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Utility Maximization Under Trading Constraints with Discontinuous Utility*

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Abstract. This paper investigates a utility maximization problem in a Black–Scholes market, in which trading is subject to a convex cone constraint and the utility function is not necessarily continuous or concave. The problem is initially formulated as a stochastic control problem, and a partial differential equation method is subsequently used to study the associated Hamilton–Jacobi–Bellman equation. The value function is shown to be discontinuous at maturity (with the exception of trivial cases), and its lower-continuous envelope is shown to be concave before maturity. The comparison principle shows that the value function is continuous and coincides with that of its concavified problem.

Key words. discontinuous utility function, convex cone constraint, variational inequality, viscosity solution, stochastic control

AMS subject classifications. 35R35, 60H30, 91B70, 93E20

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1. Introduction.

As one of the most predominant investment decision rules in the portfolio selection theory in financial economics, the expected utility (EU) theory has been extensively investigated in the literature. In a generalized EU model, the aim of an investor is to optimize

$$\mathbb{E}[U(X)] \tag{1}$$

over a set of possible payoffs X for a certain utility function U under a certain (linear or nonlinear) expectation $\mathbb{E}$.

In the classical case, in which U is a concave and smooth utility function and $\mathbb{E}$ is the linear mathematical expectation, the EU model and its solution, in both complete and incomplete markets, are well-known; see, e.g., Karatzas and Zitkovic [21], Kramkov and Schachermayer [22], Hugonnier and Kramkov [18], Biagini and Frittelli [2], and Bian and Zheng [4] and the references therein. The model has also been widely investigated in a nonstandard case, in which U is concave, but not smooth; see, e.g., Bouchard, Touzi, and Zeghal [6], Westray and Zheng [28], and Bian, Miao, and Zheng [3]. In the presence of transaction costs, the EU model is more involved; a closed-form solution is, generally, not necessarily available. We refer to Deelstra, Pham, and Touzi [13] and Dai and Yi [12] for recent development along this direction.

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Many well-known financial models can also be interpreted as special EU models wherein U is concave but $\mathbb{E}$ is not a necessary linear expectation. For instance, if $\mathbb{E}$ is chosen as a certain Choquet expectation, then the model (1) becomes the one with the rank-dependent utility; see Xia and Zhou [29] and Xu [30]. While if $\mathbb{E}$ is the infimum of a set of linear mathematical expectations, it becomes a model with ambiguity; see Gilboa and Schmeidler [14], Hansen and Sargent [16], Chen and Epstein [10], and Bordigoni, Matoussi, and Schweizer [5].

Meanwhile, as pointed out by Reichlin [26], there is considerable empirical evidence showing that agents, in practice, tend to switch between risk-averse and risk-seeking behaviors, depending on the context. This fact partially accounts for the study of nonconcave EU models; see Berkelaar, Kouwenberg, and Post [1], Larsen [24], Rieger [27], and Carpenter [9]. Some well-known behavioral finance models can also be regarded as nonconcave EU models. For instance, if $U(x) = \mathbb{1}_{x \geqslant \mathrm{goal}}$ (which is neither continuous nor concave) in (1), the model becomes a goal-reaching problem; see Kulldorff [23], Browne [7, 8], and He and Zhou [17]. While if the utility function is S-shaped (that is, convex on the left of a reference point and concave on the right), the model becomes a cumulative prospect theory (CPT) model *without* probability weighting. (If one further takes $\mathbb{E}$ as a certain Choquet expectation, it becomes a standard CPT model *with* probability weighting; see Jin and Zhou [19], He and Zhou [17], Xu and Zhou [32], and Xu [30] for more on this.)

Reichlin [26] studied a utility maximization problem for a not necessarily concave utility function in a complete market setting via a delicate probabilistic argument. He showed that whether the underlying probability space is atomic or atomless crucially affects the result of the portfolio selection problem. If the underlying probability space is atomless and there are no trading constraints, then the concave envelope of the value function is the value function of the *concavified* problem, namely, the one defined by replacing the utility function with its concave envelope in the old problem.

This paper, along with the research of [3, 28, 26], studies an EU model in which the utility function is not necessarily continuous or concave. There are at least two important differences between our model and that of Reichlin [26]. Economically speaking, in our model, the investment strategy is subject to a convex cone constraint and the market may not be complete, whereas in [26], there is no investment constraint and the market is complete. Mathematically speaking, we adopt the stochastic analysis and viscosity solution approach in contrast to the probabilistic argument approach used in [26]. We show that the value function is discontinuous at maturity (with the exception of trivial cases). We also provide a comparison principle instead of a verification theorem to guarantee the uniqueness of the viscosity solution.

Using stochastic analysis techniques, we first derive the upper and lower bounds for the value function when time approaches maturity. Next, we adopt the viscosity solution method to study the value function. By using the supersolution property, we prove the concavity of the value function before maturity. The concavity reveals that the value function is not continuous at maturity (unless the utility function is a classical concave one). This feature distinguishes our results from those of the classical ones. Next, by applying Ishii's lemma, we obtain a comparison principle for the value function. Finally, we derive the second-order smoothness of the value function before maturity by showing that it coincides with that of its *concavified* problem when the market is complete.

This paper is organized as follows. In section 2, we formulate the problem. In section 3, we study the limits of the value function at maturity and prove the concavity of the value function before maturity. We also deduce a comparison principle for the value function. In section 4, we study the smoothness of the value function, and in section 5, we conclude the paper.

Notation. We use the following notation throughout this paper:

- M^T, the transpose of a matrix or vector M;
- $\|M\| = \sqrt{\sum_{i,j} m_{ij}^2}$, the L^2 norm for a matrix or vector $M = (m_{ij})$;
- $\mathbb{R}^m$, m-dimensional real Euclidean space;
- $\mathbb{R}^m_+$, the subset of $\mathbb{R}^m$ consisting of elements with nonnegative components.

The underlying uncertainty of the financial market is generated by a standard $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted m-dimensional Brownian motion $B(\cdot) \equiv (B^1(\cdot), \dots, B^m(\cdot))^T$ defined on a fixed filtered complete probability space $(\Omega, \mathbf{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geqslant 0})$.

