Available online at www.sciencedirect.com

ScienceDirect

Transportation Research Procedia 23 (2017) 922–941

Transportation Research Procedia

www.elsevier.com/locate/procedia

22nd International Symposium on Transportation and Traffic Theory

Robust Perimeter Control for Two Urban Regions with Macroscopic Fundamental Diagrams: A Control-Lyapunov Function Approach

R.X. Zhong [a], C. Chen [a], Y.P. Huang [a], A. Sumalee [b], W.H.K. Lam [b], D.B. Xu [c,*]

[a] *Guangdong Key Laboratory of Intelligent Transportation Systems, School of Engineering, Sun Yat-Sen University, Guangzhou, China.*
[b] *Department of Civil and Environmental Engineering, The Hong Kong Polytechnic University, Hong Kong SAR, China.*
[c] *School of Automation, Nanjing University of Science and Technology, Nanjing, China.*

Abstract

The Macroscopic Fundamental Diagram (MFD) framework has been widely utilized to describe traffic dynamics in urban networks as well as to design perimeter flow control strategies under stationary (constant) demand and deterministic settings. In real world, both the MFD and demand however suffer from various intrinsic uncertainties while travel demand is of time-varying nature. Hence, *robust control* for traffic networks with uncertain MFDs and demand is much appealing and of greater interest in practice. In literature, there would be a lack of robust control strategies for the problem. One major hurdle is of requirement on model linearization that is actually a basis of most existing results.

The main objective of this paper is to explore a new robust perimeter control framework for dynamic traffic networks with parameter uncertainty (on the MFD) and exogenous disturbance induced by travel demand. The disturbance in question is in general time-varying and stochastic. Our main contribution focuses on developing a control-Lyapunov function (CLF) based approach to establishing a couple of universal control laws, one is almost smooth and the other is Bang-bang like, for different implementation scenarios. Moreover, it is indicated that the almost smooth control is more suited for road pricing while the Bang-bang like is for signal timing. In sharp contrast to existing methods, in which adjusting extensive design parameters are usually needed, the proposed methods can determine the control in an automatic manner. Furthermore, numerical results demonstrate that the control can drive the system dynamics towards a desired equilibrium under various scenarios with uncertain MFDs and travel demand. Both stability and robustness can be substantially observed. As a major consequence, the proposed methods achieve not only global asymptotic stability but also appealing robustness for the closed-loop traffic system.

Peer-review under responsibility of the scientific committee of the 22nd International Symposium on Transportation and Traffic Theory.

Keywords: Macroscopic fundamental diagram; demand and supply uncertainty; robust perimeter control; control-Lyapunov function; universal controller.

1. Introduction

Severe pollution and congestion during peak periods, induced by traffic, have been recognized as serious challenges for metropolises around the world. Efficient utilization of existing infrastructures through various forms of

* Corresponding author. Tel:+86 25 8431 5872.
E-mail address: xu.dabo@gmail.com

Peer-review under responsibility of the scientific committee of the 22nd International Symposium on Transportation and Traffic Theory.
10.1016/j.trpro.2017.05.051

traffic control schemes is a crucial ingredient towards sustainable urban mobility in terms of efficiency, safety and environment-friendly.

Despite a number of achievements in the field of urban traffic signal control, most methods are uncoordinated w.r.t. optimizing the goals of efficiency and equity for multiple heterogeneous networks. For example, widely deployed real-time signal control systems such as SCOOT (Hunt et al., 1982) and SCATS (Lowrie, 1982) are much less efficient under oversaturated traffic conditions with long queues and spillbacks. To tackle congested traffic conditions, some recent methods employ computationally intensive optimization algorithms that unfortunately obstruct their real-time feasibility (see e.g. Aboudolas and Geroliminis, 2013; Keyvan-Ekbatani et al., 2015a, for a discussion). A noteworthy practicable method is the feedback control strategy TUC (Diakaki et al., 2002, 2003; Aboudolas et al., 2009, 2010; Kouvelas et al., 2011). As a gating or metering control, the TUC, by means of limiting access to the link when operating near its capacity, can realize the protection of downstream links from over-saturation as well as minimization of spillback risk of link queues. For network-wide access control, Zhong et al. (2011, 2012) considered both congestion and vehicle emission metrics under the umbrella of dynamic traffic assignment with side constraints wherein the link traffic volume is restricted to less or equal to its capacity.

Note that the aforementioned gating and access controls are implemented at the link level while Gayah et al. (2014) showed that the locally (link-level) adaptive traffic signals are ineffective at preventing premature gridlock from occurring due to downstream congestion and queue spillbacks in an extremely congested network. Hence, under heavily saturated traffic conditions, other control strategies should be pursued to mitigate the instability. Remarkably, the MFD, depicting a reproducible relationship between the network accumulation and the trip completion rate under certain conditions (e.g. homogeneous spatial distribution of congestion), has gained a major development and a high impact in this direction. This notion, pioneered by Godfrey (1969), has been evidenced by simulation-based experiments (Gartner and Wagner, 2004) and real data based investigations (Geroliminis and Daganzo, 2008). Daganzo (2007); Daganzo and Geroliminis (2008); Helbing (2009); Gayah and Daganzo (2011) further considered some analytic aspects. Other interesting empirical and simulation studies have verified the existence of well-defined MFDs for homogeneous urban traffic networks (Geroliminis and Daganzo, 2008; Daganzo et al., 2011; Ji et al., 2010; Feng, 2011). Through the lens of the MFD, urban traffic dynamics can be modeled by traffic conservation law in conjunction with MFDs for a heterogeneous network as long as it can be partitioned into several homogenous subregions (Geroliminis et al., 2013; Haddad and Geroliminis, 2012; Aboudolas and Geroliminis, 2013; Haddad et al., 2013; Haddad, 2015). The MFD provides an analytically simple and computationally efficient framework for aggregate modeling of urban traffic network dynamics (which is denoted as MFD framework hereafter) as details of individual links are not necessary to describe the congestion level and its evolution.

Daganzo (2007) applied the MFD framework to devise a control rule that maximizes the network trip completion rate. Thenceforth, the idea of perimeter control on the borders of urban regions has attracted considerable interest. Recent studies showed that the feedback-based gating/perimeter control is efficient in mitigating congestion in urban networks by exploiting such a framework. For a short survey, some relevant studies are summarized in Table 1. Most previous studies in this direction for urban traffic networks aim at seeking suitable supply (e.g. signal timing) or demand (e.g., road pricing) management strategies for maximizing the network total trip completion rate under several restrict assumptions, e.g., exact knowledge on the network from both demand and supply sides whilst linearizing the MFD-based traffic dynamics. However, the MFD and travel demand are subject to various uncertainties which would render the flow evolution of practical urban networks unstable or even induce gridlocks. For instance, inherent errors in MFDs may result from asymmetric origin and destination (OD) and route choices as well as the calibration of MFDs (Geroliminis and Sun, 2011; Knoop et al., 2012), while noises in the demand may result from recurrent variations of day-to-day travel demand or nonrecurrent events such as special events and traffic accidents (Geroliminis et al., 2013). Uncertainty in MFDs is usually regarded as parametric uncertainty (or model mis-specification) (Courbon and Leclercq, 2011; Leclercq and Geroliminis, 2013; Haddad and Shraiber, 2014), while travel demand uncertainty is usually classified as exogenous disturbance in dynamic traffic models (Zhong et al., 2014). Moreover, it is well known in control theory that optimality does not imply stability in general. Nevertheless, the efficiency of the strategies based on optimal control, e.g., the MPC, increasingly deteriorate with increasing disturbance prediction and model errors, which may be a serious impediment for real-world deployment. Regardless of their importance, few existing studies, as reviewed in Table 1, have focused on the robust control for urban traffic networks with either uncertain demand or uncertain MFDs in terms of perimeter control.

Table 1: Summary of relevant perimeter control design based on the MFD

	Objective	Methodology	MFD function	Uncertainty	Disturbance	Linearization	Boundary queue	Time delay
Aboudolas and Geroliminis (2013)	Set-point control	LQI[1] & LQR[2]	✓Linearization required	×	×	✓	×	×
Keyvan-Ekbatani et al. (2012)	Set-point control	PI[3]	✓Linearization required	✓(MFD)	✓	✓	✓	×
Geroliminis et al. (2013)	Maximize throughput	MPC[4] (solved by numerical NLP[5])	✓	✓(MFD)	✓	×	×	×
Haddad and Geroliminis (2012)	Stability characteristics & maximize throughput	Location of the eigenvalue	✓Linearization required	×	×	✓	×	×
Haddad et al. (2013)	Minimize network delay	MPC & on-ramp metering (ALINEA)	✓Linearization required	✓(MFD)	✓	✓	✓	✓
Haddad (2015)	Set-point control	LMI[6] & LP[7]	✓Linearization required	✓(MFD)	×	✓	×	×
Keyvan-Ekbatani et al. (2013)	Set-point control	PI	✓Linearization required	×	✓	✓	✓	×
Keyvan-Ekbatani et al. (2015a)	Set-point control	PI	✓Linearization required	×	✓	✓	✓	✓
Keyvan-Ekbatani et al. (2015b)	Set-point control	PI	✓Linearization required	×	×	✓	✓	×
Ramezani et al. (2015)	Maximize throughput	Hierarchical control (MPC & FHC)[8]	✓	×	×	×	×	×
Haddad (2017)	Maximize throughput with boundary queue constraint	Optimal control	✓	×	×	×	✓	×
Hajiahmadi et al. (2015)	Maximize throughput	MPC	✓	✓(MFD)	✓	✓	×	×
Haddad and Shraiber (2014)	Set-point control	R-PI[9] & S-PI[10]	✓Linearization required	✓(MFD)	×	✓	×	×
Haddad and Mirkin (2016)	Set-point control	Transfer function	✓Linearization required	✓(MFD)	✓	✓	×	✓
Kouvelas et al. (2017)	Set-point control	LQR & LQI	×	✓	✓	✓	✓	×

[1] Linear-Quadratic-Integral
[2] Linear-Quadratic-Regulator
[3] Proportional Integral
[4] Model Predictive Control
[5] Nonlinear Program
[6] Linear Matrix Inequalities
[7] Linear Program
[8] Feedback Homogeneity Controller
[9] Robust Proportional Integral
[10] Standard Proportional Integral

To tackle the aforementioned challenges, this paper explores the robust perimeter control for dynamic traffic networks with inherent parametric uncertainties and exogenous disturbances. Distinguished from previous linearization based methods, we develop an effective CLF based approach to establish universal (explicit formula for obtaining the feedback controller from the system dynamics and the CLF chosen) robust controllers that make closed-loop traffic systems asymptotically stable in spite of uncertainties. Our results are substantially applicable to most of MFD based robust perimeter control problems. This new approach gives rise to two classes of controllers, i.e., almost smooth and Bang-bang like controls, constructed for different implementation scenarios. In particular, one is more suited for road pricing while the other is for signal timing. The proposed methods achieve not only global asymptotic stability but also appealing robustness for the closed-loop traffic system.

Paper Organization. Section 2 introduces the dynamic system representation of two urban regions under the MFD framework. Section 3 addresses the main results of designing control laws for both nominal system and system with demand and supply uncertainties. Section 4 gives several numerical examples and their implications for demonstration. Finally, Section 5 closes the paper with a few concluding remarks. For more materials, we shall refer the interested reader to the online appendix at https://www.dropbox.com/s/nmj2hrti2aywfhe/Online%20Appendix.pdf?dl=0. For a better grasp of this work, we shall refer to Figure 1 for exposition of the core notions and theorems. At the end of this section, we use Table 2 below to collect key variables throughout this paper.

