A discrete day-to-day link flow dynamic model considering travelers' heterogeneous inertia patterns

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Abstract

This study models and analyzes how the heterogeneous psychological inertia of travelers affects the link flow evolution process on a day-to-day basis. The psychological inertia of a traveler is defined as his/her reluctance to reconsider his/her route choice, and is characterized by a sequence of binary parameters along time. All travelers are grouped into different classes by their inertia patterns. Based on these classes, a variational inequality formulation for the multi-class user equilibrium problem is presented. We proceed to develop a generic day-to-day link flow dynamic model by considering the heterogeneous inertia patterns of the travelers. The convergence properties of the model are rigorously demonstrated in the spirit of the essential cyclic type Gauss-Seidel decomposition algorithm. The developed model is formulated in a general form under mild assumptions, making it appealing for the implementation in practice. We further consider a special case of the generic model by specifying the inertia pattern, the distance function and link flow cost function. We examine properties of this special case model, and investigate its relationship with existing models in the literature. Numerical experiments are conducted to demonstrate our theoretical results.

Keywords: inertia pattern, class-based formulation, heterogeneous psychological inertia, day-to-day, link flow adjustment model.

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1. Introduction

Day-to-day flow dynamic models are used to simulate travelers' learning behaviors and traffic flow fluctuations over a transportation network from one day to another. (Horowitz, 1984; Cantarella and Cascetta, 1995; He et al., 2010; Guo et al., 2013; Smith et al., 2014; Smith and Watling, 2016; Watling and Cantarella, 2013; Xiao et al., 2017; Shang et al., 2017; Zhang et al., 2018). With increasing applications of advanced intelligent transport systems (ATIS), day-to-day flow dynamic models have received extensive attentions over the past decades. For example, Bifulco et al. (2009) explored the role of uncertainty in ATIS application through the day-to-day flow dynamic model proposed in Cantarella and Cascetta (1995). To evaluate the impact of AITS, Lo and Szeto (2004) formulated the dynamic traffic assignment problem as a cell-based variational inequality. Han et al. (2011) examined day-to-day evolution of the traffic network with ATIS and a pricing strategy was proposed to mitigate the instability of the traffic system. Cantarella (2013) analyzed the effects of the introduction of ATIS on total user surplus based on a day-to-day dynamic model. Bifulco et al. (2016) further investigated the equilibrium and stability properties of a day-to-day dynamic model under ATIS for recurrent traffic conditions.

The underlying assumption behind many day-to-day dynamic models is that if a transportation network falls into some disequilibrium or unstable state, travelers will adjust their routes, either with their experiences or with information provided by an ATIS, so that the resultant link or route flows evolve to some fixed point (which corresponds to equilibria) or attractors. Therefore, investigations into day-to-day dynamic models could facilitate not only a better understanding of the flow adjustment mechanism toward equilibrium in urban traffic networks but also a better utilization of various ATIS from the practical point of view.

Generally speaking, the day-to-day flow dynamic models can be classified into different groups according to different criteria. For example, based on the stochasticity of evolution process, the day-to-day flow dynamic models can be divided into deterministic or stochastic process models. Deterministic process models assume that the flow adjustment mechanism in each day is deterministic (Cantarella, 2013; Cantarella and Watling, 2016a). This type of models can be used to predict the flow evolution trajectory and study the equilibrium properties of the system. On the contrary, stochastic models allow an explicit simulation of the randomness of both demand and supply (Watling and Cantarella, 2015; Cantarella and Watling, 2016b, Watling and Hazelton, 2018.). They focus on providing the probability distribution of flow states and/or the expected flow state. Obviously, stochastic models are more general than deterministic models. However, the computational burden of stochastic models is much higher than deterministic models.

The day-to-day flow dynamic models can also be sorted into deterministic or probabilistic flow dynamic models based on different route choice behavior of travelers. The deterministic route choice models rely on the assumption that travelers evaluate their route costs without random variation (Yang and Zhang, 2009; He et al. 2010), whereas the probabilistic route choice models assume that travelers select their routes according to the random utility theory (Balijepalli and Watling, 2005). From a theoretical point of view, the probabilistic route choice model seems more realistic than the deterministic route choice model. However, some empirical studies have justified the rationality of the deterministic route choice model using the real traffic data. For example, He and Liu (2012) calibrated and validated a deterministic day-to-day dynamic model with the field data collected after the collapse of I-35W Bridge in Minneapolis, Minnesota. The results imply that the Wardrop route choice behavior is appropriate to characterize the day-to-day traffic evolution process after a real-world disruption scenario. Besides, several models also consider the bounded rationality of travelers in their route choice. For example, a boundedly rational user equilibrium (BRUE)-based day-to-day framework has been developed by Guo and Liu (2011) to characterize the impact of the irreversible network change on flow dynamics. The collapse and reopening of the I-35W Bridge in the Twin Cities network in Minnesota was used as a case study to illustrate the flow evolution under irreversible network change. Di et al. (2015) further examined the stability of Guo and Liu (2011)'s model under a special case in a parallel-link and single origin-destination (OD) network with separable and linear link time functions. Wu et al. (2013) investigated the path flow evolution in urban railway networks. In their study, the passengers were assumed to follow a bounded rational behavior in learning perceived travel costs. Ye and Yang (2017) integrated BRUE into the conceptual framework of rational behavior adjustment process (RBAP), and analyzed the stability of BRUE-RBAP with separable and link non-separable link travel cost functions.

If the criterion of the level of aggregation of the travel demands is employed, the day-to-day flow dynamic models can be categorized into aggregate or disaggregate models. Travelers in the aggregated model are assumed to be infinitesimal (non-atomic), and the traffic flows on the network are therefore be treated as a continuum (Watling and Hazelton, 2003; Bifulco et al., 2016). Conversely, the disaggregated models explicitly consider individual travelers, so the demand and flows of this model are discrete (Yang and Liu, 2007). It has been proved that as the number of individual travelers becomes large, the aggregated model can be approximated by the disaggregated models. In addition, there are also some models between the above two extremes. For example, if travelers are classified into a finite number of different classes, each class is treated as a continuum, then the aggregation level of this type of models is intermediate.

Among the above three ways of classification of the day-to-day dynamic models, this study focuses on the aggregated deterministic process model with deterministic route choice. This type of models usually assumes that travelers' route choice decisions for the next day are only dependent on the road condition of the current day, which enables it to deal with global convergence. The reason is that if travelers make their decisions according to today's network traffic conditions, then the day-to-day flow adjustment process is similar to the evolution of a descent iterative algorithm. As a result, a variety of algorithmic methodologies in the field of mathematical programming can be employed to investigate the convergence of the day-to-day dynamic model. Therefore, this type of models is more mathematically tangible when analyzing the convergence properties, which is also one of the main contributions of this study.

The aggregate deterministic process model with deterministic route choice can be further divided into continuous or discrete models, path-based or link-based models according to whether they are continuous or discrete, and whether they are formulated in terms of path flow or link flow. The path-based day-to-day dynamic model can explicitly describe the adjustment process of travelers' route choice behavior. Examples of this type of models include the simplex gravity flow dynamics (Smith, 1983), the proportional-switch adjustment process (Smith, 1984), the network tatonnement process (Friesz et al., 1994), the projected dynamical system (Zhang and Nagurney, 1996), and the evolutionary traffic dynamics (Sandholm, 2001). Yang and Zhang (2009) proposed a rational behavior adjustment process and demonstrated that their model includes as its special cases the aforementioned five continuous day-to-day path flow dynamic models.

The path-based flow dynamic models, however, suffer from the path-flow-nonuniqueness and path-overlapping problems, which were successfully addressed by a link-based flow dynamic model developed by He et al. (2010). Han and Du (2012) extended the model developed by He et al. (2010) to the case where the link travel cost functions are asymmetric, and they conducted a stability analysis of the continuous link flow adjustment model by using Lyapunov stability theory. Guo et al. (2013) developed a generic discrete link-based day-to-day dynamic model called the discrete rational adjustment process. Guo et al. (2015) further proposed a continuous link-based day-to-day dynamical system model. Several properties were established in their study, including the invariance of its evolutionary trajectories, the uniqueness and stability of its stationary points. Their model covered four existing models: the link-based version of the network tatonnement process (Friesz et al., 1994), the link based version of the projected dynamical system (Zhang and Nagurney, 1996), the models developed by He et al. (2010) and the link flow splitting mechanism proposed by Smith and Mounce (2011).

In another branch of the literature, the concept of inertia has also been widely considered by many transportation studies over the past decades. Generally speaking, it is used to describe many different phenomena that related to a resistance to change. The specific meanings of inertia may vary in different study contexts, based on which many traffic equilibrium models have been proposed (Chorus and Dellaert, 2012; Choudhury and Ben-Akiva 2013; Xie and Liu, 2014; Zhang and Yang, 2015; Liu et al., 2017). Particularly, in the context of day-to-day dynamic models, the presence of travelers' intrinsic inertia would inevitably affect the evolution process of traffic flows, which makes this concept well accepted in this field. Most day-to-day dynamic models in the literature focus on two types of inertia. One was proposed in He et al. (2010). Their model assumed that travelers are reluctant to make significant route changes. They tend to form a target flow that is 'closest' to the current flow. Since this type of inertia belongs to the category of travelers' route choice habits (c.f. Fig 1 in Section 2), we name it as 'route choice inertia'. The other was discussed in Cantarella (2013) and Guo et al. (2013). They assumed that only a portion of travelers would finally switch to their target routes, even if these routes have already been activated in their minds. This type of inertia in essence reflects travelers' behavioral response, therefore we refer to it as 'behavioral inertia'.

Clearly, the route choice inertia occurs when travelers are choosing their target routes, while the behavioral inertia occurs after their target routes have been determined. Both of these studies reckoned that travelers are active enough to reconsider their routes, so that they are willing to make a decision about what target route to choose every day. However, this assumption seems to be restrictive in reality. As is known, making a target route decision includes tedious procedures of evaluating different route costs and choosing an appropriate target route according to some criterion. For some travelers, making such a decision every day is boring and exhausting. Although an ATIS can alleviate their decision making effort by providing some information of route or link travel costs, it cannot eliminate the mentally consuming process to select the target routes (c.f., the target flow choosing criteria proposed in different studies). As a result, travelers (especially commuters) who are familiar with their routes tend to avoid the effort of making a new route decision, and hence are usually not willing to reconsider their route every day. This is another type of inertia that occurs before travelers make their target route decisions. Since this type of inertia reflects travelers' psychological willingness to reconsider their routes, we name it as 'psychological inertia' in this study. Yang and Liu (2007) introduced a similar concept to characterize travelers 'route switching rate'. By applying the evolutional game theory, Yang and Liu proposed a Markov model to study travelers' stochastic behavior in their day-to-day route choice adjustment process. They showed that when the demand is large, the mean route flow dynamic over finite time spans follows almost deterministic trajectory. The route flow Markov model is very general in its form, in the sense that it may evolve to equilibrium or disequilibrium states depending on different behavioral rules of route-switching rate and route choice probability. As a result, the convergence of the route flow Markov model was not further discussed in Yang and Liu's work. Furthermore, the underlying route reconsidering assumption for the route flow Markov model assumes that the interval between two consecutive reconsideration behavior follows the exponential distribution, which seems a little restrictive from a practical point of view.

