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Optimal Insurance with Rank-Dependent Utility and Increasing Indemnities

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Abstract

Bernard et al. (2015) study an optimal insurance design problem where an individual's preference is of the rank-dependent utility (RDU) type, and show that in general an optimal contract covers both large and small losses. However, their results suffer from the unrealistic assumption that the random loss has no atom, as well as a problem of moral hazard for paying more compensation for a smaller loss. This paper addresses these setbacks by removing the non-atomic assumption, and by exogenously imposing the constraint that both the indemnity function and the insured's retention function be increasing with respect to the loss. We characterize the optimal solutions via calculus of variations, and then apply the result to obtain explicitly expressed contracts for problems with Yaari's dual criterion and general RDU. Finally, we use numerical examples to compare the results between ours and that of Bernard et al. (2015).

Keywords: optimal insurance design, rank-dependent utility theory, Yaari's dual criterion, probability weighting function, moral hazard, indemnity function, retention function, quantile formulation.

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1 Introduction

Risk sharing is a method of reducing risk exposure by spreading the burden of loss among several parties. Mathematically, risk sharing can be generally formulated as a multi-optimization problem in which a Pareto optimality is sought with respect to each party's well-being modelled as a preference functional.

In the context of insurance, the primary risk sharing problem is that of designing an insurance contract between an insurer and an insured that achieves Pareto optimal for the two parties. Specifically, given an upfront premium that the insured pays the insurer, the problem is to determine the amount of loss $I(X)$ covered by the insurer – called indemnity – for a random, typically nonhedgeable loss X . The premium usually includes a safety loading on top of the actuarial value of the contract in order for the insurer to have sufficient incentive to offer the contract - this is called the participation constraint of the insurer.

Optimal insurance contract design is an important problem, manifested not only in theory but also in insurance and financial practices. In the insurance literature, most of the work assume that the insurer is risk neutral¹ while the insured is a risk-averse expected utility (EU) maximizer; see e.g. Arrow (1963), Raviv (1979), and Gollier and Schlesinger (1996). The problem is formulated as one that maximizes the insured's expected *concave* utility function of his net wealth subject to the insurer's participant constraint being satisfied. Technically, it is a constrained convex optimization problem that can be solved by standard optimization techniques. It has been shown in the aforementioned papers that the optimal contract is in general a deductible one that covers part of the loss in excess of a deductible level. This theoretical result is consistent with most of the insurance contracts available in practice. As a result, the problem is reduced to a one-dimensional optimization problem that determines the optimal deductible. Another important implication of this classical result is that the insurer and insured shares of risk are both increasing² functions of the risk; in other words, there is no incentive for either party to hide risk and thus there is a genuine risk sharing.

However, the EU theory has received many criticisms, for it fails to explain numerous experimental observations and theoretical puzzles. For example, it fails to explain the famous Allais Paradox or

¹This assumption is motivated by the fact that an insurer typically has many independent insureds as its clients, hence its risk is adequately diversified.

²Throughout this paper, by an “increasing” function we mean a “non-decreasing” function, namely f is increasing if $f(x) \geq f(y)$ whenever $x > y$. We say f is “strictly increasing” if $f(x) > f(y)$ whenever $x > y$. Similar conventions are used for “decreasing” and “strictly decreasing” functions.

the reason why a same person may buy both lottery and insurance. Other paradoxes/puzzles that EU theory cannot explain include common ratio effect (Allais, 1953), Friedman and Savage puzzle (Friedman and Savage, 1948), Ellsberg paradox (Ellsberg, 1961), and the equity premium puzzle (Mehra and Prescott, 1985). In the context of insurance contracting, the classical EU-based models again fail to account for some behaviors in insurance demand. Sydnor (2010) investigate how people choose the deductible decisions between \$100, \$250, \$500, and \$1,000. The major finding is that the households choosing a \$500 deductible pay an average premium of \$715 per year, yet these households all rejected a policy with a \$1,000 deductible whose average premium was just \$615. Since the claim rate is about 5 percent, effectively these households were willing to pay \$100 to protect against a 5 percent possibility of paying an additional \$500! As explained by Barberis (2013), this choice can only be explained by unreasonably high levels of risk aversion within the EU framework. Another insurance phenomenon that cannot be explained by the EU theory is demand for protection of small losses (e.g. demand for warranties); see Bernard et al. (2015) for a detailed discussion.

In order to overcome this drawback of the EU theory, different measures of evaluating uncertain outcomes have been put forward to depict human behaviors. A notable one is the rank-dependent utility (RDU) proposed by Quiggin (1982). In this theory, the preference measure of a final (random) wealth $W \geq 0$ is defined as

$$V^{rd\mu}(W) = \int_{\mathbb{R}^+} u(W)d(T \circ \mathbb{P}) := \int_{\mathbb{R}^+} u(x)d[-T(1 - F_W(x))], \quad (1)$$

where $u : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a (usual) utility function, $T : [0, 1] \mapsto [0, 1]$ is called a probability *weighting* function, and $F_W(\cdot)$ is the cumulative distribution function (CDF) of W . Clearly, if $T(x) \equiv x$ then $V^{rd\mu}(W) = E[u(W)]$, the classical EU. To see what a non-identity function T brings about, we rewrite assuming that T is differentiable:

$$V^{rd\mu}(W) = \int_{\mathbb{R}^+} u(x)T'(1 - F_W(x))dF_W(x). \quad (2)$$

Thus, $T'(1 - F_W(x))$ serves as a weight on the outcome x of W when evaluating the expected utility. This weight depends on $1 - F_W(x)$, the decumulative probability or the *rank* of the outcome x of W , hence the name of the rank-dependent utility.³ In particular, if T is inverse- S shaped, that is, it is first concave and then convex; see Figure 1, then $T'(1 - F_W(x)) > 1$ when x is both sufficiently large and sufficiently small. This captures the common observation that people tend to exaggerate small probabilities of extremely good and bad outcomes (hence people buy both insurances and lotteries).

³On the other hand, the RDU preference reduces to Yaari's dual criterion (Yaari 1987) when the utility function is the identity one.

From the optimization point of view, maximizing the RDU preference (1) has a clear challenge: with the presence of a general weighting function T , (1) is no longer concave even if u is concave. With the development of advanced mathematical tools, the RDU preference has been applied to many areas of finance, including portfolio choice and option pricing. In particular, the approach of the so-called quantile formulation has been developed to deal with the non-convex optimization involved in solving RDU portfolio choice models (e.g. Jin and Zhou 2008, He and Zhou 2011). The key idea is to change the decision variable from the wealth W to its quantile function, which miraculously leads to a concave optimization problem. On the other hand, Barseghyan et al. (2013) use data on households' insurance deductible decisions in auto and home insurance to demonstrate the relevance and importance of the probability weighting and suggest the possibility of generalizing their conclusions to other insurance choices.

There have been also studies in the area of insurance contract design within the RDU framework; see for example Chateauneuf, Dana and Tallon (2000), Dana and Scarsini (2007), and Carlier and Dana (2008). However, all these papers assume that the probability weighting function is convex. Bernard et al. (2015) are probably the first to study RDU-based insurance contracting with *inverse-S shaped* weighting functions, using the quantile formulation. They derive optimal contracts that not only insure large losses above a deductible level but also cover small ones. However, their results suffer from two major problems. One is the assumption that the random loss X has no atom, which is not realistic in the insurance context. The reason is that 0 is typically an atom of X , as it is plausible that $\mathbb{P}(X = 0) > 0$. The second is that their contracts pose a severe problem of moral hazard, since they are not increasing with respect to the losses. As a consequence, insureds may be motivated to hide their true losses in order to obtain additional compensations; see a discussion on pp. 175–176 of Bernard et al. (2015).

This paper aims to address these setbacks. We consider the same insurance model as in Bernard et al. (2015), but removing the non-atomic assumption on the loss, and adding an explicit constraint that both the indemnity function and the insured's retention function (i.e. the part of the losses to be born by the insured) must be globally increasing with respect to the losses - this latter constraint will rule out completely the aforementioned behaviour of moral hazard. However, mathematically we encounter substantial difficulty. The approach used in Bernard et al. (2015) no longer works. We develop a general approach to overcome this difficulty. Specifically, we first derive the necessary and sufficient conditions for optimal solutions via calculus of variations. While calculus of variations

is a rather standard technique for infinite-dimensional optimization,⁴ deducing explicitly expressed optimal contracts based on these conditions requires a fine and involved analysis. An interesting finding is that, for a good and reasonable range of parameters specifications, there are only two types of optimal contracts, one being the classical deductible one and the other a “three-fold” one covering both small and large losses.

The remainder of the paper is organized as follows. Section 2 presents the optimal insurance model under the RDU framework including its quantile formulation. Section 3 applies the calculus of variations to derive a general necessary and sufficient condition for optimal solutions. We then derive optimal contracts for Yaari’s criterion and the general RDU in Sections 4 and 5, respectively. Section 6 provides a numerical example to illustrate our results. Finally, we conclude with Section 7. All the proofs and some auxiliary results are placed in an Appendix.

2 The Model

In this section, we present the optimal insurance contracting model in which the insured has the RDU type of preferences, followed by its quantile formulation that will facilitate deriving the solutions.

2.1 Problem formulation

We follow Bernard et al. (2015) for the problem formulation except for two critical differences, which we will highlight. Let $(\Omega, \mathbf{F}, \mathbb{P})$ be a probability space. An insured, endowed with an initial wealth W_0 , faces a non-negative random loss X , *possibly having atoms* and supported in $[0, M]$, where M is a given positive scalar. He chooses an insurance contract to protect himself from the loss, by paying a premium π to the insurer in return for a compensation (or *indemnity*) in the case of a loss. This compensation is to be determined as a function of the loss X , denoted by $I(\cdot)$ throughout this paper. The *retention* function $R(X) := X - I(X)$ is thereby the part of the loss to be borne by the insured.

For a given X , the insured aims to choose an insurance contract that provides the best tradeoff between the premium and compensation based on his risk preference. In this paper, we consider the case when insured’s preference on the final random wealth $W > 0$ is dictated by the RDU functional (1), where $u : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and $T : [0, 1] \mapsto [0, 1]$. On the other hand, if the insurer is risk-neutral and the cost of offering the compensation is proportional to the expectation of the indemnity, then the

⁴Calculus of variations has also been applied in the insurance context. For example, Spence and Zeckhauser (1971) employ calculus of variations to solve an insurance contracting problem in the setting of expected utility theory.

premium to be charged for an insurance contract should satisfy the participation constraint

$$\pi \geq (1 + \rho)E[I(X)],$$

where the constant ρ is the *safety loading* of the insurer.

It is natural to require an indemnity function to satisfy

$$I(0) = 0, \quad 0 \leq I(x) \leq x, \quad \forall 0 \leq x \leq M, \quad (3)$$

a constraint that has been imposed in most insurance contracting literature. If the insured's preference is dictated by the classical EU theory, then the optimal contract is typically a deductible contract which automatically renders the indemnity function increasing; see e.g. Arrow (1971) and Raviv (1979). However, for the RDU preference the resulting optimal indemnity may not be an increasing function, as shown in Bernard et al. (2015). This may potentially cause moral hazard as pointed out earlier. Similarly, a non-monotone retention function may also lead to moral hazard. To incorporate the increasing constraint on the contract has been an outstanding open question.

In this paper, we require *both* the indemnity function and the retention function to be globally increasing. Economically speaking, this means the insurer and insured wealths are comonotone, both bearing more when a bigger loss happens. Mathematically speaking, we require

$$I(y) \leq I(x), \quad R(y) \leq R(x), \quad \forall 0 \leq y \leq x \leq M. \quad (4)$$

As $R(x) \equiv x - I(x)$, it is easily seen that the joint constraint of (3) and (4) is equivalent to the following one

$$I(0) = 0, \quad 0 \leq I(x) - I(y) \leq x - y, \quad \forall 0 \leq y \leq x \leq M. \quad (5)$$

We can now formulate our insurance contracting problem as

$$\begin{aligned} \max_{I(\cdot)} \quad & V^{rd\mu}(W_0 - \pi - X + I(X)) \\ \text{s.t.} \quad & (1 + \rho)E[I(X)] \leq \pi, \\ & I(\cdot) \in \mathbb{I}, \end{aligned} \quad (6)$$

where

$$\mathbb{I} := \{I(\cdot) : I(0) = 0, 0 \leq I(x) - I(y) \leq x - y, \forall 0 \leq y \leq x \leq M\}, \quad (7)$$

and W_0 and π are fixed scalars.

For any random variable $Y \geq 0$ a.s., define the quantile function of Y as

$$F_Y^{-1}(t) := \inf\{x \in \mathbb{R}^+ : P(Y \leq x) \geq t\}, \quad t \in [0, 1].$$

Note that any quantile function is nonnegative, increasing and left-continuous (ILC).

