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Research Article

On Travelling Wave Modes of Axially Moving String and Beam

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The traditional vibrational standing-wave modes of beams and strings show static overall contour with finite number of fixed nodes. The travelling wave modes are investigated in this study of axially moving string and beam although the solutions have been obtained in the literature. The travelling wave modes show time-varying contour instead of static contour. In the model of an axially moving string, only backward travelling wave modes are found and verified by experiments. Although there are $n-1$ fixed nodes in the n^{th} order mode, similar to the vibration of traditional static strings, the presence of travelling wave phenomenon is still spotted between any two adjacent nodes. In contrast to the stationary nodes of string modes, the occurrence of galloping nodes of axially moving beams is discovered: the nodes oscillate periodically during modal motions. Both forward and backward travelling wave phenomena are detected for the axially moving beam case. It is found that the ranges of forward travelling wave modes increase with the axially moving speed. It is also concluded that backward travelling wave modes can transform to the forward travelling wave modes as the transport speed surpasses the buckling critical speed.

1. Introduction

Axially moving string and beam can be found in many engineering devices, such as robot manipulators, magnetic belts, and power transmission chains, which have been widely applied in various industries. Due to the initial disturbance and eccentricity of structures, vibration always accompanies with the motion of axially moving structures, which leads to the damage of such systems. Hence, a better understanding of the working mechanism of axially moving continua plays a significant role in the process of vibration control and isolation. Although the investigation of axially moving continua has been initiated for a long time, many scholars have focused on studying the frequencies, critical velocities, and stabilities. The phenomenon and rules of complex modes in travelling manner have never been investigated in detail for axially moving continua, which are apparently the base of the vibration analysis of such gyroscopic systems.

As a classical gyroscopic system, the investigation of axially moving string has been conducted to analyze the natural frequencies and responses. In an early effort, a comprehensive review for axially moving strings was presented [1, 2]. Many studies have focused on the classical vibration of axially moving continua including free and forced responses [2–4]. For string model problems, the vibration modes can be computed by Laplace transform methods [5]. Park et al. analyzed the deploying and retracting axially moving beam and derived the dynamic responses of the longitudinal and transverse vibrations [6]. Modal analysis is one of the classical methods to derive the response of gyroscopic systems [7–10]. Using the Galerkin truncation method to discretize the gyroscopic system is an effective method to truncate the partial differential equations into a set of ordinary differential equations [11–13].

Wave propagation, described by the dispersive equation, is usually related to infinitely long structures. Miranker decomposed the general solution of a tape into travelling wave form by extending the boundary condition periodically [14]. The transverse vibration of a tensioned string is a classical example, which can be solved by the method of d'Alembert [15–18]. But the d'Alembert technique for the finite string is very difficult as claimed by Swope and Ames [19]. If boundary conditions are supplemented, the natural frequencies and mode shapes arise. The natural frequencies and modes are corresponding to the eigenvalues and eigenvectors of the mathematical model. If the axially moving velocity is considered, the gyroscopic effect should be studied, which makes the eigenvectors complex instead of real for static structures. Swope and Ames derived and solved the linear mathematical model for the vibration of axially moving strings [19]. It is found that the natural frequency of each mode is closely related to the transport speed. The axially moving beam may suffer from buckling instability and flutter instability when the transport speed increases to a sufficiently high speed [20, 21]. The stability of the axially moving beam was investigated by many researchers, and the effects of axial speed and the system parameters were presented [22–24]. Ding et al. applied the discrete Fourier transform to analyze the natural frequencies of supercritical axially moving Timoshenko beams. They also reported that the natural frequencies are highly sensitive to the bending stiffness [25]. The bending stiffness plays an important role in the wave propagation of the transverse vibration excited by the point load moving in a beam model [26]. Based on the modal analysis, further investigations, such as vibration suppressions, become possible [27–30].

In the previous literature, there are only limited results focusing on the gyroscopic modes. Recently, the complex modes of an axially moving beam have received more attentions [31–33]. Furthermore, the travelling wave phenomenon of axially moving continua with gyroscopic modes has been studied [34–36]. However, the phenomena and patterns of complex modes are still not clear. Further investigations of the travelling wave modes of gyroscopic systems are still in demand.

From the perspective of complex modes, we apply the Galerkin truncation and modal analysis methods to investigate the modal motions of the axially moving string and beam. The relationship of travelling wave modes with the transport speed, tension, and bending stiffness is revealed. The interesting modal motions of gyroscopic systems are presented for both moving string and beam models.

2. Problem Formulation

2.1. Axially Moving String Model. Consider the transverse vibration of an axially moving string between two fixed eyelets and assume that the string is a uniform and flexible one with the transverse displacement $U(X, T)$, linear density ρ, and tension P, where L represents the distance between the two eyelets and V is the transport speed. According to Newton's second law, the motion of the axially moving string is governed by the following linear and hyperbolic second-order partial differential equation:

$$\rho\left(U_{TT} + 2VU_{XT} + V^2U_{XX}\right) - PU_{XX} = 0, \tag{1}$$

where the subscripts denote the differentiation with respect to the corresponding variable. By introducing the following notations,

$$\begin{aligned} x &= \frac{X}{L}, \\ u &= \frac{U}{L}, \\ t &= T\sqrt{\frac{P}{\rho L^2}}, \\ v &= V\sqrt{\frac{\rho}{P}}, \end{aligned} \tag{2}$$

one can obtain the dimensionless form of equation (1) as

$$u_{tt} + 2vu_{xt} - \left(1 - v^2\right)u_{xx} = 0. \tag{3}$$

For fixed supports, the boundary conditions are

$$u(0, t) = 0, u(1, t) = 0. \tag{4}$$

2.2. Axially Moving Euler–Bernoulli Beam Model. The governing fourth-order partial differential equation of transverse motion for an axially moving, tensioned Euler–Bernoulli beam is given by

$$\rho\left(U_{TT} + 2VU_{XT} + V^2U_{XX}\right) - PU_{XX} + EIU_{XXXX} = 0. \tag{5}$$

Herein, EI stands for the bending stiffness, without which the equation governing the moving beam recovers to that governing the moving string. The remaining parameters share the same meaning as those presented in equation (1). The dimensionless form of equation (5) can be written in the form

$$u_{tt} + 2vu_{xt} - \left(\mu^2 - v^2\right)u_{xx} + u_{xxxx} = 0, \tag{6}$$

by using the following notations:

$$\begin{aligned} x &= \frac{X}{L}, \\ u &= \frac{U}{L}, \\ t &= T\sqrt{\frac{EI}{\rho L^4}}, \\ v &= V\sqrt{\frac{\rho L^2}{EI}}, \\ \mu^2 &= \frac{PL^2}{EI}. \end{aligned} \tag{7}$$