In this research, we adopt another way to model travelers' psychological inertia and explicitly investigate the issue of convergence. Specifically, we consider a more general route reconsidering behavior by assuming that the intervals between two consecutive reconsideration behavior can be any irregular values, provided that the inertia scenarios follow some recurrent patterns. To facilitate the incorporation of travelers' heterogeneous inertia patterns into the day-to-day flow dynamic framework, we first formulate a class-based variational inequality for the multi-class user equilibrium problem. A generic inertia-based multi-class link flow adjustment model is developed subsequently. Since our model is formulated in a general form under mild assumptions, it offers great ease and flexibility for practical implementations. Moreover, the convergence properties of the generic model are rigorously established. The corresponding proof in essence extends the essential cyclic type Gauss-Seidel decomposition algorithm (Tseng, 1991; Patriksson, 1998) to the case where multiple subproblems are allowed to be solved in an irregular way at each iteration step. Therefore, the methodology proposed in this study is not only appropriate to investigate the evolution of link flow with travelers' heterogeneous inertia patterns, but also suitable to theoretically demonstrate the convergence of a board class of parallel and distributed computing problems.

The remainder of this study is organized as follows. The notion of psychological inertia and the multi-class user equilibrium model are presented in Section 2. Section 3 proposes a generic day-to-day link flow dynamic model considering heterogeneous psychological inertia. The convergence of the proposed model is rigorously demonstrated under mild conditions in Section 4. Section 5 discusses a special case of the generic model. Numerical experiments are conducted in Section 6 to evaluate the performance of proposed day-to-day link flow dynamic model. Conclusions and future research directions are presented in Section 7.

2. Notations, assumptions and problem description

Travelers are a fundamental component of road traffic. Their decisions and actions influence the flows on the network every day. Day-to-day dynamic models are developed to characterize traveler's daily route choice behavior from an evolutionary perspective. Traditionally, these models assume that based on the current day's situation, travelers are active enough to reconsider their routes for the next day, such that a decision about what target route should be chosen is made. The target route decision process involves evaluating different route costs and choosing an appropriate target route according to some criterion, which allows for various interpretations in different models. However, in reality, everyone has some degree of psychological inertia. A recent survey conducted by our team found that some travelers (especially commuters) are not accustomed to making such a decision every day, since they regard it as a boring and mentally consuming procedure. They may use the same route for a few days before reconsidering their routes. This phenomenon aligns with the basic concept of inertia, i.e., once the habit is formed, it is difficult to get rid of it.

Following on from the discussions in the introduction, there are three main factors that affect the daily traffic flows on the network. The first factor is travelers' psychological inertia. Given a traffic situation at some day, not all travelers are willing to reconsider their routes immediately. Some travelers may make their new decisions several days later. The second factor is travelers' route changing habit, i.e., what criterion do travelers adopt to change their route. For example, Smith (1984) assumed that travelers on a higher cost route switch to other lower cost routes on the next day in a rate that is proportional to the cost difference between them. Zhang and Nagurney (1996) prescribed the traffic flow changing rate as a projection of negative cost function to the network feasible set. Note that different route changing habits can be described by a unified framework based on the difference between the target flow and the current flow (Yang and Zhang 2009, Guo et al. 2015). The third factor is travelers' behavioral inertia. Even a traveler is active enough to evaluate and compare different routes so that a target route has already formed in his mind, he/she may not use the target route due to behavioral inertia. In the literature, this type of inertia appears in discrete day-to-day flow dynamic models (Cantarella 2013, Guo et al. 2013). It is characterized by the proportion of travelers who finally travel on their target routes, which is essentially a step size parameter between [0,1] from the modeling point of view.

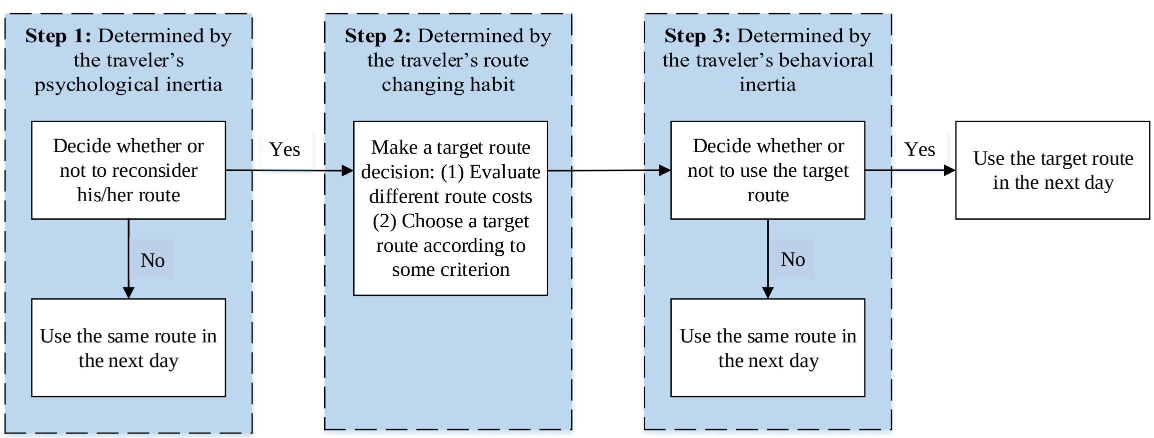


Fig. 1. The route change decision process of a typical traveler

Fig. 1 shows the route change decision process of a typical traveler. It consists of three steps, each of which corresponds to a factor that is discussed above. As we can see, in the first step, the traveler decides whether or not to reconsider his/her route on the next day. This step is determined by the traveler's psychological inertia. If the answer is 'Yes', then the decision process will switch to the second step. On the contrary, if the answer is ‘No’, the second step will not be activated. He/she will use the same route on the next day. The second step is related to the traveler’s target route decision, which concerns with the evaluation of different route costs on the current day, and selection of a target route according to some criterion. Clearly, the selection of the target route reflects his/her route changing habit. The third step is influenced by the traveler’s behavioral inertia. In this step, the traveler will make a final decision whether or not to travel on the target route. If the answer is ‘Yes’, then the target route will be used on the next day. If the answer is ‘No’, the traveler will still use the same route as before.

It is known that most existing day-to-day dynamic models only considered travelers’ habitual and behavioral inertia in the second and third step of the above route change decision process. However, ignoring the first step may lead to an incomplete understanding of travelers’ decision process. According to our survey, most travelers who do not change their routes usually serve the purpose of avoiding the effort to make a new (target route) decision. Therefore, compared with the behavioral inertia in Step 3, the psychological inertia in Step 1 is probably the main reason that prevents travelers from changing their routes. In addition, we would like to point out that even if a traveler is willing to reconsider his/her route, he/she is not necessary to travel on a different one, because after evaluation and comparison, he/she may finally use the same route as before. On the contrary, if the traveler is not willing to reconsider his/her route, he/she will definitely travel on the same route as before.

In order to describe the evolution process of travelers’ psychological inertia, next we introduce the definition of inertia pattern. It is apparent that the outcome of the first step in Fig. 1 can be represented by a binary variable. This variable characterizes the traveler’s psychological inertia at some day. We name it as a decision variable. At each day, if a traveler is willing to reconsider his route, the decision variable is marked as 1, if not, it is marked as 0. Therefore, for an ordinary traveler, the evolution of his decision variable would be a 0-1 sequence. We call this sequence an inertia pattern, which is formally defined as follows:

Definition 1 An inertia pattern is a sequence of (binary) decision variables representing the evolution of a traveler’s (or a class of travelers’) route change willingness over time.

For example, Fig. 2 illustrates the inertia pattern of three typical travelers. Traveler 1’s inertia pattern is represented by the sequence (1,0,1,0,1,0,1,0,1,0,…), which means this traveler is willing to reconsider his/her route choice every 2 days. Traveler 2's inertia pattern is described by the sequence (1,0,0,1,0,0,1,0,0,…), which implies he/she would like to reconsider his/her route choice every 3 days. Following the same logic, Traveler 3's inertia pattern is (1,0,1,1,1,1,0,1,0,1,…), which indicates his/her positive willingness occurs irregularly on different days.

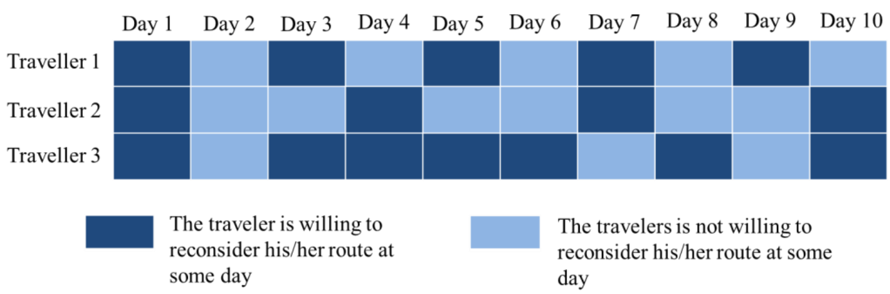


Fig. 2. Inertia patterns for 3 typical travelers.

The above definition enables us to construct a novel class-specific formulation for the traditional multi-class user equilibrium problem. Consider a transportation network $G(N, A)$, where N is the set of nodes and A is the set of directed links. Let W be the set of all OD pairs in the network, and R_w be the set of routes between OD pair $w \in W$. Assume that there are m classes of travelers with a typical class denoted by i. Travelers in each class have the same inertia pattern. The set of all the class indexes is represented by $M = \{1, 2, ...m\}$ and we have $i \in M$.

Let $(d_w)_i$ denote the demand of class i for OD pair $w \in W$, and $(f_{rw})_i$ the flow of class i on route $r \in R_w$, $w \in W$. The multi-class demand and route flows satisfy the following conservation conditions:

$$\sum_{r \in R_w} (f_{rw})_i = (d_w)_i, \ \forall w \in W, i \in M. \tag{1}$$

Let $(x_a)_i$ be the flow of user class i on link a and x_a the total flow on link a, then the following relationship holds between the link flows and route flows of class i:

$$(x_a)_i = \sum_{w \in W} \sum_{r \in R_w} (f_{rw})_i \delta_{ar}, \forall a \in A, i \in M, \tag{2}$$

where $\delta_{ar} = 1$ if path r traverses link a, and 0 otherwise. In addition, the total flow on link a can be expressed by the following equation:

$$x_a = \sum_{i \in M} (x_a)_i, \forall a \in A. \quad (3)$$

Let $c_a(x_a)$, $a \in A$ denote the link cost function and $\left(c_a(x_a)\right)_i \triangleq (c_a)_i$ be the link travel cost of class i. Assume that $c_a(x_a)$ is continuously differentiable and strictly increasing with respect to link flow x_a. Clearly, for any link in the network, its travel cost is the same for all classes of travelers, i.e.,

$$(c_a)_1 = (c_a)_2 = \ldots = (c_a)_m = c_a, \forall a \in A. \quad (4)$$

The route cost function, denoted by p_{rw}, $r \in R_w$, $w \in W$, can be expressed as follows:

$$(p_{rw})_i = \sum_{a \in A} (c_a)_i \delta_{ar}, \forall a \in A. \quad (5)$$

The multi-class user equilibrium conditions are given by

$$(p_{rw})_i = \begin{cases} = (\mu_w)_i & if \ \ (f_{rw})_i > 0 \\ \geq (\mu_w)_i & if \ \ (f_{rw})_i = 0 \end{cases}, \ \forall r \in R_w, \ w \in W, \ i = 1, 2, \ldots m, \quad (6)$$

where $(\mu_w)_i$ equals the minimum travel time of OD pair w for travelers of class i. These conditions imply that, at the optimum, for any class $i \in M$, all used paths have the same and minimum travel times.