We now introduce the following assumptions that will be used hereafter.

Assumption 2.1 *The random loss X has a strictly increasing distribution function F_X . Moreover, F_X^{-1} is absolutely continuous on $[0, 1]$.*

Assumption 2.2 *(Concave Utility) The utility function $u : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is strictly increasing and continuously differentiable. Furthermore, u' is decreasing.*

Assumption 2.3 *(Inverse-S Shaped Weighting) The probability weighting function T is a continuous and strictly increasing mapping from $[0, 1]$ onto $[0, 1]$ and twice differentiable on $(0, 1)$. Moreover, there exists $b \in (0, 1)$ such that $T'(\cdot)$ is strictly decreasing on $(0, b)$ and strictly increasing on $(b, 1)$. Furthermore, $T'(0+) := \lim_{z \downarrow 0} T'(z) > 1$ and $T'(1-) := \lim_{z \uparrow 1} T'(z) = +\infty$.*

The first part of Assumption 2.1, crucial for the quantile formulation, is standard; see e.g. Raviv (1979). As noted, a significant difference from Bernard et al. (2015) is that here we allow X to have atoms. For example, let $F_X(x) = \frac{1-\gamma e^{-\eta x}}{1-\gamma e^{-\eta M}}$ for $x \in [0, M]$, where $\gamma \in (0, 1)$ and $\eta > 0$. Then, X satisfies Assumption 2.1, and has an atom at 0 with the probability $\mathbb{P}(X = 0) = \frac{1-\gamma}{1-\gamma e^{-\eta M}} > 0$. This assumption also ensures that $F_X^{-1}(F_X(x)) \equiv x, \forall x \in [0, M]$, a fact that will be used often in the subsequent analysis. Next, Assumption 2.2 is standard for a utility function. Finally, Assumption 2.3 is satisfied for many weighting functions proposed or used in the literature, e.g. the one proposed by Tversky and Kahneman (1992) (parameterized by θ):

$$T_\theta(x) = \frac{x^\theta}{(x^\theta + (1-x)^\theta)^{\frac{1}{\theta}}}. \quad (8)$$

Figure 1 displays this (inverse-S shaped) weighting function (in blue) when $\theta = 0.5$.

In practice, most of the insurance contracts are not tailor-made for individual customers. Instead, an insurance company usually has contracts with different premiums to accommodate customers with different needs. Each contract is designed with the best interest of a representative customer in mind so as to stay marketable and competitive, while maintaining the desired profitability (the participation constraint). An insured can then choose one from the menu of contracts to cater for individual needs. The problem (6) is therefore motivated by the insurer's making of this menu.

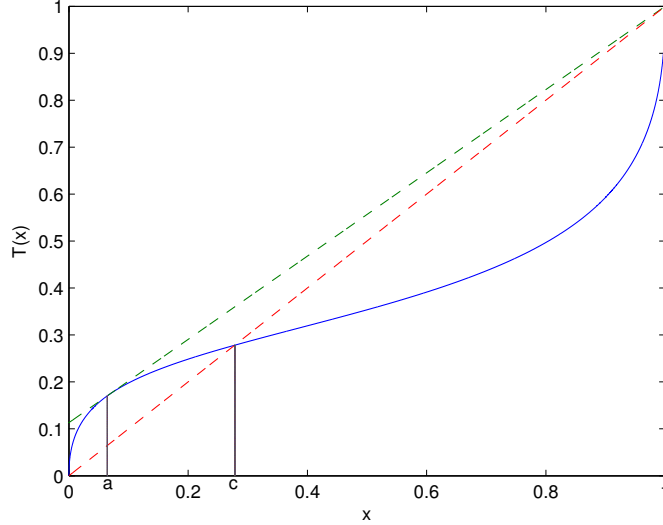


Figure 1: An inverse-S shaped weighting function (in blue) satisfying Assumption 2.3. The marked points a and c will be explained later.

If the premium $\pi \geq (1 + \rho)E[X]$, then $I^*(x) \equiv x$ (corresponding to a full coverage) is feasible and maximizes the objective function in the problem (6) pointwisely; hence optimal. To rule out this trivial case, henceforth we restrict $0 < \pi < (1 + \rho)E[X]$. Moreover, we assume

$$W_0 - (1 + \rho)E[X] - M \geq 0, \quad (9)$$

to ensure that the policyholder will not go bankrupt because $W_0 - \pi - M > 0$ for all $0 < \pi < (1 + \rho)E[X]$.

It is more convenient to consider the retention function $R(x) = x - I(x)$ instead of $I(x)$ in our study below. Letting

$$\Delta := E[X] - \frac{\pi}{1 + \rho} \in (0, E[X]),$$

$$W := W_0 - (1 + \rho)E[X] > 0,$$

$$W_\Delta := W + (1 + \rho)\Delta \equiv W_0 - \pi,$$

one can easily reformulate (6) in terms of $R(\cdot)$:

$$\begin{aligned} \max_{R(\cdot)} \quad & V^{rd}u(W_\Delta - R(X)) \\ \text{s.t.} \quad & E[R(X)] \geq \Delta, \\ & R(\cdot) \in \mathcal{R}, \end{aligned} \quad (10)$$

where

$$\mathcal{R} := \{R(\cdot) : R(0) = 0, 0 \leq R(x) - R(y) \leq x - y, \forall 0 \leq y \leq x \leq M\}.$$

2.2 Quantile Formulation

The objective function in (10) is not concave in $R(X)$ (due to the nonlinear weighting function T), leading to a major difficulty in solving (10). However, under Assumption 2.3, we have

$$\begin{aligned} V^{rd}u(W_\Delta - R(X)) &= \int_{\mathbb{R}^+} u(x) d[-T(1 - F_{W_\Delta - R(X)}(x))] \\ &= \int_0^1 u(F_{W_\Delta - R(X)}^{-1}(z)) T'(1 - z) dz = \int_0^1 u(W_\Delta - F_{R(X)}^{-1}(1 - z)) T'(1 - z) dz \\ &= \int_0^1 u(W_\Delta - F_{R(X)}^{-1}(z)) T'(z) dz, \end{aligned}$$

where the third equality is because

$$F_{W_\Delta - R(X)}^{-1}(z) = W_\Delta - F_{R(X)}^{-1}(1 - z)$$

except for an at most countable set of z . Moreover, $E[R(X)] \geq \Delta$ is equivalent to $\int_0^1 F_{R(X)}^{-1}(z) dz \geq \Delta$.

The above suggests that we may change the decision variable from the random variable $R(X)$ to its quantile function $F_{R(X)}^{-1}$, with which the objective function of (10) becomes concave and the first constraint is linear. It remains to rewrite the monotonicity constraint (represented by the constraint set \mathcal{R}) also in terms of $F_{R(X)}^{-1}$. To this end, the next lemma plays an important role.

Lemma 2.1 *Under Assumption 2.1, for any given $R(\cdot) \in \mathcal{R}$, we have*

$$R(x) = F_{R(X)}^{-1}(F_X(x)), \quad \forall x \in [0, M].$$

In view of the above results, we can rewrite (10) as the following problem, in which the decision variable is $F_{R(X)}^{-1}(\cdot)$ (denoted by $G(\cdot)$ for simplicity):

$$\begin{aligned} \max_{G(\cdot)} \quad & \int_0^1 u(W_\Delta - G(z)) T'(z) dz, \\ \text{s.t.} \quad & \int_0^1 G(z) dz \geq \Delta, \\ & G(\cdot) \in \mathbb{G}, \end{aligned} \tag{11}$$

where $\mathbb{G} := \{F_{R(X)}^{-1}(\cdot) : R(\cdot) \in \mathcal{R}\}$.

In the absence of an explicit expression the constraint set \mathbb{G} is hard to deal with. The following result addresses this issue. Note the major technical difficulty arises from the possible existence of the atoms of X .

Lemma 2.2 *Under Assumption 2.1, we have*

$$\mathbb{G} = \{G(\cdot) : G(\cdot) \text{ is absolutely continuous, } G(0) = 0, 0 \leq G'(z) \leq (F_X^{-1})'(z), \text{ a.e. } z \in [0, 1]\}. \quad (12)$$

To solve (11), we apply the Lagrange dual method to remove the constraint $\int_0^1 G(z)dz - \Delta \geq 0$ and consider the following auxiliary problem:

$$\begin{aligned} \max_{G(\cdot)} \quad & U_\Delta(\lambda, G(\cdot)) := \int_0^1 [u(W_\Delta - G(z))T'(z) + \lambda G(z)]dz - \lambda\Delta, \\ \text{s.t.} \quad & G(\cdot) \in \mathbb{G}. \end{aligned} \quad (13)$$

The existence of the optimal solutions to (11) and (13) (for each given $\lambda \in \mathbb{R}^+$) is established in Appendix B, while the uniqueness is straightforward when the utility function u is strictly concave.

To derive the optimal solution to (11), we first solve (13) to obtain an optimal solution, denoted by $\tilde{G}_\lambda(\cdot)$. Then we determine $\lambda^* \in \mathbb{R}^+$ by binding the constraint $\int_0^1 \tilde{G}_{\lambda^*}(z)dz = \Delta$. A standard duality argument then deduces that $G^*(\cdot) := \tilde{G}_{\lambda^*}(\cdot)$ is an optimal solution to (11). Finally, an optimal solution to (10) is given by $R^*(z) = G^*(F_X(z)) \forall z \in [0, M]$ and that to (6) by $I^*(z) = z - R^*(z) \forall z \in [0, M]$.

So our problem boils down to solving (13). However, in doing so the *convex* constraint that $0 \leq G'(z) \leq (F_X^{-1})'(z)$ in \mathbb{G} poses the major difficulty compared with Bernard et al. (2015) in which the constraint is a convex cone.

3 Characterization of Solutions

In this section, we derive a necessary and sufficient condition for a function to be optimal to (13).

Assume $\tilde{G}_\lambda(\cdot)$ solves (13) with a fixed λ . Let $G(\cdot) \in \mathbb{G}$ be arbitrary and fixed. For any $\varepsilon \in (0, 1)$, set $G^\varepsilon(\cdot) = (1 - \varepsilon)\tilde{G}_\lambda(\cdot) + \varepsilon G(\cdot)$. Then $G^\varepsilon(\cdot) \in \mathbb{G}$. By the optimality of $\tilde{G}_\lambda(\cdot)$ and the concavity of u , we have

$$\begin{aligned} 0 &\geq \frac{1}{\varepsilon} \left\{ \int_0^1 [u(W_\Delta - G^\varepsilon(z))T'(z) + \lambda G^\varepsilon(z)] dz - \int_0^1 [u(W_\Delta - \tilde{G}_\lambda(z))T'(z) + \lambda \tilde{G}_\lambda(z)] dz \right\} \\ &= \frac{1}{\varepsilon} \left\{ \int_0^1 [(u(W_\Delta - G^\varepsilon(z)) - u(W_\Delta - \tilde{G}_\lambda(z)))T'(z) + \lambda(G^\varepsilon(z) - \tilde{G}_\lambda(z))] dz \right\} \\ &\geq \frac{1}{\varepsilon} \left\{ \int_0^1 [(u'(W_\Delta - G^\varepsilon(z)))(W_\Delta - G^\varepsilon(z) - W_\Delta + \tilde{G}_\lambda(z))T'(z) + \lambda(G^\varepsilon(z) - \tilde{G}_\lambda(z))] dz \right\} \\ &\xrightarrow{\varepsilon \downarrow 0} \int_0^1 [(u'(W_\Delta - \tilde{G}_\lambda(z)))(\tilde{G}_\lambda(z) - G(z))T'(z) + \lambda(G(z) - \tilde{G}_\lambda(z))] dz \\ &= \int_0^1 [u'(W_\Delta - \tilde{G}_\lambda(z))T'(z) - \lambda] (\tilde{G}_\lambda(z) - G(z))dz. \end{aligned} \quad (14)$$

Define

$$N_\lambda(z) := - \int_z^1 \left[u'(W_\Delta - \tilde{G}_\lambda(t))T'(t) - \lambda \right] dt, \quad z \in [0, 1]. \quad (15)$$

Then (14) yields

$$\begin{aligned} 0 &\geq \int_0^1 \left[u'(W_\Delta - \tilde{G}_\lambda(z))T'(z) - \lambda \right] (\tilde{G}_\lambda(z) - G(z)) dz = \int_0^1 \int_0^z (\tilde{G}'_\lambda(t) - G'(t)) dt dN_\lambda(z) \\ &= \int_0^1 \int_t^1 (\tilde{G}'_\lambda(t) - G'(t)) dN_\lambda(z) dt = \int_0^1 N_\lambda(t) (G'(t) - \tilde{G}'_\lambda(t)) dt, \end{aligned}$$

leading to

$$\int_0^1 N_\lambda(z) G'(z) dz \leq \int_0^1 N_\lambda(z) \tilde{G}'_\lambda(z) dz, \quad \forall G(\cdot) \in \mathbb{G}.$$

In other words, $\tilde{G}'_\lambda(\cdot)$ maximizes $\int_0^1 N_\lambda(z) G'(z) dz$ over $G(\cdot) \in \mathbb{G}$. Therefore, a necessary condition for $\tilde{G}_\lambda(\cdot)$ to be optimal for (13) is

$$\tilde{G}'_\lambda(z) \stackrel{a.e.}{=} \begin{cases} 0, & \text{if } N_\lambda(z) = \int_z^1 [\lambda - u'(W_\Delta - \tilde{G}_\lambda(t))T'(t)] dt < 0, \\ \in [0, (F_X^{-1})'(z)], & \text{if } N_\lambda(z) = \int_z^1 [\lambda - u'(W_\Delta - \tilde{G}_\lambda(t))T'(t)] dt = 0, \\ (F_X^{-1})'(z), & \text{if } N_\lambda(z) = \int_z^1 [\lambda - u'(W_\Delta - \tilde{G}_\lambda(t))T'(t)] dt > 0. \end{cases} \quad (16)$$

It turns out that (16) completely characterizes the optimal solutions to (13).