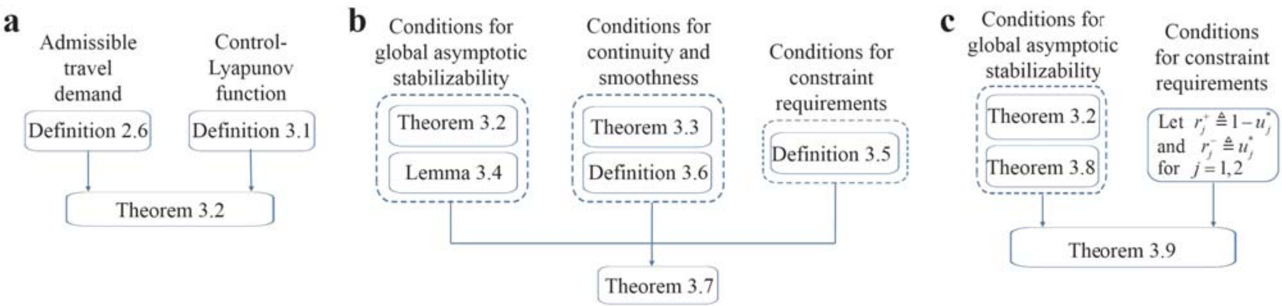


Figure 1: Gallery of this study

Table 2: List of key notation

Symbol	*Meaning*
$n_{ij}(t)$	Number of vehicles in R_i with destination to R_j at time t for $i, j = 1, 2$
$n_i(t)$	Accumulation or total number of vehicles in R_i at time t, and $n_i = \sum_j n_{ij}(t)$ for $i, j = 1, 2$
$q_{ii}(t)$	Endogenous travel demand defined as a flow in which its origin and destination are the same region for $i = 1, 2$
$q_{ij}(t)$	Exogenous travel demand defined as a flow in which its origin and destination are not the same region for $i, j = 1, 2$ and $i \neq j$
$u_i(t)$	Perimeter controllers controlling the ratio of the transfer flow that transfer from one region to the other at time t for $i = 1, 2$
$G_i(n_i(t))$	MFD that maps the network accumulation $n_i(t)$ to trip completion rate for region i at time t for $i = 1, 2$

2. Traffic dynamics for two urban regions with MFD

2.1. Two-region MFD framework

We consider an urban traffic network that can be partitioned into 2 homogeneous sub-networks with each region admits a well-defined MFD in line with Haddad and Geroliminis (2012); Geroliminis et al. (2013). An endogenous travel demand is defined as a flow whose origin and destination are in the same region. As a contrast, an exogenous travel demand is with the OD located in different regions. For the two-region system, there are two endogenous travel demands in R_1, denoted by $q_{11}(t)$ (veh/s), and in R_2, denoted by $q_{22}(t)$ (veh/s), and two exogenous travel demands generated in R_1 and R_2 with destination to R_2 and R_1, denoted by $q_{12}(t)$ (veh/s) and $q_{21}(t)$ (veh/s), respectively. Corresponding to the endogenous and exogenous travel demands, four accumulation states are used to model the network dynamics, $n_{ij}(t)$ (veh) for $i, j = 1, 2$, where $n_{ij}(t)$ is the number of vehicles (or uncompleted trips) in R_i with destination to R_j at time t. Let us denote $n_i(t)$ (veh) as the accumulation or the total number of vehicles in R_i at time t, i.e., $n_i(t) = \sum_j n_{ij}(t)$. The perimeter controls, denoted by $u_1(t)$ and $u_2(t)$, are introduced on the border between the two regions whose purpose is to control the cross boundary flows such that the network state of the two-region MFDs system can be regulated to the desired traffic equilibrium. The MFD $G_i(n_i(t))$ (veh/s) is defined as function that maps the network state $n_i(t)$ of region i to its trip completion rate at time t. The transfer flow from i with destination to j is calculated corresponding to the ratio between the numbers of uncompleted trips with different destinations, i.e., $M_{ij}(t) = \frac{n_{ij}(t)}{n_i(t)} G_i(n_i(t))$ for $i \neq j$, whereas the internal flow from i with destination to i is calculated by $M_{ii}(t) = \frac{n_{ii}(t)}{n_i(t)} G_i(n_i(t))$. These relationships assume that the trip lengths for all trips within a region (internal or external) are similar, i.e., the distance traveled per vehicle inside a region is independent of the OD.

Let $u_{\min}$ and $u_{\max}$ be the lower and upper bounds for $u_1(t)$ and $u_2(t)$, respectively. The two-region MFDs dynamics is formulated following the flow conservation law in line with Geroliminis et al. (2013):

$$\frac{dn_{11}(t)}{dt} = -\frac{n_{11}(t)}{n_1(t)} G_1(n_1(t) + \frac{n_{21}(t)}{n_2(t)} G_2(n_2(t)) u_2(t) + q_{11}(t) \tag{1a}$$

$$\frac{dn_{12}(t)}{dt} = -\frac{n_{12}(t)}{n_1(t)} G_1(n_1(t) u_1(t) + q_{12}(t) \tag{1b}$$

$$\frac{dn_{21}(t)}{dt} = -\frac{n_{21}(t)}{n_2(t)} G_2(n_2(t) u_2(t) + q_{21}(t) \tag{1c}$$

$$\frac{dn_{22}(t)}{dt} = -\frac{n_{22}(t)}{n_2(t)} G_2(n_2(t) + \frac{n_{12}(t)}{n_1(t)} G_1(n_1(t) u_1(t) + q_{22}(t) \tag{1d}$$

subject to

$$0 \leq n_{11}(t) + n_{12}(t) \leq n_1^{jam} \tag{2a}$$

$$0 \leq n_{21}(t) + n_{22}(t) \leq n_2^{jam} \tag{2b}$$

$$0 \leq u_{\min} \leq u_1(t) \leq u_{\max} \leq 1 \tag{2c}$$

$$0 \leq u_{\min} \leq u_2(t) \leq u_{\max} \leq 1 \tag{2d}$$

$$n_1(t) = n_{11}(t) + n_{12}(t), \quad n_2(t) = n_{21}(t) + n_{22}(t). \tag{2e}$$

2.2. Feasible and admissible travel demand

A central goal of traffic science is to characterize the travel demand and the performance of road infrastructure through some functional forms of appropriate macroscopic variables, e.g., the fundamental diagram (FD) which expresses flow as a function of density and is regarded as supply function in the literature, and the demand-to-supply reaction law for propagating the traffic flow while describing the congestion level. Furthermore, as revealed by the behavior analysis of the cell transmission model, the demand would also affect the equilibrium of the system as well as the convergence and stability properties of the system. Similar to the fundamental diagram for a roadway, the MFD defines the supply function of a homogeneous traffic network, e.g., network capacity (or maximum throughput). However, in the MFD literature, the demand-to-supply reaction law is rarely discussed. Moreover, the dependency between the travel demand and the network behavior (e.g. existence of equilibrium and its convergence and stability properties) is not well studied in the literature. Nevertheless, how would the travel demand affects the controllability of the network is never clear. To fill this gap, Zhong et al. (2016) conducted a behavior analysis on the MFD system in light of Gomes et al. (2008). As travel demand affects the equilibrium behavior and the controllability of the network, we summarize the keys points in this section.

Definition 2.1. (Feasible travel demand) For the two-region MFD system (1a)-(1d) with unimodal right-skewed shapes, a travel demand $q(t) = [q_1(t), q_2(t)]^T$ is said to be *feasible* if it satisfies the following conditions:

$$G_1(\bar{n}_1) + G_2(\bar{n}_2) \geq \bar{q}_1 + \bar{q}_2 \text{ and } n_i^0 + \int_0^T q_i(t)dt \leq \int_0^T G_i^{\max} dt, \quad i = 1, 2$$

where n_1^0 and n_2^0 denote the initial state of each region, respectively; $G_1^{\max}$ and $G_2^{\max}$ denote the maximal trip completion flow rate of each region, respectively; and $\bar{q}$ and $\bar{n}$ denote the steady-state of the demand and the corresponding accumulation, respectively (see Figure 2).

Remark 2.2. These inequalities imply that the demand and network initial accumulation should not exceed the maximum network throughput. From a system theory prospective, the energy of the input signal to a dynamical system should be bounded (otherwise, the signal cannot physically exist) or absolutely integrable, see e.g., Khalil (2002). Since the (non-negative) demand patterns are exogenous inputs to the MFD system, this requirement implies

$$\int_0^\infty q_1(t)dt < \infty, \int_0^\infty q_2(t)dt < \infty. \tag{3}$$

Noting that $q(t)$ is quantified by flow rate, the physical meaning of this constraint is that the travel demand should be finite. It seems that the simplest case—a constant demand q (flow rate) would violate the above inequalities. However, a constant demand q could not exist forever but in a finite time interval. Therefore, the restriction on demand (3) is general. Although the control design methods depicted in this paper do not explicitly require this finite demand property, let us keep in mind that physical quantities should be of finite values.

Given a desired traffic equilibrium, for a single-region MFD system, Haddad and Shraiber (2014) derived a sufficient condition[1] for controllability w.r.t. travel demand during the transient period, i.e., the time period wherein the initial state is being brought to the steady state. In order to increase or decrease the state to the steady-state state, the condition $\frac{dn_i}{dt} > 0$ or $\frac{dn_i}{dt} < 0$ should be satisfied, respectively. Introducing time-varying parameters $\upsilon_{ii}(t) = \frac{n_{ii}(t)}{n_i(t)}$ and $\upsilon_{ij}(t) = \frac{n_{ij}(t)}{n_i(t)}$, denoting the desired steady traffic state (target equilibrium) by $\bar{n}$, one can easily obtain the following proposition for a sufficient condition for the stabilizability[2] of the two-region MFD system as follows:

Proposition 2.3. Given the target equilibrium $\bar{n} = [\bar{n}_1, \bar{n}_2]^T$ of the two-region MFD system described by (1a)-(1d), provided for the transient period, if the travel demand $q = [q_1, q_2]^T$ satisfies, for $i, j = 1, 2$,

$$q_i < G_i(n_i) - \upsilon_{ji}G_j(n_j)u_j - \upsilon_{ij}G_i(n_i)(1 - u_i), \text{ if } n_i > \bar{n}_i \tag{4a}$$

$$q_i > G_i(n_i) - \upsilon_{ji}G_j(n_j)u_j - \upsilon_{ij}G_i(n_i)(1 - u_i), \text{ if } n_i < \bar{n}_i \tag{4b}$$

then the MFD system is stabilizable.

Proof of Proposition 2.3. See the online Appendix. □

A steady-state network outflow would result in two steady-state network flows $\bar{n}^s$ and $\bar{n}^u$, see e.g., Figure 2. Similar to the cell transmission model (CTM) case by Gomes et al. (2008), it is proven in Zhong et al. (2016) that $\bar{n}^s$ is a stable equilibrium while $\bar{n}^u$ is an unstable saddle point (see e.g., Haddad and Geroliminis, 2012). On the other hand, $\bar{n}^s$ in the non-congested region of the MFD is more likely to be a desired steady state for the sake of urban mobility, e.g., the ramp metering control always pursue such an equilibrium. The objectives of existing operational pricing strategies are similar. Zheng et al. (2012); Zheng and Geroliminis (2013); Zheng et al. (2016) claimed that with pricing the system should be operated at a state (e.g., $\bar{n}^s$) less than but close to the critical accumulation that maximizes outflow from the viewpoint of total travel time saving. Following these observations and *Proposition* 2.3, we can define the admissible travel demand in accordance to the MFD which governs the demand that can be received by the network as follows.