Given a Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|_{\mathcal{H}}$, we can define a Banach space as follows:

$$L^2_{\mathcal{F}}(a,b;\mathcal{H}) = \left\{ \varphi(\cdot) \;\middle|\; \begin{array}{l} \varphi(\cdot) \text{ is an } \{\mathcal{F}_t\}_{t\geqslant 0}\text{-adapted, } \mathcal{H}\text{-valued progressively measurable} \\ \text{process defined on } [a,b] \text{ and that satisfies } \|\varphi(\cdot)\|_{\mathcal{F}} < +\infty \end{array} \right\}$$

with the norm

$$\|\varphi(\cdot)\|_{\mathcal{F}} = \left(\mathbb{E}\left[\int_a^b \|\varphi(t,\omega)\|^2_{\mathcal{H}} \,\mathrm{d}t \right] \right)^{\frac{1}{2}}.$$

2. Problem formulation.

Let $T > 0$ be a fixed investment maturity throughout the paper. Consider an arbitrage-free financial market in which $n+1$ assets are traded continuously over the investment horizon $[0,T]$. One of the assets is a *bond*, whose price $S_0(\cdot)$ evolves according to the ordinary differential equation

$$\begin{cases} \mathrm{d}S_0(t) = rS_0(t)\,\mathrm{d}t, \quad t \geqslant 0, \\ S_0(0) = s_0 > 0, \end{cases}$$

where r is the interest rate of the bond. The remaining n assets are *stocks*, and their prices $S_i(\cdot)$, $i = 1, 2, \dots, n$, are modeled by the system of stochastic differential equations

$$\begin{cases} \mathrm{d}S_i(t) = S_i(t)\{b_i\,\mathrm{d}t + \sum_{j=1}^m \sigma_{ij}\,\mathrm{d}B^j(t)\}, \quad t \geqslant 0, \\ S_i(0) = s_i > 0, \end{cases}$$

where b_i is the appreciation rate of the stock i and σ_{ij} is the volatility coefficient. We define the volatility matrix $\sigma := (\sigma_{ij})$ and the excess return vector $\mu = (b_1 - r, \dots, b_n - r)^T$. The parameters r, μ, and σ are all (deterministic) constants. As usual, we assume there exists a vector θ such that $\mu = \sigma\theta$, and it has the minimum L^2 norm over all such vectors. This assumption ensures that the market is free of arbitrage opportunity.

The number of stocks should be no more than that of the uncertainties, namely, $n \leqslant m$; otherwise some of the stocks could get redundant and could be removed from the market. The market is incomplete when $n < m$. In section 4, we will assume $m = n$ and $\mathrm{rank}(\sigma) = n$ to ensure the smoothness of the value function.

Suppose an agent has an initial wealth $x_0 > 0$ to invest in the market and her total wealth at time t is denoted by $X(t)$. Let $\pi_i(t)$ denote her total market value in the stock i at time t, $i = 1, \ldots, n$. We refer to $\pi(\cdot) := (\pi_1(\cdot), \ldots, \pi_n(\cdot))^T \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ as a portfolio. We only consider self-financing portfolios so that the wealth process $X(\cdot)$ follows (see Karatazas and Shreve [20])

$$\begin{cases} \mathrm{d}X(t) = [rX(t) + \pi(t)^T\mu]\,\mathrm{d}t + \pi(t)^T\sigma\,\mathrm{d}B(t), \quad t \geqslant 0, \\ X(0) = x_0. \end{cases}$$

An important restriction considered in this paper is the convex cone portfolio constraint, that is,

$$\pi(t) \in \mathcal{C} \quad \forall\, t \in [0, T],$$

where $\mathcal{C} \subseteq \mathbb{R}^n$ is a nonempty closed convex cone. This model covers many important practical cases, for instance, shorting is not allowed in the market when $\mathcal{C} = \mathbb{R}^n_+$, and there are no trading constraints when $\mathcal{C} = \mathbb{R}^n$. We assume that $\mathcal{C}^T\sigma$ is not identical to zero; otherwise $\pi(t)^T\sigma = 0$, and $\pi(t)^T\mu = \pi(t)^T\sigma\theta = 0$ for all $t \geqslant 0$, so that the problem is not interesting at all.

Another important restriction considered in this paper is the prohibition on bankruptcy, namely,

$$X(t) \geqslant 0 \quad \forall\, t \in [0, T].$$

Let $\mathcal{A}$ denote the set of all the admissible portfolios satisfying the aforementioned constraints, namely,

$$\mathcal{A} := \{\pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) : \pi(t) \in \mathcal{C},\ X(t) \geqslant 0 \ \ \forall\, t \in [0, T]\}.$$

It is easily seen that $\mathcal{A}$ is a convex set.

Let $U : [0, \infty) \mapsto [0, \infty)$ be a nonconstant, nondecreasing utility function with $U(0) = 0$. The agent's value function is defined as

$$V(x, t) = \sup_{\pi \in \mathcal{A}} \mathbb{E}_{t,x}[U(X_T)] \quad \forall\, (x, t) \in \mathcal{S} := [0, \infty) \times [0, T),$$

where $\mathbb{E}_{t,x}$ denotes the condition expectation given that $X_t = x$. The main aim of this paper is to study the property of this value function.

Let $\widehat{U}$ denote the concave envelope function of U that is the smallest concave function dominating U on $[0, \infty)$. See Figure 1 for a demonstration of the utility function U and its concave envelope $\widehat{U}$.

We impose the following assumption throughout the paper.

Assumption 1. The function $\widehat{U}$ satisfies $\widehat{U}(0) = 0$ and is Hölder continuous, namely,

$$|\widehat{U}(x) - \widehat{U}(y)| \leqslant L|x - y|^p \quad \forall\, x, y \geqslant 0$$

for some constants $L > 0$ and $0 < p < 1$.

As U is nonconstant and nondecreasing, so is $\widehat{U}$. Moreover $U \equiv \widehat{U}$ if and only if U is concave and continuous.

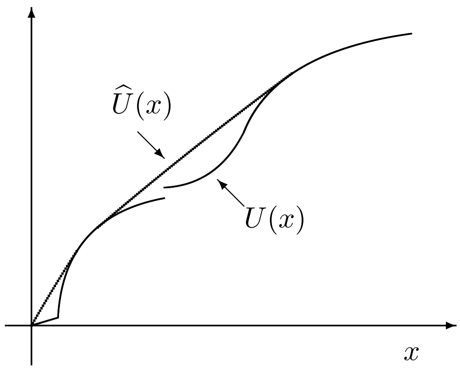

Figure 1. *The utility function U and its concave envelope $\widehat{U}$.*

Example 1. Assumption 1 holds true for the following utility functions (where a and b are positive constants):

- $U(x) = ((x-a)_+ + b)^\alpha - b^\alpha$ for $0 < \alpha < 1$;
- $U(x) = \log(b(x-a)_+ + 1)$;
- $U(x) = u((x-a)_+)$ where u is continuous increasing and concave with $u(0) = 0$ and $u(x) \leqslant Cx^p$ for some $C > 0$ and $0 < p < 1$.

3. Concavity and comparison principle.

In this section, we study the properties of the value function V.