Theorem 3.1 *A function $\tilde{G}_\lambda(\cdot)$ is an optimal solution to (13) if and only if $\tilde{G}_\lambda(\cdot) \in \mathbb{G}$ and $\tilde{G}_\lambda(\cdot)$ satisfies (16).*

The above theorem establishes a general characterization result for the optimal solutions of (13). This result, however, is only implicit as an optimal $\tilde{G}_\lambda(\cdot)$ appears on both sides of (16). Moreover, the derivative of $\tilde{G}_\lambda(z)$ is undetermined when $N_\lambda(z) = 0$. In the next two sections, we will apply this general result to derive the solutions.

4 Model with Yaari's Dual Criterion

When $u(x) \equiv x$, the corresponding V^{rd} reduces to the so-called Yaari's dual criterion (Yaari 1987). Doherty and Eeckhoudt (1995) studied an optimal insurance problem under Yaari's dual criterion by considering only coinsurance or deductible contracts. In this section, by utilizing Theorem 3.1, we are able to solve our insurance problem under Yaari's criterion without restricting the set of contracts.

When $u(x) \equiv x$, the condition (16) is greatly simplified. Indeed, (16) reduces to

$$\tilde{G}'_{\lambda}(z) \stackrel{a.e.}{=} \begin{cases} 0, & \text{if } \int_z^1 (\lambda - T'(t)) dt = \lambda(1-z) - (1-T(z)) < 0, \\ \in [0, (F_X^{-1})'(z)], & \text{if } \int_z^1 (\lambda - T'(t)) dt = \lambda(1-z) - (1-T(z)) = 0, \\ (F_X^{-1})'(z), & \text{if } \int_z^1 (\lambda - T'(t)) dt = \lambda(1-z) - (1-T(z)) > 0. \end{cases} \quad (17)$$

It should be noted that although $u(x) \equiv x$ is not strictly concave here, the uniqueness of optimal solution to (13) is implied by the characterizing condition (17).

To apply (17), we need to compare λ and $\frac{1-T(z)}{1-z}$. Define

$$f(z) := \frac{1-T(z)}{1-z}, \quad z \in [0, 1).$$

Lemma 4.1 *The function $f(\cdot)$ is a continuous function on $[0, 1)$. Moreover, under Assumption 2.3, there exists a unique $a \in (0, b)$ such that $f(\cdot)$ is strictly decreasing on $[0, a]$ and strictly increasing on $[a, 1)$.*

Clearly, $f(0) = 1, f(1-) = +\infty$. Let a be defined as in Lemma 4.1. From the proof of Lemma 4.1, it is easily seen that a is uniquely determined by

$$T'(a) = \frac{1-T(a)}{1-a}. \quad (18)$$

Set

$$\hat{\lambda} := f(a) < f(0) = 1.$$

Let $c \in (a, 1]$ be the unique scalar such that $f(c) = 1$, or equivalently, $T(c) = c$. See Figure 1 for the locations of the points a and c .

Now, we proceed by considering three cases based on the value of λ .

Case I $\lambda \leq \hat{\lambda}$. In this case,

$$N_{\lambda}(z) = (1-z)(\lambda - f(z)) < 0 \quad \forall z \in [0, a) \cup (a, 1].$$

It then follows from (17) that $\tilde{G}'_{\lambda}(z) \stackrel{a.e.}{=} 0$; hence $\tilde{G}_{\lambda}(z) = 0 \forall z \in [0, 1]$. Thus the corresponding retention $\tilde{R}_{\lambda}(z) = 0 \forall z \in [0, M]$ and indemnity $\tilde{I}_{\lambda}(z) = z \forall z \in [0, M]$, namely, the optimal contract is a full insurance contract.

Case II $\widehat{\lambda} < \lambda < 1$. By Lemma 4.1, there exist unique $x_0 \in (0, a)$ and $y_0 \in (a, c)$ such that $f(x_0) = f(y_0) = \lambda$. Accordingly, we have

$$N_\lambda(z) = \begin{cases} < 0, & \text{if } 0 < z < x_0, \\ > 0, & \text{if } x_0 < z < y_0, \\ < 0, & \text{if } y_0 < z < 1. \end{cases}$$

Hence, (17) leads to the following function:

$$\widetilde{G}_\lambda(z) = \begin{cases} 0, & \text{if } 0 \leq z < x_0, \\ F_X^{-1}(z) - F_X^{-1}(x_0), & \text{if } x_0 \leq z < y_0, \\ F_X^{-1}(y_0) - F_X^{-1}(x_0), & \text{if } y_0 \leq z \leq 1. \end{cases} \quad (19)$$

The corresponding retention and indemnity functions are, respectively,

$$\widetilde{R}_\lambda(z) \equiv \widetilde{G}_\lambda(F_X(z)) = \begin{cases} 0, & \text{if } 0 \leq z < F_X^{-1}(x_0), \\ z - F_X^{-1}(x_0), & \text{if } F_X^{-1}(x_0) \leq z < F_X^{-1}(y_0), \\ F_X^{-1}(y_0) - F_X^{-1}(x_0), & \text{if } F_X^{-1}(y_0) \leq z \leq M, \end{cases}$$

and

$$\widetilde{I}_\lambda(z) \equiv z - \widetilde{R}_\lambda(z) = \begin{cases} z, & \text{if } 0 \leq z < F_X^{-1}(x_0), \\ F_X^{-1}(x_0), & \text{if } F_X^{-1}(x_0) \leq z < F_X^{-1}(y_0), \\ z - F_X^{-1}(y_0) + F_X^{-1}(x_0), & \text{if } F_X^{-1}(y_0) \leq z \leq M. \end{cases} \quad (20)$$

The corresponding indemnity function is schematically illustrated by Figure 2. Qualitatively, the insurance covers not only large losses (when $z \geq F_X^{-1}(y_0)$) but also *small* losses (when $z < F_X^{-1}(x_0)$), and the compensation is a constant for the median range of losses. We term such a contract a *threefold* one. The need for small loss coverage along with its connection to the probability weighting are amply discussed in Bernard et al. (2015). However, in Bernard et al. (2015) the optimal indemnity is strictly decreasing in some ranges of the losses. Such a contract may incentivize the insured to hide partial losses in order to get more compensations. In contrast, both our indemnity and retention are increasing functions of the loss, which will rule out this sort of moral hazard.

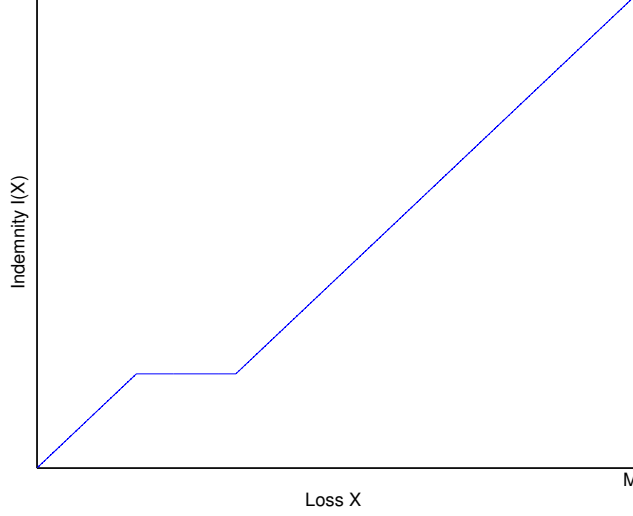


Figure 2: A schematic illustration of a threefold contract: it covers small losses as well as large losses in excess of a deductible.

Case III $1 \leq \lambda < +\infty$. By Lemma 4.1, there exists a unique $z_0 \in [c, 1]$ such that $f(z_0) = \lambda$. Thus

$$N_\lambda(z) = \begin{cases} > 0, & \text{if } 0 < z < z_0, \\ < 0, & \text{if } z_0 < z < 1. \end{cases}$$

By (17), we have

$$\tilde{G}_\lambda(z) = \begin{cases} F_X^{-1}(z), & \text{if } 0 \leq z < z_0, \\ F_X^{-1}(z_0), & \text{if } z_0 \leq z \leq 1. \end{cases} \quad (21)$$

So

$$\tilde{I}_\lambda(z) \equiv z - \tilde{R}_\lambda(z) = \begin{cases} 0, & \text{if } 0 \leq z < F_X^{-1}(z_0), \\ z - F_X^{-1}(z_0), & \text{if } F_X^{-1}(z_0) \leq z \leq M. \end{cases} \quad (22)$$

This contract is a standard deductible contract in which only losses above a deductible point will be covered.

Define

$$\bar{G}(z) = \begin{cases} F_X^{-1}(z), & \text{if } 0 \leq z < c, \\ F_X^{-1}(c), & \text{if } c \leq z \leq 1, \end{cases} \quad (23)$$

and let

$$K_c := \int_0^1 \bar{G}(z) dz,$$

and

$$\pi_c := (1 + \rho)(E[X] - K_c).$$

Clearly $K_c \leq \int_0^1 F_X^{-1}(z) dz = E[X]$.

We are now in the position to state our main result in terms of the premium π and the indemnity function $I(\cdot)$.

Theorem 4.2 *Under Yaari's criterion, $u(x) \equiv x$, and Assumptions 2.1 and 2.3, the optimal indemnity function $I^*(\cdot)$ to the problem (6) is given as*

(i) *If $\pi = (1 + \rho)E[X]$, then $I^*(z) = z \forall z \in [0, M]$.*

(ii) *If $\pi_c < \pi < (1 + \rho)E[X]$, then*

$$I^*(z) = \begin{cases} z, & \text{if } 0 \leq z < F_X^{-1}(d), \\ F_X^{-1}(d), & \text{if } F_X^{-1}(d) \leq z < F_X^{-1}(e), \\ z - F_X^{-1}(e) + F_X^{-1}(d), & \text{if } F_X^{-1}(e) \leq z \leq M, \end{cases} \quad (24)$$

where (d, e) is the unique pair satisfying $0 \leq d < a < e \leq c$, $f(d) = f(e)$ and $E[I^*(X)] = \frac{\pi}{1+\rho}$.

(iii) *If $0 \leq \pi \leq \pi_c$, then*

$$I^*(z) = \begin{cases} 0, & \text{if } 0 \leq z < F_X^{-1}(q), \\ z - F_X^{-1}(q), & \text{if } F_X^{-1}(q) \leq z \leq M, \end{cases} \quad (25)$$

where q is the unique scalar satisfying $c \leq q$ and $E[I^*(X)] = \frac{\pi}{1+\rho}$.

The economic interpretation of this result is clear. When the premium is small ($0 \leq \pi \leq \pi_c$), the contract only compensates large losses in excess of certain amount. When the premium is in middle range ($\pi_c < \pi < (1 + \rho)E[X]$), the contract is a threefold one, covering both small and large losses. When the premium is sufficiently large ($\pi \geq (1 + \rho)E[X]$), it is a full coverage.

It is interesting to investigate the comparative statics of the point π_c (in terms of c) that triggers the coverage for small losses. In fact, as

$$K_c = \int_0^c F_X^{-1}(z) dz + F_X^{-1}(c)(1 - c),$$

we have

$$\frac{\partial K_c}{\partial c} = (1 - c)(F_X^{-1})'(c).$$

However, $\pi_c = (1 + \rho)(E[X] - K_c)$; hence

$$\frac{\partial \pi_c}{\partial c} = (1 + \rho)(c - 1)(F_X^{-1})'(c) < 0.$$

This implies that the insurer is more willing to be protected against small losses if his weighting function has a bigger c . This is consistent with the fact that a bigger c renders a larger concave domain of the probability weighting that overweighs small losses (refer to Figure 1).