[1] Haddad and Shraiber (2014) stated this as a necessary condition. However, the condition $\frac{dn_i}{dt} > 0$ or $\frac{dn_i}{dt} < 0$ is sufficient to increase or decrease the initial state to the steady-state state at any time instant.

[2] Roughly speaking, controllability describes the ability of an external input (or the vector of control variables) to move the internal state of a dynamic system from any initial state to any other final state in a finite time interval. It does not mean that once a state is reached, that state can be maintained while stabilizability implies the system under control is stable and thus the desired state can be maintained.

Definition 2.4. (Strictly admissible travel demand) Given the two-region MFD system (1a)-(1d) with unimodal right-skewed shapes and a feasible demand, the *strictly admissible* travel demand $\tilde{q}_i$ for Region i for $i = 1, 2$, is

$$\tilde{q}_i = \begin{cases} \text{mid}\{q_i, G_i(n_i) - \upsilon_{ji}G_j(n_j)u_j - \upsilon_{ij}G_i(n_i)(1-u_i) + \epsilon_i, G_i(n_i)\}, & \text{if } 0 < n_i < \bar{n}_i^s \\ \min\{q_i, G_i(n_i) - \upsilon_{ji}G_j(n_j)u_j - \upsilon_{ij}G_i(n_i)(1-u_i)\}, & \text{if } \bar{n}_i^s \leq n_i < \bar{n}_i^u \\ \min\{q_i, G_i(n_i) - \upsilon_{ji}G_j(n_j)u_j - \upsilon_{ij}G_i(n_i)(1-u_i) - \epsilon_i\}, & \text{if } \bar{n}_i^u \leq n_i < n_i^{jam} \end{cases} \quad (5)$$

where $\epsilon_i \in \mathbb{R}_+$ is an arbitrary small number, "mid" denotes the middle point of the three points. Moreover, it is assumed that an admissible demand can yield a steady state for the network, i.e., the Equations (A.1) admit at least a non-negative solution.

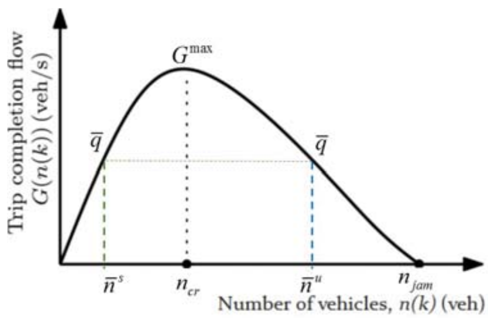


Figure 2: Relationship between steady-state demand and corresponding network flow

Remark 2.5. The definitions of feasible demand and strictly admissible demand are proposed in light of Gomes et al. (2008) for the CTM. Physical interpretation for the admissible demand can be outlined as follows. Traffic network governed by the MFD system is of limited capacity, it cannot receive too many vehicles at a time but a limited number according to the functional form of the MFD. The first equation of (5) implies (4b) while it is saturated by the throughput of the network $G_i(n_i)$. The second and third equations state that the inflow to the network should be the minimum of the demand and the available space. The constant $\epsilon > 0$ is used to guarantee the strict feasibility of (4a) and (4b). Note that if the travel demand fulfills the properties in *Definition* 2.4, the urban traffic network could be well protected from gridlock for which lots of control strategies might break down plus the MFD would not be well-defined (Mahmassani et al., 2013). Hereafter, $\tilde{q}$ denotes the *admissible* travel demand per *Definition* 2.4.

Note that in the first equation of the above definition, when $n_i < \bar{n}_i^s$, the *admissible* travel demand loaded into the network would be $G_i(n_i) - \upsilon_{ji}G_j(n_j)u_j - \upsilon_{ij}G_i(n_i)(1-u_i) + \epsilon_i$ if $q_i < G_i(n_i) - \upsilon_{ji}G_j(n_j)u_j - \upsilon_{ij}G_i(n_i)(1-u_i) + \epsilon_i$. That is the actual demand is too small to render the system to reach the desired equilibrium. So the "demand" equals to the available space rather than the actual demand is loaded into the network. Intuitively, it is of no physical meaning. For instance, when the demand is constant, loading a value larger than the demand is meaningless. On the other hand, this requirement was stemmed from the unreasonable choice of the desired equilibrium $\bar{n}_i^s$, i.e., setting $\bar{n}_i^s$ larger than the steady state that the demand pattern yields. This is not necessary and without practical consideration because the control objective is to make the network more dense than it should be without any control. It is a waste of network capacity and control effect while causing more delay to the network. As a similar case, the ramp metering, Hegyi et al. (2005) claimed that ramp metering is only useful when traffic is not too light (otherwise ramp metering is not needed) and not too dense (otherwise breakdown will happen anyway). Therefore, perimeter control is not necessary when the traffic is too light. For mathematical completeness, we use this condition.

2.3. Formulation of feedback control problem

We are now ready to formulate the main feedback control problem undertaken in this paper.

Control Goal/Problem: Given an admissible travel demand, if possible, devise a control input $u(t)$ such that the traffic dynamics, from arbitrary initial state within the feasible region, will evolve to a desired equilibrium.

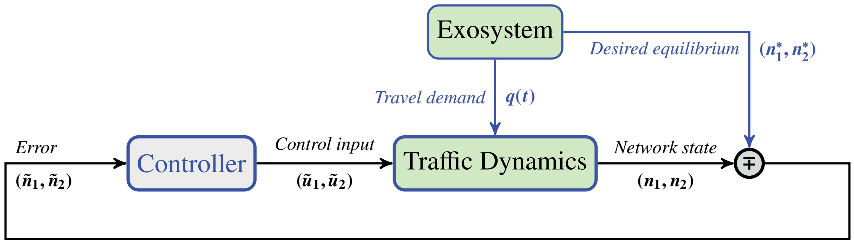


Figure 3: Control architecture illustration

Remark 2.6. Note that if the steady state n^* is chosen to be the critical value on the MFD, the control input will also maximize the total number of vehicles that complete their trips, i.e., maximizing the throughput. It is of interest to note that, as it will be proven in Subsection 3.2, through the lens of CLF, the two-region MFD framework can be globally stabilized even when heavily traffic congestion happens for all *strictly admissible* travel demand.

Remark 2.7. To the best of our knowledge, based on the same MFD framework, most existing researches have been dedicated to the local stabilization of the traffic dynamics or locally optimal control based on MPC. Noteworthily, Haddad and Geroliminis (2012) calculated the region of attraction boundaries for different levels of constant controls and travel demands, implying the non-stabilizability of the traffic system (or the network gridlock) under the condition of heavy congestion with high travel demand. Haddad and Geroliminis (2012); Haddad (2015) invited future investigation to this problem. Regarding the demand as exogenous input to the MFD system, the non-stabilizability for heavily congested situations might not be an inherent nature of the MFD system but due to the network loads in unreasonable large inflow (or demand). As it will be shown later, the MFD system with strictly admissible demand per *Definition* 2.4 implies that the global stabilizability of the two-region network dynamics even under heavily congested conditions.

Discriminating the internal factors from the exogenous ones is important to access the controllability of a dynamic system. To this end, we illustrate the architecture for closed-loop controller design in terms of block diagram as depicted in Figure 3. The exosystem, which can be regarded as the source for the urban network, generates the travel demand (usually regarded as disturbance to the network traffic dynamics) as well as the desired equilibrium (possibly specified by the traffic manager). The strictly admissible travel demand is input to the traffic network which influences the traffic state n (*veh*). The actual traffic state is compared with its reference (i.e., the desired equilibrium) n^* (*veh*) to evaluate the error vector. It is then used as feedback signal to drive the controller. Finally, a control law is implemented to enforce the actual traffic state closer to the desired equilibrium. This control process continues over time till the desired steady traffic state is reached.

3. CLF-based feedback control regulators

In this section, two forms of universal perimeter feedback controllers, i.e., the almost smooth controller and the Bang-bang like controller, are constructed through the lens of CLF to regulate the cross-boundary traffic such that the network states of the two regions may evolve towards the desired equilibrium. To achieve that, we first convert this control (or regulation) problem into a conventional stabilization problem, i.e., the controller will force the system state governed by (1a)-(1d) to the origin.

3.1. Converting the control problem to a stabilization problem

The control objective is to find a feasible control input $u(t)$ such that the system state $n(t)$ converges asymptotically to the desired steady state n^* as $t \to \infty$. Given a desired network equilibrium, the steady state n^* can be solved from the steady-state equations (A.1). Since the state governing internal and cross-boundary traffic is virtual (or not accessible) while noting that only network traffic states n_1 and n_2 are physically well defined, a simplification of the original MFD dynamics (1a)-(1d) by aggregating the virtual state can simplify the analysis (Haddad and Shraiber, 2014; Haddad, 2015).

Let $n_i(t)$ for $i = 1, 2$ denote the total number of vehicles in R_i at time t. By introducing time-varying parameters $\upsilon_{ii}(t) = \frac{n_{ii}(t)}{n_i(t)}$ and $\upsilon_{ij}(t) = \frac{n_{ij}(t)}{n_i(t)}$ and summing (1a), (1b) and (1c), (1d) for $i, j = 1, 2$, respectively, we obtain

$$\frac{dn_1(t)}{dt} = -\upsilon_{11}(t)G_1(n_1(t)) - \upsilon_{12}(t)G_1(n_1(t))u_1(t) + \upsilon_{21}(t)G_2(n_2(t))u_2(t) + \tilde{q}_1(t) \quad (6a)$$

$$\frac{dn_2(t)}{dt} = -\upsilon_{22}(t)G_2(n_2(t)) - \upsilon_{21}(t)G_2(n_2(t))u_2(t) + \upsilon_{12}(t)G_1(n_1(t))u_1(t) + \tilde{q}_2(t) \quad (6b)$$

where

$$n_1(t) = n_{11}(t) + n_{12}(t), \quad n_2(t) = n_{21}(t) + n_{22}(t) \quad (7a)$$

$$\tilde{q}_1(t) = \tilde{q}_{11}(t) + \tilde{q}_{12}(t), \quad \tilde{q}_2(t) = \tilde{q}_{21}(t) + \tilde{q}_{22}(t). \quad (7b)$$