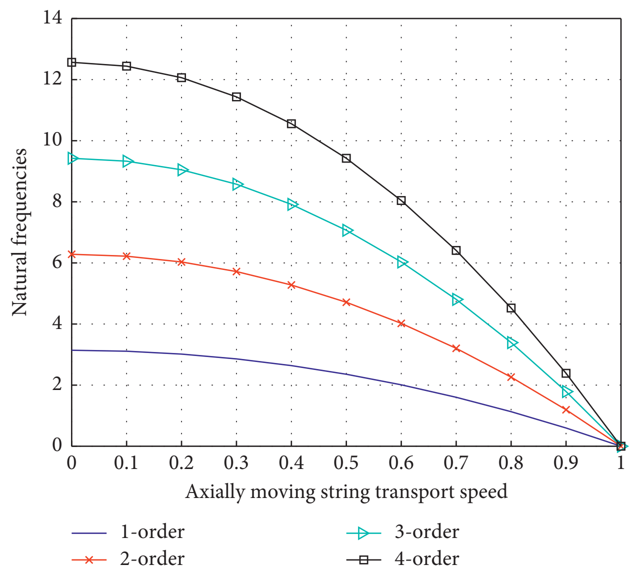


Figure 1: First four-order mode natural frequencies derived from analytical solutions.

Similarly, the boundary conditions of the beam model are

$$\begin{aligned} u(0,t) &= u(1,t) = 0, \\ u_{xx}(0,t) &= u_{xx}(1,t) = 0, \end{aligned} \tag{8}$$

for simply supported conditions and

$$\begin{aligned} u(0,t) &= u(1,t) = 0, \\ u_{x}(0,t) &= u_{x}(1,t) = 0, \end{aligned} \tag{9}$$

for fixed conditions.

Although many dynamic investigations have appeared after Mote's work [3] for both string and beam models, there are rare studies on the feature of mode shapes or modal motions under various transport speeds. In Sections 3 and 4, we will study the interesting modal motions of such gyroscopic systems and the effects of transport speed and bending stiffness on mode shapes.

3. Travelling Wave Modes of Axially Moving String Model

The purpose of this section is to discuss the contour of modal motions of the axially moving string model. The mode functions can be derived by the analytical method [37, 38], and the exact solutions of the string model in real mode form has been obtained [39, 40]. The solutions to equation (3) are assumed as

$$u(x,t) = \varphi(x)e^{i\omega t}. \tag{10}$$

Substituting equation (10) into equation (3) and using the boundary conditions equation (4), one obtains the analytical solution

$$u(x,t) = Ce^{ik\pi vx}\sin(k\pi x)e^{ik\pi(1-v^2)t}, \tag{11}$$

from which the natural frequencies can be extracted as

$$\omega_k = k\pi\left(1 - v^2\right), \quad k = 1, 2, \ldots, \tag{12}$$

where C is a complex value constant and k denotes the order of the natural frequency. The mode functions can also be derived by equation (11) as $e^{ik\pi vx}\sin(k\pi x)$, which is complex instead of a real function of x. When the axially moving velocity v is set zero, the frequencies and real-value sine mode functions will be obtained as a static vibrating string problem. The complex modes cannot be described by a particular shape as the real modes can be plotted. It can be found that the exact physical meaning of the complex modes is travelling wave modal motions, which are mathematically related to the periodic switch of the real and imaginary parts of the complex modes.

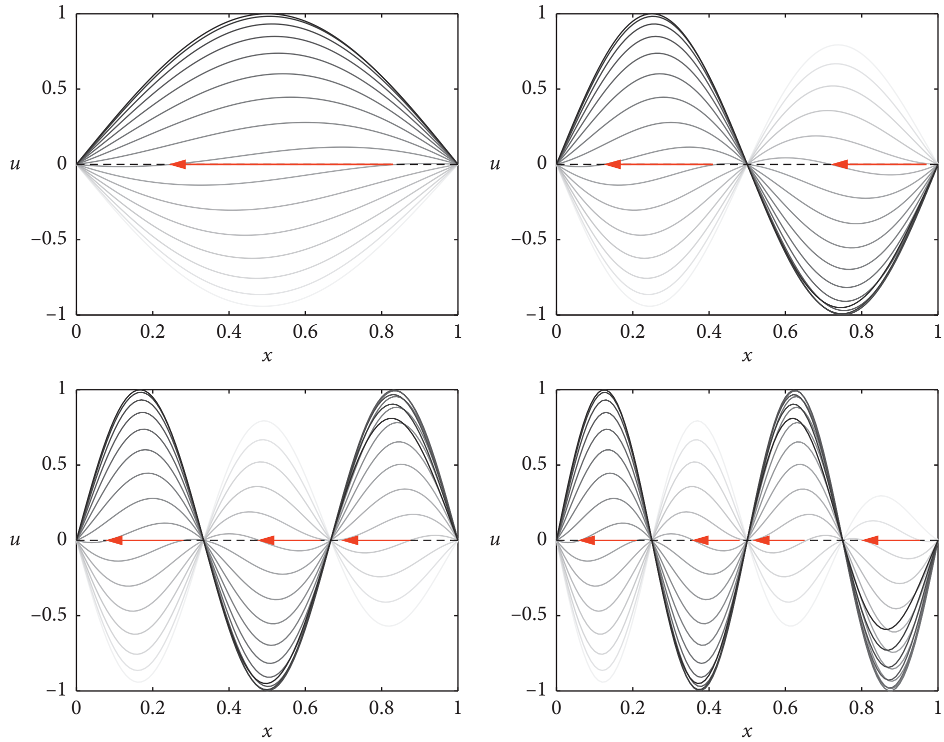
The solutions to equation (3) can also be obtained by using the Galerkin truncation method, and the validation of the approximate method has been examined by the author [35]. Taking the advantage of orthogonality, one expresses the trial functions in terms of trigonometric polynomials as

$$u(x,t) = \left(\sum_{k-1}^{n} A_k \sin k\pi x\right)e^{i\omega_j t}. \tag{13}$$