5 Model with the RDU Criterion

In this section we study the general RDU model in which the utility function is strictly concave. Compared with the Yaari model, solving the corresponding insurance problem calls for a more delicate analysis.

For any twice differentiable function f with $f'(x) \neq 0$, define its *Arrow-Pratt measure of absolute risk aversion*

$$A_f(x) := -\frac{f''(x)}{f'(x)}.$$

We now introduce the following assumptions.

Assumption 5.1 (*Strictly Concave Utility*) *The utility function $u : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is strictly increasing and twice differentiable. Furthermore, u' is strictly decreasing.*

Assumption 5.2 (i) *The function $A_u(z)$ is decreasing on $(0, \infty)$.*

$$(ii) \quad A_T(z) > A_u(W - F_X^{-1}(z))(F_X^{-1})'(z), \quad \forall z \in (0, a].$$

Assumption 5.1 is to replace Assumption 2.2, ensuring a genuine RDU criterion. Assumption 5.2-(i) requires that the absolute risk aversion measure of the utility function u is decreasing, which holds true for many frequently used utility functions including logarithmic, power and exponential utilities. In general, experimental and empirical evidences are consistent with the decreasing absolute risk aversion; see e.g. Friend and Blume (1975). On the other hand, $A_T(z)$, $z \in (0, a]$, measures the level of probability weighting for small losses. The economical interpretation of Assumption 5.2-(ii) is, therefore, that the degree of the insured's concern for small losses is sufficiently large relative to the absolute risk aversion of the utility function. Note that Assumption 5.2-(ii) is automatically satisfied

when $F_X^{-1}(z) = 0, \forall z \in [0, a]$, which is equivalent to $\mathbb{P}(X = 0) \geq a$. In practice, $\mathbb{P}(X = 0) \geq 0.5$ is a plausible assumption for many insurance products such as automobile and house insurance. On the other hand, a is very small for many commonly used inverse- S shaped weighting functions. Take Tversky and Kahneman's weighting function (8) as an example, $a \approx 0.013$ when $\theta = 0.3$, $a \approx 0.07$ when $\theta = 0.5$, and $a \approx 0.166$ when $\theta = 0.8$. In these cases, Assumption 5.2-(ii) holds automatically.

The problem (11) has trivial solutions in the following two cases. When $\Delta = 0$, the optimal solution is $G^*(z) = 0 \forall z \in [0, 1]$, corresponding to a full coverage. When $\Delta = E[X]$, the optimal solution is $G^*(z) = F_X^{-1}(z) \forall z \in [0, 1]$ as it is the only feasible solution, corresponding to no coverage.

So we are interested in only the case $0 < \Delta < E[X]$. It follows from Proposition C.1 in Appendix C that there exists λ^* such that $\tilde{G}_{\lambda^*}(\cdot)$ is optimal solution to (13) under λ^* and $\int_0^1 \tilde{G}_{\lambda^*}(z) dz = \Delta$. Furthermore, recall that we have proved that (13) has a unique solution when u is strictly concave and (16) provides the necessary and sufficient condition for the optimal solution.

Lemma 5.1 *For any $G(\cdot) \in \mathbb{G}$, if there exists $z \in (0, 1)$ such that*

$$\lambda - u'(W_\Delta - G(z))T'(z) = \int_z^1 [\lambda - u'(W_\Delta - G(t))T'(t)] dt = 0,$$

then $z \leq a$.

Lemma 5.2 *Under Assumption 5.2, for any $G(\cdot) \in \mathbb{G}$, $u'(W_\Delta - G(z))T'(z)$ is a strictly decreasing function of z on $[0, a]$.*

Now, for any $\lambda \leq \hat{\lambda}u'(W_\Delta)$, we have

$$\begin{aligned} \int_z^1 [\lambda - u'(W_\Delta - \tilde{G}_\lambda(t))T'(t)] dt &\leq \int_z^1 [\hat{\lambda}u'(W_\Delta) - u'(W_\Delta)T'(t)] dt \\ &= u'(W_\Delta) \int_z^1 [\hat{\lambda} - T'(t)] dt = u'(W_\Delta)(1 - z) \left[\hat{\lambda} - \frac{1 - T(z)}{1 - z} \right] < 0, \end{aligned}$$

where the last inequality is due to Lemma 4.1. Hence $\tilde{G}_\lambda(z) = 0 \forall z \in [0, 1]$ is the only solution satisfying (16). However, $\int_0^1 \tilde{G}_\lambda(z) dz = 0 < \Delta$, a contradiction. Therefore, only when $\lambda > \hat{\lambda}u'(W_\Delta)$ it is possible for (16) to hold.

Fixing $\lambda > \hat{\lambda}u'(W_\Delta)$, we now analyze the shape of the function $\tilde{G}_\lambda(\cdot)$ that satisfies (16). Suppose that $\tilde{G}_\lambda(1) = k < W_\Delta$. We then have $N_\lambda(1) = 0$ and $\lambda - u'(W_\Delta - k)T'(1-) < 0$ since $T'(1-) = +\infty$. So, $\tilde{G}'_\lambda(z) = 0$ when z is close to 1 since $N_\lambda(z) < 0$ for such z . Hence, $\tilde{G}_\lambda(z) \equiv k \forall z \in [z_1, 1]$ for some $z_1 \in [0, 1)$, at which $N_\lambda(z_1) = 0$ and $N_\lambda(z) < 0$ for $\forall z \in (z_1, 1)$. Next, we consider three cases respectively depending on the value of k and location of z_1 .

Case (A) $k > W_\Delta - (u')^{-1}(\frac{\lambda}{\hat{\lambda}})$, i.e. $\lambda < \hat{\lambda}u'(W_\Delta - k)$. In this case, we have, for any $z \in [0, 1)$,

$$\begin{aligned} \int_z^1 [\lambda - u'(W_\Delta - k)T'(t)] dt &< \int_z^1 \left[\hat{\lambda}u'(W_\Delta - k) - u'(W_\Delta - k)T'(t) \right] dt \\ &= u'(W_\Delta - k)(1 - z) \left[\hat{\lambda} - \frac{1 - T(z)}{1 - z} \right] \leq 0. \end{aligned}$$

It then follows from (16) that $\tilde{G}_\lambda(z) \equiv k = \tilde{G}_\lambda(0) = 0$. However, $0 = k > W_\Delta - (u')^{-1}(\frac{\lambda}{\hat{\lambda}})$, or $\lambda \leq \hat{\lambda}u'(W_\Delta)$, leading to a contradiction. So, this case in fact will not take place.

Case (B) $k = W_\Delta - (u')^{-1}(\frac{\lambda}{\hat{\lambda}})$. In this situation, z_1 should be a . This is because $\int_a^1 [\lambda - u'(W_\Delta - k)T'(t)]dt = 0$ and

$$\int_z^1 [\lambda - u'(W_\Delta - k)T'(t)]dt = \frac{\lambda}{\hat{\lambda}}(1 - z) \left(\hat{\lambda} - \frac{1 - T(z)}{1 - z} \right) < 0$$

for $z \in (a, 1)$ by Lemma 4.1. Moreover, $\lambda - u'(W_\Delta - k)T'(a) = 0$. By Lemma 5.2, $\lambda - u'(W_\Delta - \tilde{G}_\lambda(z))T'(z)$ strictly increases with respect to $z \in [0, a]$. It follows that

$$\lambda - u'(W_\Delta - \tilde{G}_\lambda(z))T'(z) < 0$$

for $z \in [0, a)$. Then (16) implies $\tilde{G}'_\lambda(z) = 0$ for $z \in (0, a)$. As a result, $k = \tilde{G}_\lambda(a) = \tilde{G}_\lambda(0) = 0$, or $\lambda = \hat{\lambda}u'(W_\Delta)$, which is a contradiction. So, again, this case will not occur.

Case (C) $k < W_\Delta - (u')^{-1}(\frac{\lambda}{\hat{\lambda}})$. In this case, $z_1 \in (a, 1)$ exists. By Lemma 5.1, we have $\lambda - u'(W_\Delta - k)T'(z_1) > 0$. Hence, there may or may not exist $z_2 \in (0, 1)$ such that $N_\lambda(z_2) = 0$ and $N_\lambda(z) > 0$ for $z \in (z_2, z_1)$. We now discuss four subcases depending on the existence and location of z_2 .

(C.1) If z_2 does not exist or $z_2 = 0$ (i.e. $N_\lambda(z) > 0$ for $z \in (0, z_1)$), then by (16), $\tilde{G}'_\lambda(z) = (F_X^{-1})'(z)$ for $z \in (0, z_1)$. Combined with the fact that $\tilde{G}_\lambda(0) = 0$, we have:

$$\tilde{G}_\lambda(z) = \begin{cases} F_X^{-1}(z), & \text{if } 0 \leq z < z_1, \\ F_X^{-1}(z_1), & \text{if } z_1 \leq z \leq 1. \end{cases}$$

This corresponds to a deductible contract.

(C.2) If z_2 exists and $z_2 \in (0, a]$, then $\tilde{G}'_\lambda(z) = (F_X^{-1})'(z)$ for $z \in (z_2, z_1)$ in view of (16).

Combining the property of z_1 and z_2 , we deduce

$$\lambda - u'(W_\Delta - \tilde{G}_\lambda(z_2))T'(z_2) \leq 0.$$

Then, using Lemma 5.2, we have

$$\lambda - u'(W_\Delta - \tilde{G}_\lambda(z))T'(z) < 0$$

for $z \in [0, z_2]$. It follows from (16) that $\tilde{G}'_\lambda(z) = 0$ for $z \in (0, z_2)$. In this case, we can express $\tilde{G}_\lambda(\cdot)$ as follows

$$\tilde{G}_\lambda(z) = \begin{cases} 0, & \text{if } 0 \leq z < z_2, \\ F_X^{-1}(z) - F_X^{-1}(z_2), & \text{if } z_2 \leq z < z_1, \\ F_X^{-1}(z_1) - F_X^{-1}(z_2), & \text{if } z_1 \leq z \leq 1. \end{cases}$$

This is the threefold contract, schematically depicted in Figure 2.

(C.3) If z_2 exists and $z_2 \in (b, 1)$ (recall that b is the turning point where the weighting function $T(\cdot)$ changes from being concave to convex), then a similar analysis as in Case (B) shows that $\lambda - u'(W_\Delta - \tilde{G}_\lambda(z_1))T'(z_1) > 0$ and $\lambda - u'(W_\Delta - \tilde{G}_\lambda(z_2))T'(z_2) < 0$. This means

$$u'(W_\Delta - \tilde{G}_\lambda(z_2))T'(z_2) > u'(W_\Delta - \tilde{G}_\lambda(z_1))T'(z_1).$$

However, $u'(W_\Delta - \tilde{G}_\lambda(z_1)) \geq u'(W_\Delta - \tilde{G}_\lambda(z_2)) > 0$ and $T'(z_1) > T'(z_2) > 0$, which is a contradiction. So, this case is not feasible.

(C.4) If z_2 exists and $z_2 \in (a, b]$, then $\lambda - u'(W_\Delta - \tilde{G}_\lambda(z_2))T'(z_2) < 0$. We prove $\tilde{G}_\lambda(z) \equiv \tilde{G}_\lambda(z_2) \forall z \in [0, z_2]$. In fact, if it is false, then there exists z_3 such that

$$\int_{z_3}^{z_2} [\lambda - u'(W_\Delta - \tilde{G}_\lambda(z_2))T'(t)]dt = 0 \text{ and } \int_z^{z_2} [\lambda - u'(W_\Delta - \tilde{G}_\lambda(z_2))T'(t)]dt < 0$$

for $z \in (z_3, z_2)$. However,

$$\lambda - u'(W_\Delta - \tilde{G}_\lambda(z_2))T'(z) < \lambda - u'(W_\Delta - \tilde{G}_\lambda(z_2))T'(z_2) < 0$$

for $z \in (z_3, z_2)$ since $z_2 \in (a, b]$. So,

$$\int_{z_3}^{z_2} [\lambda - u'(W_\Delta - \tilde{G}_\lambda(z_2))T'(t)]dt < (\lambda - u'(W_\Delta - \tilde{G}_\lambda(z_2))T'(z_2))(z_2 - z_3) < 0,$$

arriving at a contradiction. Therefore, $k = F_X^{-1}(z_1) - F_X^{-1}(z_2)$. From $\int_{z_1}^1 [\lambda - u'(W_\Delta - k)T'(t)]dt = 0$, it follows

$$\lambda = u'(W_\Delta - k) \frac{1 - T(z_1)}{1 - z_1} = u'(W_\Delta + F_X^{-1}(z_2) - F_X^{-1}(z_1)) \frac{1 - T(z_1)}{1 - z_1}.$$

However,

$$\begin{aligned}
& \int_{z_2}^{z_1} \left[u'(W_\Delta + F_X^{-1}(z_2) - F_X^{-1}(z_1)) \frac{1 - T(z_1)}{1 - z_1} - u'(W_\Delta + F_X^{-1}(z_2) - F_X^{-1}(t)) T'(t) \right] dt \\
& > \int_{z_2}^{z_1} \left[u'(W_\Delta + F_X^{-1}(z_2) - F_X^{-1}(z_1)) \frac{1 - T(z_1)}{1 - z_1} - u'(W_\Delta + F_X^{-1}(z_2) - F_X^{-1}(z_1)) T'(t) \right] dt \\
& = u'(W_\Delta + F_X^{-1}(z_2) - F_X^{-1}(z_1)) (z_1 - z_2) \left[\frac{1 - T(z_1)}{1 - z_1} - \frac{T(z_1) - T(z_2)}{z_1 - z_2} \right] > 0,
\end{aligned}$$

where the last inequality follows from Lemma A.1-(ii) in Appendix A. This is a contradiction. So, the current case will not occur either.