It can be easily shown that the multi-class user equilibrium flow pattern can be obtained by solving the following optimization problem:

[MC-UE]

$$\min_{\mathbf{x} \in \Omega} \ Z(\mathbf{x}) = \sum_{a \in A} \int_0^{x_a} c_a(s) ds, \quad (7)$$

where $\mathbf{x}$ is the multi-class link flow vector, and Ω is the feasible region of $\mathbf{x}$, i.e., $\Omega = \left\{ \mathbf{x} = (\ldots, (x_a)_i, \ldots)^T \mid (x_a)_i = \sum_{w \in W} \sum_{r \in R_w} (f_{rw})_i \delta_{ar}, a \in A; \sum_{r \in R_w} (f_{rw})_i = (d_w)_i, (f_{rw})_i \geq 0, \forall w \in W, r \in R_w, i \in M \right\}.$

Clearly, the objective function of [MC-UE] is strictly convex with respect to the total link flow variable x_a, while the strict convexity with respect to the class-specific link flow variable $(x_a)_i$ cannot be assured. As a result, the equilibrium flow pattern is unique in terms of total link flows, and there may exist multiple class-specific link flows patterns at equilibria.

In addition, the following proposition demonstrates that the multi-class equilibrium flow pattern can also be obtained by solving the following variational inequality problem:

$$\begin{bmatrix} \mathbf{c}_1(\mathbf{x}^*) \\ \vdots \\ \mathbf{c}_i(\mathbf{x}^*) \\ \vdots \\ \mathbf{c}_m(\mathbf{x}^*) \end{bmatrix}^T \begin{pmatrix} \mathbf{x}_1 - \mathbf{x}_1^* \\ \vdots \\ \mathbf{x}_i - \mathbf{x}_i^* \\ \vdots \\ \mathbf{x}_m - \mathbf{x}_m^* \end{pmatrix} \geq 0, \ \ \forall \mathbf{x}_i \in \Omega_i, i \in M, \tag{8}$$

where $\mathbf{x}$ is rewritten in term of the class-specific link flow vector $\mathbf{x}_i = \left((x_1)_i, (x_2)_i, ..., (x_{|A|})_i\right)^T$, i.e., $\mathbf{x} = (\underbrace{(x_1)_1, (x_2)_1, ..., (x_{|A|})_1}_{\mathbf{x}_1}, ..., \underbrace{(x_1)_m, (x_2)_m, ..., (x_{|A|})_m}_{\mathbf{x}_m})^T$, and accordingly $\mathbf{c}_i = \left((c_1)_i, (c_2)_i, ..., (c_{|A|})_i\right)^T$ and

$$\Omega_i = \left\{ \mathbf{x}_i \mid (x_a)_i = \sum_{w \in W} \sum_{r \in R_w} (f_{rw})_i \delta_{ar}, a \in A; \sum_{r \in R_w} (f_{rw})_i = (d_w)_i, (f_{rw})_i \geq 0, \forall w \in W, r \in R_w \right\}.$$

Proposition 1. A class-specific flow variable $\mathbf{x}^* = (\mathbf{x}_1^*, \mathbf{x}_2^*, ... \mathbf{x}_m^*)$ satisfies the multi-class user equilibrium conditions (6) if it solves the variational inequality problem (8).

Proof. Follows directly from the definition of $\mathbf{c}_i$ and Theorem 1 in Nagurey (2000).■

In view of the traffic flow dynamics, two natural questions arises: (i) How will the traffic flow evolve if the day-to-day flow dynamic framework incorporates the heterogeneous inertia patterns? (ii) Will the resultant day-to-day flow dynamic model converge to the multi-class user equilibrium state? The objective of this study is to answer these two question by developing an inertia-incorporated day-to-day link flow dynamic model and examining its convergence properties.

3. A generic multi-class link flow dynamic model considering heterogeneous psychological inertia

This section presents a generic link flow adjustment model considering heterogeneous psychological inertia. In this model, travelers' flow adjustment behaviors or inertia patterns are represented by a general form of a convex function or some sets satisfying certain assumptions. By specifying the function or these sets in different ways, such a framework can encompass many models as special cases.

We first introduce the definition of inertia scenario to facilitate the presentation of the related formulas and theorems. Let $M^t \subseteq M$ denote the set of class indexes of travelers whose decision variable at day t are represented by 1, i.e., the set of class indexes of travelers who are willing to reconsider their route choice. It follows that M^t is a nonempty subset of $M = \{1, 2, \ldots m\}$. i.e., $M^t \subset M$. Taking Fig. 3 as an example, at Day 1 travelers of Class1, Cass 2 and Class 3 are all willing to reconsider their routes, so $M^1 = \{1, 2, 3\}$. At Day 2 and 3, only travelers of Class 2 would like to reconsider their routes, thus $M^2 = M^3 = \{2\}$. Following the same logic, we have $M^4 = M^5 = \{1, 2\}$.

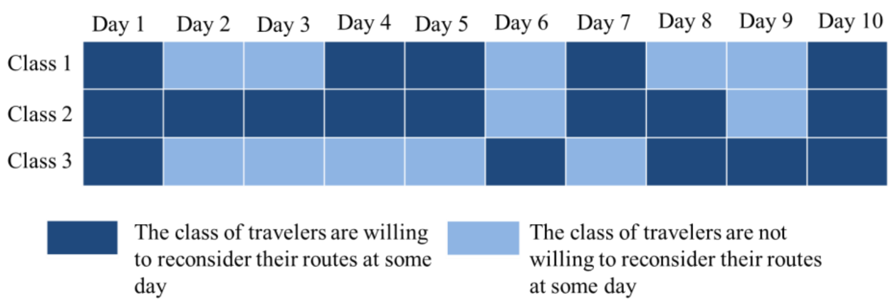


Fig. 3. Inertia patterns for 3 classes of travelers.

From the above example, we can see that the ‘value’ (which is in essence a set) of M^t may be the same at different days. This value is defined as an inertia scenario. Making such a definition captures the essence of different inertia patterns.

Definition 2. Group all possible cases of M^t into a set $\{S_1, S_2 \ldots S_n\}$ ($S_1 \neq S_2 \neq \ldots \neq S_n$). Then S_i ($i = 1, 2, \ldots n$) is referred to as an inertia scenario of M^t, and n is a finite number of all inertia scenarios that are encountered during the evolution process.

As an illustration, the inertia scenarios from day 1 to day 10 in Fig. 3 are described by six sets: $S_1 = \{2\}$, $S_2 = \{3\}$ $S_3 = \{1, 2\}$, $S_4 = \{2, 3\}$, $S_5 = \{1, 2, 3\}$, which are different from one another.

Based on the class-specific flow variable, we propose the link flow dynamic model with heterogeneous psychological inertia as follows.

$$\begin{cases} \mathbf{x}_i^{t+1} - \mathbf{x}_i^t = l^t \left(\mathbf{y}_i^t - \mathbf{x}_i^t \right), & i \in M^t \\ \mathbf{x}_i^{t+1} = \mathbf{x}_i^t, & i \notin M^t \end{cases}, \tag{9}$$

where $\mathbf{x}_i^t$ is the link flow of class i on day t, $\mathbf{y}_i^t$ is the target link flow of class i for the next day (which is determined on day t when travelers finish their trip), $l^t \in (0,1)$ is the flow changing rate, and $\mathbf{x}_i^{t+1}$ is the link flow of class i on day $t+1$. It can be found that the proposed model utilizes a discrete version of the link-based day-to-day dynamic process proposed by He et al. (2010) for the travelers of class $i \in M^t$. Since those travelers are willing to reconsider their route choice on day t, their link flows will move from the current flow $\mathbf{x}_i^t$ towards a future flow $\mathbf{x}_i^{t+1}$ on day $t+1$ with a flow changing direction $\mathbf{y}_i^t - \mathbf{x}_i^t$ and a flow changing rate l^t. On the contrary, the travelers of class $i \notin M^t$ are not willing to reconsider their route choice at day t and thus the link flows of them will remain unchanged on day $t+1$ because they will use the same route on day $t+1$.

The target flow determination in most day-to-day dynamic models captures travelers' cost minimization behaviors because it is believed that the motivation of travelers to change their route comes from their recognition of an alternative route with less time. In other words, based on today's network traffic conditions, travelers tend to choose the route with the minimal travel time on the next day. However, determining the target flow by minimizing the travel time of travelers may lead to the indeterminacy of flow dynamics when there are multiple shortest routes and travelers may choose any one of them. To overcome this problem, He et al. (2010) assumed that travelers do not make unnecessary changes when they seek to minimize their travel costs and they introduced a convex function in the cost minimization problem to ensure the uniqueness of the target link flow.

By the same spirit of He et al. (2010), we also use a convex function to regularize the target link flow. Specifically, the target flow $\mathbf{y}_i^t$ in model (9) is obtained by solving the following problem given the current link flow $\mathbf{x}^t$:

$$\min_{\mathbf{y}_i \in \Omega_i} \ \mathbf{c}_i(\mathbf{x}^t)^T \mathbf{y}_i + D_i^t(\mathbf{y}_i, \mathbf{x}_i^t), \ \forall i \in M^t, \tag{10}$$

where $D_i^t(\mathbf{y}_i, \mathbf{x}_i^t)$ is a certain function that measures the difference between the target flow $\mathbf{y}_i$ and the current flow $\mathbf{x}_i^t$ of class i. This function is used to describe travelers' route changing habits in the second step of the route change decision process (c.f. Fig.1). It has different interpretations in different studies. In He et al. (2010), for example, it is regarded as travelers' reluctance to make significant changes such that they tend to form a target flow $\mathbf{y}_i^t$ that is 'closest' to the current flow $\mathbf{x}_i^t$. This interpretation reflects a kind of route choice inertia that is distinguished from the psychological or behavioral inertia discussed above. In Han and Du (2012) and Guo et al. (2015), it is viewed as a link-based version of the network tatonnement process (Friesz et al. 1994) or projected dynamical system (Zhang and Nagurney 1996).

Note that in all these studies, the function $D(\mathbf{y},\mathbf{x})$ is time-invariant under evolution. However, in this study, the function $D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)$ depends on t. It can have different forms on different evolution days. This offers us more flexibility to characterize travelers' behaviors in different evolution stages.

Clearly, $D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)$ is a function of two vector variables conditioned on the 2nd variable $\mathbf{x}_i^t$ being known. To avoid confusion, in this paper the notation $\nabla D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)$ is referred to as the gradient of $D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)$ with respect to the 1st vector variable $\mathbf{y}_i$. As indicated by He et al. (2010), in order for the link flow dynamic model to be well defined, the following assumptions on $D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)$ should be made:

Assumption 1. $D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)$ is nonnegative and satisfies $D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)=0$ if and only if $\mathbf{y}_i=\mathbf{x}_i^t$.

Assumption 2. $D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)$ is continuously differentiable and strongly convex on Ω_i for every fixed $\mathbf{x}_i^t$. i.e.,

$$\left[\nabla D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)-\nabla D_i^t(\mathbf{z}_i,\mathbf{x}_i^t)\right]^T\left(\mathbf{y}_i-\mathbf{z}_i\right)\ge m_{D_i^t}\left\|\mathbf{y}_i-\mathbf{z}_i\right\|^2,\ \forall \mathbf{y}_i\in\Omega_i,\mathbf{z}_i\in\Omega_i,i\in M, \tag{11}$$

where $m_{D_i^t}$ is a positive, bounded modulus of the strongly convex function D_i^t.

Remark: These two assumptions have meaningful implications. From Theorem 1 in He et al. (2010), we know that $\mathbf{y}_i^t=\mathbf{x}_i^t$ can only be achieved at user equilibrium. Therefore, Assumption 1 implies that only at this equilibrium state, travelers of class i do not have any incentive to reconsider their route, such that the distance function $D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)$ becomes unnecessary. Assumption 2 ensures that the solution to problem (10) is unique, which means the target flow can be reasonably predicted. Both of these assumptions are widely adopted in the literature.