Obviously, equation (13) satisfies the boundary conditions in equation (4). Here, ω_j is the j^{th} order mode natural frequency of the axially moving string and $i = (-1)^{1/2}$. Substituting equation (13) into equation (3) and making use of the Galerkin procedure leads to

$$\mathbf{D}_n^s \mathbf{A}_n = 0, \tag{14}$$

where $\mathbf{D}_n^s$ is a coefficient matrix and $\mathbf{A}_n$ is a vector that consists of the amplitudes of trial functions in equation (13). They are presented as



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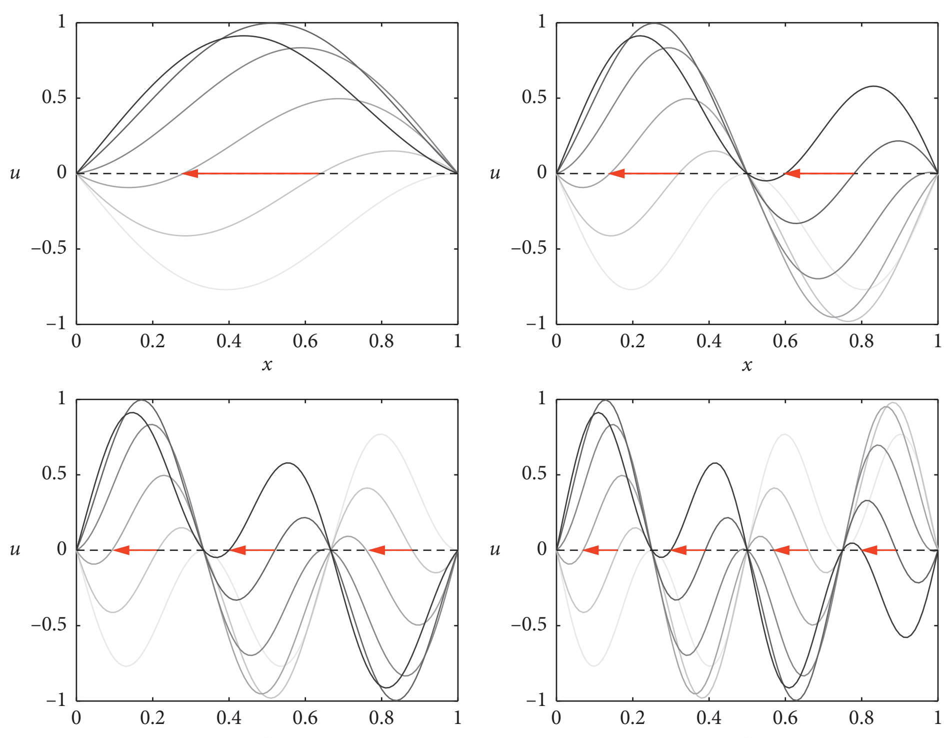
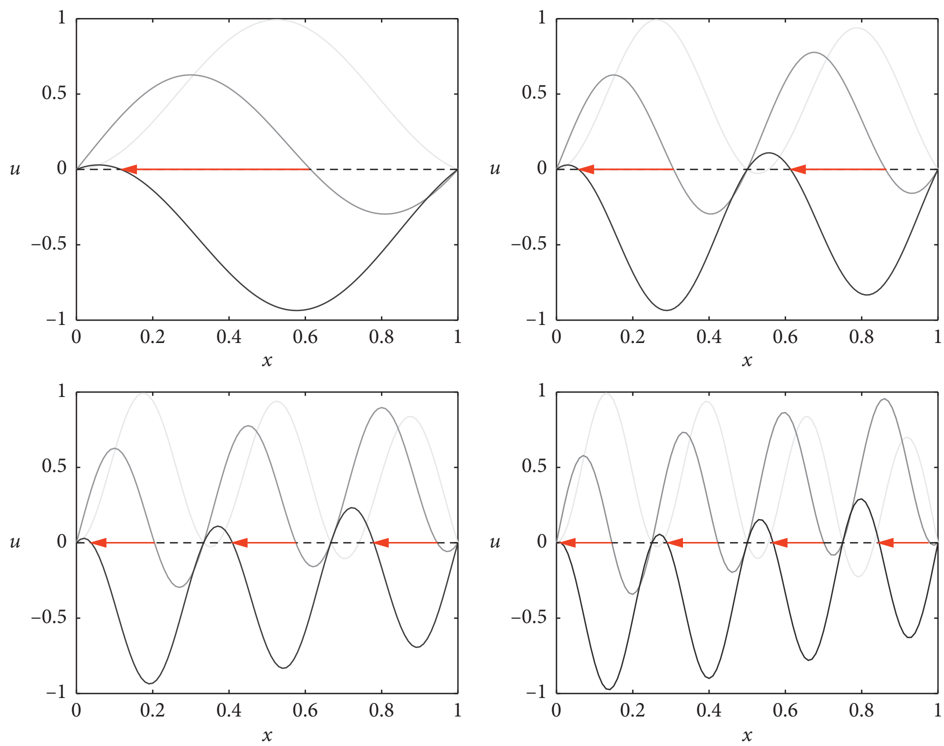


(b)

Figure 2: Continued.



(c)

Figure 2: First four-order modes of an axially moving string. Every single curve plotted by the colored levels from shallow to deep denotes the modal motion in different time. The red arrows denote the wave direction. (a) $v = 0.1$, (b) $v = 0.5$, and (c) $v = 0.9$.

$$\mathbf{A}_n = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix},$$

$$\mathbf{D}_n^s = \begin{pmatrix} \frac{1}{2}\alpha_1 & a_{12} & \cdots & a_{n1} \\ a_{12} & \frac{1}{2}\alpha_2 & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \frac{1}{2}\alpha_n \end{pmatrix}_{n\times n}, \tag{15}$$

where $\alpha_k = (1 - v^2)(k\pi)^2 - \omega_j^2$ and $a_{kl} = 4i\omega_j vkl/(k^2 - l^2)$, ($k - l$ is odd, otherwise $a_{kl} = 0$). According to the theory of matrix, if equation (14) has nontrivial solutions, the determinant of the coefficient matrix is zero. Here, $\mathbf{D}_n^s$ is a summation of one symmetric matrix and one skew symmetric matrix, which is a feature of gyroscopic systems [41].

The relationship between the natural frequencies and the axially moving speed is presented in Figure 1. The natural frequencies decrease with the axial speed, until all the natural frequencies vanish at the same critical point. The value of the critical speed can be derived by letting the natural frequencies in equation (12) equal to zero, leading to $v^2 = 1$. Beyond the critical point, the axially moving string behaves as instable [21].