To summarize, for any $\lambda > \widehat{\lambda} u'(W_\Delta)$, only deductible and threefold contracts are possibly optimal, stipulated by (C.1) and (C.2). Next, we investigate these two cases more closely.

Define a function $h_\Delta(\cdot)$ on $[a, c]$ as follows:

$$h_\Delta(z) := \int_0^z \left[\frac{u'(W_\Delta - F_X^{-1}(z))(1 - T(z))}{1 - z} - u'(W_\Delta - F_X^{-1}(t)) T'(t) \right] dt. \quad (26)$$

Then, by (18) and using Lemma 5.2, we have

$$h_\Delta(a) = \int_0^a \left[u'(W_\Delta - F_X^{-1}(a)) T'(a) - u'(W_\Delta - F_X^{-1}(t)) T'(t) \right] dt < 0.$$

Recalling that $T(c) = c$, we have

$$\begin{aligned}
h_\Delta(c) &= \int_0^c \left[\frac{u'(W_\Delta - F_X^{-1}(c))(1 - T(c))}{1 - c} - u'(W_\Delta - F_X^{-1}(t)) T'(t) \right] dt \\
&= \int_0^c \left[u'(W_\Delta - F_X^{-1}(c)) - u'(W_\Delta - F_X^{-1}(t)) T'(t) \right] dt \\
&> u'(W_\Delta - F_X^{-1}(c)) c - \int_0^c \left[u'(W_\Delta - F_X^{-1}(c)) T'(t) \right] dt = 0.
\end{aligned}$$

Moreover, we take the derivative of $h_\Delta(z)$ with respect to $z \in [a, c]$ to obtain

$$\begin{aligned}
h'_\Delta(z) &= -u'(W_\Delta - F_X^{-1}(z)) T'(z) + u'(W_\Delta - F_X^{-1}(z)) \frac{1 - T(z)}{1 - z} \\
&\quad - u''(W_\Delta - F_X^{-1}(z)) \frac{1 - T(z)}{1 - z} z (F_X^{-1})'(z) + u'(W_\Delta - F_X^{-1}(z)) z \frac{\frac{1 - T(z)}{1 - z} - T'(z)}{1 - z} \\
&= u'(W_\Delta - F_X^{-1}(z)) \left(\frac{1 - T(z)}{1 - z} - T'(z) \right) - u''(W_\Delta - F_X^{-1}(z)) \frac{1 - T(z)}{1 - z} z (F_X^{-1})'(z) \\
&\quad + u'(W_\Delta - F_X^{-1}(z)) z \frac{\frac{1 - T(z)}{1 - z} - T'(z)}{1 - z} > 0.
\end{aligned}$$

Hence, there exists a unique point $l_\Delta \in (a, c)$ such that

$$h_\Delta(z) \begin{cases} < 0, & \text{if } a \leq z < l_\Delta, \\ = 0, & \text{if } z = l_\Delta, \\ > 0, & \text{if } l_\Delta < z \leq c, \end{cases} \quad (27)$$

Define

$$\underline{G}(z) = \begin{cases} F_X^{-1}(z), & \text{if } 0 \leq z < l_\Delta, \\ F_X^{-1}(l_\Delta), & \text{if } l_\Delta \leq z \leq 1, \end{cases} \quad (28)$$

and $K_\Delta := \int_0^1 \underline{G}(z) dz$.

Proposition 5.1 *If $K_\Delta \leq \Delta < E[X]$, then the optimal solution to (11) is*

$$G^*(z) = \begin{cases} F_X^{-1}(z), & \text{if } 0 \leq z < f, \\ F_X^{-1}(f), & \text{if } f \leq z \leq 1, \end{cases} \quad (29)$$

where f is the unique scalar such that $f \geq l_\Delta$ and $\int_0^1 G^*(z) dz = \Delta$.

Lemma 5.3 *If $0 < \Delta < K_\Delta$, then the corresponding optimal contract is not a deductible one.*

It follows from Lemma 5.3 that, if $0 < \Delta < K_\Delta$, the optimal contract (which always exists) can only be a threefold one, corresponding to (C.2). We are now led to the following proposition.

Proposition 5.2 *If $0 < \Delta < K_\Delta$, then the optimal solution to (11) is given as*

$$G^*(z) = \begin{cases} 0, & \text{if } 0 \leq z < z_2, \\ F_X^{-1}(z) - F_X^{-1}(z_2), & \text{if } z_2 \leq z < z_1, \\ F_X^{-1}(z_1) - F_X^{-1}(z_2), & \text{if } z_1 \leq z \leq 1, \end{cases}$$

where z_1, z_2 satisfy $z_2 \leq a \leq z_1$,

$$\int_{z_2}^{z_1} \left[\frac{u'(W_\Delta - F_X^{-1}(z_1) + F_X^{-1}(z_2))(1 - T(z_1))}{1 - z_1} - u'(W_\Delta - F_X^{-1}(t) + F_X^{-1}(z_2))T'(t) \right] dt = 0$$

and

$$\int_0^1 G^*(z) dz = \Delta.$$

Note that any pair (z_2, z_1) satisfying the requirements in Proposition 5.2 leads to an optimal solution to (11). Therefore such a pair (z_2, z_1) is unique as the optimal solution to (11) is unique.

Proposition 5.1 and Proposition 5.2 give two qualitatively distinct optimal contracts for any given $0 < \Delta < E[X]$, and the two cases are divided depending on whether or not $\Delta < K_\Delta$. However, K_Δ in general depends on Δ in an implicit and complicated way; so it is hard to compare Δ and K_Δ . Nevertheless we are able to treat at least two cases where $A_u(z)$ is either a constant or strictly decreasing in z .

First, assume that the utility function exhibits constant absolute risk aversion, e.g. $u(z) = 1 - e^{-\alpha z}$ $\forall z \in \mathbb{R}^+$. Then it is easy to see from (26) that l_Δ is independent of Δ , and hence so is K_Δ . In this case, denote $K \equiv K_\Delta$ and $\hat{\pi} = (1 + \rho)(E[X] - K)$. Then we have the following result.

Theorem 5.4 *Assume that Assumptions 2.1, 2.3, and 5.2 hold, and that $u(\cdot)$ exhibits constant absolute risk aversion. Then the optimal indemnity function $I^*(\cdot)$ to the problem (6) is given as*

(i) *If $\pi = (1 + \rho)E[X]$, then $I^*(z) = z$ for $z \in [0, M]$.*

(ii) *If $\hat{\pi} < \pi < (1 + \rho)E[X]$, then*

$$I^*(z) = \begin{cases} z, & \text{if } 0 \leq z < F_X^{-1}(z_2), \\ F_X^{-1}(z_2), & \text{if } F_X^{-1}(z_2) \leq z < F_X^{-1}(z_1), \\ z - F_X^{-1}(z_1) + F_X^{-1}(z_2), & \text{if } F_X^{-1}(z_1) \leq z \leq M, \end{cases}$$

where (z_2, z_1) is the unique pair satisfying $z_2 \leq a \leq z_1$,

$$\int_{z_2}^{z_1} \left[\frac{u'(W_\Delta - F_X^{-1}(z_1) + F_X^{-1}(z_2))(1 - T(z_1))}{1 - z_1} - u'(W_\Delta - F_X^{-1}(t) + F_X^{-1}(z_2))T'(t) \right] dt = 0,$$

$$\text{and } E[I^*(X)] = \frac{\pi}{1 + \rho}.$$

(iii) *If $0 \leq \pi \leq \hat{\pi}$, then*

$$I^*(z) = \begin{cases} 0, & \text{if } 0 \leq z < F_X^{-1}(f), \\ z - F_X^{-1}(f), & \text{if } F_X^{-1}(f) \leq z \leq M, \end{cases}$$

where f is the unique scalar satisfying $E[I^*(X)] = \frac{\pi}{1 + \rho}$.

Now, we study the case in which $A_u(z)$ is strictly decreasing. We need the following lemma.

Lemma 5.5 *If $0 < \Delta_1 < \Delta_2 < E[X]$, then $a < l_{\Delta_1} < l_{\Delta_2} < c$.*

Define

$$\Delta(d) := \int_0^d F_X^{-1}(z) dz + \int_d^1 F_X^{-1}(d) dz = \int_0^d F_X^{-1}(z) dz + F_X^{-1}(d)(1 - d)$$

on $d \in [a, c]$. Then $\Delta'(d) = (1 - d)(F_X^{-1})'(d) > 0$. Hence, $\Delta(\cdot)$ is a continuous and strictly increasing function. Determine $l_{\Delta(a)}$ and $l_{\Delta(c)}$ by $h_{\Delta(a)}(l_{\Delta(a)}) = 0$ and $h_{\Delta(c)}(l_{\Delta(c)}) = 0$, and set $\tilde{\Delta} := \Delta(l_{\Delta(a)})$ and $\bar{\Delta} := \Delta(l_{\Delta(c)})$. Finally, define a function $g(\cdot)$ on $[a, c]$ as follows:

$$g(z) := \int_0^z \left[\frac{u'(W_0 + (1 + \rho)\Delta(z) - F_X^{-1}(z))(1 - T(z))}{1 - z} - u'(W_0 + (1 + \rho)\Delta(z) - F_X^{-1}(t))T'(t) \right] dt.$$

Now, we are ready to give the main result in terms of the premium π and the indemnity function $I(\cdot)$.

Theorem 5.6 Assume that Assumptions 2.1, 2.3, and 5.2 hold, and that $A_u(\cdot)$ is strictly decreasing. Then the optimal indemnity function $I^*(\cdot)$ to the problem (6) is given as

(i) If $\pi = (1 + \rho)E[X]$, then $I^*(z) = z \forall z \in [0, M]$.

(ii) If $(1 + \rho)(E[X] - \tilde{\Delta}) \leq \pi < (1 + \rho)E[X]$, then

$$I^*(z) = \begin{cases} z, & \text{if } 0 \leq z < F_X^{-1}(z_2), \\ F_X^{-1}(z_2), & \text{if } F_X^{-1}(z_2) \leq z < F_X^{-1}(z_1), \\ z - F_X^{-1}(z_1) + F_X^{-1}(z_2), & \text{if } F_X^{-1}(z_1) \leq z \leq M, \end{cases}$$

where (z_2, z_1) is the unique pair satisfying $z_2 \leq a \leq z_1$,

$$\int_{z_2}^{z_1} \left[\frac{u'(W_\Delta - F_X^{-1}(z_1) + F_X^{-1}(z_2))(1 - T(z_1))}{1 - z_1} - u'(W_\Delta - F_X^{-1}(t) + F_X^{-1}(z_2))T'(t) \right] dt = 0,$$

and $E[I^*(X)] = \frac{\pi}{1+\rho}$.

(iii) If $(1 + \rho)(E[X] - \bar{\Delta}) < \pi < (1 + \rho)(E[X] - \tilde{\Delta})$, then let $p \in (l_{\Delta(a)}, l_{\Delta(c)})$ such that $\Delta(p) = E[X] - \frac{\pi}{1+\rho}$. If $g(p) < 0$, then

$$I^*(z) = \begin{cases} z, & \text{if } 0 \leq z < F_X^{-1}(z_2), \\ F_X^{-1}(z_2), & \text{if } F_X^{-1}(z_2) \leq z < F_X^{-1}(z_1), \\ z - F_X^{-1}(z_1) + F_X^{-1}(z_2), & \text{if } F_X^{-1}(z_1) \leq z \leq M, \end{cases}$$

where (z_2, z_1) is the unique pair satisfying $z_2 \leq a \leq z_1$,

$$\int_{z_2}^{z_1} \left[\frac{u'(W_\Delta - F_X^{-1}(z_1) + F_X^{-1}(z_2))(1 - T(z_1))}{1 - z_1} - u'(W_\Delta - F_X^{-1}(t) + F_X^{-1}(z_2))T'(t) \right] dt = 0,$$

and $E[I^*(X)] = \frac{\pi}{1+\rho}$. If $g(p) \geq 0$, then

$$I^*(z) = \begin{cases} 0, & \text{if } 0 \leq z < F_X^{-1}(f), \\ z - F_X^{-1}(f), & \text{if } F_X^{-1}(f) \leq z \leq M, \end{cases}$$

where q is the unique number satisfying $f < l_{\Delta(c)}$ and $E[I^*(X)] = \frac{\pi}{1+\rho}$.