As reviewed in the online Appendix, it is a common practice in the control literature to investigate equilibrium behavior at the origin. For system with control, it is well known that, given a dynamic system $\dot{n} = f(n, u)$ and a pair $\bar{n} \in \mathbb{R}^n$ and $u^* \in \mathbb{R}^m$ such that $0 = f(\bar{n}, u^*)$, then a coordinate transformation is possible by introducing new variables $\tilde{n}(t) = n(t) - \bar{n}$ and $\tilde{u}(t) = u(t) - u^*$, thus achieving a new system:

$$\dot{\tilde{n}}(t) = F(\tilde{n}, \tilde{q}) + S(\tilde{n}, t)\tilde{u} \quad (8)$$

subject to

$$0 - \bar{n}_1 \leq \tilde{n}_1(t) \leq n_1^{jam} - \bar{n}_1$$
$$0 - \bar{n}_2 \leq \tilde{n}_2(t) \leq n_2^{jam} - \bar{n}_2$$
$$0 - u_1^* \leq u_{\min} - u_1^* \leq \tilde{u}_1(t) \leq u_{\max} - u_1^* \leq 1 - u_1^*$$
$$0 - u_2^* \leq u_{\min} - u_2^* \leq \tilde{u}_2(t) \leq u_{\max} - u_2^* \leq 1 - u_2^*$$

where $\tilde{n}(t) = [\tilde{n}_1(t), \tilde{n}_2(t)]^T$, $\tilde{u}(t) = [\tilde{u}_1(t), \tilde{u}_2(t)]^T$. $F : \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}_+^2 \to \mathbb{R}^2$ and $S : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ are at least continuously differentiable and of appropriate dimensions, described by

$$F \triangleq \begin{bmatrix} -\upsilon_{11}G_1(\tilde{n}_1 + \bar{n}_1) - \upsilon_{12}G_1(\tilde{n}_1 + \bar{n}_1)\bar{u}_1 + \upsilon_{21}G_2(\tilde{n}_2 + \bar{n}_2)\bar{u}_2 + \tilde{q}_1 \\ -\upsilon_{22}G_2(\tilde{n}_2 + \bar{n}_2) - \upsilon_{21}G_2(\tilde{n}_2 + \bar{n}_2)\bar{u}_2 + \upsilon_{12}G_1(\tilde{n}_1 + \bar{n}_1)\bar{u}_1 + \tilde{q}_2 \end{bmatrix}$$

$$S \triangleq \begin{bmatrix} -\upsilon_{12}G_1(\tilde{n}_1 + \bar{n}_1) & \upsilon_{21}G_2(\tilde{n}_2 + \bar{n}_2) \\ \upsilon_{12}G_1(\tilde{n}_1 + \bar{n}_1) & -\upsilon_{21}G_2(\tilde{n}_2 + \bar{n}_2) \end{bmatrix}.$$

Details on the derivation of this coordinate transformation is presented in online Appendix. At this moment, the problem is to devise a control such that the origin of (8) can be made asymptotically stable. The network traffic state and the control are restricted into some compact sets, i.e., $n(t) \in \mathcal{N}$ and $u(t) \in \mathcal{U}$, where $\mathcal{N}$ and $\mathcal{U}$ are the universal sets of $n(t)$ and $u(t)$, respectively. As a consensus in the literature, controller design for nonlinear systems subject to input constraints induces both practical significance and theoretical challenges.

3.2. A quadratic CLF

In this section we will construct a CLF for the global stabilization problem with both state and control constraints. Consider a class of affine nonlinear time-varying system of the form (8) with $F(\tilde{n}, q)$ and $S(\tilde{n})$ continuously differentiable. Let $V'(\tilde{n})$ denote the partial derivative of $V(\tilde{n})$ w.r.t. $\tilde{n}$. Define

Definition 3.1. (CLF, see Haddad and Chellaboina, 2008, pp. 438) Consider nonlinear system (8) with control input constraints $\tilde{u} \in \mathcal{K} \subseteq \mathbb{R}^2$. A continuously differentiable positive-definite function $V : \mathcal{D} \to \mathbb{R}_+$ satisfying

$$\inf_{\tilde{u}\in\mathcal{K}} V'(\tilde{n})(F + S\tilde{u}) < 0, \quad \forall \tilde{n} \neq 0, \quad \tilde{n} \in \mathcal{D} \subseteq \mathbb{R}^2 \quad (9)$$

is called a *CLF*.

It follows from *Lyapunov's Theorem* and *Converse Lyapunov's Theorem* (see Haddad and Chellaboina, 2008, chap. 3) that system (8) is feedback asymptotically stabilizable if and only if there exists a CLF. Moreover, in the case when $\mathcal{D} = \mathbb{R}^2$ and $\mathcal{K} = \mathbb{R}^2$, $\tilde{n} \equiv 0$ is globally asymptotically stabilizable if and only if $V(\tilde{n})$ is *radially unbounded*, i.e., $V(\tilde{n}) \to \infty$ as $\|\tilde{n}\| \to \infty$. The interested readers are referred to the online *Appendix* for a brief introduction of Lyapunov theory. According to the characteristics of the traffic dynamics, a quadratic function $V(\tilde{n}) = \frac{1}{2}\tilde{n}_1^2 + \frac{1}{2}\tilde{n}_2^2$ could be a CLF candidate that is verifiable by the following theorem.

Theorem 3.2. *The function*

$$V(\tilde{n}) = \frac{1}{2}\tilde{n}_1^2 + \frac{1}{2}\tilde{n}_2^2$$

is a CLF for system (8) with strictly admissible travel demand per Definition 2.4.

Proof of Theorem 3.2. See the online Appendix. □

With the CLF given in Theorem 3.2, our goal is to construct universal control laws that stabilize the nonlinear system (8). For systems with k control input where k is a known positive constant, such universal (closed-form) control laws can be obtained through *Sontag's formula* (see Sontag, 1998). However, one of the main difficulties in establishing controllers through the lens of CLFs lies in the fact that the control is restricted to a specified control value set (CVS) $\mathcal{U}$ satisfying (2c) and (2d). As for the stabilization problem subject to bounded control, Lin and Sontag (1991) obtained a universal formula for the case when the CVS is the open unit ball in Euclidean k-space, whereas recently Leyva et al. (2013) extended that result to the case when the CVS is given by the hyperbox $\mathcal{U} \triangleq [-r_1^-, r_1^+] \times \cdots \times [-r_k^-, r_k^+] \subset \mathbb{R}^k$, with $r_j^\pm > 0$ for $j = 1, \ldots, k$.

3.3. Almost smooth controller

For the two-region MFD dynamic traffic system with two state variables and two control variables, note that (8) is of the affine in control form, the stability is now referred to the nominal equilibrium $\tilde{n} = 0$. With input constraints where the CVS $\mathcal{U}$ is the unit ball of $\mathbb{R}^k$, we could construct an explicit feedback controller that asymptotically stabilizes the zero solution of the shifted system (8) as follows:

$$\mu(t) = \Phi(\tilde{n}(t)) = \phi(\alpha(\tilde{n}),\ \|\beta(\tilde{n})\|^2)\beta(\tilde{n}) \tag{10}$$

where

$$\phi(\alpha(\tilde{n}), \|\beta(\tilde{n})\|^2) \triangleq \begin{cases} -\frac{\alpha(\tilde{n})+\sqrt{\alpha^2(\tilde{n})+(\beta^T(\tilde{n})\beta(\tilde{n}))^2}}{\beta^T(\tilde{n})\beta(\tilde{n})(1+\sqrt{1+\beta^T(\tilde{n})\beta(\tilde{n})})}, & \text{if } \beta(\tilde{n}) \neq 0 \\ 0, & \text{if } \beta(\tilde{n}) = 0 \end{cases} \tag{11}$$

$V(\tilde{n})$ is a CLF for system (8), $\alpha(\tilde{n}) \triangleq V'(\tilde{n})F(\tilde{n}, q)$, $\beta(\tilde{n}) = [\beta_1(\tilde{n}), \ldots, \beta_k(\tilde{n})] \triangleq S^T(\tilde{n})V'^T(\tilde{n})$. Since $F(\cdot, q)$ and $S(\cdot)$ are smooth, it follows that $\alpha(\tilde{n})$ and $\beta(\tilde{n}), \tilde{n} \in \mathbb{R}^2$, are smooth functions, and hence, $\Phi(\tilde{n}(t))$ given by (11) is smooth for all $\tilde{n} \in \mathbb{R}^2$ if either $\beta(\tilde{n}) \neq 0$ or $\alpha(\tilde{n}) < 0$. Hence, the feedback control law given by (11) is smooth everywhere except at the origin. And the following result, concerned with an additional so-called small control property (SCP)[3], provides necessary and sufficient conditions under which the feedback control law given by (11) is guaranteed to be continuous and Lipschitz continuous at the origin in addition to being smooth everywhere else.

Theorem 3.3. *(see Haddad and Chellaboina, 2008) Consider the nonlinear dynamical system given by (8) with a radially unbounded CLF $V(\tilde{n})$. Then both the following statements are true.*

1. *The control law $\Phi(\tilde{n}(t))$ given by (11) is continuous at $\tilde{n} = 0$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 < \|\tilde{n}\| < \delta$, there exists $\mu \in \mathbb{R}^k$ such that $\|\tilde{u}\| < \varepsilon$ and $\alpha(\tilde{n}) + \beta^T(\tilde{n})\tilde{u} < 0$.*
2. *There exists a stabilizing control law $\hat{\Phi}(\tilde{n}(t))$ such that $\alpha(\tilde{n}) + \beta^T(\tilde{n})\hat{\Phi}(\tilde{n}(t)) < 0,\ \tilde{n} \in \mathbb{R}^m,\ \tilde{n} \neq 0$, and $\hat{\Phi}(\tilde{n}(t))$ is Lipschitz continuous at $\tilde{n} = 0$ if and only if the control law $\Phi(\tilde{n}(t))$ given by (11) is Lipschitz continuous at $\tilde{n} = 0$.*

[3] To ensure feedback control functions continuous at the origin, Artstein (1983) introduced the concept of *small control property*. Readers are referred to Artstein (1983) or the online *Appendix* for its definition

For presentation convenience, define $a \triangleq \alpha(\tilde{n})$, $b \triangleq \beta^T(\tilde{n})\beta(\tilde{n})$, and

$$\phi(a,b) \triangleq \begin{cases} -\frac{a+\sqrt{a^2+b^2}}{b(1+\sqrt{1+b})}, & \text{if } b > 0 \\ 0, & \text{if } b = 0 \end{cases}$$

Since $V(\tilde{n}) = \frac{1}{2}(\tilde{n}_1^2+\tilde{n}_2^2)$ is a CLF for system (8), for any subset $\mathcal{U} \subseteq \mathbb{R}^k$, there exists some $u \in \mathcal{U}$ such that $a+\langle\beta,\tilde{u}\rangle < 0$, i.e., there exists a set of feasible control such that

$$\mathfrak{D}_{\mathcal{U}} \triangleq \{(a,\beta) \in \mathbb{R} \times \mathbb{R}^2 \mid \exists u \in \mathcal{U}, a + \langle\beta,\tilde{u}\rangle < 0\},$$

where $\langle\beta,\tilde{u}\rangle = \beta^T\tilde{u}$. In particular, for the case of scalar-valued controls, and denoting $b = \beta_1$, the resulting bounded feedback law is $\Phi(\tilde{n}) = \varphi(a(\tilde{n}), b(\tilde{n}))$ where

$$\varphi(a,b) \triangleq \begin{cases} -\frac{a+\sqrt{a^2+b^4}}{b(1+\sqrt{1+b^2})}, & \text{if } b \neq 0 \\ 0, & \text{if } b = 0 \end{cases} \tag{12}$$

and the feasible control forms the following set:

$$\mathfrak{D}_{\mathcal{U}} \triangleq \{(a,b) \mid a < |b|, a, b \in \mathbb{R}\}.$$

The following lemma is useful in the subsequent development.

Lemma 3.4. (see Lin and Sontag, 1991) Assume the real numbers a, b, δ satisfy $a < \delta|b|$ and $0 < \delta \leq 1$. Then, with φ defined by (12), it holds that

$$|\varphi(a,b)| < \min\{2\delta + |b|, 1\}.$$

For our case, the set in question is

$$\mathfrak{D}_{\mathcal{U}} \triangleq \{(a,b) \mid a < \min\{1-u_1^*, u_1^*\}|b|, a, b \in \mathbb{R}\}.$$

Note that constraints (2c)-(2d) are imposed on the controls, a modification to the control formula (11) given by Lin and Sontag (1991) is necessary. To begin with, we first define a saturation function and the meaning of almost smooth.