Lemma 3.1. *Let Assumption 1 hold. Then, there exists a constant $C = C(T)$ such that $0 \leqslant V(x,t) \leqslant LCx^p$ for any $(x,t) \in \mathcal{S}$.*

Proof. Under Assumption 1, we have $U \leqslant \widehat{U} \leqslant Lx^p$. For $U(x) = Lx^p$ and $\mathcal{C} = \mathbb{R}^n$, the value function at (x,t) is well-known to be $Le^{c(T-t)}x^p$, where

$$c = \sup_{\pi \in \mathbb{R}^n} \left(-\frac{1}{2}(1-p)p\|\pi^T\sigma\|^2 + p\pi^T\mu \right) + pr < \infty,$$

giving the claimed upper bound. ∎

Lemma 3.2. *The value function $V(x,t)$ is nondecreasing in x and nonincreasing in t.*

Proof. The proof is trivial by definition. ∎

An upper bound for the value function near maturity is given as follows.

Lemma 3.3. *We have*

$$\limsup_{t \nearrow T} V(x,t) \leqslant \widehat{U}(x).$$

Proof. Let

$$\rho_t = e^{-\frac{1}{2}\int_0^t \|\theta\|^2\,\mathrm{d}s - \int_0^t \theta^T\,\mathrm{d}B(s)}. \tag{2}$$

Itô's lemma gives

$$\mathrm{d}\,(\rho_t X_t) = \rho_t(\pi(t)^T\sigma - X_t\theta^T)\,\mathrm{d}B(t),$$

so the process $\rho_t X_t$ is a local martingale. As $\rho_t X_t$ is also nonnegative, it is a supermartingale, and hence

$$\mathbb{E}_{t,x}\left[\frac{\rho_T}{\rho_t}X_T\right] \leqslant x.$$

For any $\pi \in \mathcal{A}$, by Jensen's inequality,

$$\mathbb{E}_{t,x}\left[\widehat{U}\left(\frac{\rho_T}{\rho_t}X_T\right)\right] \leqslant \widehat{U}\left(\mathbb{E}_{t,x}\left[\frac{\rho_T}{\rho_t}X_T\right]\right) \leqslant \widehat{U}(x).$$

Hence,

$$\limsup_{t\nearrow T}\sup_{\pi\in\mathcal{A}}\mathbb{E}_{t,x}\left[\widehat{U}\left(\frac{\rho_T}{\rho_t}X_T\right)\right] \leqslant \widehat{U}(x). \tag{3}$$

Next, we proceed to prove

$$\lim_{t\nearrow T}\sup_{\pi\in\mathcal{A}}\mathbb{E}_{t,x}\left|\widehat{U}(X_T) - \widehat{U}\left(\frac{\rho_T}{\rho_t}X_T\right)\right| = 0. \tag{4}$$

Indeed, for all $x,\ y \geqslant 0$, by Assumption 1,

$$|\widehat{U}(x) - \widehat{U}(y)| \leqslant L|x-y|^p.$$

Thus, for any $\pi \in \mathcal{A}$, Hölder's inequality yields

$$\begin{aligned}
\mathbb{E}_{t,x}\left[\left|\widehat{U}(X_T) - \widehat{U}\left(\frac{\rho_T}{\rho_t}X_T\right)\right|\right] &\leqslant C\mathbb{E}_{t,x}\left[\left(\frac{\rho_T}{\rho_t}X_T\right)^p\left|\frac{\rho_t}{\rho_T}-1\right|^p\right] \\
&\leqslant C\left(\mathbb{E}_{t,x}\left[\frac{\rho_T}{\rho_t}X_T\right]\right)^p\left(\mathbb{E}_{t,x}\left[\left|\frac{\rho_t}{\rho_T}-1\right|^{\frac{p}{1-p}}\right]\right)^{1-p} \\
&\leqslant Cx^p\left(\mathbb{E}_{t,x}\left[\left|\frac{\rho_t}{\rho_T}-1\right|^{\frac{p}{1-p}}\right]\right)^{1-p} \\
&= Cx^p\left(\mathbb{E}\left[\left|\frac{1}{\rho_{T-t}}-1\right|^{\frac{p}{1-p}}\right]\right)^{1-p},
\end{aligned}$$

where we used the fact that ρ is a geometric Brownian motion to obtain the last equality. Consequently,

$$\limsup_{t\nearrow T}\sup_{\pi\in\mathcal{A}}\mathbb{E}_{t,x}\left[\left|\widehat{U}(X_T) - \widehat{U}\left(\frac{\rho_T}{\rho_t}X_T\right)\right|\right] \leqslant \limsup_{t\nearrow T} Cx^p\left(\mathbb{E}\left[\left|\frac{1}{\rho_{T-t}}-1\right|^{\frac{p}{1-p}}\right]\right)^{1-p} = 0.$$

Therefore, by (3), (4), and the fact that $U \leqslant \widehat{U}$, we have

$$\begin{aligned}
\limsup_{t\nearrow T} V(x,t) &= \limsup_{t\nearrow T}\sup_{\pi\in\mathcal{A}}\mathbb{E}_{t,x}[U(X_T)] \leqslant \limsup_{t\nearrow T}\sup_{\pi\in\mathcal{A}}\mathbb{E}_{t,x}\left[\widehat{U}(X_T)\right] \\
&\leqslant \limsup_{t\nearrow T}\sup_{\pi\in\mathcal{A}}\mathbb{E}_{t,x}\left[\widehat{U}\left(\frac{\rho_T}{\rho_t}X_T\right)\right] + \limsup_{t\nearrow T}\sup_{\pi\in\mathcal{A}}\mathbb{E}_{t,x}\left[\left|\widehat{U}(X_T) - \widehat{U}\left(\frac{\rho_T}{\rho_t}X_T\right)\right|\right] \leqslant \widehat{U}(x).
\end{aligned}$$

The desired result is thus proved. ■

We next assign a lower bound for the value function near maturity.

Lemma 3.4. *We have*

$$\liminf_{t\nearrow T} V(x,t) \geqslant U(x).$$

Proof. By taking $\pi \equiv 0$, we derive a trivial lower bound $V(x,t) \geqslant U(xe^{r(T-t)}) \geqslant U(x)$. The claim follows immediately. ∎

Remark 1. We remark that the upper and the lower bounds for the value function obtained thus far are not the same unless $U \equiv \widehat{U}$, that is, the utility function U is concave. Later we will show that the lower bound inequality can be strict unless U is concave. This is one important finding from our model.

We now study the properties of the value function before maturity by using the viscosity solution approach. For the definition of a (discontinuous) viscosity solution, see Definition 4.2.1 in [25].

As in [25], let us define

$$V^*(x,t) = \limsup_{(y,s)\to(x,t)} V(y,s), \quad V_*(x,t) = \liminf_{(y,s)\to(x,t)} V(y,s).$$

Then the following holds.