From a theoretical point of view, Assumption 2 is crucial for the convergence of traditional link based day-to-day dynamic models, since it makes the potential function $Z(\mathbf{x})$ decrease every day (Theorem 9 in Han and Du 2012), such that $\mathbf{x}^t$ eventually evolves to some user equilibrium point (Theorem 3 and 4 in Guo et al. 2015). However, for the case of the inertia-based multi-class link flow adjustment model proposed in this study, using Assumption 2 only is not sufficient to ensure convergence. For example, suppose that travelers' route changing habits satisfy Assumption 2. During the evolution process, a certain class of travelers consider to change their route on only one day, after which they will stick to the same route (e.g., Route A) every day. By Proposition 2 in Section 4, although other classes of travelers make $Z(\mathbf{x})$ decrease every day, if Route A is not an equilibrium route at UE state, then it is clear that user equilibrium cannot be achieved. The reason for this non-convergence is attributed to asynchronism in their route choice. Due to psychological inertia, some travelers may not have enough opportunity to change their route, which may prevent the traffic flow from evolving to equilibrium. In order to overcome this drawback, two additional assumptions should be made, which are listed as follows:

Assumption 3. The union $S_1 \cup S_2 ... \cup S_n$ equals M .

Remark: By definition, $S_1, S_2 ... S_n$ are all inertia scenarios that we meet in the evolution process. Since S_j $j \in \{1, 2, ...n\}$ denotes the classes of travelers who are willing to reconsider their routes, $S_1 \cup S_2 ... \cup S_n$ equals M implies during the evolution process, all classes of travelers will reconsider their routes at least once.

Assumption 4. There exists a positive integer T ($T \geq n$) such that all scenarios from the set $\{S_1, S_2 ... S_n\}$ are met at least once during every T consecutive days.

Remark: Assumption 4 implies that during the evolution process, M^t ($t = 1, 2, ...$) are in some sense 'recurrent', such that different scenarios can be recurred at least T days. Since T can be chosen very long, this assumption is actually very weak. For example, we can assume that travelers' inertia pattern evolves irregularly during the evolution process. We can even assume that travelers' inertia pattern is a function of route costs, such that as the oscillation of route costs becomes small, more travelers are reluctant to reconsider their routes. The condition of Assumption 4 is very likely to be satisfied for both of these cases, provided that T is chosen sufficient long.

In view of the discussions above, we know that Assumptions 1 and 2 follow the convention of the traditional day-to-day dynamic models, while Assumptions 3 and 4 are newly proposed conditions that enable the model to be well defined and convergent. Since Assumptions 3 and 4 are very weak, they can hold for most cases in reality.

As a conclusion of this section, we describe the iterative process of the generic link flow adjustment model with heterogeneous psychological inertia as follows: Assume that the travelers can be divided into m classes, each of which has the same inertia pattern. Let $\mathbf{x}_i^t$ be the class-specific flow pattern at day t, and be M^t the set of class indexes of travelers whose decision variable at day t are represented by 1. For travelers who are willing to reconsider their route at day t, their flow pattern for the next day are given by

$$\mathbf{x}_i^{t+1} = \mathbf{x}_i^t + l^t\left(\mathbf{y}_i^t - \mathbf{x}_i^t\right) \;\; \forall i \in M^t,$$

where $\mathbf{y}_i^t$ is determined by Eq. (10);

For travelers who are not willing to reconsider their route at day t, their flow pattern for the next day will remain unchanged, i.e.,

$$\mathbf{x}_i^{t+1} = \mathbf{x}_i^t \quad \forall i \notin M^t..$$

The above evolution process iterates until the equilibrium state is achieved.

4. Convergence analysis of the generic day-to-day link flow dynamic model

In order to deal with infinitely many irregular cases that may occur in reality, Assumptions 3 and 4 are expressed through the language of set theory. This makes the convergence analysis of the generic day-to-day link flow dynamic model more complex than traditional models. In what follows, we will investigate the convergence properties of the proposed model.

We note that the link flow dynamic model (9) is similar to the evolution process of the essential cyclic type Gauss-Seidel decomposition algorithm (Tseng 1991; Patriksson 1998). However, this algorithm allows only one class of travelers to update their flow pattern at each day. While in model (9), an irregular number of classes of travelers are required to update their flow patterns every day. In order to meet these requirements, we make modifications of the essential cyclic type Gauss-Seidel decomposition algorithm and apply it to analyze the convergence of the link flow dynamic model. We will show that each limit point of the class-specific link flow sequence corresponds to an equilibrium state under mild conditions. Furthermore, the convergence of the class-specific day-to-day dynamic link flow towards the set of equilibrium points is also established.

In the first place, we will discuss the properties of the target flow. Proposition 2 below indicates that for any class of travelers who are willing to reconsider their route choice on day t, their target flow will decrease their total travel cost based on the link costs on day t.

Proposition 2. For any class of travelers who are willing to change their route on day t, their target flow $\mathbf{y}_i^t \ \forall i \in M^t$ that obtained by solving the problem (10) satisfies

$$\mathbf{c}_i(\mathbf{x}^t)^T(\mathbf{y}_i^t - \mathbf{x}_i^t) \le -m_{D_i^t} \left\| \mathbf{y}_i^t - \mathbf{x}_i^t \right\|^2 < 0, \ \forall i \in M^t. \tag{12}$$

Proof. The optimality conditions of problem (10) is given by

$$\left[\mathbf{c}_i(\mathbf{x}^t) + \nabla D_i^t(\mathbf{y}_i^t, \mathbf{x}_i^t) \right]^T (\mathbf{z}_i - \mathbf{y}_i^t) \ge 0, \ \forall \mathbf{z}_i \in \Omega_i, \forall i \in M^t. \tag{13}$$

By substituting $\mathbf{z}_i = \mathbf{x}_i^t$ in the above equation, we obtain

$$\mathbf{c}_i(\mathbf{x}^t)^T(\mathbf{y}_i^t - \mathbf{x}_i^t) \le -\nabla D_i^t(\mathbf{y}_i^t, \mathbf{x}_i^t)^T(\mathbf{y}_i^t - \mathbf{x}_i^t), \ \forall \mathbf{z}_i \in \Omega_i, \forall i \in M^t. \tag{14}$$

By Assumption 1, the minimum of $D_i^t(\mathbf{y}_i, \mathbf{x}_i^t)$ is achieved at $\mathbf{y}_i = \mathbf{x}_i^t$. Since $D_i^t(\mathbf{y}_i, \mathbf{x}_i^t)$ is differentiable with respect to $\mathbf{y}_i$, we have

$$\nabla D_i^t(\mathbf{x}_i^t, \mathbf{x}_i^t) = 0. \tag{15}$$

Assumption 2 suggests that

$$\left[\nabla D_i^t(\mathbf{y}_i, \mathbf{x}_i^t) - \nabla D_i^t(\mathbf{z}_i, \mathbf{x}_i^t) \right]^T (\mathbf{y}_i - \mathbf{z}_i) \ge m_{D_i^t} \left\| \mathbf{y}_i - \mathbf{z}_i \right\|^2, \ \forall \mathbf{y}_i \in \Omega_i, \mathbf{z}_i \in \Omega_i. \tag{16}$$

Substituting $\mathbf{y}_i = \mathbf{y}_i^t, \mathbf{z}_i = \mathbf{x}_i^t$ in Eq. (16) and using Eq.(15) , we have

$$\nabla D_i^t(\mathbf{y}_i^t, \mathbf{x}_i^t)^T(\mathbf{y}_i^t - \mathbf{x}_i^t) = \left[\nabla D_i^t(\mathbf{y}_i^t, \mathbf{x}_i^t) - \nabla D_i^t(\mathbf{x}_i^t, \mathbf{x}_i^t) \right]^T (\mathbf{y}_i^t - \mathbf{x}_i^t) \ge m_{D_i^t} \left\| \mathbf{y}_i^t - \mathbf{x}_i^t \right\|^2. \tag{17}$$

In view of Eqs. (14) and (17), we get

$$\mathbf{c}_i(\mathbf{x}^t)^T(\mathbf{y}_i^t - \mathbf{x}_i^t) \le -m_{D_i^t} \left\| \mathbf{y}_i^t - \mathbf{x}_i^t \right\|^2, \ \forall i \in M^t. \tag{18}$$

which is the desired result. ■

Remark: Although Proposition 2 only considers the class of travelers who are willing to change their routes on day t, that is, $\forall i \in M^t$, the results also apply to those who do not change their routes because $\mathbf{y}_i - \mathbf{x}_i = 0$ for all $\forall i \notin M^t$ by Eq. (9). Therefore Eq. (12) can be rewritten by

$$\mathbf{c}(\mathbf{x}^t)^T(\mathbf{y}^t - \mathbf{x}^t) \le -m_{D_i^t} \left\| \mathbf{y}^t - \mathbf{x}^t \right\|^2 < 0. \tag{19}$$

It should be pointed out that Eq. (19) coincides with the descent condition of the rational behavior adjustment process proposed by Yang and Zhang (2009) and Guo et al. (2015).

The following proposition shows that as time goes on, the class-specific target flow will approach the current class-specific flow, which implies that travelers become more and more reluctant to change their routes.

Proposition 3. Let $\mathbf{y}^t$ and $\mathbf{x}^t$ be the class-specific target flow and current flow on day t, then we have $\lim_{t\to\infty}(\mathbf{y}^t - \mathbf{x}^t) = 0$ if the flow changing rate $l^t \in (0, \min(1, \frac{2m_{D^t}}{L_{\mathbf{c}}}))$ where $m_{D^t} = \min_{i\in M^t}\left\{m_{D_i^t}\right\}$ and $L_{\mathbf{c}} = \max_{i\in M^t}\left\{L_{\mathbf{c}_i}\right\}$, and the class-specific link cost functions $\mathbf{c}_i(\mathbf{x})$ are Lipschitz continuous, i.e.,

$$\left|\mathbf{c}_i(\mathbf{x}) - \mathbf{c}_i(\mathbf{y})\right| \le L_{\mathbf{c}_i}\left\|\mathbf{x} - \mathbf{y}\right\|, \ \forall \mathbf{x}, \mathbf{y} \in \Omega, i \in M, \tag{20}$$

where $L_{\mathbf{c}_i}$ is a positive, bounded modulus of the link cost function $\mathbf{c}_i(\mathbf{x})$.

Proof. By applying Taylor's formula, the difference of objective values of model [MC-UE] between day $t+1$ and day t is calculated by

$$\begin{aligned}
Z(\mathbf{x}^{t+1}) - Z(\mathbf{x}^t) &= \int_0^{l^t} (\mathbf{y}^t - \mathbf{x}^t)^T \nabla Z\left(\mathbf{x}^t + s(\mathbf{y}^t - \mathbf{x}^t)\right) ds \\
&= \int_0^{l^t} \sum_{i\in M^t} (\mathbf{y}_i^t - \mathbf{x}_i^t)^T \nabla_i Z\left(\mathbf{x}^t + s(\mathbf{y}^t - \mathbf{x}^t)\right) ds \\
&= \sum_{i\in M^t} \int_0^{l^t} (\mathbf{y}_i^t - \mathbf{x}_i^t)^T \nabla_i Z\left(\mathbf{x}^t + s(\mathbf{y}^t - \mathbf{x}^t)\right) ds \\
&= \sum_{i\in M^t} \int_0^{l^t} (\mathbf{y}_i^t - \mathbf{x}_i^t)^T \mathbf{c}_i\left(\mathbf{x}^t + s(\mathbf{y}^t - \mathbf{x}^t)\right) ds.
\end{aligned} \tag{21}$$

Rearranging terms in Eq. (21) and taking norms, we have

$$Z(\mathbf{x}^{t+1}) - Z(\mathbf{x}^t) \le \sum_{i\in M^t} \int_0^{l^t} (\mathbf{y}_i^t - \mathbf{x}_i^t)^T \mathbf{c}_i(\mathbf{x}^t) ds + \sum_{i\in M^t} \int_0^{l^t} \left\|\mathbf{y}_i^t - \mathbf{x}_i^t\right\| \left\|\mathbf{c}_i\left(\mathbf{x}^t + s(\mathbf{y}^t - \mathbf{x}^t)\right) - \mathbf{c}_i(\mathbf{x}^t)\right\| ds. \tag{22}$$

