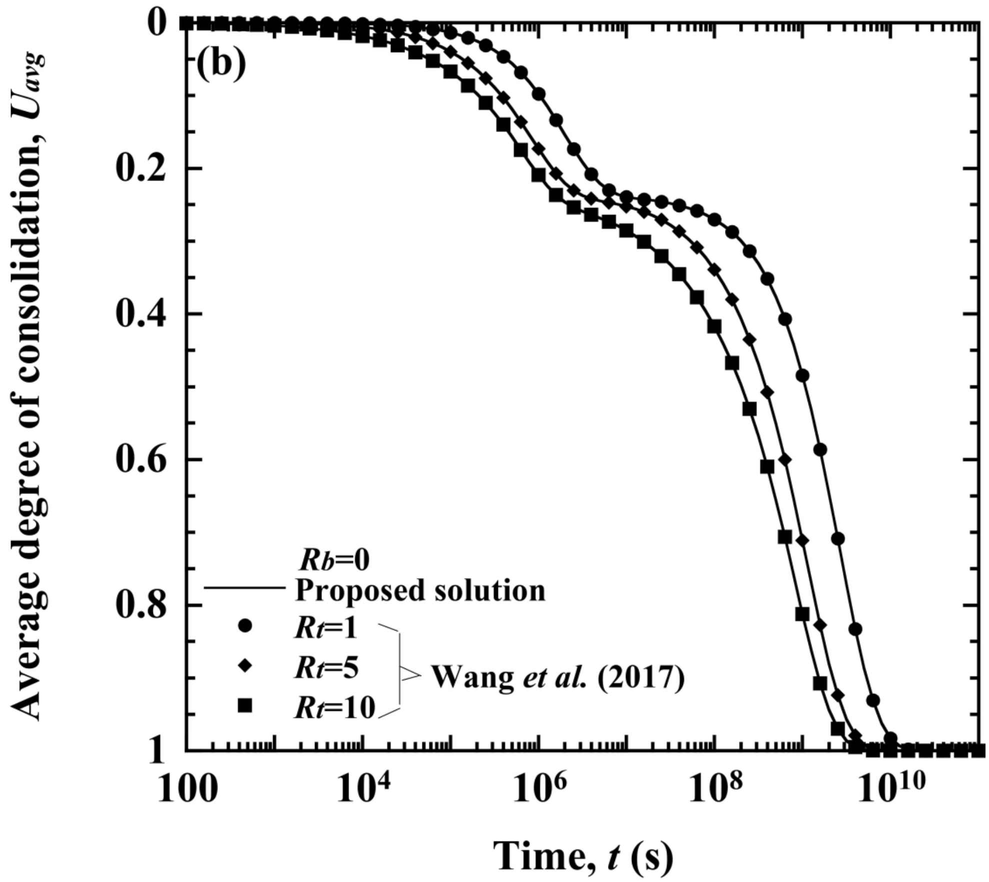

Fig.4. Verification profiles for the average degree of consolidation:(b) Rb=0, Rt=1,5, and 50;

150x136mm (600 x 600 DPI)

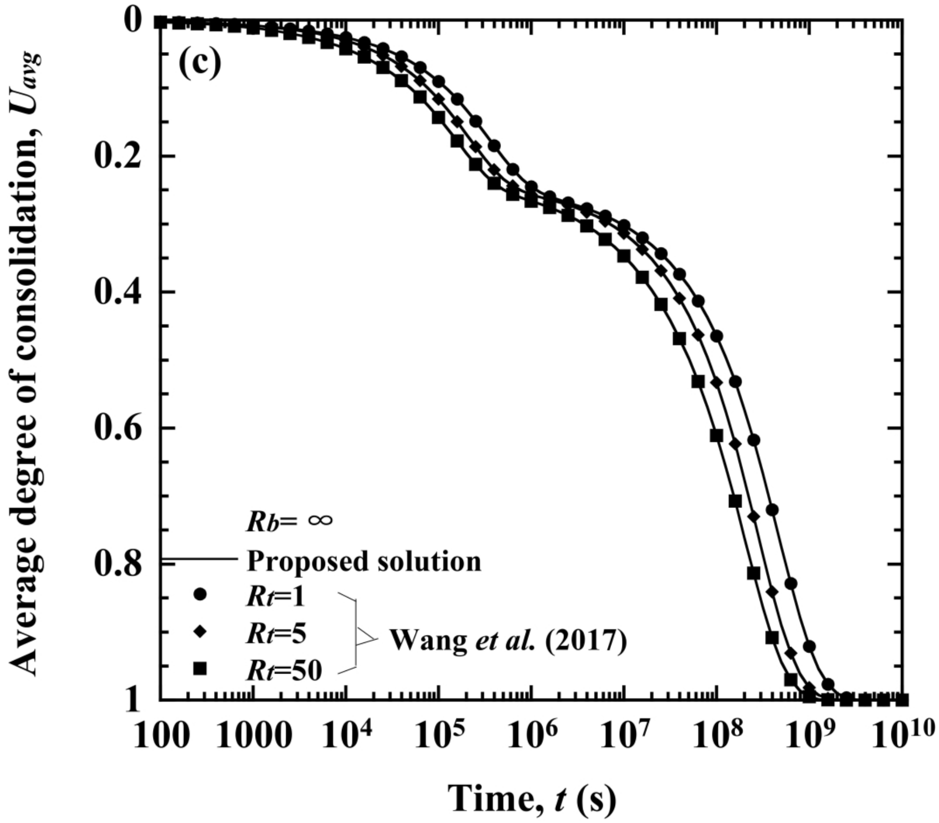

Fig.4. Verification profiles for the average degree of consolidation:(c) Rb=∞, Rt=1,5, and 50.

160x136mm (600 x 600 DPI)