Proposition 3.5. *The function $V_*(x,t)$ ($V^*(x,t)$) is a lower (upper) semicontinuous, super-(sub)-solution of*

$$V_t + \sup_{\pi\in\mathcal{C}} \left(\frac{1}{2}\|\pi^T\sigma\|^2 x^2 V_{xx} + \pi^T \mu x V_x \right) + rxV_x = 0, \quad (x,t) \in \mathcal{S}, \tag{5}$$

which is nondecreasing in x and nonincreasing in t.

Proof. The monotonicity follows from Lemma 3.2, while the other claims are from [25]. ∎

Now we are ready to present our main result.

Theorem 3.6. *Assume that $U(x,t)$ is a lower semicontinuous supersolution of (5) that is nondecreasing in x and nonincreasing in t. Then for each $t \in [0,T)$, the function $U(x,t)$ is a concave function in x, and hence, it is continuous in x.*

Proof. Suppose that U is not concave in x. Then, there exists $0 < x_1 < x_0 < x_2$, $0 \leqslant t_0 < T$ and $0 < \alpha < 1$ such that $x_0 = \alpha x_1 + (1-\alpha)x_2$ and $U(x_0,t_0) < \alpha U(x_1,t_0) + (1-\alpha)U(x_2,t_0)$. Set

$$3\delta = \alpha U(x_1,t_0) + (1-\alpha)U(x_2,t_0) - U(x_0,t_0) > 0.$$

Choose a and b such that $U(x_i,t_0) = ax_i + b$, $i = 1,2$. We note that $a \geqslant 0$ as U is nondecreasing in x. By the lower semicontinuity of U, there exists $x_1 < \overline{x}_1 < \overline{x}_2 < x_2$ and $t_0 < t_1 < T$ such that

$$U(x,t) - ax - b + \delta > 0 \quad \forall\ (x,t) \in \{[x_1,\overline{x}_1] \cup [\overline{x}_2,x_2]\} \times [t_0,t_1). \tag{6}$$

Let

$$\Omega_0 = [x_1,x_2] \times [t_0,t_1);$$

and let

$$W_d(x,t) = U(x,t) - ax - b + 2\delta + dh(x) + \frac{\varepsilon}{t_1 - t}, \quad (x,t) \in \Omega_0,$$

where

$$h(x_1) = h(x_2) = 0, \quad h''(x) < 0 \quad \forall\, x \in [x_1, x_2] \quad (\text{e.g., } h(x) = (x - x_1)(x_2 - x)).$$

Observe that $h > 0$ on (x_1, x_2) because it is strictly concave. We define

$$d^* = \inf\{d \geqslant 0 : W_d(x,t) \geqslant 0 \;\; \forall\, (x,t) \in \Omega_0\}.$$

First, we have, for $0 < \varepsilon < \delta(t_1 - t_0)$,

$$\begin{aligned}
&W_0(x_0, t_0)\\
&\quad = U(x_0, t_0) - ax_0 - b + 2\delta + \frac{\varepsilon}{t_1 - t_0}\\
&\quad = \alpha U(x_1, t_0) + (1-\alpha)U(x_2, t_0) - 3\delta - a(\alpha x_1 + (1-\alpha)x_2) - b + 2\delta + \frac{\varepsilon}{t_1 - t_0}\\
&\quad = \alpha(U(x_1, t_0) - ax_1 - b) + (1-\alpha)(U(x_2, t_0) - ax_2 - b) - \delta + \frac{\varepsilon}{t_1 - t_0}\\
&\quad = -\delta + \frac{\varepsilon}{t_1 - t_0} < 0.
\end{aligned}$$

This implies $d^* > 0$.

Second, for $(x,t) \in [x_1, \overline{x}_1] \times [t_0, t_1) \cup [\overline{x}_2, x_2] \times [t_0, t_1)$, by (6) and $h \geqslant 0$, we have

$$W_d(x,t) = U(x,t) - ax - b + 2\delta + dh(x) + \frac{\varepsilon}{t_1 - t} \geqslant \delta > 0 \quad \forall\, d \geqslant 0.$$

For $(x,t) \in [\overline{x}_1, \overline{x}_2] \times [t_0, t_1)$, using the monotonicity of U, we have

$$\begin{aligned}
W_d(x,t) &= U(x,t) - ax - b + 2\delta + dh(x) + \frac{\varepsilon}{t_1 - t}\\
&\geqslant U(\overline{x}_1, t_1) - a\overline{x}_2 - b + 2\delta + d \min_{y \in [\overline{x}_1, \overline{x}_2]}\{h(y)\}\\
&= U(\overline{x}_1, t_1) - a\overline{x}_2 - b + 2\delta + d\min\{h(\overline{x}_1), h(\overline{x}_2)\}, \qquad (7)
\end{aligned}$$

where the last equality is due to the concavity of h. Notice $h > 0$ on $[\overline{x}_1, \overline{x}_2] \subset (x_1, x_2)$, so the right-hand side of (7) is positive for a sufficiently large d. We now conclude that $0 < d^* < \infty$ and $W_{d^*} \geqslant 0$ on Ω_0. Moreover, by the lower semicontinuity of U, there exists $(x^*, t^*) \in [\overline{x}_1, \overline{x}_2] \times [t_0, t_1) \subseteq \Omega_0$ such that $W_{d^*}(x^*, t^*) = 0$. Let

$$\eta(x,t) = ax + b - 2\delta - d^* h(x) - \frac{\varepsilon}{t_1 - t}.$$

We then have $U \geqslant \eta$ on Ω_0 and $U(x^*, t^*) = \eta(x^*, t^*)$. Hence, the test function η must satisfy the supersolution property of the Hamilton–Jacobi–Bellman (HJB) equation (5) at (x^*, t^*), that is,

$$\eta_t + \sup_{\pi \in \mathcal{C}} \left(\frac{1}{2}\|\pi^T \sigma\|^2 x^2 \eta_{xx} + \pi^T \mu x \eta_x \right) + rx\eta_x \bigg|_{(x^*, t^*)} \leqslant 0.$$

However, this is impossible because $\eta_{xx}(x^*, t^*) = -d^* h''(x^*) > 0$ and $\|\pi^T\sigma\|$ can take an arbitrarily large value as $\mathcal{C}^T\sigma \not\equiv 0$ and $\mathcal{C}$ is a cone. ■

Applying Theorem 3.6, we can now derive the exact value of the value function when time approaches maturity.

Corollary 3.7. *We have*

$$\lim_{t\nearrow T} V(x,t) = \widehat{U}(x).$$

Proof. As V_* is concave in x, by Lemma 3.4, we obtain a new lower bound $\liminf_{t\nearrow T} V_*(x,t) \geqslant \widehat{U}(x)$. Together with the upper bound given in Lemma 3.3, the claim follows from the definition of V^*. ∎

Remark 2. In [25], the author considered the superreplication cost in an uncertain volatility model, which is similar to our problem. He introduced a continuous function $G(t,x,p,M)$ such that the Hamiltonian is well-defined if and only if $G \geqslant 0$. The concavity and terminal conditions of the value function were studied by introducing the function G into the HJB variational inequality. See (4.7), (4.12), and Theorem 4.3.2 in [25]. However, in our case, it is impossible to find a suitable function G to discuss the concavity and terminal condition of the value function.