(iv) If $0 \leq \pi \leq (1 + \rho)(E[X] - \bar{\Delta})$, then

$$I^*(z) = \begin{cases} 0, & \text{if } 0 \leq z < F_X^{-1}(f), \\ z - F_X^{-1}(f), & \text{if } F_X^{-1}(f) \leq z \leq M, \end{cases}$$

where q is the unique number satisfying $f \geq l_{\Delta(c)}$ and $E[I^*(X)] = \frac{\pi}{1+\rho}$.

6 Numerical Examples

In this section, we use numerical examples to illustrate our result with varying levels of the premium. We take the same numerical setting as in Bernard et al. (2015) for the comparison purpose (except for the value of the premium; see below). The loss X follows a truncated exponential distribution with the density function

$$f(x) = \frac{me^{-mx}}{1 - e^{-mM}},$$

where the intensity parameter $m = 0.1$, and $M = 10$. The initial wealth $W_0 = 15$, and $u(x) = 1 - e^{-\gamma x}$ with $\gamma = 0.02$. Moreover, the safety loading of the insurer ρ is 0.2. Finally, the weighting function is

$$T_\theta(x) = \frac{x^\theta}{(x^\theta + (1-x)^\theta)^{\frac{1}{\theta}}}$$

with $\theta = 0.5$. We can verify that the assumptions of Theorem 5.4 are satisfied under this setting. In Bernard et al. (2015), the premium is fixed at $\pi = 3$; but here we compute the optimal indemnities under $\pi = 1.5, 3$ and 4.5 respectively. These are plotted in Figure 3. The contract corresponding to $\pi = 1.5$ is a deductible one, whereas those corresponding to the two higher premiums are threefold covering smaller as well larger losses. For the latter two contracts, the one with the higher premium covers more smaller losses and has a lower deductible. Clearly, these features are all intuitive and sensible.

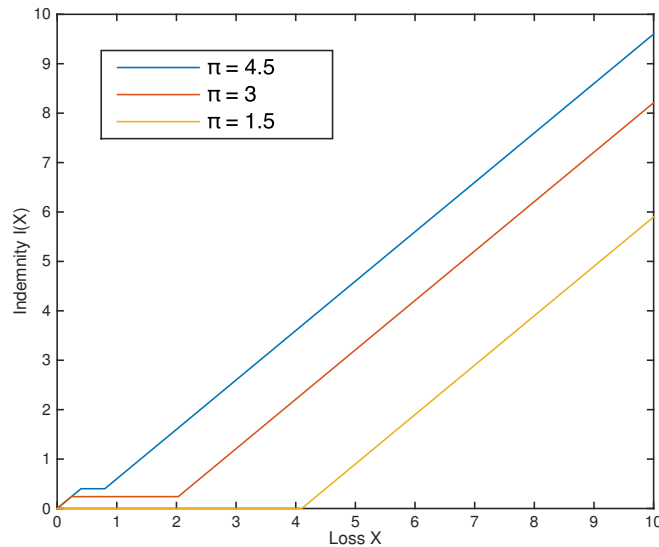


Figure 3: Optimal contracts under different premiums: The higher the premiums, the lower the deductibles and the more smaller losses covered.

When $\pi = 3$, the optimal indemnity obtained by Bernard et al. (2015) (without the monotone constraint) is plotted in blue in Figure 4. We note that in the result of Bernard et al. (2015), in some range of the loss the insured has the incentive to hide part of the loss in order to be paid with a larger compensation. By contrast, our indemnity function, depicted in red, is increasing and any increment in compensations is always less than or equal to the increment in losses. It effectively rules out the aforementioned behavior of moral hazard. In addition, in this example, we calculate the optimal RDU value of Bernard et al. (2015) to be 0.19 and that of our model to be 0.187. The difference, 0.003, is the “cost” of the additional monotonicity constraint, i.e. the loss in RDU value compared to the unconstrained case.

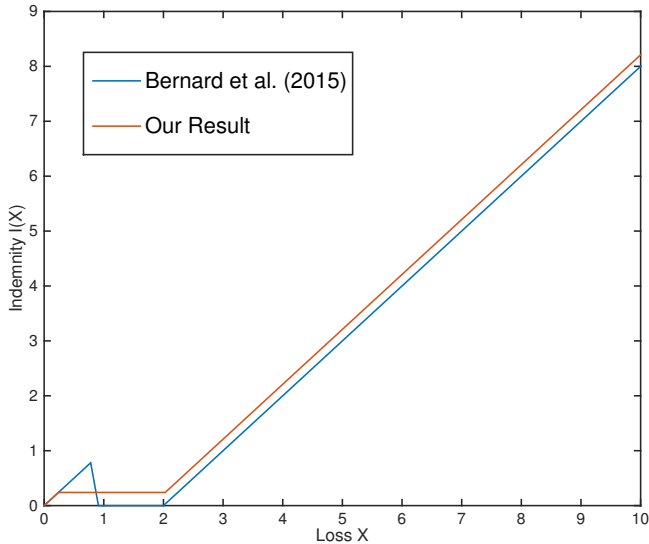


Figure 4: A comparison between our contract and Bernard et al. (2015): Our contract is a monotone, threefold contract, whereas theirs has a decreasing part causing potential moral hazard.

7 Conclusion

In this paper, we have studied an optimal insurance design problem where the insured uses the RDU preference. There are documented evidences proving that this preference captures human behaviors better than the EU preference. The main contribution of our work is that our optimal contracts are monotone with respect to losses, thereby eliminating the potential problem of moral hazard associated with the existing results.

An interesting conclusion from our results is that, under our assumptions (in particular Assump-

tion 5.2-(ii)), there are only two types of non-trivial optimal contracts possible, one being the classical deductible and the other the threefold contract covering both small and large losses. On the other hand, while we have demonstrated that Assumption 5.2-(ii) holds for many economically interesting cases, removing this assumption remains a mathematically outstanding open problem.

A Two Lemmas

In this part, we prove some lemmas which have been used in Section 5.

Lemma A.1 *Assume $T(\cdot) : [0, 1] \mapsto [0, 1]$ satisfies Assumption 2.3. Then*

(i) *If $a < z$, then $T'(z) < \frac{1-T(z)}{1-z}$.*

(ii) *If $a \leq z_2 < z_1 < 1$, then $\frac{1-T(z_1)}{1-z_1} > \frac{T(z_1)-T(z_2)}{z_1-z_2}$.*

PROOF: (i) If $a < z \leq b$, then $T'(z) < T'(a) < \frac{1-T(z)}{1-z}$. If $b < z$, then $T'(z) < \frac{1-T(z)}{1-z}$ since $T(\cdot)$ is convex and strictly increasing on $[b, 1]$.

(ii) Since $1 - T(z_1)$, $1 - z_1$, $T(z_1) - T(z_2)$, and $z_1 - z_2$ are all strictly positive, we have

$$\begin{aligned} \frac{1 - T(z_1)}{1 - z_1} > \frac{T(z_1) - T(z_2)}{z_1 - z_2} &\iff \frac{1 - T(z_1)}{1 - z_1} > \frac{(1 - T(z_1)) + (T(z_1) - T(z_2))}{(1 - z_1) + (z_1 - z_2)} \\ &\iff \frac{1 - T(z_1)}{1 - z_1} > \frac{1 - T(z_2)}{1 - z_2}. \end{aligned}$$

However, $\frac{1-T(z_1)}{1-z_1} > \frac{1-T(z_2)}{1-z_2}$ follows from Lemma 4.1. □

For fixed $x > 0$, define $q(z) := u'(x+z)u'(x-z)$ on $z \in (0, x)$.

Lemma A.2 *If $-\frac{u''(z)}{u'(z)}$ is strictly decreasing, then $q(z)$ is a strictly increasing function on $z \in (0, x)$.*

PROOF: We take derivative:

$$\begin{aligned} q'(z) &= u''(x+z)u'(x-z) - u'(x+z)u''(x-z) \\ &= u'(x+z)u'(x-z) \left[\left(-\frac{u''(x-z)}{u'(x-z)}\right) - \left(-\frac{u''(x+z)}{u'(x+z)}\right) \right] > 0. \end{aligned}$$

Hence, we get the result. □

B Existence of Optimal Solutions to (11) and (13)

We first prove that the constraint set \mathbb{G} is compact under some norm. We consider all the continuous functions on $[0, 1]$, denoted as $C[0, 1]$. Define a metric between $x(\cdot), y(\cdot) \in C[0, 1]$ as

$$\rho(x(\cdot), y(\cdot)) = \max_{0 \leq t \leq 1} |x(t) - y(t)|.$$

Clearly, $C[0, 1]$ is a metric space under ρ . By Arzela–Ascoli’s theorem, for any sequence $(G_n(\cdot))_{n \in \mathbb{N}}$ in \mathbb{G} , there exists a subsequence $G_{n_k}(\cdot)$ that converges in $C[0, 1]$ under ρ .

Lemma B.1 *The feasible set \mathbb{G} is compact under ρ .*

PROOF: For any sequence $(G_n(\cdot))_{n \in \mathbb{N}}$ in \mathbb{G} , there exists a subsequence $G_{n_k}(\cdot)$ that uniformly converges in $G^*(\cdot) \in C[0, 1]$. We now prove that $G^*(\cdot) \in \mathbb{G}$. If there exist $a > b$ such that $G^*(b) - G^*(a) = \eta > 0$, then take $\varepsilon := \frac{1}{3}\eta$. It follows from the uniform convergence that there exists K such that

$$\rho(G_{n_k}(\cdot), G^*(\cdot)) \leq \varepsilon$$

for any $k \geq K$. Hence,

$$0 < \eta = G^*(b) - G^*(a) = G^*(b) - G_{n_k}(b) + G_{n_k}(b) - G_{n_k}(a) + G_{n_k}(a) - G^*(a) \leq \varepsilon + 0 + \varepsilon = \frac{2}{3}\eta$$

for any $k \geq K$, which is a contradiction. This proves that $0 \leq G^*(a) - G^*(b)$, for all $a > b$. Similarly, we can prove that $G^*(a) - G^*(b) \leq F_X^{-1}(a) - F_X^{-1}(b)$. \square

The existence of optimal solutions to (11) and (13) can be established now. For example, for (13), let $v_\lambda(\Delta)$ be the optimal value of (13) under given λ and Δ . We can take a sequence $(G_n(\cdot))_{n \in \mathbb{N}}$ in \mathbb{G} such that $v_\lambda(\Delta) = \lim_{n \uparrow +\infty} U_\Delta(\lambda, G_n(\cdot))$. Then, according to Lemma B.1, there exists a subsequence $G_{n_k}(\cdot)$ converging to $G^*(\cdot)$ in \mathbb{G} and $G^*(\cdot)$ is optimal solution to (13). For (11), the proof is similar.

C Existence of Lagrangian Multiplier to (11)

For the following lemma, refer to Komiya (1988) for an elementary proof.

Lemma C.1 (*Sion’s Minimax Theorem*) *Let X be a compact convex subset of a linear topological space and Y a convex subset of a linear topological space. If f is a real-valued function on $X \times Y$ such that $f(x, \cdot)$ is continuous and concave on $Y \forall x \in X$, and $f(\cdot, y)$ is continuous and convex on $X \forall y \in Y$, then, $\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y)$.*

Proposition C.1 For any $0 < \Delta < E[X]$, there is λ^* such that $\tilde{G}_{\lambda^*}(\cdot)$ is optimal solution to (13) under λ^* and $\int_0^1 \tilde{G}_{\lambda^*}(z)dz = \Delta$.

PROOF: Let Δ be given with $0 < \Delta < E[X]$. Denote by $G^*(\cdot)$ the optimal solution to (11) under Δ (it is easy to show $\int_0^1 G^*(z)dz = \Delta$) and by $\tilde{G}_\lambda(\cdot)$ the optimal solution to (13) under λ and Δ . Denote by $v(\Delta)$ and $v(\lambda, \Delta)$ be respectively the optimal values of (11) and (13).