According to the wave theory, the wave velocity is defined as $W = \lambda f$, in which λ is the wavelength and f is the wave frequency. Similarly, each order travelling wave mode velocity can be calculated in the following equation:

$$W_k = \lambda_k f_k, \tag{16}$$

where λ_k denotes the k^{th} order mode travelling wavelength, $f_k = \omega_k/2\pi$ is the frequency, and ω_k is the circular frequency as previously stated. It should be noted that here the modal motions show the phenomenon of travelling wave, and such travelling wave is not the traditional one with continuous wave frequencies and continuous wavelengths. The travelling wave modes are "quantized" here, which stem from the static modes.

Since $\lambda_k = 2/k$, where k is the wavenumber, one obtains dimensionless wave velocity

$$W_k = 1 - v^2. \tag{17}$$

Obviously, the speeds of all the travelling wave modes in different orders share the same value for a given transport speed v which is a nature of an axially moving string. The invariance of wave speed to different orders can be explained by dispersion character of the system. The solutions to the axially moving string can be expressed as

$$u(x, t) = ae^{i(kx - \omega t)}. \tag{18}$$

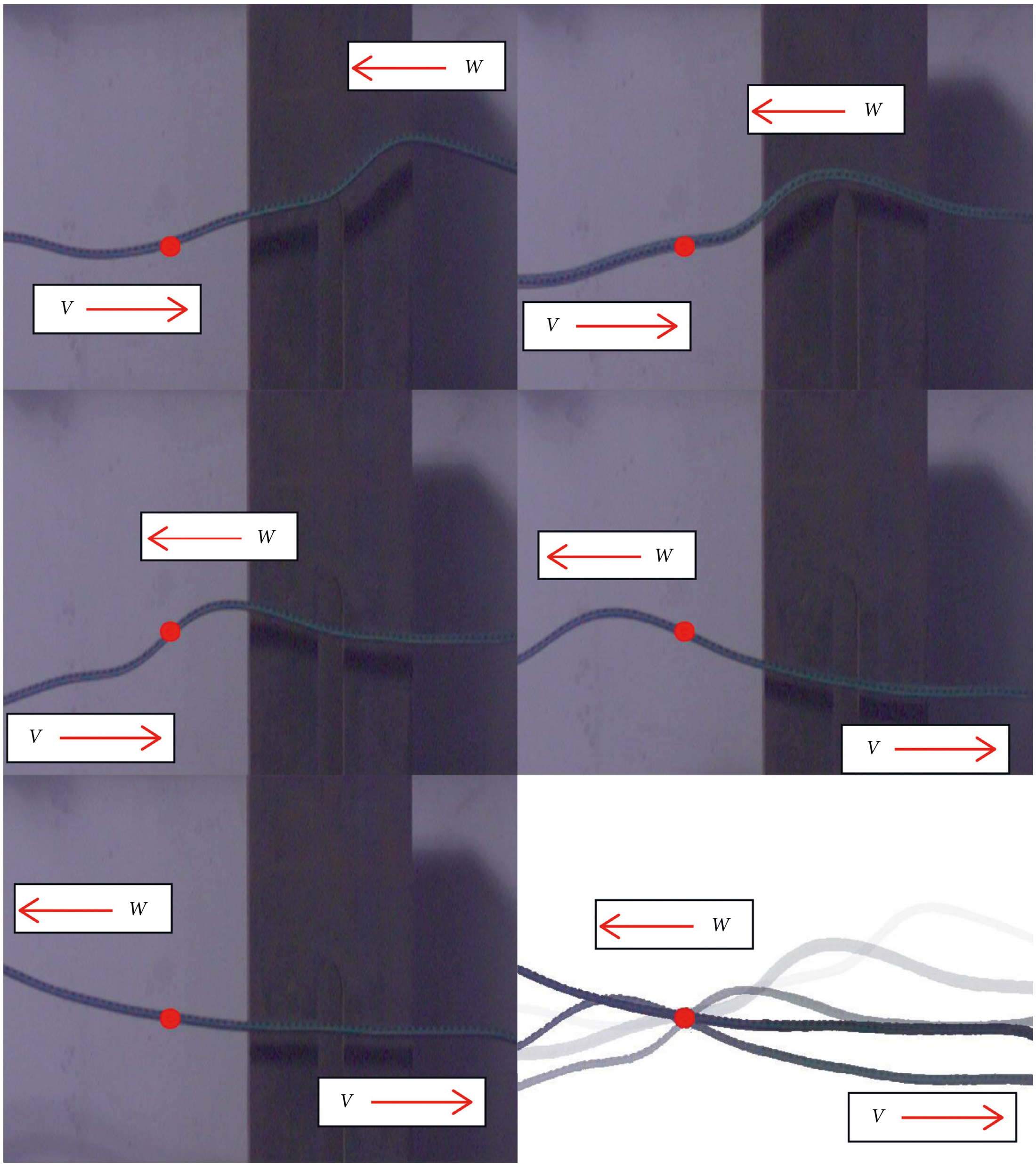


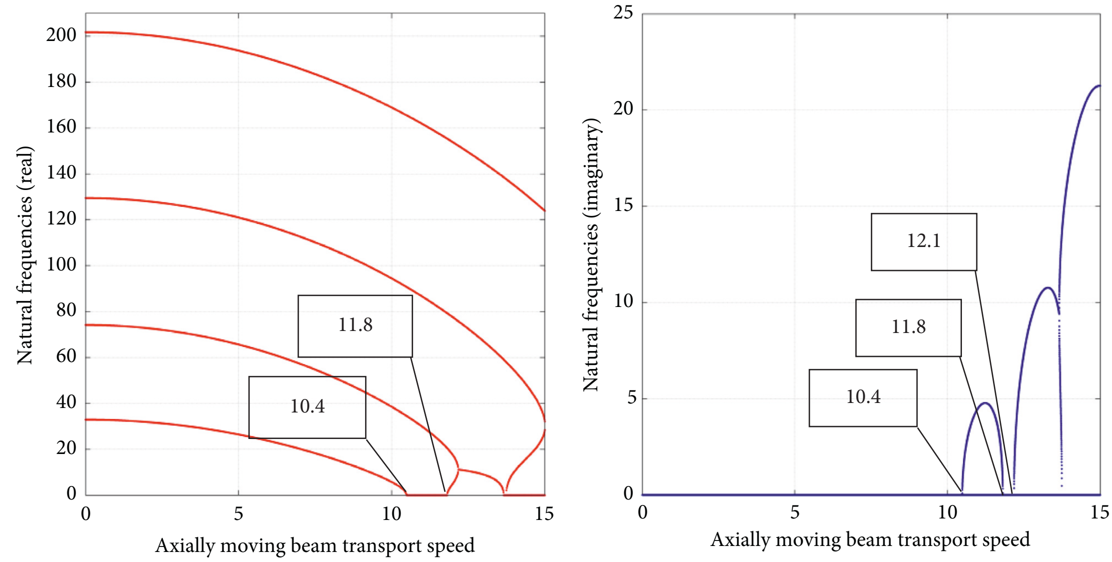
Figure 3: Snapshots recorded by a high-speed camera for an axially moving string. The red point is one node. The last picture is a compound one drawn from preceding snapshots. V denotes the transport speed and W denotes the backward travelling wave speed. Their directions are opposite.