We derive the following crucial comparison principle, which ensures the uniqueness of the viscosity solution and the continuity of the value function.

Theorem 3.8 (comparison principle). *Let u^* be the upper semicontinuous subsolution, and v_* be the lower semicontinuous supersolution of (5), respectively. Suppose that*

$$u^*(x,T) \leqslant v_*(x,T), \quad u^*(0,t) \leqslant v_*(0,t), \tag{8}$$

and $|u^*(x,t)| + |v_*(x,t)| \leqslant C(1+x^p)$ for all $(x,t) \in \mathcal{S}$ with some $0 < p < 1$ and $C > 0$. Then

$$u^*(x,t) \leqslant v_*(x,t) \quad \forall\ (x,t) \in \mathcal{S}. \tag{9}$$

Proof. Suppose that (9) is not true; then, by boundary conditions (8), there exists a $\delta > 0$ and $(x_0,t_0) \in \mathcal{S}$ such that

$$u^*(x_0,t_0) - v_*(x_0,t_0) = 2\delta > 0.$$

First fix an arbitrary constant q in $(p,1)$. Then choose a small constant $\varepsilon > 0$ such that

$$u^*(x_0,t_0) - v_*(x_0,t_0) - 2\varepsilon x_0^q > \delta$$

and a large constant β such that

$$\beta > \frac{\|\theta\|^2 q}{2(1-q)} + rq. \tag{10}$$

Define the test function

$$\varphi(x,y,t,s) = (k((x-y)^2 + (t-s)^2) + \varepsilon(x^q + y^q))e^{\beta(s-t)}$$

on $\mathbb{R}_+^2 \times [0,T]^2$ for each $k > 0$. Let

$$W(x,y,t,s) = u^*(x,t) - v_*(y,s) - \varphi(x,y,t,s),$$

then

$$W(x,y,t,s) \leqslant C(2+x^p+y^p) - \varepsilon(x^q+y^q)e^{-\beta T} \to -\infty,$$

as $x+y \to +\infty$, and we conclude from the boundary condition (8) that W attains its maximum value on $(0,\infty)^2 \times [0,T]^2$ at some point $(\overline{x},\overline{y},\overline{t},\overline{s})$, for instance, which of course depends on ε and k. Immediately,

$$W(\overline{x},\overline{y},\overline{t},\overline{s}) \geqslant W(x_0,x_0,t_0,t_0) = u^*(x_0,t_0) - v_*(x_0,t_0) - 2\varepsilon x_0^q > \delta, \tag{11}$$

and consequently,

$$u^*(\overline{x},\overline{t}) - v_*(\overline{y},\overline{s}) > \delta + \varphi(\overline{x},\overline{y},\overline{t},\overline{s}),$$

that is,

$$u^*(\overline{x},\overline{t}) - v_*(\overline{y},\overline{s}) > \delta + (k((\overline{x}-\overline{y})^2 + (\overline{t}-\overline{s})^2) + \varepsilon(\overline{x}^q+\overline{y}^q))e^{\beta(\overline{s}-\overline{t})}. \tag{12}$$

This, together with the growth condition

$$u^*(\overline{x},\overline{t}) - v_*(\overline{y},\overline{s}) \leqslant C(2+\overline{x}^p+\overline{y}^p),$$

yields

$$|\overline{x}| + |\overline{y}| + k((\overline{x}-\overline{y})^2 + (\overline{t}-\overline{s})^2) \leqslant C_\varepsilon, \tag{13}$$

where C_ε does not depend on k. Here, the assumption $p<q<1$ plays a crucial role. By the compactness of the real set, we may assume that $\overline{x}$ and $\overline{y}$ go to x_ε, and $\overline{t}$ and $\overline{s}$ go to t_ε, as $k \to \infty$. Then

$$\begin{aligned} u^*(x_\varepsilon,t_\varepsilon) - v_*(x_\varepsilon,t_\varepsilon) &\geqslant \limsup_{k\to\infty} u^*(\overline{x},\overline{t}) - \liminf_{k\to\infty} v_*(\overline{y},\overline{s}) \\ &\geqslant \limsup_{k\to\infty}(u^*(\overline{x},\overline{t}) - v_*(\overline{y},\overline{s})) \geqslant \limsup_{k\to\infty} W(\overline{x},\overline{y},\overline{t},\overline{s}) \geqslant \delta, \end{aligned}$$

by (11), which implies $x_\varepsilon > 0$ and $t_\varepsilon < T$ by the boundary condition (8). Hence,

$$\overline{x} > x_\varepsilon/2 > 0, \quad \overline{t} < T, \tag{14}$$

for all sufficiently large k. From $W(\overline{x},\overline{y},\overline{t},\overline{s}) \geqslant W(x_\varepsilon,x_\varepsilon,t_\varepsilon,t_\varepsilon)$, we obtain

$$\begin{aligned} k((\overline{x}-\overline{y})^2 + (\overline{t}-\overline{s})^2)e^{-\beta T} &\leqslant k((\overline{x}-\overline{y})^2 + (\overline{t}-\overline{s})^2)e^{\beta(\overline{s}-\overline{t})} \\ &\leqslant (u^*(\overline{x},\overline{t}) - u^*(x_\varepsilon,t_\varepsilon)) - (v_*(\overline{y},\overline{s}) - v_*(x_\varepsilon,t_\varepsilon)) + 2\varepsilon x_\varepsilon^q - \varepsilon(\overline{x}^q+\overline{y}^q)e^{\beta(\overline{s}-\overline{t})}, \end{aligned}$$

which leads to

$$\limsup_{k\to\infty} k((\overline{x}-\overline{y})^2 + (\overline{t}-\overline{s})^2) = 0. \tag{15}$$

Letting $k \to \infty$ in (12), we deduce a lower bound for

$$u^*(x_\varepsilon,t_\varepsilon) - v_*(x_\varepsilon,t_\varepsilon) \geqslant \delta + 2\varepsilon x_\varepsilon^q. \tag{16}$$

Next, we shall use Ishii's lemma. (See Theorem 8.3 in the user's guide [11].)