We first prove that $v(\lambda, \Delta)$ is a convex function in λ for given Δ . Noting that $U_\Delta(\lambda, G(\cdot))$ is linear in λ for any given $G(\cdot)$, we have

$$\begin{aligned} v(\alpha\lambda_1 + (1-\alpha)\lambda_2, \Delta) &= \max_{G(\cdot)} U_\Delta(\alpha\lambda_1 + (1-\alpha)\lambda_2, G(\cdot)) \\ &= \max_{G(\cdot)} \{\alpha U_\Delta(\lambda_1, G(\cdot)) + (1-\alpha)U_\Delta(\lambda_2, G(\cdot))\} \\ &\leq \max_{G(\cdot)} \{\alpha U_\Delta(\lambda_1, G(\cdot))\} + \max_{G(\cdot)} \{(1-\alpha)U_\Delta(\lambda_2, G(\cdot))\} \\ &= \alpha \max_{G(\cdot)} \{U_\Delta(\lambda_1, G(\cdot))\} + (1-\alpha) \max_{G(\cdot)} \{U_\Delta(\lambda_2, G(\cdot))\} \\ &= \alpha v(\lambda_1, \Delta) + (1-\alpha)v(\lambda_2, \Delta). \end{aligned}$$

Moreover, by Sion's minimax theorem, the following equality holds:

$$\max_{0 \leq \lambda} \min_{G(\cdot) \in \mathbb{G}} -U_\Delta(\lambda, G(\cdot)) = \min_{G(\cdot) \in \mathbb{G}} \max_{0 \leq \lambda} -U_\Delta(\lambda, G(\cdot));$$

hence

$$\min_{0 \leq \lambda} \max_{G(\cdot) \in \mathbb{G}} U_\Delta(\lambda, G(\cdot)) = \max_{G(\cdot) \in \mathbb{G}} \min_{0 \leq \lambda} U_\Delta(\lambda, G(\cdot)).$$

Finally, we have

$$v(\Delta) = \inf_{0 \leq \lambda} v(\lambda, \Delta),$$

namely,

$$\min_{0 \leq \lambda} \max_{G(\cdot) \in \mathbb{G}} U_\Delta(\lambda, G(\cdot)) = U_\Delta(G^*(\cdot)).$$

Let us denote

$$\bar{\lambda} := \frac{v(\Delta) + 1}{\int_0^1 F_X^{-1}(z)dz - \Delta} = \frac{U_\Delta(G^*(\cdot)) + 1}{E[X] - \Delta}.$$

For any $\lambda \geq \bar{\lambda}$, we have

$$\begin{aligned} v(\lambda, \Delta) &= \max_{G(\cdot) \in \mathbb{G}} U_\Delta(\lambda, G(\cdot)) \geq U_\Delta(\lambda, F_X^{-1}(z)) \\ &= \int_0^1 u(W_\Delta - F_X^{-1}(z))T'(z)dz + \lambda \left(\int_0^1 F_X^{-1}(z)dz - \Delta \right) \\ &\geq \lambda \left(\int_0^1 F_X^{-1}(z)dz - \Delta \right) \end{aligned}$$

$$\begin{aligned}
&\geq \bar{\lambda} \left(\int_0^1 F_X^{-1}(z) dz - \Delta \right) \quad (\text{since } \int_0^1 F_X^{-1}(z) dz > \Delta) \\
&= v(\Delta) + 1,
\end{aligned}$$

which yields

$$v(\Delta) = \inf_{0 \leq \lambda} v(\lambda, \Delta) = \inf_{0 \leq \lambda \leq \bar{\lambda}} v(\lambda, \Delta).$$

Therefore, by the convexity of $v(\lambda, \Delta)$, we can find the optimal $\lambda^* \in [0, \bar{\lambda}]$ minimizes the right part, and satisfies that $v(\Delta) = v(\lambda^*, \Delta)$. Moreover,

$$\begin{aligned}
v(\lambda^*, \Delta) &\geq U_\Delta(\lambda^*, G^*(\cdot)) = \int_0^1 u(W_\Delta - G^*(z))T'(z) dz + \lambda^* \left(\int_0^1 G^*(z) dz - \Delta \right) \\
&= \int_0^1 [u(W_\Delta - G^*(z))T'(z)] dz = U_\Delta(G^*(\cdot)) = v(\Delta).
\end{aligned}$$

The second equality comes from the fact that $G^*(\cdot)$ is the optimal solution to (11) under Δ ; hence $\int_0^1 G^*(z) dz = \Delta$. By $v(\Delta) = v(\lambda^*, \Delta)$ and $v(\lambda^*, \Delta) \geq U_\Delta(\lambda^*, G^*(\cdot)) = v(\Delta)$, we have $G^*(\cdot)$ is optimal solution to (13) under given λ^* . And, by uniqueness of optimal solutions to (13), we know that $G^*(\cdot)$ is the unique optimal solution to (13) under given λ^* and satisfying $\int_0^1 G^*(z) dz = \Delta$. \square

D Proofs

Proof of Lemma 2.1:

First, by the monotonicity of $R(\cdot)$, we have

$$\mathbb{P}(R(X) \leq R(x)) \geq \mathbb{P}(X \leq x) = F_X(x),$$

so by the definition of $F_{R(X)}^{-1}(F_X(x))$, we conclude that

$$F_{R(X)}^{-1}(F_X(x)) \leq R(x).$$

It suffices to prove the reverse inequality. There are two possible cases.

- $R(x) = 0$. In this case, we have $F_{R(X)}^{-1}(F_X(x)) = 0$ as quantile functions are always nonnegative by definition.

- $R(x) > 0$. It suffices to prove that $\mathbb{P}(R(X) \leq z) < F_X(x)$ for any $z < R(x)$. Take z_1 such that $z < z_1 < R(x)$. By the continuity and monotonicity of $R(\cdot)$, there exists $y < x$ such that $R(y) = z_1$. Then,

$$\mathbb{P}(R(X) \leq z) \leq \mathbb{P}(R(X) < z_1) = \mathbb{P}(R(X) < R(y)) \leq \mathbb{P}(X \leq y) = F_X(y) < F_X(x),$$

where we have used the fact that F_X is strictly increasing under Assumption 2.1.

The claim is thus proved.

Proof of Lemma 2.2:

We denote the right hand side of (12) by \mathbb{G}_1 . For any $G(\cdot) \in \mathbb{G}$, there exists $R(\cdot) \in \mathcal{R}$ such that $G(\cdot) = F_{R(X)}^{-1}(\cdot)$. For any $0 \leq b < a \leq 1$, define

$$\begin{aligned} \underline{a} &= \inf\{x \in [0, M] : R(x) = G(a)\}, \\ \bar{a} &= \sup\{x \in [0, M] : R(x) = G(a)\}, \end{aligned}$$

define \underline{b} and \bar{b} similarly. Let us show that $\underline{a} \leq F_X^{-1}(a) \leq \bar{a}$. In fact, by definition,

$$\begin{aligned} F_X^{-1}(a) &= \inf\{x \in \mathbb{R}^+ : F_X(x) \geq a\} \geq \inf\{x \in \mathbb{R}^+ : G(F_X(x)) \geq G(a)\} \\ &= \inf\{x \in \mathbb{R}^+ : R(x) \geq G(a)\} = \underline{a}. \end{aligned}$$

Suppose $F_X^{-1}(a) - \varepsilon > \bar{a}$ for some $\varepsilon > 0$. Then by monotonicity,

$$G(a) = R(\bar{a}) < R(F_X^{-1}(a) - \varepsilon) = G(F_X(F_X^{-1}(a) - \varepsilon)) \leq G(a),$$

where we have used the fact that $F_X(F_X^{-1}(a) - \varepsilon) < a$ to get the last inequality. This leads to a contradiction; hence it must hold that $F_X^{-1}(a) \leq \bar{a}$. Similarly, we can prove $\underline{b} \leq F_X^{-1}(b) \leq \bar{b}$. Then we have

$$0 \leq G(a) - G(b) = R(\underline{a}) - R(\bar{b}) \leq \underline{a} - \bar{b} \leq F_X^{-1}(a) - F_X^{-1}(b).$$

This inequality shows that G is absolutely continuous since F_X^{-1} is an absolutely continuous function under Assumption 2.1. Furthermore, it also implies

$$0 \leq G'(z) \leq (F_X^{-1})'(z),$$

a.e. $z \in [0, 1]$. So we have established that $\mathbb{G} \subseteq \mathbb{G}_1$.

To prove the reverse inclusion, take any $G(\cdot) \in \mathbb{G}_1$ and define $R(\cdot) = G(F_X(\cdot))$. It follows from Assumption 2.1 that

$$0 \leq R(0) = G(F_X(0)) - G(0) \leq F_X^{-1}(F_X(0)) - F_X^{-1}(0) = 0$$

and

$$0 \leq R(a) - R(b) = G(F_X(a)) - G(F_X(b)) \leq F_X^{-1}(F_X(a)) - F_X^{-1}(F_X(b)) = a - b, \quad \forall 0 \leq b < a \leq 1.$$

Hence $R(\cdot) \in \mathcal{R}$. It now suffices to show $G(a) = F_{R(X)}^{-1}(a)$ for any $0 \leq a \leq 1$. If $G(a) = 0$, then $G(a) \leq F_{R(X)}^{-1}(a)$ holds. Otherwise, for any $s < G(a)$, there exists y such that $s < R(y) = G(F_X(y)) < G(a)$ by the continuity of $R(\cdot)$. Then by the monotonicity of $R(\cdot)$ and $G(\cdot)$, we have

$$\mathbb{P}(R(X) \leq s) \leq \mathbb{P}(R(X) < R(y)) \leq \mathbb{P}(X \leq y) = F_X(y) < a,$$

which means $G(a) \leq F_{R(X)}^{-1}(a)$. Using the same notation, \bar{a} , as above, and noting that $G(a) = R(\bar{a}) = G(F_X(\bar{a}))$, we have $a \leq F_X(\bar{a})$ by the definition of \bar{a} and the continuity of $R(\cdot)$. Moreover, it follows from

$$\mathbb{P}(R(X) \leq G(a)) = \mathbb{P}(R(X) \leq R(\bar{a})) = \mathbb{P}(X \leq \bar{a}) = F_X(\bar{a})$$

that $F_{R(X)}^{-1}(F_X(\bar{a})) \leq G(a)$. Therefore,

$$G(a) \leq F_{R(X)}^{-1}(a) \leq F_{R(X)}^{-1}(F_X(\bar{a})) \leq G(a)$$

holds by monotonicity. The desired result follows.

Proof of Theorem 3.1:

We only need to prove the "if" part. For any feasible $G(\cdot)$ in \mathbb{G} , we have

$$\begin{aligned} & U_\Delta(\lambda, \tilde{G}_\lambda(\cdot)) - U_\Delta(\lambda, G(\cdot)) \\ &= \int_0^1 [u(W_\Delta - \tilde{G}_\lambda(z)) - u(W_\Delta - G(z))]T'(z)dz + \int_0^1 \lambda(\tilde{G}_\lambda(z) - G(z))dz \\ &\geq \int_0^1 u'(W_\Delta - \tilde{G}_\lambda(z))(G(z) - \tilde{G}_\lambda(z))T'(z)dz - \int_0^1 \lambda(G(z) - \tilde{G}_\lambda(z))dz \\ &= \int_0^1 N'_\lambda(z)(G(z) - \tilde{G}_\lambda(z))dz = \int_0^1 N_\lambda(t)(\tilde{G}'_\lambda(t) - G'(t))dt \geq 0. \end{aligned}$$

Hence, $\tilde{G}_\lambda(\cdot)$ is optimal for (13).

Proof of Lemma 4.1:

We have

$$f'(z) = \frac{(1 - T(z)) - T'(z)(1 - z)}{(1 - z)^2} = \frac{p(z)}{(1 - z)^2},$$

where

$$p(z) := (1 - T(z)) - T'(z)(1 - z).$$

Since

$$p'(z) = -T'(z) + T'(z) - T''(z)(1 - z) = -T''(z)(1 - z),$$

it follows from Assumption 2.3 that $p'(z) > 0$ for $z \in (0, b)$ and $p'(z) < 0$ for $z \in (b, 1)$. Moreover, $p(0+) = 1 - T'(0+) < 0$,

$$p(b) = (1 - T(b)) - T'(b)(1 - b) = \left(\frac{1 - T(b)}{1 - b} - T'(b) \right) (1 - b) > 0,$$

and

$$p(1-) = \lim_{z \uparrow 1} \left(\frac{1 - T(z)}{1 - z} - T'(z) \right) (1 - z) \geq 0,$$

as $T(\cdot)$ is strictly convex on $[b, 1]$. So, there exists $a \in (0, b)$ such that $p(z) < 0$ for $z \in [0, a)$ and $p(z) > 0$ for $z \in (a, 1)$. The desired result follows.