Substituting equation (18) into equation (3) yields to the dispersion relation

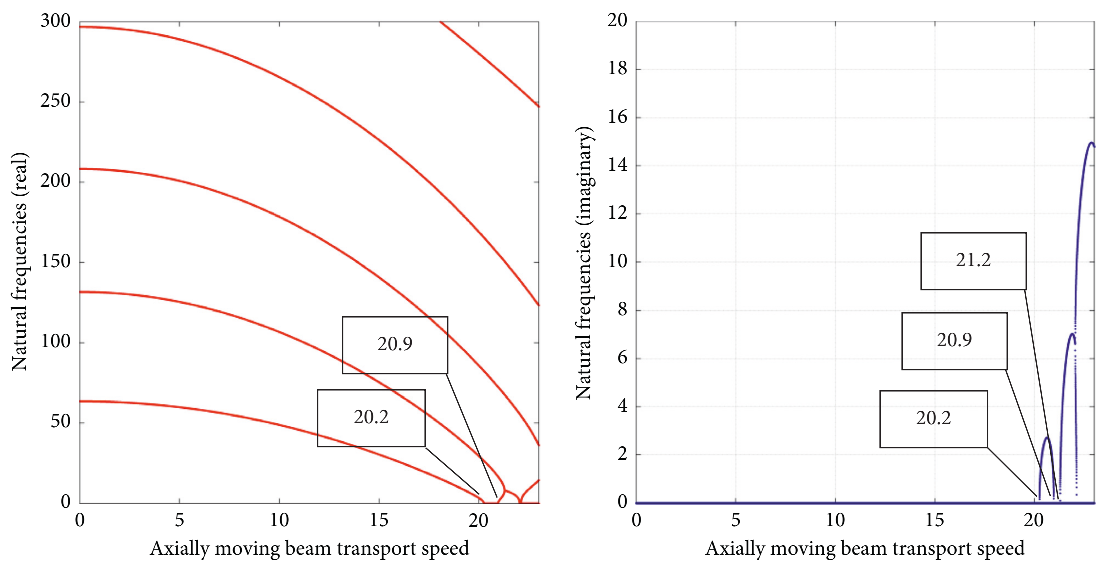
$$\omega = (v \pm 1)k. \tag{19}$$

The phase speed $\omega/k = v \pm 1$ is independent of the wavenumber k, and the wave transmission of the axially moving string is nondispersive.

Now, we inspect the travelling wave feature of the modal motions by considering a flexible string moving along its longitudinal direction dragged by an axial tension. Taking the first four-order modes at the transport speeds $v = 0.1$, $v = 0.5$, and $v = 0.9$, the travelling wave modes are depicted in Figure 2. Since they are not static, the series of snapshots are plotted by the color levels from shallow to deep to display the trajectory of the periodic motion. It can be found that there are only backward travelling wave modes detected between any adjacent stationary nodes in each order. The modes of the axially moving string present remarkable features as travelling waves between any nodes. However, the nodes are fixed and time independent. It will be found later that the axially moving beam model shows galloping nodes instead of fixed nodes in the next section. Such analytical results of the axially moving string are confirmed by the visible moving string experiment as shown in Figure 3. The quantitative comparison has not



(a)



(b)

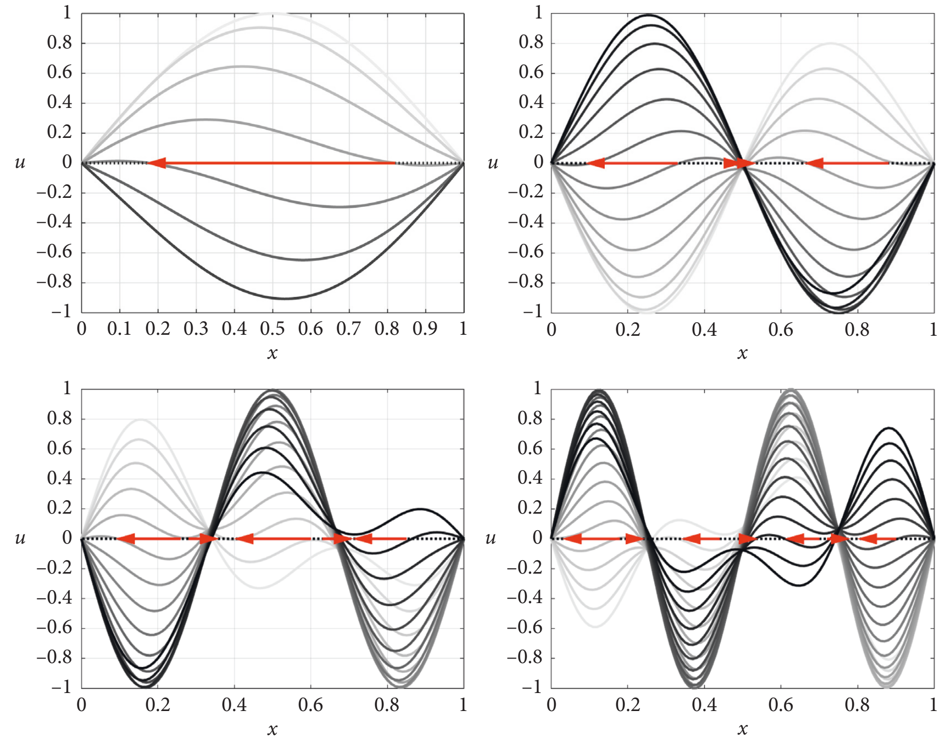
FIGURE 4: Real and imaginary parts of frequencies vs. axially moving speed. (a) $\mu = 10$ and (b) $\mu = 20$.