Since $m_{D^t} = \min_{i\in M^t}\left\{m_{D_i^t}\right\}$ and $L_{\mathbf{c}} = \max_{i\in M^t}\left\{L_{\mathbf{c}_i}\right\}$, it follows from Proposition 2 (i.e., Eq. (12)) and Eq. (20) that

$$Z(\mathbf{x}^{t+1}) - Z(\mathbf{x}^t) \le -l^t \sum_{i\in M^t} m_{D^t} \left\|\mathbf{y}_i^t - \mathbf{x}_i^t\right\|^2 + \sum_{i\in M^t} \left\|\mathbf{y}_i^t - \mathbf{x}_i^t\right\| \int_0^{l^t} L_{\mathbf{c}} \left\|\mathbf{y}_i^t - \mathbf{x}_i^t\right\| sds$$
$$= \sum_{i\in M^t} (-m_{D^t} l^t + \frac{L_{\mathbf{c}}}{2} l^{t2}) \left\|\mathbf{y}_i^t - \mathbf{x}_i^t\right\|^2. \tag{23}$$

On the other hand, Eq. (9) implies that

$$\mathbf{y}_i^t - \mathbf{x}_i^t = 0, \forall i \notin M^t. \tag{24}$$

Hence, Eq. (23) can be rewritten as

$$Z(\mathbf{x}^{t+1}) - Z(\mathbf{x}^t) \le (-m_{D^t} l^t + \frac{L_{\mathbf{c}}}{2} l^{t2}) \left\|\mathbf{y}^t - \mathbf{x}^t\right\|^2. \tag{25}$$

Since $l^t \in (0, \min(1, \frac{2m_{D^t}}{L_{\mathbf{c}}}))$, we have

$$(-m_{D^t} l^t + \frac{L_{\mathbf{c}}}{2} l^{t2}) \left\|\mathbf{y}^t - \mathbf{x}^t\right\|^2 \le 0, \tag{26}$$

which, together with Eq. (25), implies that $Z(\mathbf{x}^t)$ is monotone decreasing. Since $Z(\mathbf{x}^t)$ is lower bounded on Ω, the sequence $Z(\mathbf{x}^t)$ is convergent. By Cauchy's criterion, we have

$$\lim_{t\to\infty} \left[Z(\mathbf{x}^{t+1}) - Z(\mathbf{x}^t) \right] \to 0. \tag{27}$$

In view of Eqs. (25) and (27), we can conclude that

$$\lim_{t\to\infty} (\mathbf{y}^t - \mathbf{x}^t) = 0. \tag{28}$$

The proof is complete. ■

We now analyze the convergence of the inertia-based multi-class link flow adjustment model. Since the objective function of problem [MC-UE] is not strictly convex with respect to the class-specific link flow variable $\mathbf{x}^t$, it may have multiple equilibrium points. On the other hand, the boundedness of Ω implies the existence of at least one accumulation point for the sequence $\{\mathbf{x}^t\}$. The relationship between the accumulation point and equilibrium point is given by the next proposition. The proof is adapted from Patriksson (1998) and Tseng (1991).

Proposition 4. Assume that $\nabla D_i^t(\mathbf{y}_i, \mathbf{x}_i^t)$ is Lipchitz continuous on Ω_i and the assumptions of Proposition 3 hold, then every limit point of the class-specific link flow sequence $\{\mathbf{x}^t\}$ generated by the proposed day-to-day dynamic model satisfies the multi-class user equilibrium condition

(6).

Proof. Since Ω is bounded, it follows that there exists a limit point $\mathbf{x}^\infty$ and a subsequence $\{\mathbf{x}^t\}_{t\in\Upsilon}$, such that $\{\mathbf{x}^t\}_{t\in\Upsilon} \to \mathbf{x}^\infty$. Then by Proposition 3, we have that $\{\mathbf{y}^t\}_{t\in\Upsilon} \to \mathbf{x}^\infty$.

Let ι_t denote the index of inertia scenario that occurs on day t. Since both the number of different inertia scenarios (which is denoted by n) and the value of T are finite, there exists a subsequence $\bar{\Upsilon}$ of Υ, such that $(\iota_t, \iota_{t+1}, ..., \iota_{t+T-1})$ is the same for all $t \in \bar{\Upsilon}$. Without loss of generality, we assume that $(\iota_t, \iota_{t+1}, ..., \iota_{t+T-1}) = (\iota_0, \iota_1, ..., \iota_{T-1})$ for all $t \in \bar{\Upsilon}$.

The boundedness of l^t in Eq. (9) implies that there exists some $\delta > 0$ such that

$$\left\|\mathbf{x}^{t+1} - \mathbf{x}^t\right\| \le \delta \left\|\mathbf{y}^t - \mathbf{x}^t\right\| \tag{29}$$

holds for all $t \in \bar{\Upsilon}$.

It follows from Eq. (29) and Proposition 3 that

$$\left\{\mathbf{x}^{t+1} - \mathbf{x}^t\right\}_{t\in\bar{\Upsilon}} \to 0,$$

and consequently,

$$\left\{\mathbf{x}^{t+1}\right\}_{t\in\bar{\Upsilon}} \to \mathbf{x}^\infty.$$

Repeating the proceeding argument leads to the following results:

$$\left\{\mathbf{x}^{t+j}\right\}_{t\in\bar{\Upsilon}} \to \mathbf{x}^\infty \text{ and } \left\{\mathbf{y}^{t+j}\right\}_{t\in\bar{\Upsilon}} \to \mathbf{y}^\infty \text{ for all } j = 0,1,...T-1.$$

In the next, we will show that as $t \in \bar{\Upsilon}$ tends to infinity, the multi-class user equilibrium condition (6) is satisfied.

The Lipchitz continuity of $\nabla D_i^t(\mathbf{y}_i, \mathbf{x}_i^t)$ suggests that

$$\left\|\nabla D_i^{t+j}(\mathbf{y}_i^{t+j}, \mathbf{x}_i^{t+j}) - \nabla D_i^{t+j}(\mathbf{x}_i^{t+j}, \mathbf{x}_i^{t+j})\right\| \le L_{\nabla D_i^{t+j}} \left\|\mathbf{y}_i^{t+j} - \mathbf{x}_i^{t+j}\right\|, \forall t \in \bar{\Upsilon}, i \in M^{t+j}, j = 0,1,...T-1, \tag{30}$$

where $L_{\nabla D_i^{t+j}}$ is the bounded modulus of the function $\nabla D_i^{t+j}(\mathbf{y}_i, \mathbf{x}_i^t)$.

Since $\nabla D_i^{t+j}(\mathbf{x}_i^{t+j}, \mathbf{x}_i^{t+j}) = 0$ by Eq. (15), it follows that

$$\left\|\nabla D_i^{t+j}(\mathbf{y}_i^{t+j}, \mathbf{x}_i^{t+j})\right\| \le L_{\nabla D_i^{t+j}} \left\|\mathbf{y}_i^{t+j} - \mathbf{x}_i^{t+j}\right\|, \forall t \in \bar{\Upsilon}, i \in M^{t+j}, j = 0,1,...T-1. \tag{31}$$

By using the optimality conditions of the problem (10), we find that the points $\mathbf{y}_i^{t+j}$ satisfy the following inequality:

$$\left[\ \mathbf{c}_i(\mathbf{x}^{t+j}) + \nabla D_i^{t+j}(\mathbf{y}_i^{t+j}, \mathbf{x}_i^{t+j})\ \right]^T (\mathbf{z}_i - \mathbf{y}_i^{t+j}) \geq 0,\ \ \forall t \in \overline{\Upsilon}, \mathbf{z}_i \in \Omega_i, i \in M^{t+j},\ j = 0,1,...T-1. \tag{32}$$

In view of Eqs. (31) and (32), we have

$$\begin{aligned} \mathbf{c}_i(\mathbf{x}^{t+j})^T(\mathbf{z}_i - \mathbf{y}_i^{t+j}) &\geq -\nabla D_i^{t+j}(\mathbf{y}_i^{t+j}, \mathbf{x}_i^{t+j})^T(\mathbf{z}_i - \mathbf{y}_i^{t+j}) \\ &\geq -\left\|\nabla D_i^{t+j}(\mathbf{y}_i^{t+j}, \mathbf{x}_i^{t+j})\right\|\left\|\mathbf{z}_i - \mathbf{y}_i^{t+j}\right\| \\ &\geq -L_{\nabla D_i^{t+j}}\left\|\mathbf{y}_i^{t+j} - \mathbf{x}_i^{t+j}\right\|\left\|\mathbf{z}_i - \mathbf{y}_i^{t+j}\right\|,\ \forall t \in \overline{\Upsilon}, \mathbf{z}_i \in \Omega_i, i \in M^{t+j},\ j = 0,1,...T-1. \end{aligned} \tag{33}$$

Note that $L_{\nabla D_i^{t+j}}$ is bounded. Taking limits with respect to $t \in \overline{\Upsilon}$ in the above inequality, we have

$$\mathbf{c}_i(\mathbf{x}^\infty)^T(\mathbf{z}_i - \mathbf{y}_i^\infty) \geq 0 \quad \forall \mathbf{z}_i \in \Omega_i, i \in M^{t+j},\ j = 0,1,...T-1. \tag{34}$$

By Assumption 4,

$$M^t \cup M^{t+1} ... \cup M^{t+T-1} = S_1 \cup S_2 ... \cup S_n = M. \tag{35}$$

Then it follows from Eq.(34) and Eq. (35) that

$$\mathbf{c}_i(\mathbf{x}^\infty)^T(\mathbf{z}_i - \mathbf{y}_i^\infty) \geq 0 \quad \forall \mathbf{z}_i \in \Omega_i, i \in M, \tag{36}$$

which implies that Eq. (8) holds. Hence, by Proposition 1, $\mathbf{x}^\infty$ is a multi-class user equilibrium point. ■

Note that Proposition 4 only proves that if $\mathbf{x}^\infty$ is an accumulation point of the class-specific link flow sequence $\{\mathbf{x}^t\}$, then it is an equilibrium point. This assertion, however, does not imply that the whole sequence will converge to $\mathbf{x}^\infty$. In fact, since there may exist multiple non-isolated equilibrium points and $\{\mathbf{x}^t\}$ may be attracted by any of them, the convergence of the whole sequence of $\{\mathbf{x}^t\}$ towards a unique point may not be guaranteed. In what follows we proceed to show that $\{\mathbf{x}^t\}$ in some sense approaches the equilibrium point set.

According to Proposition B.11 (b) in Bertsekas (1999), the variational inequality problem (8) is equivalent to the following group projection equations:

$$\mathbf{x}_i = P_{\Omega_i}\left[\mathbf{x}_i - \mathbf{c}_i(\mathbf{x})\right], \forall i \in M. \tag{37}$$

Let

$$\mathbf{e}(\mathbf{x})=\begin{bmatrix}\mathbf{e}_1(\mathbf{x})\\ \vdots\\ \mathbf{e}_i(\mathbf{x})\\ \vdots\\ \mathbf{e}_m(\mathbf{x})\end{bmatrix}=\begin{bmatrix}\mathbf{x}_1-P_{\Omega_1}\left[\mathbf{x}_1-\mathbf{c}_1(\mathbf{x})\right]\\ \vdots\\ \mathbf{x}_i-P_{\Omega_i}\left[\mathbf{x}_i-\mathbf{c}_i(\mathbf{x})\right]\\ \vdots\\ \mathbf{x}_m-P_{\Omega_m}\left[\mathbf{x}_m-\mathbf{c}_m(\mathbf{x})\right]\end{bmatrix}\tag{38}$$

denote the residual function of the above equations, then the multi-class user equilibrium flow pattern can be obtained by finding a zero point of $\mathbf{e}(\mathbf{x})$. Let $\mathbf{X}^*$ be the class-specific equilibrium point set. Then $\|\mathbf{e}(\mathbf{x})\|$ can be viewed as the error bound that measures how much $\mathbf{x}$ fails to be in the solution set $\mathbf{X}^*$. Proposition 5 below shows that $\mathbf{e}(\mathbf{x}^t)\rightarrow 0$ as $t\rightarrow\infty$. The proof of this proposition is adapted from Tseng (1991).