Ishii's Lemma. For any $\eta > 0$, there exist M and N such that

$$(\varphi_t(\overline{x},\overline{y},\overline{t},\overline{s}), M, \varphi_x(\overline{x},\overline{y},\overline{t},\overline{s})) \in \overline{J}^{2,+}u^*(\overline{x},\overline{t}),$$
$$(-\varphi_s(\overline{x},\overline{y},\overline{t},\overline{s}), N, -\varphi_y(\overline{x},\overline{y},\overline{t},\overline{s})) \in \overline{J}^{2,-}v_*(\overline{y},\overline{s}),$$

and

$$\begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \leqslant \mathcal{D}^2\varphi + \eta(\mathcal{D}^2\varphi)^2 = \begin{pmatrix} \varphi_{xx} & \varphi_{xy} \\ \varphi_{yx} & \varphi_{yy} \end{pmatrix} + \eta \begin{pmatrix} \varphi_{xx} & \varphi_{xy} \\ \varphi_{yx} & \varphi_{yy} \end{pmatrix}^2.$$

A simple calculation yields

$$\begin{aligned} &\varphi_x = (2k(x-y) + q\varepsilon x^{q-1})e^{\beta(s-t)}, \qquad \varphi_y = (-2k(x-y) + q\varepsilon y^{q-1})e^{\beta(s-t)}, \\ &\varphi_t = -\beta\varphi + 2k(t-s)e^{\beta(s-t)}, \qquad \varphi_s = \beta\varphi - 2k(t-s)e^{\beta(s-t)}, \\ &\mathcal{D}^2\varphi = 2ke^{\beta(s-t)}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + q(q-1)\varepsilon e^{\beta(s-t)}\begin{pmatrix} x^{q-2} & 0 \\ 0 & y^{q-2} \end{pmatrix}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} (\mathcal{D}^2\varphi)^2 = {} & 8k^2e^{2\beta(s-t)}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + 2kq(q-1)\varepsilon e^{2\beta(s-t)}\begin{pmatrix} x^{q-2} & -y^{q-2} \\ -x^{q-2} & y^{q-2} \end{pmatrix} \\ & + 2kq(q-1)\varepsilon e^{2\beta(s-t)}\begin{pmatrix} x^{q-2} & -x^{q-2} \\ -y^{q-2} & y^{q-2} \end{pmatrix} + q^2(q-1)^2\varepsilon^2 e^{2\beta(s-t)}\begin{pmatrix} x^{2q-4} & 0 \\ 0 & y^{2q-4} \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} Mx^2 - Ny^2 &= (x\ y)\begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &\leqslant (x\ y)\,\mathcal{D}^2\varphi\begin{pmatrix} x \\ y \end{pmatrix} + \eta\,(x\ y)\,(\mathcal{D}^2\varphi)^2\begin{pmatrix} x \\ y \end{pmatrix} \\ &= 2ke^{\beta(s-t)}(x-y)^2 + q(q-1)\varepsilon e^{\beta(s-t)}(x^q + y^q) \\ &\quad + \eta(8k^2(x-y)^2 + 4kq(q-1)\varepsilon(x-y)(x^{q-1} - y^{q-1}) \\ &\quad + q^2(q-1)^2\varepsilon^2(x^{2q-2} + y^{2q-2}))e^{2\beta(s-t)} \\ &\leqslant 2ke^{\beta T}(x-y)^2 + q(q-1)\varepsilon e^{\beta(s-t)}(x^q + y^q) \\ &\quad + \eta(8k^2(x-y)^2 + q^2(q-1)^2\varepsilon^2(x^{2q-2} + y^{2q-2}))e^{2\beta T}. \end{aligned}$$

Taking $\eta = e^{-k}$, this gives, by (13), (14), and (15),

$$M\overline{x}^2 - N\overline{y}^2 \leqslant -q(1-q)\varepsilon e^{\beta(\overline{s}-\overline{t})}(\overline{x}^q + \overline{y}^q) - I_{k,\varepsilon} < 0$$

for sufficiently large k, where

$$\limsup_{k\to\infty} |I_{k,\varepsilon}| = 0. \tag{18}$$

By (5), Ishii's lemma, and (17),

$$-\beta u^* + 2k(t-s)e^{\beta(s-t)} + \sup_{\pi\in\mathcal{C}}\left(\frac{1}{2}\|\pi^T\sigma\|^2 x^2 M + \pi^T\mu x\varphi_x + rx\varphi_x\right)\Big|_{(\overline{x},\overline{y},\overline{t},\overline{s})} \geqslant 0,$$
$$-\beta v_* + 2k(t-s)e^{\beta(s-t)} + \sup_{\pi\in\mathcal{C}}\left(\frac{1}{2}\|\pi^T\sigma\|^2 y^2 N - \pi^T\mu y\varphi_y - ry\varphi_y\right)\Big|_{(\overline{x},\overline{y},\overline{t},\overline{s})} \leqslant 0.$$

Subtracting the first inequality from the second one, after rearrangement, gives

$$\begin{aligned}
\beta(u^*-v_*)\Big|_{(\overline{x},\overline{y},\overline{t},\overline{s})} &\leqslant \sup_{\pi\in\mathcal{C}}\left(\frac{1}{2}\|\pi^T\sigma\|^2 x^2 M + \pi^T\mu x\varphi_x + rx\varphi_x\right)\Big|_{(\overline{x},\overline{y},\overline{t},\overline{s})}\\
&\quad - \sup_{\pi\in\mathcal{C}}\left(\frac{1}{2}\|\pi^T\sigma\|^2 y^2 N - \pi^T\mu y\varphi_y - ry\varphi_y\right)\Big|_{(\overline{x},\overline{y},\overline{t},\overline{s})}\\
&\leqslant \sup_{\pi\in\mathcal{C}}\left(\frac{1}{2}\|\pi^T\sigma\|^2(Mx^2-Ny^2) + (\pi^T\mu+r)(x\varphi_x+y\varphi_y)\right)\Big|_{(\overline{x},\overline{y},\overline{t},\overline{s})}\\
&\leqslant \sup_{\pi\in\mathcal{C}}\bigg(-\frac{1}{2}\|\pi^T\sigma\|^2\big(q(1-q)\varepsilon e^{\beta(\overline{s}-\overline{t})}(\overline{x}^q+\overline{y}^q)+I_{k,\varepsilon}\big)\\
&\qquad + (\pi^T\sigma\theta+r)(2k(\overline{x}-\overline{y})^2+q\varepsilon(\overline{x}^q+\overline{y}^q))\bigg)\\
&\leqslant \sup_{\alpha\in\mathbb{R}_+}\bigg(-\frac{1}{2}\alpha^2\big(q(1-q)\varepsilon e^{\beta(\overline{s}-\overline{t})}(\overline{x}^q+\overline{y}^q)+I_{k,\varepsilon}\big)\\
&\qquad + (\alpha\|\theta\|+r)(2k(\overline{x}-\overline{y})^2+q\varepsilon(\overline{x}^q+\overline{y}^q))\bigg)\\
&\leqslant \frac{\|\theta\|^2}{2}\frac{(2k(\overline{x}-\overline{y})^2+q\varepsilon(\overline{x}^q+\overline{y}^q))^2}{q(1-q)\varepsilon e^{\beta(\overline{s}-\overline{t})}(\overline{x}^q+\overline{y}^q)+I_{k,\varepsilon}} + r(2k(\overline{x}-\overline{y})^2+q\varepsilon(\overline{x}^q+\overline{y}^q)).
\end{aligned}$$