TABLE 1: Effect of tension parameter (μ) on critical speeds.

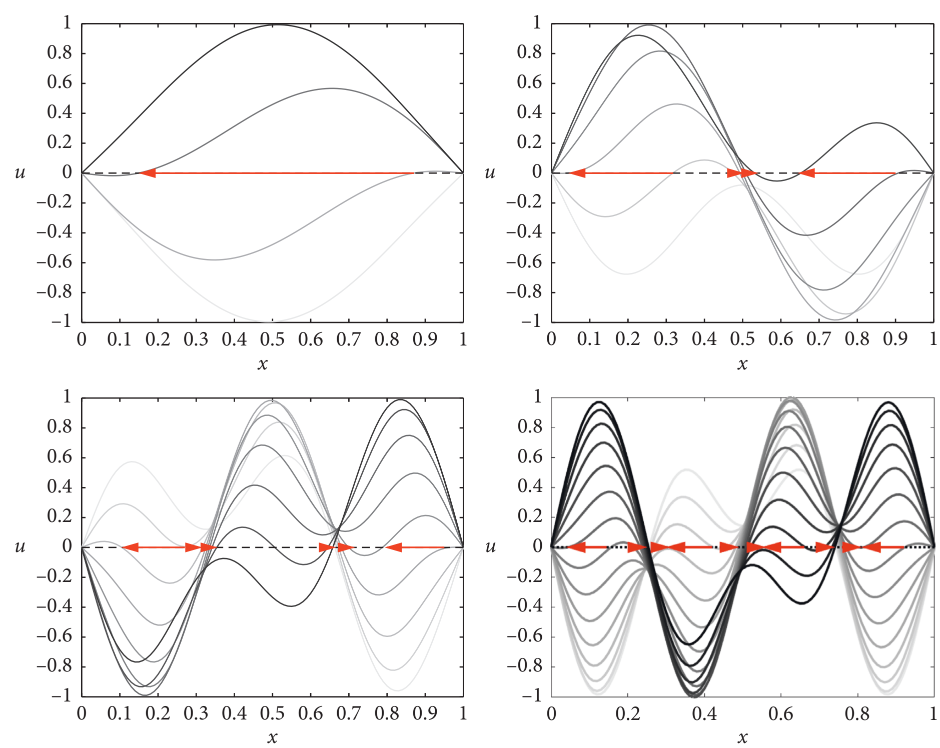
	Stable region I	Buckling instability critical speed	Instable region due to divergence	Second stability critical speed	Stable region II	Flutter instability critical speed	Instable region due to flutter
$\mu = 10$		10.48		11.81		12.18	
$\mu = 20$		20.25		20.96		21.28	

been given because the pulley-belt impacts give rise to coarse vibrations.

From the explicit expression of solution (11), the stationary nodes can also be easily found by the sine function which leads to fixed zero points independent of time. For the n^{th} order mode, there are $n-1$ points from 0 to 1 on the x-axis, similar to the static case. It means that there are $n-1$ stationary nodes in the n^{th} order mode, distributed uniformly on the points $x_k = k/n$, $k = 1, 2, \ldots, n-1$. The property of stationary nodes at those points stands in a stark contrast to that of the backward travelling wave between those points. Comparing different transport speeds in Figure 2, faster transport speed makes the travelling wave pattern between two nodes more drastic. All of the travelling waves of the first four-order modes are backward, i.e., the direction of the travelling waves is

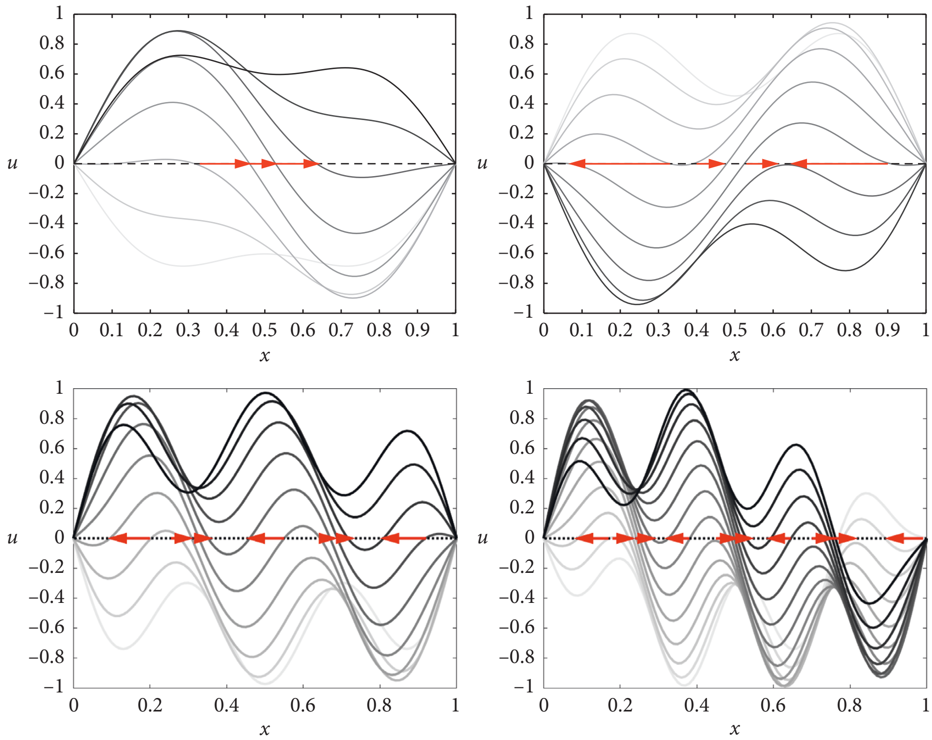


(a)



(b)

Figure 5: Continued.