Proposition 5. The sequence of the class-specific link flow $\{\mathbf{x}^t\}$ converges towards the equilibrium point set in the sense that $\lim_{t\rightarrow\infty}\|\mathbf{e}(\mathbf{x}^t)\|=0$.

Proof. The proof involves two steps. First, fix a certain class index $i\in M$, we will show that at any day t such that $i\in M^t$, the following group of equations hold:

$$\lim_{t\rightarrow\infty}\left[P_{\Omega_i}[\mathbf{x}_i^t-\mathbf{c}_i(\mathbf{x}^t)]-\mathbf{x}_i^t\right]=0,\ \forall t\ \textit{with}\ i\in M^t.\tag{39}$$

Taking norms and rearranging terms, we have

$$\begin{aligned}&\left\|P_{\Omega_i}[\mathbf{x}_i^t-\mathbf{c}_i(\mathbf{x}^t)]-\mathbf{x}_i^t\right\|\\ &=\left\|P_{\Omega_i}[\mathbf{x}_i^t-\mathbf{c}_i(\mathbf{x}^t)]-P_{\Omega_i}[\mathbf{y}_i^t-\mathbf{c}_i(\mathbf{x}^t)-\nabla D_i^t(\mathbf{y}_i^t,\mathbf{x}_i^t)]+P_{\Omega_i}[\mathbf{y}_i^t-\mathbf{c}_i(\mathbf{x}^t)-\nabla D_i^t(\mathbf{y}_i^t,\mathbf{x}_i^t)]-\mathbf{y}_i^t+\mathbf{y}_i^t-\mathbf{x}_i^t\right\|\\ &\le\left\|P_{\Omega_i}[\mathbf{x}_i^t-\mathbf{c}_i(\mathbf{x}^t)]-P_{\Omega_i}[\mathbf{y}_i^t-\mathbf{c}_i(\mathbf{x}^t)-\nabla D_i^t(\mathbf{y}_i^t,\mathbf{x}_i^t)]\right\|+\left\|P_{\Omega_i}[\mathbf{y}_i^t-\mathbf{c}_i(\mathbf{x}^t)-\nabla D_i^t(\mathbf{y}_i^t,\mathbf{x}_i^t)]-\mathbf{y}_i^t\right\|+\left\|\mathbf{y}_i^t-\mathbf{x}_i^t\right\|,\\ &\qquad\forall t\ \textit{with}\ i\in M^t.\end{aligned}\tag{40}$$

By Proposition B.11 (b) in Bertsekas (1999), the optimality conditions for problem (10) are equivalent to the following group of projection equations

$$\mathbf{y}_i^t=P_{\Omega_i}[\mathbf{y}_i^t-\mathbf{c}_i(\mathbf{x}^t)-\nabla D_i^t(\mathbf{y}_i^t,\mathbf{x}_i^t)],\forall i\in M^t.\tag{41}$$

Substituting Eq. (41) into Eq. (40), and considering the fact that the projection mapping is non-expansive, we have

$$\begin{aligned}\left\|P_{\Omega_i}[\mathbf{x}_i^t-\mathbf{c}_i(\mathbf{x}^t)]-\mathbf{x}_i^t\right\|&\le\left\|\mathbf{x}_i^t-\mathbf{y}_i^t+\nabla D_i^t(\mathbf{y}_i^t,\mathbf{x}_i^t)\right\|+0+\left\|\mathbf{y}_i^t-\mathbf{x}_i^t\right\|\\ &\le\left\|\nabla D_i^t(\mathbf{y}_i^t,\mathbf{x}_i^t)\right\|+2\left\|\mathbf{y}_i^t-\mathbf{x}_i^t\right\|,\ \forall t\ \textit{with}\ i\in M^t.\end{aligned}\tag{42}$$

It follows from the Lipchitz continuity of $\nabla D_i^t(\mathbf{y}_i, \mathbf{x}_i^t)$ that

$$\left\|\nabla D_i^t(\mathbf{y}_i, \mathbf{x}_i^t) - \nabla D_i^t(\mathbf{z}_i, \mathbf{x}_i^t)\right\| \le L_{\nabla D_i^t} \left\|\mathbf{y}_i - \mathbf{z}_i\right\|, \ \ \forall \mathbf{y}_i \in \Omega_i, \mathbf{z}_i \in \Omega_i. \tag{43}$$

By substituting $\mathbf{z}_i = \mathbf{x}_i^t$, $\mathbf{y}_i = \mathbf{y}_i^t$ in Eq. (43) and referring to Eq. (15), we have

$$\left\|\nabla D_i^t(\mathbf{y}_i^t, \mathbf{x}_i^t)\right\| \le L_{\nabla D_i^t} \left\|\mathbf{y}_i^t - \mathbf{x}_i^t\right\|.$$

Then it follows from Eqs. (42) and (43) that

$$\left\|P_{\Omega_i}[\mathbf{x}_i^t - \mathbf{c}_i(\mathbf{x}^t)] - \mathbf{x}_i^t\right\| \le (L_{\nabla D_i^t} + 2)\left\|\mathbf{y}_i^t - \mathbf{x}_i^t\right\|, \forall t \textit{ with } i \in M^t. \tag{44}$$

Based on Proposition 3 and the boundedness of $L_{\nabla D_i^t}$, we have

$$\lim_{t\to\infty} P_{\Omega_i}\left[[\mathbf{x}_i^t - \mathbf{c}_i(\mathbf{x}^t)] - \mathbf{x}_i^t\right] = 0, \forall t \textit{ with } i \in M^t. \tag{45}$$

Next, we prove that Eq. (39) essentially holds for all day t, that is

$$\lim_{t\to\infty} P_{\Omega_i}\left[[\mathbf{x}_i^t - \mathbf{c}_i(\mathbf{x}^t)] - \mathbf{x}_i^t\right] = 0, \forall t. \tag{46}$$

For the fixed class index i, let τ_t denote the nearest day greater than or equal to day t such that $i \in M^{\tau_t}$. Then Eq. (39) can be rewritten as

$$\lim_{t\to\infty} P_{\Omega_i}[\mathbf{x}_i^{\tau_t} - \mathbf{c}_i(\mathbf{x}^{\tau_t})] - \mathbf{x}_i^{\tau_t} = 0, \ \ \forall t. \tag{47}$$

In view of Eq. (9), we have

$$\mathbf{x}_i^t = \mathbf{x}_i^{\tau_t}, \ \forall t. \tag{48}$$

Using Eq.(48), for the fixed i and for any t, we have

$$\left\|P_{\Omega_i}[\mathbf{x}_i^t - \mathbf{c}_i(\mathbf{x}^t)] - \mathbf{x}_i^t\right\| \le \sum_{k=t}^{\tau_t - 1}\left\|P_{\Omega_i}[\mathbf{x}_i^t - \mathbf{c}_i(\mathbf{x}^k)] - P_{\Omega_i}[\mathbf{x}_i^t - \mathbf{c}_i(\mathbf{x}^{k+1})]\right\| + \left\|P_{\Omega_i}[\mathbf{x}_i^{\tau_t} - \mathbf{c}_i(\mathbf{x}^{\tau_t})] - \mathbf{x}_i^{\tau_t}\right\|, \ \forall t. \tag{49}$$

The Lipchitz continuity of the function $\mathbf{c}_i(\mathbf{x})$ and the non-expansive property of the projection mapping indicate that

$$\begin{aligned}\left\|P_{\Omega_i}[\mathbf{x}_i^t - \mathbf{c}_i(\mathbf{x}^t)] - \mathbf{x}_i^t\right\| &\le \sum_{k=t}^{\tau_t - 1}\left\|\mathbf{c}_i(\mathbf{x}^k) - \mathbf{c}_i(\mathbf{x}^{k+1})\right\| + \left\|P_{\Omega_i}[\mathbf{x}_i^{\tau_t} - \mathbf{c}_i(\mathbf{x}^{\tau_t})] - \mathbf{x}_i^{\tau_t}\right\| \\ &\le \sum_{k=t}^{\tau_t - 1} L_{\mathbf{c}}\left\|\mathbf{x}^k - \mathbf{x}^{k+1}\right\| + \left\|P_{\Omega_i}[\mathbf{x}_i^{\tau_t} - \mathbf{c}_i(\mathbf{x}^{\tau_t})] - \mathbf{x}_i^{\tau_t}\right\|.\end{aligned} \tag{50}$$

Since l^t in Eq. (9) is upper bounded, there exists some $\delta > 0$ such that

$$\left\|\mathbf{x}^{t+1}-\mathbf{x}^{t}\right\| \le \delta \left\|\mathbf{y}^{t}-\mathbf{x}^{t}\right\| \tag{51}$$

holds for all t.

Then it follows from Proposition 3 and Eq. (51) that

$$\lim_{t\to\infty}(\mathbf{x}^{t+1}-\mathbf{x}^{t})=0. \tag{52}$$

By Assumptions 3 and 4, $\tau_t - t \le T$,which implies that $\tau_t - t$ is bounded.

In view of Eq.(47), Eq. (52) and the above fact, we have

$$\lim_{t\to\infty} P_{\Omega_i}\left[[\mathbf{x}_i^t-\mathbf{c}_i(\mathbf{x}^t)]-\mathbf{x}_i^t\right]=0, \forall t. \tag{53}$$

Since the class index i is arbitrarily chosen, Eq. (53) essentially holds for any class $i \in \{1,2,...m\}$. That is,

$$\lim_{t\to\infty} P_{\Omega_i}\left[[\mathbf{x}_i^t-\mathbf{c}_i(\mathbf{x}^t)]-\mathbf{x}_i^t\right]=0, \forall i \in M. \tag{54}$$

from which we conclude that

$$\lim_{t\to\infty}\left\|\mathbf{e}(\mathbf{x}^t)\right\|=0, \tag{55}$$

and the proposition is proved. ■

Remark: Since there are infinitely many inertia patterns that can be considered in the generic model, one cannot expect that all these situations evolve to only one equilibrium state. Hence, it is practically reasonable that $\{\mathbf{x}^t\}$ converges towards the equilibrium point set instead of a single point. However, once travelers' inertia patterns and the flow changing rate are determined in advance, the evolutionary trajectories of $\{\mathbf{x}^t\}$ will be uniquely determined and $\{\mathbf{x}^t\}$ will finally evolve to a unique equilibrium state.

5. A special case of the generic model

The day-to-day link flow dynamic model proposed in Section 3 is very general in its form. In fact, travelers' inertia patterns, the formulations of $D_i^t(\mathbf{y}_i, \mathbf{x}_i^t)$ and $\mathbf{c}_i(\mathbf{x})$ are not specified in the model. This provides us sufficient flexibility to simulate different psychological inertias in the real world. By specifying these terms in the generic model, we obtain a special case of the model, which is suitable for application in practice. In the special case model, $D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)$ is expressed by the following quadratic proximal function (Rockafellar, 1976):

$$D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)=\left\|\mathbf{y}_i-\mathbf{x}_i^t\right\|^2, \tag{56}$$

which offers a compromise between choosing the shortest route and being near to travelers' current flow $\mathbf{x}_i^t$. With $D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)$ specified by Eq. (56), Assumptions 1 and 2 hold. Since $\left\|\mathbf{y}_i-\mathbf{x}_i^t\right\|^2$ is a quadratic function, and Ω_i is a compact set, it is easy to verify that the Hessian $\nabla^2 D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)$ is bounded over Ω_i. Consequently, $\nabla D_i^t(\mathbf{y}_i,\mathbf{x}_i^t)$ is Lipchitz continuous with some $L_{\nabla D_i^t}>0$.