Letting $k\to\infty$, it follows from (14), (15), and (18) that

$$\beta(u^*(x_\varepsilon,t_\varepsilon)-v_*(x_\varepsilon,t_\varepsilon)) \leqslant \left(\frac{\|\theta\|^2 q}{2(1-q)}+rq\right)2\varepsilon x_\varepsilon^q. \tag{19}$$

Comparing with (16) and using $x_\varepsilon>0$, we obtain

$$\beta < \frac{\|\theta\|^2 q}{2(1-q)}+rq,$$

contradicting (10). The proof is complete. ■

Remark 3. In [25], the author assumed that the Hamiltonian $H(t,x,p,M)$ is Lipschitz continuous in x, uniformly in $\pi\in\mathcal{C}$, and proved the comparison theorem, 4.4.5. In our singular case, Hamiltonian $H(t,x,p,M)$ is not continuous uniformly. The p-growth condition is the key point for our approach.

Applying Proposition 3.5, Lemma 3.1, and Corollary 3.7, we deduce that $V_*\geqslant V^*$ by comparison principle. It yields $V=V^*=V_*$ by the definitions of V^* and V_*. To conclude, we have, by Lemma 3.2, the following.

Theorem 3.9. *The value function V is the unique continuous viscosity solution of the HJB equation (5) in the class of concave and nondecreasing functions in x with boundary conditions*

$$\lim_{x\searrow 0} V(x,t) = 0, \quad \lim_{t\nearrow T} V(x,t) = \widehat{U}(x).$$

4. Smoothness of the value function.

As the value function V is close to $\widehat{U}$ near maturity, we naturally consider the related *concavified* problem

$$\widehat{V}(x,t) := \sup_{\pi\in\mathcal{A}} \mathbb{E}_{t,x}[\widehat{U}(X_T)], \quad (x,t)\in\mathcal{S}.$$

This is a standard expected utility maximization problem except for the fact that $\widehat{U}$ is only Hölder continuous and may not be strictly concave.

Corollary 4.1. *We have*

$$V(x,t) = \widehat{V}(x,t) \quad \forall\,(x,t)\in\mathcal{S}.$$

Proof. By Theorem 3.9, V is a viscosity solution of (5). It is easily seen that $\widehat{V}$ is also a solution of (5). As they have the same boundary and terminal values, by the comparison principle 3.8, they are equal on $\mathcal{S}$. ∎

Remark 4. Using the stochastic duality method, Reichlin [26] studied a utility maximization problem for a not necessarily concave utility function in a complete market setting and obtained the same result. He showed that the underlying probability space crucially affects the optimization problem. The two value functions do not necessarily coincide if the underlying probability space is atomic. In our setting, the underlying probability space is atomless so that the two value functions are the same. However, Reichlin did not study the properties of the value function near maturity.

Theorem 4.2. *If $\mu \neq 0$ and $\operatorname{rank}(\sigma) = m = n$, then V is in $C^{2,1}(\mathcal{S})$.*

The proof is given in the appendix.

Carpenter [9] considered a nonconcave utility portfolio choice problem in a complete market setting.[1] In contrast to our PDE approach to show Corollary 4.1, Carpenter showed it by proving the optimal payoff for the concavified objective function is also optimal for the true objective function because it only takes on values where the two functions agree. In fact, using the dual method, he showed that the optimal final payoff (given initial time 0) is

$$X_T^* = (\widehat{U}')^{(-1)}(\lambda\rho_T) = \begin{cases} 0 & \text{if } \rho_T \text{ is bigger than a threshold;} \\ (\widehat{U}')^{(-1)}(\lambda\rho_T) & \text{otherwise.} \end{cases}$$

Here ρ_T is the so-called pricing kernel defined in (2) and $(\widehat{U}')^{(-1)}$ stands for the left-inverse of the function $\widehat{U}'$. The utility function and its concave envelope in [9] are demonstrated in the

[1] Guan et al. [15] considered a similar problem but with a nonsmooth utility and mixed choice of investment strategies and investment horizon.

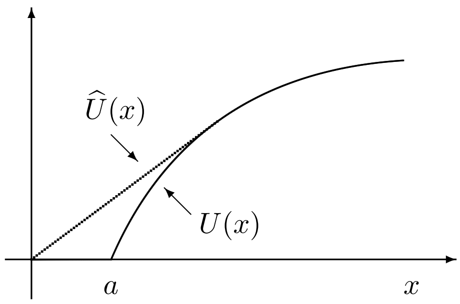

Figure 2. *The utility function U and its concave envelope $\widehat{U}$ in* [9].

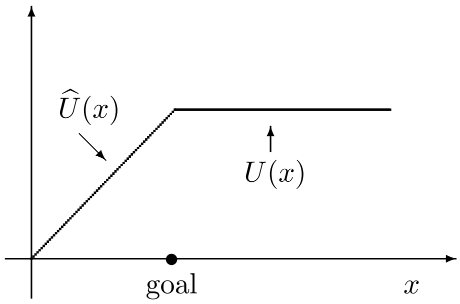

Figure 3. *The utility function $U(x) = \mathbb{1}_{x \geqslant goal}$ and its concave envelope $\widehat{U}$.*

Figure 2. As demonstrated in the picture, it can be seen that

$$U(X_T^*) = \widehat{U}(X_T^*).$$

This coincides with Corollary 4.1.

If we consider the goal-reaching problem in a complete market setting, then the utility function $U(x) = \mathbb{1}_{x \geqslant \mathrm{goal}}$ is not continuous, and its concave envelope is demonstrated in the Figure 3. In this case, although the concave envelope function is only Hölder continuous with $p = 1$, our method still works. In the absence of trading constraints, the optimal final payoff is two-point (with one point being zero) distributed, which is consistent with [17, Theorem 3.2].

5. Concluding remarks.

The main contributions of this paper are as follows. First, in our model, neither the utility function nor the value function is necessarily continuous. Second, the limit of the value function, as time approaches maturity, has been proved to be the concave envelope of the utility function. Third, the value function has been proved to be concave before maturity. Finally, we have proved the *second-order* smoothness of the value function at any point before maturity when the market is complete.

In this paper, we have assumed that the trading constraint set is a convex cone. A new method is called for in which it is not a cone.

Appendix.