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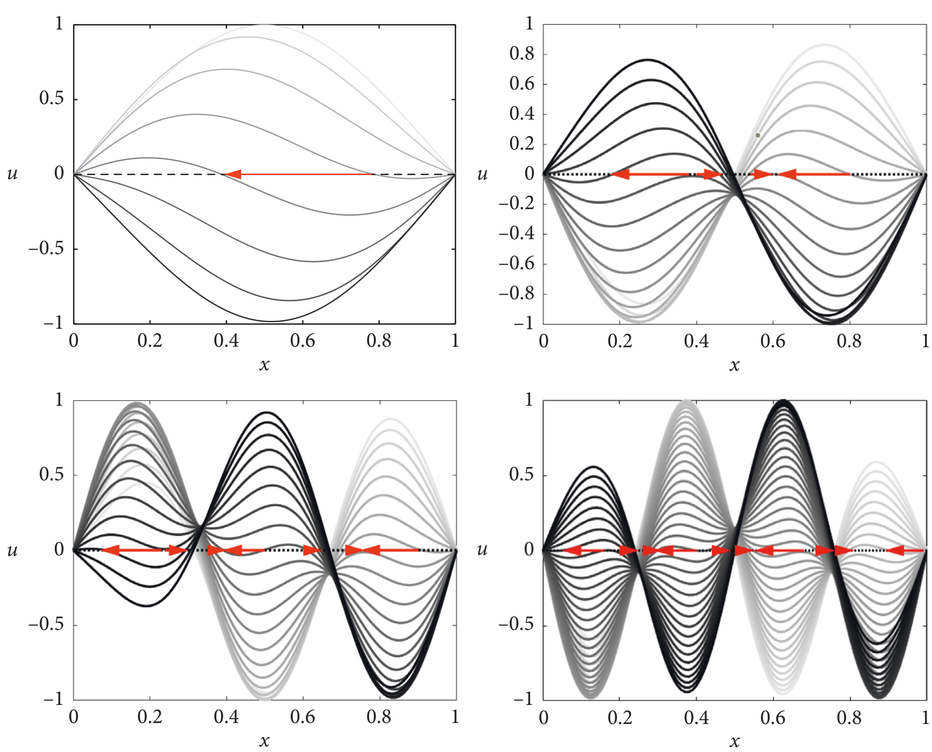
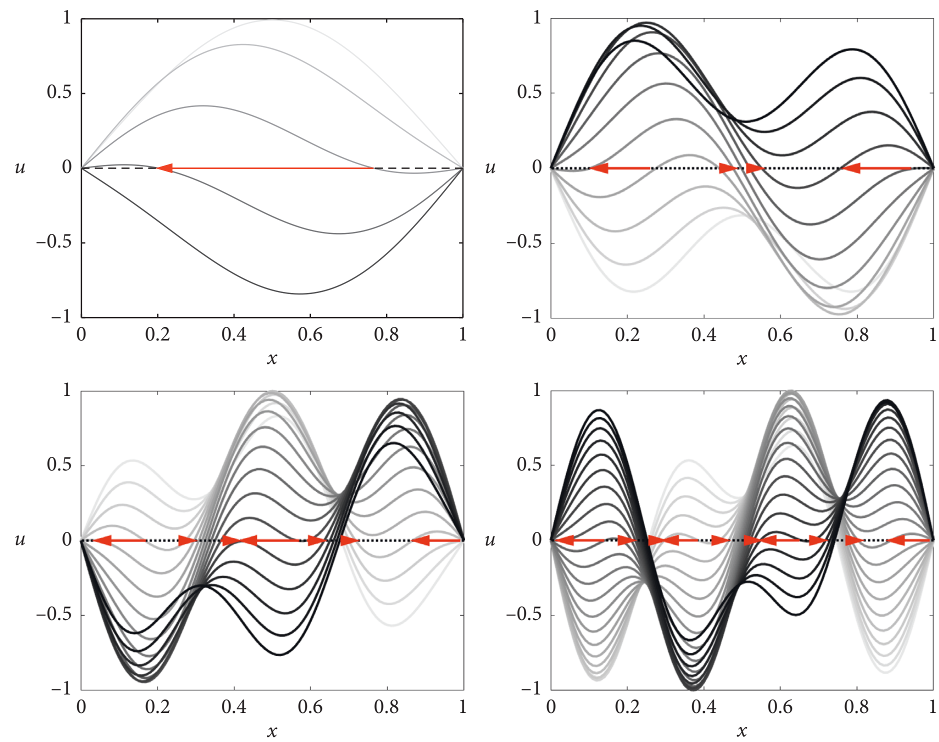
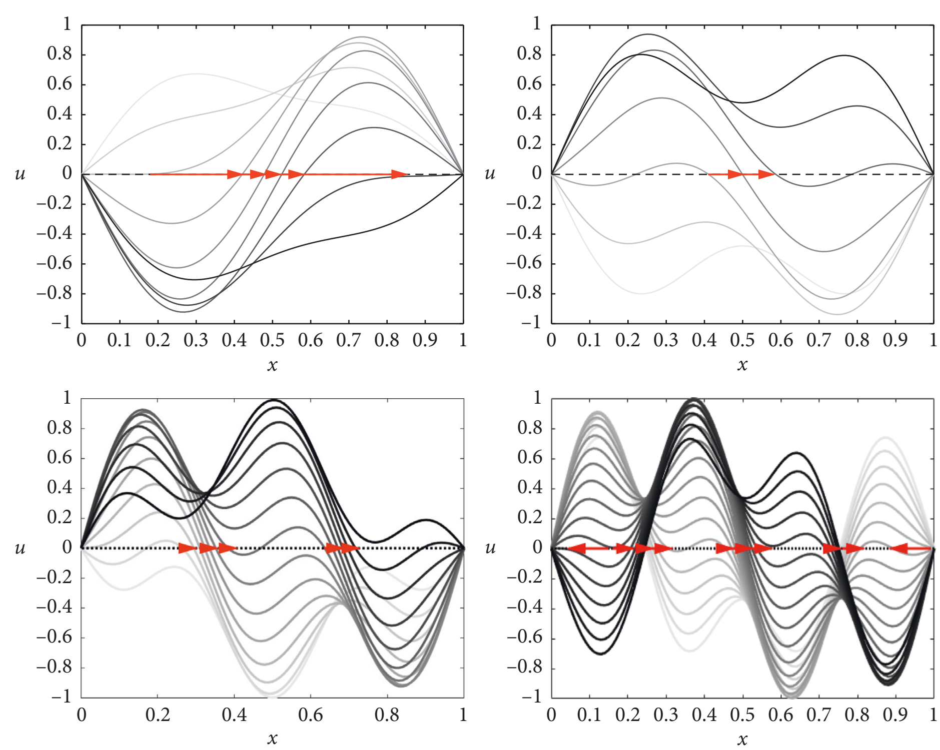


(d)

Figure 5: Continued.