Definition 3.5. (Saturation function) Given the vector bound $-r^- \leq r \leq r^+$, the concerned saturation function is element-wise defined as

$$[\text{sat}(r)]_j \triangleq \begin{cases} r_j^+, & \text{if } r_j > r_j^+ \\ r_j, & \text{if } r_j^- \leq r_j \leq r_j^+ \\ -r_j^-, & \text{if } r_j < -r_j^- \end{cases}$$

Definition 3.6. (Almost smooth) The function $\text{sat}(r) \triangleq [[\text{sat}(r)]_1, [\text{sat}(r)]_2]^T$ is called almost smooth if, it is smooth almost everywhere but finite limit points such as the origin and the inflection points of the saturation function whereas the function is continuous at these points.

Theorem 3.7. *Denote the steady-state control as $u^* = [u_1^*, u_2^*]^T$ and let $\mu(t)$ be defined by (10) and (11), $r_j^- \triangleq u_j^*$, $r_j^+ \triangleq 1 - u_j^*$, and*

$$u(t) = \tilde{u}(t) + u^* \quad \text{with} \quad \tilde{u}(t) \triangleq \text{sat}(\mu(t)) = \text{sat}(\Phi(\tilde{n})).$$

Then $\tilde{u}(t)$ is an explicit almost smooth feedback controller that globally asymptotically stabilizes the system (8) (at the origin) with strictly admissible travel demand per Definition 2.4. Therefore, $u(t)$ is an explicit almost smooth feedback controller that regulates the original traffic dynamics to the desired equilibrium asymptotically meanwhile satisfying $0 \leq u_1(t) \leq 1$ and $0 \leq u_2(t) \leq 1$.

Proof of Theorem 3.7. See the online Appendix. □

This almost smooth control is potentially applicable for calculating the price for dynamic road pricing, a discussion on this can be found in Zhong et al. (2016).

3.4. Bang-bang like controller for signal-timing schemes

Traffic signals are commonly equipped for gating control in many cities such as Guangzhou, to our familiarity. Considerable interest in literature has been devoted to devising gating control schemes through traffic signals. In respect of its specific nature, traffic signal is basically non-smooth similar to Bang-bang like control. Recently, such control schemes have been commonly used for hierarchical control in urban networks, e.g., Daganzo (2007); Haddad et al. (2013); Geroliminis et al. (2013); Ramezani et al. (2015).

To adapt to the nature of traffic signal, we thus present a more realistic Bang-bang like control scheme in what follows. The result is built on the CLF developed by Leyva et al. (2013) which aims to globally stabilize the underlying nonlinear system (8) with control restricted in a hyperbox set given by $\mathcal{U} \triangleq [-r_1^-, r_1^+] \times \cdots \times [-r_k^-, r_k^+] \subset \mathbb{R}^k$, with $r_j^\pm > 0$ for $j = 1, \ldots, k$, in terms of Bang-bang like control strategies.

For stabilizing (8) at the origin, we propose an ε-parameterized ($\varepsilon > 0$) family of feedback control functions:

$$\tilde{u}^\varepsilon(\tilde{n}) \triangleq [\tilde{u}_1^\varepsilon(\tilde{n}), \tilde{u}_2^\varepsilon(\tilde{n})]^T \ \text{ with } \ \tilde{u}_j^\varepsilon(\tilde{n}) = \varrho_j^\varepsilon(\alpha(\tilde{n}), \eta(\tilde{n}))\bar{\omega}_j(\tilde{n}) \tag{13}$$

where

$$\varrho_j^\varepsilon(\alpha(\tilde{n}), \eta(\tilde{n})) = \begin{cases} 1 - \left(1 - \frac{|\alpha|+\alpha}{2\eta}\frac{\eta_j}{\eta}\right)\exp\left(\tau_j^\varepsilon \frac{\eta_j}{\eta}\right), & \text{if } \eta_j > 0 \\ 0, & \text{if } \eta_j = 0 \end{cases}, \quad \tau_j^\varepsilon(\tilde{n}) = \begin{cases} k\frac{\ln(\lambda(\tilde{n}))}{\lambda(\tilde{n})} - \varepsilon\eta_j, & \text{if } \eta > 0 \\ 0, & \text{if } \eta = 0 \end{cases}, \quad j = 1, 2$$

with $\lambda(\tilde{n}) = 1 - \frac{1}{2}(|\alpha(\tilde{n})| + \alpha(\tilde{n}))/\eta(\tilde{n})$, $\eta(\tilde{n}) = \sum_j \eta_j(\tilde{n})$, and

$$\eta_j(\tilde{n}) \triangleq |-\beta_j|r_j = \begin{cases} |-\beta_j|r_j^+, & \text{if } \beta_j < 0 \\ -|-\beta_j|r_j^-, & \text{if } \beta_j \geq 0 \end{cases}, \quad \bar{\omega}_j(\tilde{n}) \triangleq r_j(\beta_j) = \begin{cases} r_j^+, & \text{if } \beta_j < 0 \\ -r_j^-, & \text{if } \beta_j \geq 0 \end{cases}, \quad \beta_j(\tilde{n}) = S_j(\tilde{n})V_j'(\tilde{n}) \in \mathbb{R}.$$

Theorem 3.8. *(see Leyva et al., 2013) Assume that the* CVS *is the hyperbox $\mathcal{U} = [-r_1^-, r_1^+] \times \cdots \times [-r_k^-, r_k^+]$ with $r_j^\pm > 0$, $V(\tilde{n})$ is a* CLF *(w.r.t. system (8) and controls taking values in $\mathcal{U}$) satisfying the* SCP*. Then $\tilde{u}^\varepsilon(\tilde{n})$ given by (13) is a formula for an ε-family ($\varepsilon > 0$) control functions that renders system (8) globally asymptotically stable.*

Theorem 3.9. *Denote the steady-state control as $u^* = [u_1^*, u_2^*]^T$, $r_j^+ \triangleq 1 - u_j^*$ and $-r_j^- \triangleq -u_j^*$ for $j = 1, 2$, then*

$$u^\varepsilon(t) = \tilde{u}^\varepsilon(t) + u^*$$

is an explicit Bang-bang like feedback controller that drives the traffic dynamics (with strictly admissible travel demand per Definition 2.4) asymptotically to the desired equilibrium. Meanwhile, the following restrictions $0 \leq u_1^\varepsilon(t) \leq 1, 0 \leq u_2^\varepsilon(t) \leq 1$ are fulfilled.

Proof of Theorem 3.9. See the online Appendix. □

Note that the control $\bar{\omega}_j(\tilde{n})$ is piecewise constant and hence it is a Bang-bang type of control which defines the signal phase. The function $\varrho_j^\varepsilon(\tilde{n})$ is a rescaling function used to regularize each entry $\bar{\omega}_j(\tilde{n})$. For example, consider a gating site, to be of six lanes with each lane equipped with a signal. Then $\varrho_j^\varepsilon(\tilde{n}) = 0.5$ and $\bar{\omega}_j(\tilde{n}) = 1$ for some time means that three lanes can be served for cross boundary traffic.

3.5. Robustness

In this section we generalize the preceding results to systems subject to disturbances. As claimed, the uncertain travel demand is usually regarded as disturbance in traffic systems. Regarding the nominal system (8), the demand pattern $\tilde{q}$ is viewed as "disturbances" perturbing the system. For convenience, we rewrite the dynamics with disturbance as

$$\dot{\tilde{n}} = F(\tilde{n}, \tilde{q}) + S(\tilde{n})\tilde{u}. \tag{14}$$

Assume that the disturbances are of measurable functions taking values in an arbitrary and fixed compact subset $\mathbb{D}$ of $\mathbb{R}^l$. Following Lin and Sontag (1995), we define $a(\tilde{n}, \tilde{q}) = V'(x)F(\tilde{n}, \tilde{q})$, and $\hat{a}(\tilde{n}) = \max_{\tilde{q} \in \mathbb{D}}\{a(\tilde{n}, \tilde{q})\}$.

Definition 3.10. (Uniform CLF) A proper and positive definite smooth function $V(x)$ is said a uniform CLF w.r.t. the CVS $\mathcal{K}$ if the following inequality holds.

$$\inf_{\tilde{u}\in\mathcal{K}} \{a(\tilde{n},\tilde{q}) + \beta(\tilde{n})\tilde{u}\} < 0, \quad \forall\tilde{n} \neq 0, \quad \forall\tilde{q} \in \mathbb{D} \tag{15}$$

The function V is said to satisfy the uniform SCP if for any $\epsilon > 0$, there exists a δ such that if $\tilde{n} \neq 0$ satisfies $|\tilde{n}| < \delta$, then there is some $\tilde{u} \in \mathcal{K}$ with $|\tilde{u}| < \epsilon$ such that $a(\tilde{n},\tilde{q}) + \beta(\tilde{n})\tilde{u} < 0$, for all $\tilde{q}$.

By virtue of compactness of $\mathbb{D}$, (15) implies that

$$\sup_{\tilde{u}\in\mathcal{K}} \{\hat{a}(\tilde{n}) + \beta(\tilde{n})\tilde{u}\} < 0, \quad \forall\tilde{n} \neq 0.$$

Lemma 3.11. (see Lin and Sontag, 1995) Let V be a uniform CLF for system (14) satisfying the uniform SCP with the CVS $\tilde{u} \in \mathcal{K}$. Then the control feedback law $\tilde{u}$ defined in *Theorem* 3.7, but with a replaced by $\hat{a}$, is almost smooth, continuous everywhere, and robustly stabilizes the system at the origin.

The same routine can be applied to the Bang-bang like control.

To show the closed-loop system has the robust stability margin (RSM) property, consider the following system

$$\dot{x} = f(x) + \Delta f$$

where $\|\Delta f\|_\infty = \{x \in \mathbb{R}^n : \sup \|\Delta f(x)\|\}$ is bounded as the additive unmodeled uncertainty. The RSM γ is defined as the size of the smallest destabilizing perturbation, i.e., $\gamma = \inf \|\Delta f(x)\|_\infty$ such that the system is not globally asymptotically stable at the origin. A method for approximating the RSM is based on the existence of a Lyapunov function $V(x)$ for the nominal system $\dot{x} = f(x)$ such that $\dot{V}(x) < 0$ for all $x \neq 0$ where we have assumed the origin is the equilibrium of the system.

Given an affine system (14), suppose that there exists a CLF $V(\tilde{n})$ for the hyperbox CVS as previously discussed. Given a feedback control $\tilde{u}(\tilde{n})$ satisfying $a(\tilde{n})-\beta(\tilde{n})\tilde{u}(\tilde{n}) < 0$, where $a(\tilde{n})$ and $\beta(\tilde{n})$ are the same with those in the above. The corresponding RSM is thus defined as

$$\gamma_V = \inf_{\tilde{n}\neq 0} \left\{ -\frac{a(\tilde{n}) - \beta(\tilde{n})\tilde{u}(\tilde{n})}{\|\nabla V(\tilde{n})\|} \right\}, \quad \tilde{n} \in \mathbb{R}^n.$$

It is easy to show that if $\|\Delta F\| \leq \|\gamma_V\|$ then the system

$$\dot{\tilde{n}} = F(\tilde{n},\tilde{q}) + \Delta F + S(\tilde{n})\tilde{u}$$

is robustly stable at the origin.