As for the class-specific link cost vector $\mathbf{c}_i(\mathbf{x})=\left(c_1(x_1),c_2(x_2)...,c_{|A|}(x_{|A|})\right)^T$, $\forall i\in M$, we employ the widely used BPR link travel time function given by

$$c_a(x_a)=c_a^0\left[1+\beta\left(\frac{x_a}{C_a}\right)^n\right]. \tag{57}$$

where C_a and c_a^0 are the capacity and free-flow travel time of link a respectively; β and n are deterministic parameters. We can see that $c_a(x_a)$ defined by Eq. (57) is a polynomial function and it is easy to show that $\mathbf{c}_i(\mathbf{x})$ are Lipschitz continuous with some $L_{\mathbf{c}_i}>0$ (Bertsekas, 1999, p47).

We define a special inertia pattern of a traveler, denoted by H, if he/she is willing to reconsider his/her route choice every H successive days. This specific definition assumes that the interval between two positive willingnesses is fixed, which implies travelers will keep their inertia pattern in a periodic way during the evolution process. This assumption is reasonable considering the inherently habitual nature of inertia pattern of travelers, especially the morning commuters in urban cities. For example, the traffic conditions on weekdays and weekends are quite different in many cities, these travelers tend to use the same route on working days, and reconsider their routes on weekends. According to our survey, this specific definition includes most travelers' inertia patterns in reality.

Assume that there are m classes of travelers with heterogeneous inertia patterns characterized by a series of finite positive integers H_i, $i\in M=\{1,2,...m\}$. Since H_i, $i\in M$ is finite, all classes of travelers will reconsider their route choice at least once during the infinite evolution process. Therefore, Assumption 3 holds. In addition, according to Theorem 13.1 in Finan (2017), the least common multiple of $\{H_i, i \in M\}$, represented by $lcm\{H_i, i \in M\}$, always exists. Therefore, Assumption 4 holds with $T = lcm\{H_i, i \in M\}$.

From the discussion above, we can find that with the above specifications of $D_i^t(\mathbf{y}_i, \mathbf{x}_i^t)$, $\mathbf{c}_i(\mathbf{x})$ and the inertia patterns, the relevant assumptions required to prove the convergence of the proposed generic model are satisfied. This demonstrates the well-definedness of this special case in depicting the evolution process of traffic flow towards the user equilibrium state. Next, we give some illustrative instances of this special case model.

Let the inertia patterns of four all classes of travelers be $H_i = 1$ for $i \in \{1, 2, ...m\}$. In this instance, all classes of travelers are willing to reconsider their route choice every day, which is taken as the premise by most of the existing day-to-day dynamic models in the literature. In fact, the special case model has reduced to a Jacobi type link flow evolution model, which is a discrete version of the day-to-day traffic assignment model proposed in He et al. (2010).

Fig. 4 illustrates another instance with $H_i = m$ for $i \in \{1, 2, ...m\}$ and $M^t = \{(t-1) \bmod m + 1\}$. It means that the travelers of each class reconsider their route choice in a cyclical way and there is exactly one class of travelers with positive willingness at each day. In particular, we have $M^1 = \{1\}$, $M^2 = \{2\}, \ldots M^m = \{m\}$, $M^{m+1} = \{1\}$, $M^{m+2} = \{2\}, \ldots M^{m+m} = \{m\} \ldots$. The special case model reduces to a Gauss-Seidel type link flow evolution model under such specification of the inertia pattern.

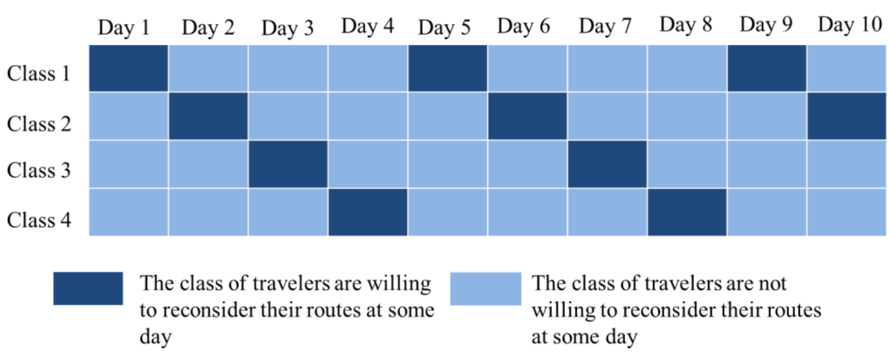


Fig. 4. Inertia pattern for 4 classes of travelers in the 2nd instance.

Fig. 5 shows the inertia patterns of four classes of travelers with $H_i = 2$ for $i \in \{1, 2, ...m\}$, and

$$M^t = \begin{cases} 1, 2, 3, ... \lfloor m/2 \rfloor & \text{for odd days} \\ \lfloor m/2 \rfloor + 1, \lfloor m/2 \rfloor + 2 ... m & \text{for even days} \end{cases}.$$

In this situation, some travelers reconsider their routes in odd days, while the others reconsider their routes in even days. The interval for all the travelers to change their routes is 2.

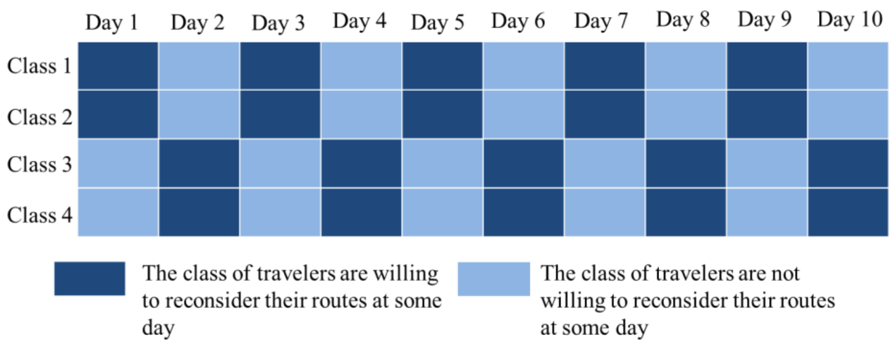


Fig. 5. Inertia pattern for 4 classes of travelers in the 3rd instance.

Fig. 6 illustrates a more general situation with $H_i =$ any positive integer for $i \in \{1, 2, \ldots m\}$. It implies travelers have different inertia patterns, whose values may range from 1 to N (N is a sufficient large positive integer). This situation includes all possible inertia patterns that are defined in this special case model. It is a situation that is most likely to occur in reality.

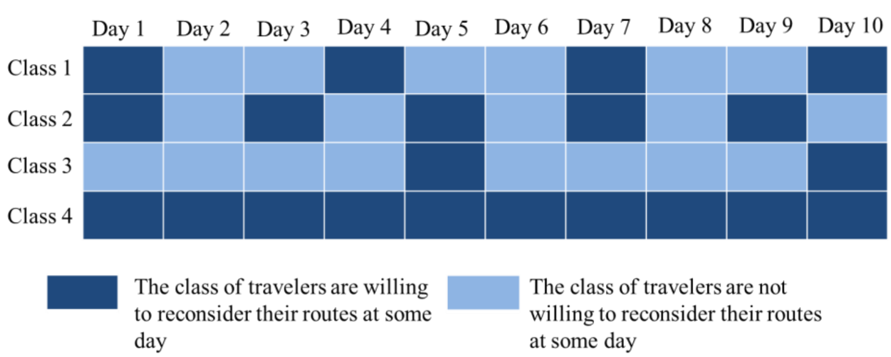


Fig. 6. Inertia patterns for 4 classes of travelers in the 4th instance.

In view of the above instances, even the special case model can have a number of different formulations. Therefore, it can be expected that the general model proposed in Section 4 is more flexible and powerful in describing travelers' inertia patterns when implemented in practice.

6. Numerical experiments

In this section, we test the convergence performance of the inertia-based multi-class link flow day-to-day dynamic model in a small network shown in Fig. 7. The network is taken from

Hearn and Ramana (1998) and it consists of 18 links, 8 nodes and 4 OD pairs. The link travel time function is given by Eq. (57) with $\beta = 0.15$ and $n = 4$. The values in the parenthesis beside each link represent its free-flow travel time and capacity. The total demands of the four OD pairs are d_{13}=10, d_{14}=20, d_{23}=30, d_{24}=40.

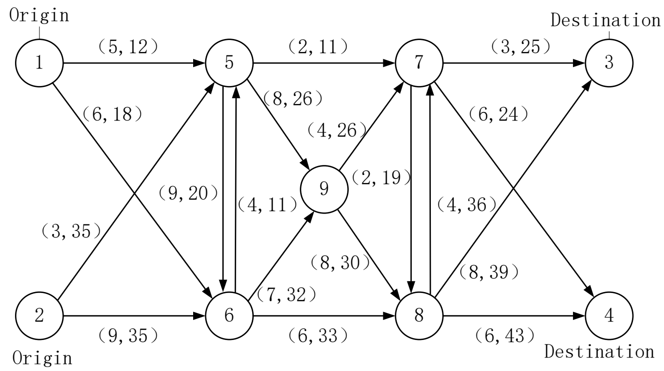


Fig. 7. The network used in the numerical example.

There are four classes of travelers in the network. The class-specific demands for each OD pair are given by

$$(d_{13}, d_{14}, d_{23}, d_{24})_1 = (d_{13}, d_{14}, d_{23}, d_{24})_3 = \frac{1}{8}(d_{13}, d_{14}, d_{23}, d_{24}) = (1.25, 2.5, 3.75, 5),$$

$$(d_{13}, d_{14}, d_{23}, d_{24})_2 = (d_{13}, d_{14}, d_{23}, d_{24})_4 = \frac{3}{8}(d_{13}, d_{14}, d_{23}, d_{24}) = (3.75, 7.5, 11.25, 15).$$

In the numerical experiments, the target flow for each class is determined by Eqs. (10) and (56), and the flow changing rate in Eq. (9) is set to be $l^t = 0.1$. The initial network condition is obtained by assigning the class-specific OD demands to the shortest path based on the free-flow travel time. In order to compare the impact of different inertia pattern combinations on the flow evolution process, we consider two cases below:

Case 1: The inertia patterns for 4 classes of travelers are given by (1,0,0,1,0,0,1,0,0,1…), (1,0,1,0,1,0,1,0,1,0,...), (1,1,0,1,1,0,1,1,0,1,…) and (1,1,1,1,1,1,1,1,1,1,…) as depicted in Fig. 8.

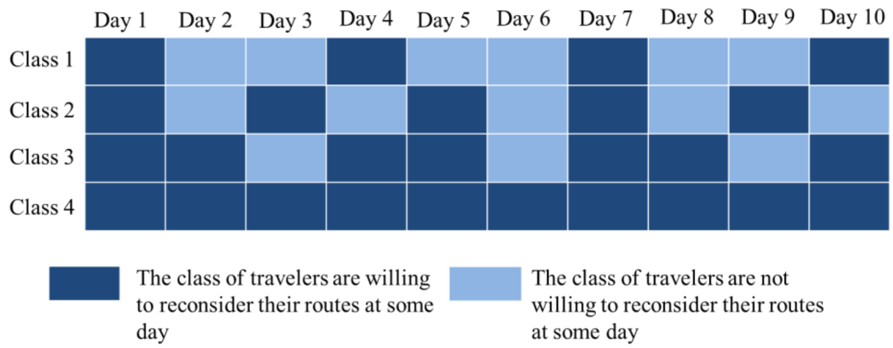


Fig. 8. Inertia patterns for 4 classes of travelers in Case 1.