(e)



(f)

Figure 5: First four-order modes of an axially moving beam with different transport speeds and tensions. Every single curve plotted by the colored levels from shallow to deep denotes the modal motion in different time. The red arrows are the wave direction. Plots of (a), (b), (d), and (e) belong to the case in the stable region I and (c) and (f) belong to the case in the stable region II. (a) $\mu = 20$, $v = 5$, (b) $\mu = 20$, $v = 10$, (c) $\mu = 20$, $v = 21.2$, (d) $\mu = 10$, $v = 5$, (e) $\mu = 10$, $v = 10$, and (f) $\mu = 10$, $v = 12$.

opposite to the transport speed of the string. It will be found that the modal motions of an axially moving beam are different, which can be affected by bending stiffness and buckling instability. In the next section, we discuss the features of travelling wave modes for the beam model.

4. Travelling Wave Modes of an Axially Moving Euler–Bernoulli Beam Model

Similar to the operations to the string model, the beam model of equation (6) can also be discretized by the Galerkin procedure. The solutions to equation (6) have the same form as equation (13), and substituting the assumed solutions into equation (6), one can obtain similar equations as

$$\mathbf{D}_n^b \mathbf{A}_n = 0. \tag{20}$$

Here, the elements α_k in matrix $\mathbf{D}_n^b$ turn into $\alpha_k' = (k\pi)^4 + (\mu^2 - v^2)(k\pi)^2 - \omega_j^2$ and the remaining elements are the same as stated in equation (15). The first four-order modes are investigated for different values of parameter μ that stands for the dimensionless tension. The relationship between the natural frequency and the transport speed for $\mu = 10$ and $\mu = 20$ is shown in Figure 4. One of the eigenvalues is usually described by a complex $\omega = \mathrm{Re}(\omega) + i\mathrm{Im}(\omega)$, where the real part $\mathrm{Re}(\omega)$ stands for the natural frequency and the imaginary part $\mathrm{Im}(\omega)$ is related to the variety of the amplitude. There are usually three cases accounting for the stability for such a conservative system. Case 1: if the imaginary parts of all the eigenvalues are zero, the system is stable. Case 2: if the real parts of one or more eigenvalues are zero and the corresponding eigenvalues have positive imaginary parts, the divergence occurs and the system is unstable. Case 3: if one or more of the eigenvalues have both positive real part and positive imaginary part, the system will undergo flutter and lead to instability. Clearly, with the increasing speed, the stable system becomes unstable due to divergence and regains stability, and further loses stability by flutter.

By inserting $\omega_j = 0$ in the nontrivial solutions condition $|\mathbf{D}_\mathrm{n}^b| = 0$, one can obtain numerically the critical speeds as presented in Table 1.

Consider the smallest three critical speeds, which correspond to the points of buckling instability, second stability, and flutter instability, respectively. These three critical speeds divide the whole velocity range into four regions: stable region I, instable region due to divergence, stable region II, and instable region due to flutter. The first four-order travelling wave modes will be investigated in the two stable regions with different tensions.

In Figure 5, for each order mode, the stationary nodes become to oscillate [35]. The centers of the galloping nodes in the beam model coincide exactly with the stationary nodes in the string model. The n^{th} order mode of the string model has $n-1$ stationary nodes, while the n^{th} order mode of the beam model has $n-1$ galloping nodes. The galloping nodes are related to the interaction between the forward travelling wave and the backward travelling wave around the equilibrium position as in the case of stable region I. The intensity of galloping of the nodes is dependent on the transport speed and the bending stiffness EI, i.e., the domains of the galloping node contours increase with the transport speed and the bending stiffness. For the case of stable region II, the galloping phenomena of nodes become even more intensive and the nodes are hard to recognize due to the strong forward travelling wave.

By varying the parameter μ from 20 to 10 in Figure 5, one can see that the range of domains increases as the bending stiffness increases.

As shown in Figures 2 and 5, the forward travelling wave modes in the axially moving beam are accompanied by the bending stiffness. Comparing Figures 1 and 4 and studying equation (7), it is found that the beam system can degenerate to the string one if let $EI \longrightarrow 0$, and $\mu \longrightarrow \infty$.

Due to the bending stiffness of axially moving beams, there exist transverse vibration modes coupled by the forward and backward travelling wave modes. The domains of galloping nodes composed by the forward travelling wave modes increase with both transport speed and bending stiffness. The forward travelling wave mode moves within the domain limit. The range of these domains also increases with the transport speed but decreases with the tension. In the first-order mode, with the increasing of the transport speed, the domain occupies the whole x-axis when the transport speed surpasses the buckling instability critical speed. Finally, the system mode changes from the backward travelling wave mode to the forward travelling wave mode.

Similar to the string model, substituting equation (18) into equation (6) yields to the dispersion relation of the axially moving beam:

$$\omega = kv \pm k\sqrt{\mu^2 + k^2}, \tag{21}$$

where the phase speed ω/k is not independent of the wavenumber k, i.e., the wave transmission of the axially moving beam is dispersive. This is different from the axially moving string whose travelling wave speed is a constant.

The first-order mode shows backward travelling wave for the first-stable region (Figures 5(a), 5(b), 5(d) and 5(e)) and becomes forward travelling wave for the second-stable region (Figures 5(c) and 5(f)).

5. Conclusions

We consider the first four-order modes of transverse vibration in the axially moving beam and string by the Galerkin truncation method and the modal analysis method. Although the solutions to the axially moving strings are known for years, the modal analysis is still in demand. In this study, the travelling wave mode properties are discovered and the mechanism of travelling wave modes in the axially moving string and beam is given. In this work, the three important findings are as follows:

(1) In the axially moving string model, there are no forward travelling wave modes, but backward travelling wave modes occur in each order mode. The speed of backward travelling wave modes in different

orders is the same when the transport speed and the tension are both given. There are $n-1$ stationary nodes in the n^{th} order mode, and the backward travelling wave modes move between two adjacent stationary nodes.

(2) In the axially moving beam model, both forward and backward travelling wave modes can exist. The forward travelling wave modes in each order mode are generated by the bending stiffness of the axially moving beam.

(3) In each order mode, the nodes in the string model are stationary and the nodes in the beam model are galloping. There are $n-1$ domains composed by the forward travelling wave modes in the n^{th} order mode of axially moving beams. With the increasing of the transport speed, the forward travelling wave modes conquer the backward travelling wave modes after the transport speed surpasses the buckling critical speed. The backward travelling wave modes transform to the forward travelling wave modes.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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