Remark 3.12. It is necessary that the MFD is well defined, otherwise the MFD-based dynamics is ill-posed. As indicated in the proofs of the main theorems on the controllability and the CLF-based controller designs, we do not rely on a specific functional form of the MFD. Indeed, we do allow parametric uncertainty in the MFD as we can infer from the robust controller design. The uncertainty structure can be arbitrary as long as the RSM property is well satisfied. This will be verified by several numerical simulations in the forthcoming section.

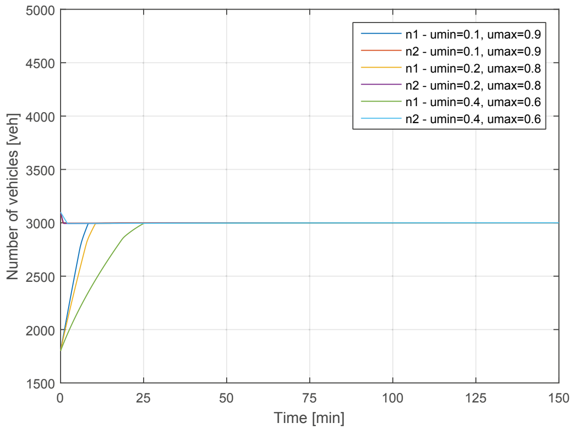
4. Simulations and discussions

4.1. Scenario I: Fixed demand case

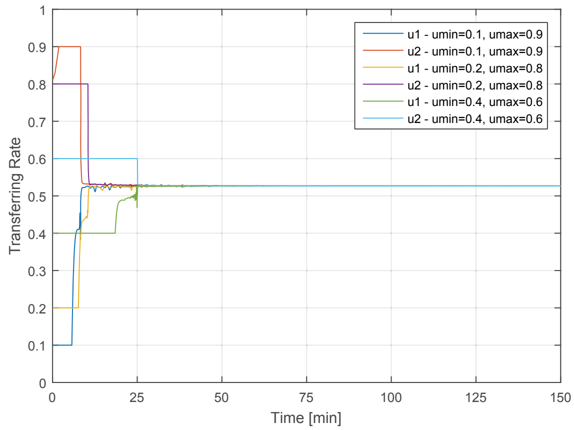
In this case, we examine the proposed control laws to the controller design scenarios of Example 1 and Example 3 presented in Haddad (2015). Note that the MFD shapes for both regions in Subsection 4.1 are assumed to be the same as those in Haddad (2015), i.e., $G_i(n_i) = (1.4877 \cdot 10^{-7} n_i^3 - 2.9815 \cdot 10^{-3} n_i^2 + 15.0912 n_i)/3600$, with $n_{i,cr} = 3400$ (veh), $G_i(n_{i,cr}) = 6.3$ (veh/s), and $n_{i,jam} = 10000$ (veh). The remaining settings are also in line with Example 1 and Example

3 respectively in Haddad (2015) while we obtain the four-state equilibrium n^* as well as the corresponding control input u^* by solving the steady-state equations.

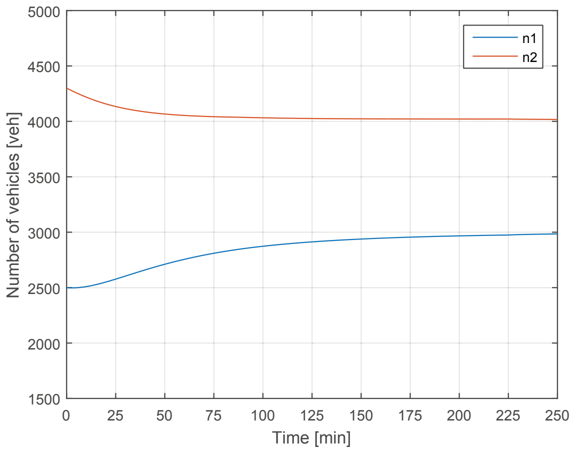
In the first example, we aim at designing a controller to regulate the traffic state to a prescribed equilibrium (i.e., the set points) $\bar{n} = [3000, 3000]^T$ in the uncongested regime. The four-state equilibrium $n^* = [1538.9, 1461.1, 1461.1, 1538.9]^T$ with the corresponding steady-state control input $u^* = [0.5267, 0.5267]^T$ are obtained by solving the steady-state equations. As expected, the simulation results in Figure 4 verify that the MFD-based dynamic system can be stabilized by both controllers which is consistent with the theoretical counterparts, i.e., *Theorem* 3.7 and *Theorem* 3.9, respectively. Figure 4(a) illustrate the effect of the lower and upper bounds of the perimeter control on the convergent rate of the dynamics. Although all the controls can regulate the traffic state to the desired equilibrium, it is found that the time to reach steady-state is longer if the gap between the lower and upper bounds is smaller. The proposed controller drives the traffic state to the set point (in 6 minutes) **in a much faster manner** than the control scheme in Haddad (2015) (which takes more than 116 minutes). From Figure 4(b), we can see that the almost smooth control converges asymptotically to its steady state soon after the state variable does.



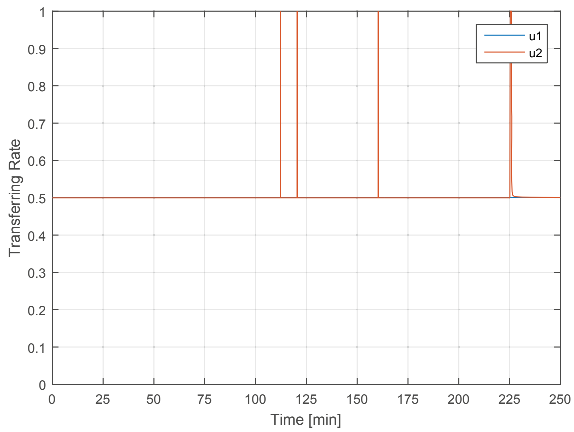
(a) State trajectories with almost smooth control



(b) Almost smooth control input over time



(c) State trajectories with Bang-bang like control



(d) Bang-bang like control input over time

Figure 4: State and control evolution over time for the constant demand case

In the second example, the set point of one region is in the uncongested regime while that of the other region is in the congested regime, i.e., $\bar{n} = [3000, 4000]^T$. The four-state equilibrium is $n^* = [1500.5, 1499.5, 2000.5, 1999.5]^T$ with the corresponding control input as $u^* = [0.5003, 0.4997]^T$. As reported in Haddad (2015), several existing methods, e.g., the MPC and LMI, have been tested to fail to deliver a feasible control law to this scenario. They encouraged future research effort such as nonlinear controller design to handle this challenge. In line with the literature, we apply Bang-bang like control to this scenario. Simulation results are depicted in Figure 4(c) and Figure 4(d). As shown in Figure 4(c), the control regulates the system trajectories asymptotically to the desired equilibrium $\bar{n}_1 = 3000$ (*veh*) and $\bar{n}_2 = 4000$ (*veh*), respectively. As expected, the control also asymptotically converges to its steady state. Since the initial accumulation of Region 1 is less than its set point, less control effort is needed as indicated by the figure. The control input sequence of u_2 has a very limited chattering behavior which indicates the amount of control effort is small. Both control input sequences during the steady-state period are constant. Due to its closed-form in conjunction with a simple CLF, the computational complexity of the proposed controllers is low. All the experiments reported in this section are performed on Matlab 2015a. It only takes a few seconds to accomplish the simulations. Unlike some existing methods that have number of parameters to be tuned carefully case-by-case and should be also tuned if the model parameters change in the future, e.g., the prediction horizon length of the MPC, **the proposed controllers are automatically designed** given the calibrated MFD network traffic model and demand pattern. These might be considered as the advantages of the proposed controllers.

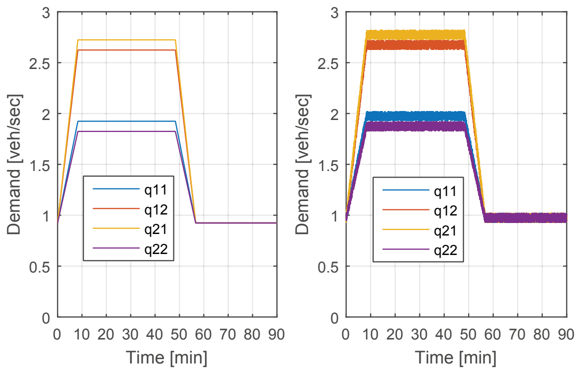
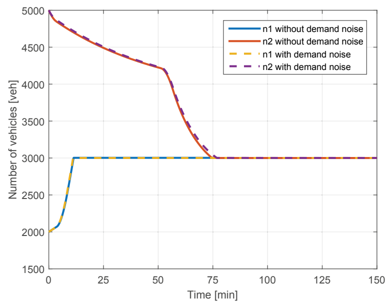


Figure 5: Time varying travel demand pattern (piecewise)

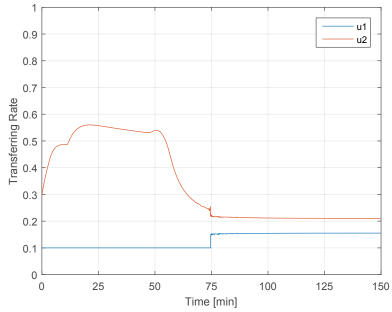
4.2. Scenario II: Piecewise travel demand case with non-identical third-order MFD

The piecewise time-varying demands shown in Figure 5 simulate a morning peak hour with high cross-boundary demand. The demand function is thus designed to excite congestion onset, stationary congestion and congestion dissolving processes such that the traffic dynamics can cover both congested and uncongested regions of the MFD. The MFDs for the two regions are different as shown in Figure 7(a). The desired accumulations are chosen as $\bar{n}_1 = 3000$ (*veh*) and $\bar{n}_2 = 3000$ (*veh*). The steady-state equations suggest that $\bar{q} = [0.924, 0.924, 0.924, 0.924]^T$, $n^* = [720, 2280, 2123, 877]^T$, $u^* = [0.1581, 0.2066]^T$. The initial accumulations are randomly chosen to be $[2000, 5000]^T$.