Proof of Theorem 4.2. By Corollary 4.1, it suffices to prove $\widehat{V}$ is in $C^{2,1}(\mathcal{S})$. By Theorem 3.9 and Corollary 4.1, $\widehat{V}$ is concave and nondecreasing in x and is the unique continuous viscosity solution of the HJB equation

$$\widehat{V}_t + \sup_{\pi\in\mathcal{C}}\left(\frac{1}{2}\|\pi^T\sigma\|^2 x^2 \widehat{V}_{xx} + \pi^T\mu x \widehat{V}_x\right) + rx\widehat{V}_x = 0, \quad (x,t)\in\mathcal{S}, \tag{20}$$

with boundary conditions $\widehat{V}(0,t)=0$, $\widehat{V}(x,T)=\widehat{U}(x)$. If we can find a classical (which must be a viscosity) solution in $C^{2,1}(\mathcal{S})$ for the above HJB equation, then it must be $\widehat{V}$ and the claim follows.

Now let us construct a $C^{2,1}(\mathcal{S})$ solution for (20) by a convex dual argument (see, e.g., Pham [25], [3, Theorems 3.8 and 5.1], and [31]).

Let

$$\alpha = \max_{\substack{\pi\in\mathcal{C}\\ \|\pi^T\sigma\|=1}} \pi^T\sigma\theta = \max_{\substack{\pi\in\mathcal{C}\\ \|\pi^T\sigma\|=1}} \pi^T\mu.$$[2]

It is positive as $\mathcal{C}^T\mu \neq \{0\}$. Furthermore, let ϕ be the solution of

$$\begin{cases}\phi_t + \frac{\alpha^2}{2}y^2\phi_{yy} - ry\phi_y = 0, & (y,t)\in\mathcal{S},\\ \phi(y,T) = \sup_{x>0}(\widehat{U}(x) - xy). \end{cases} \tag{21}$$

By the Feynman–Kac formula,

$$\phi(y,t) = \int_{\mathbb{R}} \phi(ye^{-(r+\frac{\alpha^2}{2})(T-t)+\alpha z\sqrt{T-t}}, T)\,\mathrm{d}z.$$

Under Assumption 1, $\phi(y,T)$ is a nonconstant, decreasing, and convex function in y, so one can show that $\phi\in C^{2,1}(\mathcal{S})$ is strictly decreasing and strictly convex in y in $\mathcal{S}$. This implies $\phi_y<0$. Furthermore, by the strong maximum principle, we have $\phi_{yy}>0$ in $\mathcal{S}$.

Using the facts that $\phi_y<0$ and $\phi_{yy}>0$, $\mathcal{C}$ is a cone, $\mu=\sigma\theta$, and $\alpha>0$, we have

$$\begin{aligned}\sup_{\pi\in\mathcal{C}}\left(-\frac{1}{2}\|\pi^T\sigma\|^2\frac{\phi_y^2}{\phi_{yy}} - \pi^T\mu y\phi_y\right) &= \sup_{\beta\geqslant 0}\sup_{\substack{\pi\in\mathcal{C}\\ \|\pi^T\sigma\|=1}}\left(-\frac{1}{2}\beta^2\|\pi^T\sigma\|^2\frac{\phi_y^2}{\phi_{yy}} - \beta\pi^T\sigma\theta y\phi_y\right)\\ &= \sup_{\beta\geqslant 0}\sup_{\substack{\pi\in\mathcal{C}\\ \|\pi^T\sigma\|=1}}\left(-\frac{1}{2}\beta^2\frac{\phi_y^2}{\phi_{yy}} - \beta\pi^T\sigma\theta y\phi_y\right)\\ &= \sup_{\beta\geqslant 0}\left(-\frac{1}{2}\beta^2\frac{\phi_y^2}{\phi_{yy}} - \beta\alpha y\phi_y\right) = \frac{\alpha^2}{2}y^2\phi_{yy}.\end{aligned}$$

[2]In general, one has $0\leqslant\alpha\leqslant\|\theta\|$. In particular, if the market is complete, i.e., $\mathcal{C}=\mathbb{R}^n$ and $\mathrm{rank}(\sigma)=n$, then $\alpha=\|\theta\|$.

Therefore, ϕ is a classical solution of the following equation:

$$\begin{cases} \phi_t + \sup_{\pi\in\mathcal{C}} \left(-\frac{1}{2}\|\pi^T\sigma\|^2 \frac{\phi_y^2}{\phi_{yy}} - \pi^T\mu y\phi_y\right) - ry\phi_y = 0, & (y,t)\in\mathcal{S}, \\ \phi(y,T) = \sup_{x>0}(\widehat{U}(x) - xy). \end{cases} \tag{22}$$

Define the concave dual

$$\widehat{\phi}(x,t) = \inf_{y>0}(\phi(y,t) + xy), \quad x>0.$$

In particular, since $\widehat{U}$ is concave, we have $\widehat{\phi}(x,T) = \widehat{U}(x)$. Because ϕ is convex in y, we have the dual relation

$$\phi(y,t) = \sup_{x>0}(\widehat{\phi}(x,t) - xy), \quad y>0.$$

By Assumption 1, we have

$$\phi(y,T) = \sup_{x>0}(\widehat{U}(x) - xy) \leqslant \sup_{x>0}(Lx^p - xy) \ll y^{p/(p-1)},$$

and thus

$$\phi(y,t) = \int_{\mathbb{R}} \phi(ye^{-(r+\frac{\alpha^2}{2})(T-t)+\alpha z\sqrt{T-t}}, T)\,\mathrm{d}z \ll y^{p/(p-1)}.$$

It follows that

$$\widehat{\phi}(x,t) = \inf_{y>0}(\phi(y,t)+xy) \ll \inf_{y>0}(y^{p/(p-1)} + xy) \ll x^p,$$

which implies

$$\widehat{\phi}(0,t) = 0.$$

By definition and the convexity of ϕ,

$$\widehat{\phi}(-\phi_y(y,t),t) = \phi(y,t) - y\phi_y(y,t).$$

From this one can deduce from (22) that $\widehat{\phi}\in C^{2,1}(\mathcal{S})$ satisfies

$$\widehat{\phi}_t + \sup_{\pi\in\mathcal{C}}\left(\frac{1}{2}\|\pi^T\sigma\|^2 x^2\widehat{\phi}_{xx} + \pi^T\mu x\widehat{\phi}_x\right) + rx\widehat{\phi}_x = 0, \quad (x,t)\in\mathcal{S}.$$

We conclude that $\widehat{\phi}\in C^{2,1}(\mathcal{S})$ satisfies the HJB equation (20) with boundary conditions $\widehat{\phi}(0,t)=0$ and $\widehat{\phi}(x,T)=\widehat{U}(x)$. By the uniqueness of the solution, we conclude that $\widehat{V}\equiv\widehat{\phi}\in C^{2,1}(\mathcal{S})$. ■

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