Proof of Theorem 4.2:

We note that $\Delta = E[X] - \frac{\pi}{1+\rho}$ and the binding constraint $E[R(X)] \equiv \int_0^1 G_{R(X)}(z) dz = \Delta$ is equivalent to that $E[I(X)] = \frac{\pi}{1+\rho}$.

- (i) If $\pi = (1 + \rho)E[X]$, then $\Delta = 0$. Therefore, the optimal solution to (11) is trivially $G^*(z) = 0 \forall z \in [0, 1]$, or $I^*(z) = z \forall z \in [0, M]$.
- (ii) If $\pi_c < \pi < (1 + \rho)E[X]$, then $0 < \Delta < K_c$. In this case, there exists a unique pair (d, e) such that $0 \leq d < a < e \leq c$, $f(d) = f(e)$ and $\int_0^1 G^*(z) dz = \Delta$ where G^* is defined as follows

$$G^*(z) = \begin{cases} 0, & \text{if } 0 \leq z < d, \\ F_X^{-1}(z) - F_X^{-1}(d), & \text{if } d \leq z < e, \\ F_X^{-1}(e) - F_X^{-1}(d), & \text{if } e \leq z \leq 1. \end{cases}$$

The existence of this pair follows from the condition that $\Delta < K_c$ and the definition of K_c , whereas the uniqueness comes from the requirement that $f(d) = f(e)$ and $\int_0^1 G^*(z)dz = \Delta$. Letting $\lambda = f(d)$, it is easy to show that $G^*(\cdot)$ satisfies (17) under λ , corresponding to the aforementioned **Case II**. This implies that $G^*(\cdot)$ is optimal for (11) under Δ . The optimal indemnity function is therefore $I^*(z) = z - G^*(F_X(z))$ for $z \in [0, M]$, leading to the desired expression.

(iii) If $0 \leq \pi \leq \pi_c$, then $K_c \leq \Delta \leq E[X]$, a case corresponding to **Case III**. The desired result can be derived similarly as in (ii) where $\lambda = f(q)$.

The proof is completed.

Proof of Lemma 5.1:

From $\lambda - u'(W_\Delta - G(z))T'(z) = 0$, it follows

$$u'(W_\Delta - G(z)) = \frac{\lambda}{T'(z)}.$$

Hence, if $z > a$, then

$$\begin{aligned} 0 &= \int_z^1 [\lambda - u'(W_\Delta - G(t))T'(t)] dt \\ &\leq \int_z^1 [\lambda - u'(W_\Delta - G(z))T'(t)] dt = \frac{\lambda}{T'(z)}(1-z) \left[T'(z) - \frac{1-T(z)}{1-z} \right] < 0, \end{aligned}$$

where the last inequality is due to Lemma A.1-(i) in Appendix A. This is a contradiction.

Proof of Lemma 5.2:

Noting $W_\Delta \geq W$, it follows from Assumption 5.2 that

$$A_T(z) > A_u(W - F_X^{-1}(z))(F_X^{-1})'(z) \geq A_u(W_\Delta - F_X^{-1}(z))(F_X^{-1})'(z), \quad \forall z \in (0, a].$$

This leads to

$$\begin{aligned} &\frac{d}{dz}(u'(W_\Delta - G(z))T'(z)) \\ &= u'(W_\Delta - G(z))T'(z) [A_u(W_\Delta - G(z))G'(z) - A_T(z)] \\ &< u'(W_\Delta - G(z))T'(z) [A_u(W_\Delta - G(z))(F_X^{-1})'(z) - A_u(W_\Delta - F_X^{-1}(z))(F_X^{-1})'(z)] \end{aligned}$$

$$\begin{aligned}
&= u'(W_\Delta - G(z))T'(z) [A_u(W_\Delta - G(z)) - A_u(W_\Delta - F_X^{-1}(z))] (F_X^{-1})'(z) \\
&\leq 0,
\end{aligned}$$

where the last inequality is due to the fact that A_u is decreasing and $G(z) \leq F_X^{-1}(z)$. The proof is complete.

Proof of Proposition 5.1:

The existence of f follows from the monotonicity of G^* with respect to f immediately. Denoting

$$\lambda_\Delta := u'(W_\Delta - F_X^{-1}(f)) \frac{1 - T(f)}{1 - f},$$

we need to show that $G^*(\cdot)$ satisfies (16) with $\lambda = \lambda_\Delta$. First, it is straightforward that

$$\int_f^1 \left[\frac{u'(W_\Delta - F_X^{-1}(f))(1 - T(f))}{1 - f} - u'(W_\Delta - F_X^{-1}(t))T'(t) \right] dt = 0.$$

Next, we are to prove that

$$\int_z^f \left[\frac{u'(W_\Delta - F_X^{-1}(f))(1 - T(f))}{1 - f} - u'(W_\Delta - F_X^{-1}(t))T'(t) \right] dt > 0, \quad \forall z \in (0, f).$$

We divide the proof into three cases.

- If $z \in [a, f)$, then

$$\begin{aligned}
&\int_z^f \left[\frac{u'(W_\Delta - F_X^{-1}(f))(1 - T(f))}{1 - f} - u'(W_\Delta - F_X^{-1}(t))T'(t) \right] dt \\
&\geq \int_z^f \left[\frac{u'(W_\Delta - F_X^{-1}(f))(1 - T(f))}{1 - f} - u'(W_\Delta - F_X^{-1}(f))T'(t) \right] dt \\
&= u'(W_\Delta - F_X^{-1}(f))(f - z) \left[\frac{1 - T(f)}{1 - f} - \frac{T(f) - T(z)}{f - z} \right] > 0,
\end{aligned}$$

where the last inequality is due to Lemma A.1-(ii).

- If $z \in (0, a)$ and $u'(W_\Delta - F_X^{-1}(z))T'(z) \leq \frac{u'(W_\Delta - F_X^{-1}(f))(1 - T(f))}{1 - f}$, then by Lemma 5.2 and the result above, we have

$$\begin{aligned}
&\int_z^f \left[\frac{u'(W_\Delta - F_X^{-1}(f))(1 - T(f))}{1 - f} - u'(W_\Delta - F_X^{-1}(t))T'(t) \right] dt \\
&> \int_z^a \left[\frac{u'(W_\Delta - F_X^{-1}(f))(1 - T(f))}{1 - f} - u'(W_\Delta - F_X^{-1}(t))T'(t) \right] dt > 0.
\end{aligned}$$

- If $z \in (0, a)$ and $u'(W_\Delta - F_X^{-1}(z))T'(z) > \frac{u'(W_\Delta - F_X^{-1}(f))(1-T(f))}{1-f}$, then, using $h_\Delta(l_\Delta) = 0$,

$$\begin{aligned}
& \int_z^f \left[\frac{u'(W_\Delta - F_X^{-1}(f))(1-T(f))}{1-f} - u'(W_\Delta - F_X^{-1}(t))T'(t) \right] dt \\
& \geq \int_z^{l_\Delta} \left[\frac{u'(W_\Delta - F_X^{-1}(l_\Delta))(1-T(l_\Delta))}{1-l_\Delta} - u'(W_\Delta - F_X^{-1}(t))T'(t) \right] dt \\
& = - \int_0^z \left[\frac{u'(W_\Delta - F_X^{-1}(l_\Delta))(1-T(l_\Delta))}{1-l_\Delta} - u'(W_\Delta - F_X^{-1}(t))T'(t) \right] dt > 0,
\end{aligned}$$

where the last inequality is due to

$$\begin{aligned}
u'(W_\Delta - F_X^{-1}(z))T'(z) & > \frac{u'(W_\Delta - F_X^{-1}(f))(1-T(f))}{1-f} \\
& \geq \frac{u'(W_\Delta - F_X^{-1}(l_\Delta))(1-T(l_\Delta))}{1-l_\Delta},
\end{aligned}$$

as $a < l_\Delta \leq f$ and the fact that $u'(W_\Delta - F_X^{-1}(z))T'(z)$ is strictly decreasing on $[0, a]$.

The claim follows now.

Proof of Lemma 5.3:

There exists λ^* such that $\tilde{G}_{\lambda^*}(\cdot)$ satisfies (16) under λ^* and $\int_0^1 \tilde{G}_{\lambda^*}(z) dz = \Delta$ (see Appendix C). If $\tilde{G}_{\lambda^*}(\cdot)$ corresponds to a deductible contract, then there exists \bar{z} (since $\Delta < K_\Delta$, we have $\bar{z} < l_\Delta$) such that

$$\tilde{G}_{\lambda^*}(z) = \begin{cases} F_X^{-1}(z), & \text{if } 0 \leq z < \bar{z}, \\ F_X^{-1}(\bar{z}), & \text{if } \bar{z} \leq z \leq 1. \end{cases}$$

Since $\tilde{G}_{\lambda^*}(\cdot)$ satisfies (16), we have

$$\int_{\bar{z}}^1 [\lambda^* - u'(W_\Delta - F_X^{-1}(\bar{z}))T'(t)] dt = 0,$$

or

$$\lambda^* = u'(W_\Delta - F_X^{-1}(\bar{z})) \frac{1 - T'(\bar{z})}{1 - \bar{z}}.$$

On the other hand,

$$M(z) = \int_z^{\bar{z}} \left[u'(W_\Delta - F_X^{-1}(\bar{z})) \frac{1 - T'(\bar{z})}{1 - \bar{z}} - u'(W_\Delta - F_X^{-1}(t))T'(t) \right] dt \geq 0$$

for $z \in [0, \bar{z}]$. However, by the definition of l_Δ ,

$$h_\Delta(\bar{z}) \equiv M(0) = \int_0^{\bar{z}} \left[\frac{u'(W_\Delta - F_X^{-1}(\bar{z}))(1 - T(\bar{z}))}{1 - \bar{z}} - u'(W_\Delta - F_X^{-1}(t))T'(t) \right] dt < 0$$

as $\bar{z} < l_{\Delta}$. Since $M(\cdot)$ is a continuous function, a contradiction arises.

Proof of Proposition 5.2:

The conclusion is a direct consequence of Lemma 5.3.

Proof of Theorem 5.4:

The result follows from Propositions 5.1, 5.2 and the fact that K_{Δ} is a constant for any $0 < \Delta < E[X]$.

Proof of Lemma 5.5:

According to definition of l_{Δ_1} , we have

$$h_{\Delta_1}(l_{\Delta_1}) = \int_0^{l_{\Delta_1}} \left[\frac{u'(W_{\Delta_1} - F_X^{-1}(l_{\Delta_1}))(1 - T(l_{\Delta_1}))}{1 - l_{\Delta_1}} - u'(W_{\Delta_1} - F_X^{-1}(t))T'(t) \right] dt = 0.$$

Since $W_{\Delta_1} < W_{\Delta_2}$, we have

$$\frac{u'(W_{\Delta_2} - F_X^{-1}(l_{\Delta_1}))}{u'(W_{\Delta_1} - F_X^{-1}(l_{\Delta_1}))} < \frac{u'(W_{\Delta_2} - F_X^{-1}(t))}{u'(W_{\Delta_1} - F_X^{-1}(t))}$$

for $t \in [0, l_{\Delta_1})$ by Lemma A.2 in Appendix A. Hence

$$h_{\Delta_2}(l_{\Delta_1}) = \int_0^{l_{\Delta_1}} \left[\frac{u'(W_{\Delta_2} - F_X^{-1}(l_{\Delta_1}))(1 - T(l_{\Delta_1}))}{1 - l_{\Delta_1}} - u'(W_{\Delta_2} - F_X^{-1}(t))T'(t) \right] dt < 0.$$

As a result $h_{\Delta_2}(l_{\Delta_1}) < 0$, $h_{\Delta_2}(c) > 0$. Since $h'_{\Delta_2}(z) > 0$ for $z \in [l_{\Delta_1}, c)$, we get $l_{\Delta_2} \in (l_{\Delta_1}, c)$.

Proof of Theorem 5.6:

According to the fact that $\Delta = E[X] - \frac{\pi}{1+\rho}$, (i), (ii) and (iv) are direct consequences of Propositions 5.1, 5.2 and Lemma 5.5. For (iii), if $(1 + \rho)(E[X] - \bar{\Delta}) < \pi < (1 + \rho)(E[X] - \tilde{\Delta})$, then $\tilde{\Delta} < \Delta < \bar{\Delta}$ and there is a unique $p \in (l_{\Delta(a)}, l_{\Delta(c)})$ such that $\Delta(p) = \Delta$, which follows from the definition of $\tilde{\Delta}$, $\bar{\Delta}$ and the fact that $\Delta(\cdot)$ is a continuous and strictly increasing function. If $g(p) < 0$, then $h_{\Delta}(p) < 0$; hence $l_{\Delta} > p$. Therefore, $\Delta < K_{\Delta}$. The desired result follows from Proposition 5.2. The proof for $g(p) \geq 0$ is similar.

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