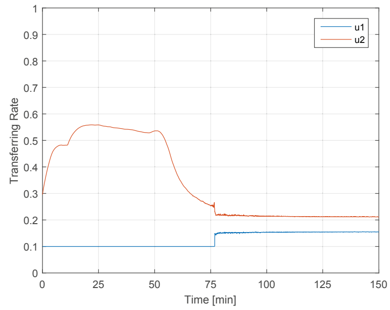
The simulation results are depicted in Figure 6. Both the almost smooth and the Bang-bang like controllers regulate the traffic state to the desired equilibrium asymptotically while the almost smooth controller delivers outperformance over the Bang-bang like controller in terms of convergent rate for certain and uncertain demand cases. If the lower bound of the control is zero and since the initial accumulation of Region 1 is less than the set point, the controller u_1 holds the cross boundary flow from Region 1 to Region 2 so as to increase the accumulation to the set point fast. Such a "behavior" seems the controller intends to create congestion in accordance to the objective described by the desired equilibrium which is discussed in *Remark* 2.5. One can change the lower bound of the control from zero to a prescribed value $u_{\min} > 0$ to prevent this from happening. Once the accumulation is higher than the desired equilibrium, the control is activated to suppress the traffic state from increasing. The controller u_2, on the other hand, acts in a different way to let go the cross boundary traffic first and decreases the gain gradually when the traffic state is approaching to the desired equilibrium. Specifically, the almost smooth control, as expected, shows a very small chattering behavior when the states are close to the set points. This is because the almost smooth control admits continuous but non-smooth points when the state is close to the desired equilibrium (or the control gain hits the saturation boundaries) as stated in Subsection 3.3. Similar phenomenon is observed for the Bang-bang like control



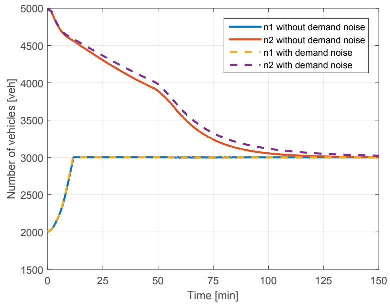
(a) State trajectories with almost smooth control



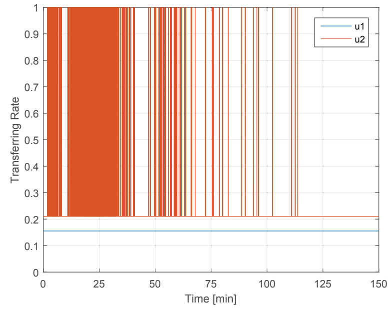
(b) Almost smooth control without disturbance



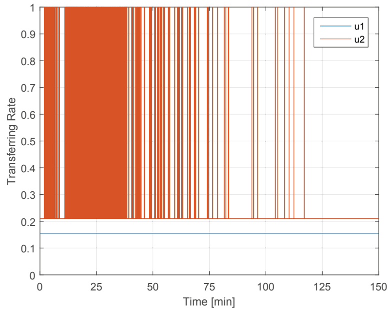
(c) Almost smooth control with disturbance



(d) State trajectories with Bang-bang like control



(e) Bang-bang like control without disturbance



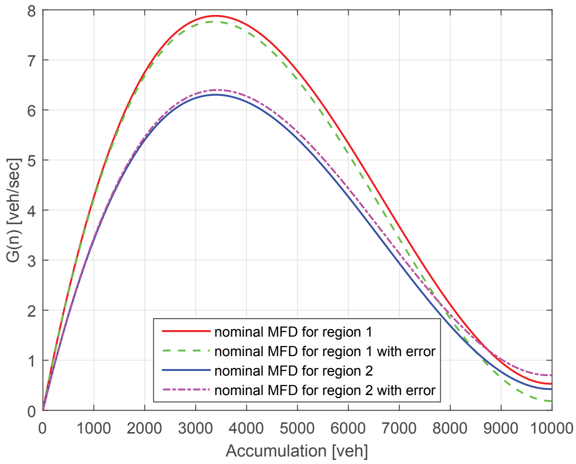
(f) Bang-bang like control with disturbance

Figure 6: Non-identical MFD case: with and without demand noise

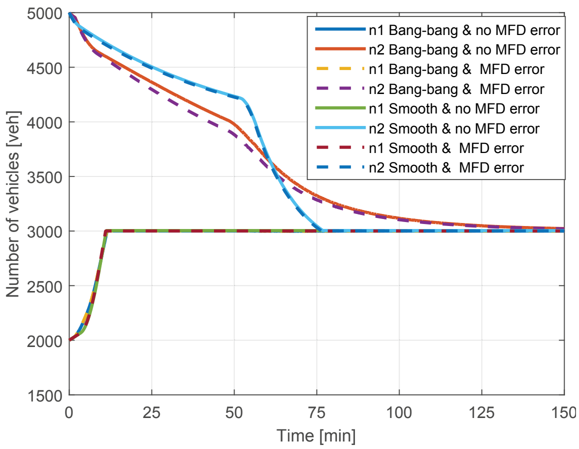
case. The Bang-bang like controller, shows a fast chattering behavior at the beginning and enters the steady-state in accordance with the state. In Guangzhou, such control is implemented by the lane based signal control. For example a gating site be of six lanes with each lane equipped with a signal, then $\varrho_j^{\varepsilon}(\tilde{n}) = 1/3$ and $\bar{\omega}_j(\tilde{n}) = 1$ for some time means two lanes can be served for cross boundary traffic. When the state is sufficiently close to the desired equilibrium (i.e., in the steady-state period), both the control input sequences are constant defined by steady-state inputs. This is the so-called SCP.

Next we assume the travel demand function is subject to a biased noise, i.e., where $\xi(t) = [\xi_{11}(t), \xi_{12}(t), \xi_{21}(t), \xi_{22}(t)]^T$ with $\xi_{ij}(t) \sim \mathbb{U}(0, 0.1)$ uniformly distributed. The almost smooth control and the Bang-bang like control are respectively applied to regulate the traffic dynamics to the desired equilibrium with a very small steady-state error caused by the biased noise in the demand pattern. The convergent rate under uncertain demand case is a bit slower than that under certain demand case for both controls. As we can see that both controllers behave very similar to the cases without disturbance except the chattering behavior continues even if the traffic states attain their steady states. This is because further control effort is required to suppress the fluctuation effect of disturbance on the traffic state so as to maintain the steady state. Such phenomenon was also observed in the literature, see e.g., Geroliminis et al. (2013). When there is no noise in the demand pattern, the SCP is well satisfied. When the biased noise presents, a small chattering is induced by the noise.

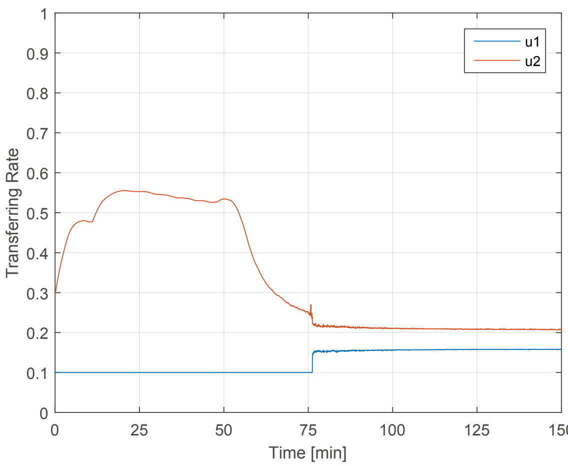
Finally, we study the effects of calibration errors of the MFD and the impacts of the noise in travel demands on the performance of the proposed schemes. The settings of this scenario are identical to that of the previous case but with additional MFD uncertainties introduced in the simulation model, see Figure 7(a). Observe from Figure 7(b) that the almost smooth controller outperforms the Bang-bang like control in terms of convergent rate. Both the almost smooth and Bang-bang like controls behave similar to the previous case. The simulation results depicted in Figure 7(b) indicate that the calibration errors of both regions do not significantly affect the state trajectory evolution when compared with the case without neither demand nor supply uncertainty case. This confirms that when the uncertainties are bounded by certain compact sets, the closed-loop system is robustly stable by the uniform CLF (regarding the demand uncertainty) and the RSM property (regarding the supply uncertainty).



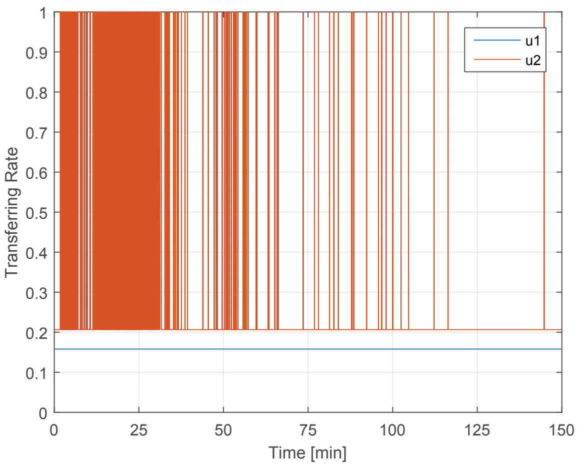
(a) Third-order MFD error



(b) State trajectories with demand noise: with and without MFD error



(c) Almost smooth control with MFD error and demand noise



(d) Bang-bang like control with MFD error and demand noise

Figure 7: Time-varying demand with both MFD error and noise case

5. Conclusion

The MFD based traffic network gating control problem in presence of inherent demand and MFD parametric uncertainties was investigated. Based on the feasible and admissible demand concepts, we have developed a CLF based approach to establish two universal control laws, both of which can globally stabilize the nonlinear traffic system. Then we showed that under certain assumptions on the demand and parameter uncertainties the closed loop system is robustly stable. Distinguished from most of existing studies, the proposed results never rely on linearization of the underlying traffic dynamics and constant demand assumption.

The almost smooth and Bang-bang like controls can be applied to different implementation scenarios, respectively. Specifically, the almost smooth control can be used for road pricing while the Bang-bang like control can be used for signal timing. Numerical results indicate that the control can drive the system dynamics towards the desired equilibrium under various uncertain MFDs and travel demand wherein the stability and robustness can still be guaranteed.

The future direction is as follows. On one hand, it is of greater interest to extend the present study to more general multi-region networks. On the other hand, the admissible demand suggests that for congested networks with exceeding travel demand, both demand and supply regulation schemes should be simultaneously implemented so as to better manage the network traffic.

Acknowledgements

Financial support from the National Natural Science Foundation of China under Grant No. 51308559 and No. 61673216, the Research Grants Council of the Hong Kong Special Administrative Region (under Grant No. PolyU 152074/14E) is gratefully acknowledged.

Appendix A.

This appendix presents the derivation of the steady-state values of n^*_{11}, n^*_{12}, n^*_{21}, n^*_{22}, u^*_1 and u^*_2 by extending the analysis in Haddad and Shraiber (2014); Luspay et al. (2015). During the steady-state period it holds that the time derivatives of the variables n_{11}, n_{12}, n_{21} and n_{22} are equal to zero, i.e.,

$$\begin{aligned} &0 = -\frac{n^*_{11}}{\bar{n}_1}G_1(\bar{n}_1) + \frac{n^*_{21}}{\bar{n}_2}G_2(\bar{n}_2)u^*_2 + \bar{q}_{11}, \quad 0 = -\frac{n^*_{12}}{\bar{n}_1}G_1(\bar{n}_1)u^*_1 + \bar{q}_{12}, \quad \bar{n}_1 = n^*_{11} + n^*_{12} \\ &0 = -\frac{n^*_{22}}{\bar{n}_2}G_2(\bar{n}_2) + \frac{n^*_{12}}{\bar{n}_1}G_1(\bar{n}_1)u^*_1 + \bar{q}_{22}, \quad 0 = -\frac{n^*_{21}}{\bar{n}_2}G_2(\bar{n}_2)u^*_2 + \bar{q}_{21}, \quad \bar{n}_2 = n^*_{21} + n^*_{22} \end{aligned} \tag{A.1}$$

Then by summation of suitable equations in (A.1), we have

$$0 = -\frac{n^*_{11}}{\bar{n}_1}G_1(\bar{n}_1) + \bar{q}_{21} + \bar{q}_{11}, \;\; 0 = -\frac{n^*_{22}}{\bar{n}_2}G_2(\bar{n}_2) + \bar{q}_{12} + \bar{q}_{22}. \tag{A.2}$$

By using $\bar{n}_i = n^*_{ii} + n^*_{ij}$ for $i, j = 1, 2$, we have

$$n^*_{11} = \frac{\bar{n}_1(\bar{q}_{11} + \bar{q}_{21})}{G_1(\bar{n}_1)}, \;\; n^*_{12} = \bar{n}_1\frac{G_1(\bar{n}_1) - (\bar{q}_{11} + \bar{q}_{21})}{G_1(\bar{n}_1)}, \;\; n^*_{21} = \bar{n}_2\frac{G_2(\bar{n}_2) - (\bar{q}_{12} + \bar{q}_{22})}{G_2(\bar{n}_2)}, \;\; n^*_{22} = \frac{\bar{n}_2(\bar{q}_{12} + \bar{q}_{22})}{G_2(\bar{n}_2)}. \tag{A.3}$$

Further from (A.1), we obtain

$$u^*_1 = \frac{\bar{q}_{12}\cdot\bar{n}_1}{n^*_{12}\cdot G_1(\bar{n}_1)} = \frac{\bar{q}_{12}}{G_1(\bar{n}_1) - \bar{q}_{21} - \bar{q}_{11}}, \;\; u^*_2 = \frac{\bar{q}_{21}\cdot\bar{n}_2}{n^*_{21}\cdot G_2(\bar{n}_2)} = \frac{\bar{q}_{21}}{G_2(\bar{n}_2) - \bar{q}_{12} - \bar{q}_{22}}.$$

References

Aboudolas, K., Geroliminis, N., 2013. Perimeter and boundary flow control in multi-reservoir heterogeneous networks. Transportation Research Part B 55, 265–281.