Case 2: The inertia patterns for 4 classes of travelers are given by (1,0,0,0,1,0,0,0,1,0,…), (0,1,0,0,0,1,0,0,0,1,…), (0,0,1,0,0,0,1,0,0,0,…) and (0,0,0,1,0,0,0,1,0,0,…), which are illustrated in Fig. 9.

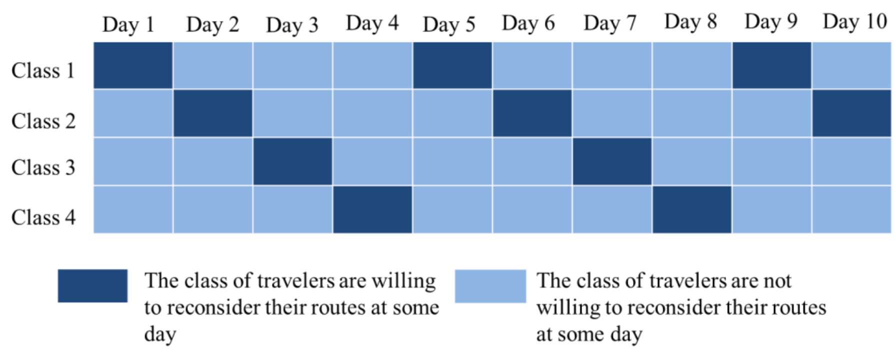


Fig. 9. Inertia patterns for 4 classes of travelers in Case 2.

We first compare the total flows on different links at stationary states in Case 1 and Case 2. The results are presented in Table 1. It can be found that the stationary link flows for both cases are almost identical, both of which satisfy the user equilibrium principle. This table validates the fact that if the flows on each link are treated as a whole, the equilibrium link flow pattern for our model is unique.

Table 1. Total flows on each link at the stationary states of Case 1 and Case 2

Link	Stationary link flows		Link	Stationary link flows	
	Case 1	Case 2		Case 1	Case 2
1-5	8.16	8.16	6-9	0.00	0.00
1-6	21.84	21.84	7-3	38.16	38.16
2-5	47.37	47.37	7-4	17.37	17.37
2-6	22.63	22.63	7-8	0.00	0.00

5-6	0.00	0.00	8-3	1.84	1.84
5-7	27.84	27.84	8-4	42.63	42.63
5-9	27.69	27.70	8-7	0.00	0.00
6-5	0.00	0.00	9-7	27.69	27.70
6-8	44.47	44.46	9-8	0.00	0.00

In order to obtain more information about flows for each class, we need to elaborate the class-specific link flows. Fig. 10 shows the evolution processes of the class-specific flows on link 2-5 and link 5-7 in Case 1. We can see that different inertia patterns yield different flow trajectories, and each trajectory evolves to a stable state corresponding to the multi-class user equilibrium state. In addition, it is interesting to note that each class of travelers evolves to their equilibrium state nearly around the same time. For example, on link 2-5 it requires about 20 days for the 4 classes of travelers to reach equilibria. This may be explained by the Wooden Bucket Theory which suggests that the evolution speed of the whole system is determined by the user class who has the highest degree of inertia.

Fig. 11 illustrates the evolution processes of the class-specific flows on link 2-5 and link 5-7 in Case 2. Similar to the discussions above, the flow trajectories in Case 2 also evolve to stationary states which follow the multi-class user equilibrium principle. Moreover, if we compare Figs. 10 and 11 carefully, several important insights can be obtained. First, we can find that the flow trajectories in Case 1 converge faster than that in Case 2. The reason for this phenomenon is as follows: In Figs. 8 and 9, the light blue square represents the day on which a certain class of travelers who are not willing to reconsider their route choice. Clearly, there are more light blue squares in Case 2 than in Case 1. This means that on average travelers in Case 2 are more inertial than the travelers in Case 1. As a result, it requires more time for travelers in Case 2 to reach equilibria. Second, we observe that on each link, the flow trajectory of a certain user class in Case 1 is quite different from that in Case 2. Since the demands for each user class in both cases are the same, we guess that inertia pattern may be a decisive factor for the evolution process. In addition, the class-specific equilibrium flow patterns in Case 1 and Case 2 are also different. This is consistent with the fact that there exists multiple equilibrium points for the inertia-based multi-class link flow adjustment model.

Finally, we elaborate the evolution of the total link flows. Fig. 12 displays the evolution processes of total flows on links 2-5 and 5-7 in Case 1 and Case 2. By comparing the flow evolution on link 2-5 (or link 5-7) in these two cases, we can see that the trajectories are different from each other. However, both of them evolve to a unique link flow value. This phenomenon is similar to the numerical results presented in He et al. (2010), in which distinct link flow evolution trajectories converge to the same equilibrium state under different circumstances.

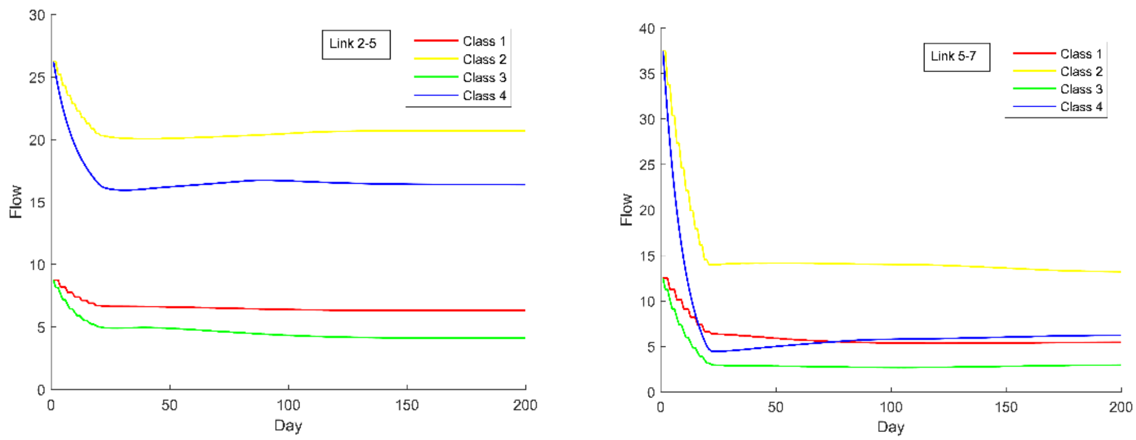


Fig. 10. Evolutionary trajectories of class-specific flows on link 2-5 and link 5-7 in Case 1

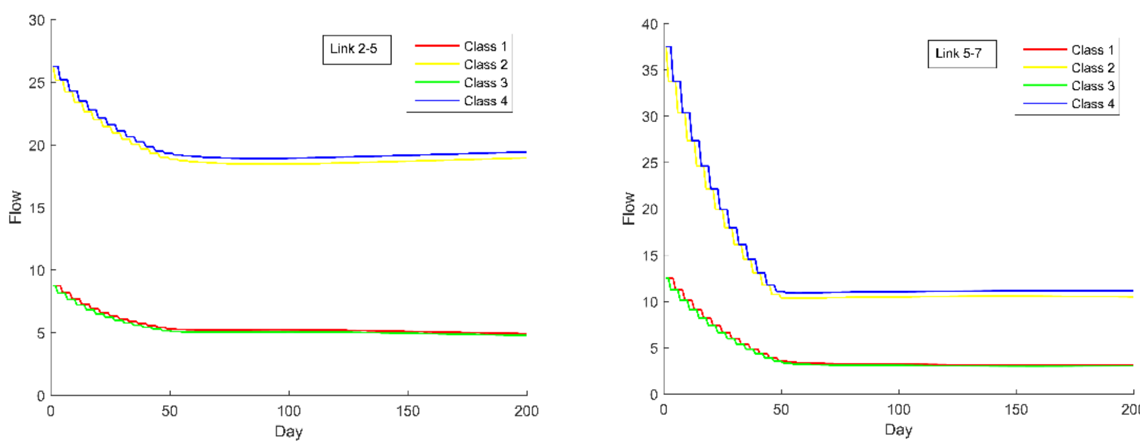
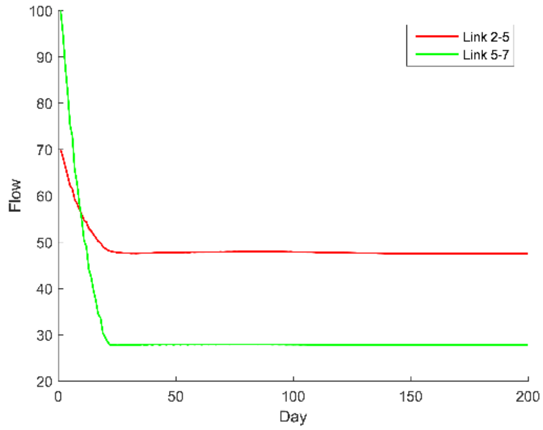


Fig. 11. Evolutionary trajectories of class-specific flows on link 2-5 and link 5-7 in Case 2



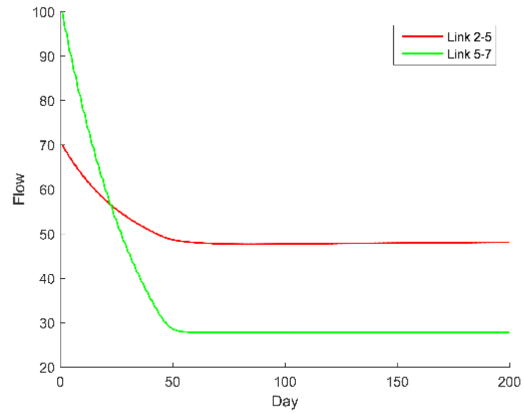


Fig. 12. Evolutionary trajectories of total flows on links 2-5 and 5-7 in Case 1 (left) and Case 2 (right).

7. Conclusions

This study investigated the heterogeneous psychological inertia and its effect on the day-to-day link flow evolution process. A new concept of psychological inertia has been introduced to describe travelers' reluctance to reconsider their route choice. Based on class-specific variables, a variational inequality for the multi-class user equilibrium problem is formulated. By incorporating heterogeneous psychological inertia into travelers' day-to-day route choice behavior, an inertia-based multi-class link flow dynamic model was proposed. We explored the convergence properties of the proposed model and established its convergence towards the multi-class user equilibrium points set. The proposed model encompasses a number of special cases, some of which are well known in the literature. This study may serve as a useful guide to simulating the flow dynamics of traffic networks in practice.

In this research, the inertia-based multi-class link flow dynamic model only considers separable link cost functions. Extending it to the case of asymmetric link cost functions is a much more challenging task. The reason is that the convergence of the proposed model requires the link cost function to be a gradient of a certain potential function, so that some descent properties can be utilized. However, for the case of asymmetric link cost functions, it is not easy to find such a potential function, especially when travelers' route choice evolves in an asynchronous manner. In fact, existing mathematical researches about essential cyclic type Gauss-Seidel decomposition algorithm (c.f. Tseng 1991; Patriksson 1998) all focus on solving a nonlinear programming problem. The extension for variational inequality formulations is an unsolved problem that beyond the scope of this paper and we therefore leave it for future study. Other possible future research directions are also highly desirable. For example, the current study only employs one rule to classify travelers. It is not difficult to extend the proposed model to the case where other classification rules (e.g. Sun et al. 2019) are also used. Furthermore, our flow adjustment process is formulated on link flows. It is interesting to develop a route flow adjustment model that incorporates travelers' heterogeneous inertia patterns. In addition, dynamic pricing or robust pricing schemes (Cheng et al. 2019; Liu et al. 2019) are expected to be designed, which takes into account the inertia-based day-to-day flow adjustment process.

Disclosure statement

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