Aboudolas, K., Papageorgiou, M., Kosmatopoulos, E., 2009. Store-and-forward based methods for the signal control problem in large-scale congested urban road networks. Transportation Research Part C 17 (2), 163–174.

Aboudolas, K., Papageorgiou, M., Kouvelas, A., Kosmatopoulos, E., 2010. A rolling-horizon quadratic-programming approach to the signal control problem in large-scale congested urban road networks. Transportation Research Part C 18 (5), 680–694.

Artstein, Z., 1983. Stabilization with relaxed controls. Nonlinear Analysis: Theory, Methods & Applications 7 (11), 1163–1173.

Courbon, T., Leclercq, L., 2011. Cross-comparison of macroscopic fundamental diagram estimation methods. Procedia-Social and Behavioral Sciences 20, 417–426.

Daganzo, C. F., 2007. Urban gridlock: macroscopic modeling and mitigation approaches. Transportation Research Part B 41 (1), 49–62.

Daganzo, C. F., Gayah, V. V., Gonzales, E. J., 2011. Macroscopic relations of urban traffic variables: Bifurcations, multivaluedness and instability. Transportation Research Part B 45 (1), 278–288.

Daganzo, C. F., Geroliminis, N., 2008. An analytical approximation for the macroscopic fundamental diagram of urban traffic. Transportation Research Part B 42 (9), 771–781.

Diakaki, C., Dinopoulou, V., Aboudolas, K., Papageorgiou, M., Ben-Shabat, E., Seider, E., Leibov, A., 2003. Extensions and new applications of the traffic-responsive urban control strategy: Coordinated signal control for urban networks. Transportation Research Record (1856), 202–211.

Diakaki, C., Papageorgiou, M., Aboudolas, K., 2002. A multivariable regulator approach to traffic-responsive network-wide signal control. Control Engineering Practice 10 (2), 183–195.

Feng, G., 2011. Macroscopic fundamental diagrams for Stockholm using FCD data. Master's thesis, KTH, Transportvetenskap.

Gartner, N., Wagner, P., 2004. Analysis of traffic flow characteristics on signalized arterials. Transportation Research Record (1883), 94–100.

Gayah, V. V., Daganzo, C. F., 2011. Clockwise hysteresis loops in the macroscopic fundamental diagram: an effect of network instability. Transportation Research Part B 45 (4), 643–655.

Gayah, V. V., Gao, X. S., Nagle, A. S., 2014. On the impacts of locally adaptive signal control on urban network stability and the macroscopic fundamental diagram. Transportation Research Part B 70, 255–268.

Geroliminis, N., Daganzo, C. F., 2008. Existence of urban-scale macroscopic fundamental diagrams: Some experimental findings. Transportation Research Part B 42 (9), 759–770.

Geroliminis, N., Haddad, J., Ramezani, M., 2013. Optimal perimeter control for two urban regions with macroscopic fundamental diagrams: A model predictive approach. IEEE Transactions on Intelligent Transportation Systems 14 (1), 348–359.

Geroliminis, N., Sun, J., 2011. Hysteresis phenomena of a macroscopic fundamental diagram in freeway networks. Transportation Research Part A: Policy and Practice 45 (9), 966–979.

Godfrey, J. W., 1969. The mechanism of a road network. Traffic Engineering & Control 11, 323–327.

Gomes, G., Horowitz, R., Kurzhanskiy, A. A., Varaiya, P., Kwon, J., 2008. Behavior of the cell transmission model and effectiveness of ramp metering. Transportation Research Part C 16 (4), 485–513.

Haddad, J., 2015. Robust constrained control of uncertain macroscopic fundamental diagram networks. Transportation Research Part C 59, 323–339.

Haddad, J., 2017. Optimal perimeter control synthesis for two urban regions with aggregate boundary queue dynamics. Transportation Research Part B: Methodological 96, 1–25.

Haddad, J., Geroliminis, N., 2012. On the stability of traffic perimeter control in two-region urban cities. Transportation Research Part B 46 (9), 1159–1176.

Haddad, J., Mirkin, B., 2016. Adaptive perimeter traffic control of urban road networks based on mfd model with time delays. International Journal of Robust and Nonlinear Control.

Haddad, J., Ramezani, M., Geroliminis, N., 2013. Cooperative traffic control of a mixed network with two urban regions and a freeway. Transportation Research Part B 54, 17–36.

Haddad, J., Shraiber, A., 2014. Robust perimeter control design for an urban region. Transportation Research Part B 68, 315–332.

Haddad, W. M., Chellaboina, V., 2008. Nonlinear dynamical systems and control: a Lyapunov-based approach. Princeton University Press.

Hajiahmadi, M., Haddad, J., De Schutter, B., Geroliminis, N., 2015. Optimal hybrid perimeter and switching plans control for urban traffic networks. IEEE Transactions on Control Systems Technology 23 (2), 464–478.

Hegyi, A., De Schutter, B., Hellendoorn, H., 2005. Model predictive control for optimal coordination of ramp metering and variable speed limits. Transportation Research Part C: Emerging Technologies 13 (3), 185–209.

Helbing, D., 2009. Derivation of a fundamental diagram for urban traffic flow. The European Physical Journal B 70 (2), 229–241.

Hunt, P. B., Robertson, D. I., Bretherton, R. D., Royle, M. C., 1982. The SCOOT on-line traffic signal optimisation technique. Traffic Engineering & Control 23 (4), 190–192.

Ji, Y., Daamen, W., Hoogendoorn, S., Hoogendoorn-Lanser, S., Qian, X., 2010. Investigating the shape of the macroscopic fundamental diagram using simulation data. Transportation Research Record (2161), 40–48.

Keyvan-Ekbatani, M., Kouvelas, A., Papamichail, I., Papageorgiou, M., 2012. Exploiting the fundamental diagram of urban networks for feedback-based gating. Transportation Research Part B 46 (10), 1393–1403.

Keyvan-Ekbatani, M., Papageorgiou, M., Knoop, V. L., 2015a. Controller design for gating traffic control in presence of time-delay in urban road networks. Transportation Research Part C 59, 308–322.

Keyvan-Ekbatani, M., Papageorgiou, M., Papamichail, I., 2013. Urban congestion gating control based on reduced operational network fundamental diagrams. Transportation Research Part C 33, 74–87.

Keyvan-Ekbatani, M., Yildirimoglu, M., Geroliminis, N., Papageorgiou, M., 2015b. Multiple concentric gating traffic control in large-scale urban networks. Intelligent Transportation Systems IEEE Transactions on 16 (4), 2141–2154.

Khalil, H., 2002. Nonlinear Systems, Third ed. Prentice Hall, New Jersey.

Knoop, V., Hoogendoorn, S., Van Lint, J., 2012. Routing strategies based on macroscopic fundamental diagram. Transportation Research Record: Journal of the Transportation Research Board (2315), 1–10.

Kouvelas, A., Aboudolas, K., Papageorgiou, M., Kosmatopoulos, E. B., 2011. A hybrid strategy for real-time traffic signal control of urban road networks. IEEE Transactions on Intelligent Transportation Systems 12 (3), 884–894.

Kouvelas, A., Saeedmanesh, M., Geroliminis, N., 2017. Enhancing model-based feedback perimeter control with data-driven online adaptive optimization. Transportation Research Part B: Methodological 96, 26–45.

Leclercq, L., Geroliminis, N., 2013. Estimating MFDs in simple networks with route choice. Transportation Research Part B 57, 468–484.

Leyva, H., Solís-Daun, J., Suárez, R., 2013. Global CLF stabilization of systems with control inputs constrained to a hyperbox. SIAM Journal on Control and Optimization 51 (1), 745–766.

Lin, Y., Sontag, E. D., 1991. A universal formula for stabilization with bounded controls. Systems & Control Letters 16 (6), 393–397.

Lin, Y., Sontag, E. D., 1995. Control-Lyapunov universal formulas for restricted inputs. Control-Theory and Advanced Technology 10 (4).

Lowrie, P. R., 1982. The Sydney coordinated adaptive traffic system - principles, methodology, algorithms. In: the IEE International Conference on Road Traffic Signalling. pp. 67–70.

Luspay, T., Kulcsár, B., Péni, T., 2015. Set-theoretic analysis of the isolated ramp metering problem. International Journal of Robust & Nonlinear Control 26 (6).

Mahmassani, H. S., Saberi, M., Zockaie, A., 2013. Urban network gridlock: Theory, characteristics, and dynamics. Transportation Research Part C 36, 480–497.

Ramezani, M., Haddad, J., Geroliminis, N., 2015. Dynamics of heterogeneity in urban networks: aggregated traffic modeling and hierarchical control. Transportation Research Part B 74, 1–19.

Sontag, E. D., 1998. Mathematical Control Theory: Deterministic Finite Dimensional Systems, 2nd Edition. Vol. 6. Springer.

Zheng, N., Geroliminis, N., 2013. On the distribution of urban road space for multimodal congested networks. Transportation Research Part B: Methodological 57, 326–341.

Zheng, N., Rérat, G., Geroliminis, N., 2016. Time-dependent area-based pricing for multimodal systems with heterogeneous users in an agent-based environment. Transportation Research Part C: Emerging Technologies 62, 133–148.

Zheng, N., Waraich, R. A., Axhausen, K. W., Geroliminis, N., 2012. A dynamic cordon pricing scheme combining the macroscopic fundamental diagram and an agent-based traffic model. Transportation Research Part A: Policy & Practice 46 (8), 1291–1303.

Zhong, R., Huang, Y., Chen, C., Lam, W. H. K., Sumalee, A., 2016. Boundary conditions and behavior of the macroscopic fundamental diagram based network traffic dynamics: A control systems perspective. Preprint.

Zhong, R., Sumalee, A., Friesz, T., Lam, W. H. K., 2011. Dynamic user equilibrium with side constraints for a traffic network: theoretical development and numerical solution algorithm. Transportation Research Part B 45 (7), 1035–1061.

Zhong, R., Sumalee, A., Maruyama, T., 2012. Dynamic marginal cost, access control, and pollution charge: a comparison of bottleneck and whole link models. Journal of Advanced Transportation 46 (3), 191–221.

Zhong, R., Sumalee, A., Pan, T., Lam, W. H. K., 2014. Optimal and robust strategies for freeway traffic management under demand and supply uncertainties: an overview and general theory. Transportmetrica A: Transport Science 10 (10), 849–877.