A Mean-Field Formulation for Multi-Period Asset-Liability Mean-Variance Portfolio Selection with an Uncertain Exit Time*

Xiangyu Cui†, Xun Li‡, Xianping Wu§ and Lan Yi¶

Abstract

This paper is concerned with multi-period asset-liability mean-variance portfolio selection with an uncertain exit time. By employing the mean-field formulation to this problem involving two-dimensional state variables, we derive the analytical optimal strategy and its efficient frontier successfully. The corresponding sensitivity analysis and a real life example shed light on influences of liability and uncertain exit time to the optimal investment strategy.

Key Words: Mean-field formulation; multi-period portfolio selection; asset-liability management; uncertain exit time.

1 Introduction

Mean-variance portfolio selection refers to the design of optimal portfolios balancing between the gain and the risk, which are in expression of expectation and variance of the ternimal return, respectively. Since Markowitz published his seminal work [12] for a single-period setting, researches on dynamic mean-variance portfolio selection problems have been well developed. For instance, by an embedding technique, Li and Ng [9] extended Markowitz's model to multi-period setting and derived analytical optimal portfolio policy with its efficient frontier. Zhou and Li [19] studied continuous-time mean-variance portfolio problem using stochastic linear-quadratic control theory and the embedding technique. Fu et al. [4] investigated dynamic mean-variance portfolio selection with a borrowing constraint. Li et al. [10] and Cui et al. [2] investigated dynamic mean-variance portfolio selection with no-shorting constraint for continuous-time setting and multi-period setting, respectively.

In reality, liability plays an important role over an investment horizon. A financial institution taking liabilities into account can operate more soundly and lucratively. Thus, a judicious investor should consider assets and liabilities simultaneously. In this line of research, Sharpe and Tint [13] studied asset and liability management in a single period setting. Keel and Muller [5]

*This work was partially supported by National Natural Science Foundation of China under grants 71201094, 71272213 and 11571124, Shanghai Pujiang Program 12PJC051, China Postdoctoral Science Foundation 2016M592505, and by Research Grants Council of Hong Kong under grants 15209614, 15224215 and 15255416.

†School of Statistics and Management, Shanghai University of Finance and Economics, China. E-mail: cui.xiangyu@mail.shufe.edu.cn.

‡Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China. E-mail: malixun@polyu.edu.hk.

§Corresponding author. School of Mathematical Sciences, South China Normal University, Guangzhou, China. E-mail: pphappe@sina.com.

¶Management School, Jinan University, Guangzhou 510632, China. E-mail: tlyi@jnu.edu.cn.

investigated the portfolio selection with liabilities and concluded that the corresponding efficient frontier is affected by liabilities. Leippold et al. [7] used a geometric approach to analyze multi-period mean-variance optimization of asset and liability. Applying the stochastic optimal control theory, Chiu and Li [1] analytically solved the asset and liability management problem in a continuous-time setting. Li et al. [8] investigated time-consistent optimal investment strategy for asset and liability management.

Another important concern about an investment is the exit time. The investor realizes to never know exactly when he/she has to exit the market as the investment can be interrupted by some unexpected events. For example, the price movement of risky assets, securities markets behavior and exogenous huge consumption goes up or down when one purchases a house or faces an accident. Research of portfolio selection models on an uncertain exit time can be traced back to Yaari [14], who formulated an optimal consumption problem for an individual with uncertain date of death. Using the embedding technique, Yi et al. [17] studied a multi-period mean-variance portfolio selection problem on risky assets and liability with the uncertain exit time. Li and Xie [11] incorporated a market-related exogenous uncertain time horizon into a continuous-time optimal portfolio selection problem. Zhang and Li [18] proposed a multi-period mean-variance portfolio optimization model with the uncertain time horizon where its returns are serially correlated. Yao et al. [15] considered uncertain exit time multi-period mean-variance portfolio selection problems with endogenous liabilities in a Markov jump market.

In this paper, we aim to study multi-period asset-liability mean-variance portfolio selection with an uncertain exit time. The difficulties of this portfolio selection model are the non-separability induced by the variance term and the presence of liabilities. Due to the non-separability, dynamic programming cannot be applied directly to tackle this problem. The presence of endogenous liabilities enlarges dimension number of state variables which makes this non-separable model much more challenging. To overcome this fundamental difficulty induced by non-separability, many researchers developed some fascinating yet important methods such as the embedding technique proposed by Li and Ng [9], the parameterized method introduced by Li et al. [10], and the mean-field formulation presented by Cui et al. [3] and Yi et al. [16], etc. Compared to the embedding technique in Yi et al. [17], we are more interested in the mean-field formulation to tackle our problem involving two-dimensional state variables, and derive its analytical optimal strategy and efficient frontier. Furthermore, the corresponding sensitivity analysis and a real-life example are presented to help the investor better understand influences of liabilities and uncertain exit time.

The rest of the paper is organized as follows. In section 2, we present the mean-field formulation for the multi-period asset-liability mean-variance portfolio selection with an uncertain exit time. We derive strictly the optimal strategies and efficient frontiers for different cases in section 3. Section 4 provides sensitivity analysis and a real-life example. The final section is concluding remarks.

2 Formulation

Assume that an investor, joining the market at the beginning of period 0 with an initial wealth x_0 and initial liability l_0, plans to invest his/her wealth over an entire time horizon. He/she can reallocate his/her portfolio at the beginning of each following T consecutive periods. The capital market consists of one risk-free asset, n risky assets and one liability. At time period t, the given deterministic return of the risk-free asset, the random returns of the n risky assets, and the random return of the liability are denoted by s_t (> 1), vector $\mathbf{e}_t = (e_t^1, \cdots, e_t^n)'$

and q_t, respectively. The random vector $\mathbf{e}_t$ and the random variable q_t are defined over the probability space $(\Omega, \mathcal{F}, P)$ and are supposed to be statistically independent among different time periods. We further assume that the only information known about $\mathbf{e}_t$ and q_t are their first two unconditional moments, $\mathbb{E}[\mathbf{e}_t] = \left(\mathbb{E}[e_t^1], \cdots, \mathbb{E}[e_t^n]\right)'$, $\mathbb{E}[q_t]$ and $(n+1)\times(n+1)$ positive definite covariance

$$\mathrm{Cov}\left(\begin{pmatrix} \mathbf{e}_t \\ q_t \end{pmatrix}\right) = \mathbb{E}\left[\begin{pmatrix} \mathbf{e}_t\mathbf{e}_t' & \mathbf{e}_t q_t \\ q_t\mathbf{e}_t' & q_t^2 \end{pmatrix}\right] - \mathbb{E}\left[\begin{pmatrix} \mathbf{e}_t \\ q_t \end{pmatrix}\right]\mathbb{E}\left[\begin{pmatrix} \mathbf{e}_t' & q_t \end{pmatrix}\right] \succ 0,$$

where $A \succ 0$ to denote a positive definite matrix A and $\mathbf{v}'$ to denote the transpose of vector $\mathbf{v}$.

From the above assumptions, we have

$$\begin{pmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{e}_t'] & s_t\mathbb{E}[q_t] \\ s_t\mathbb{E}[\mathbf{e}_t] & \mathbb{E}[\mathbf{e}_t\mathbf{e}_t'] & \mathbb{E}[\mathbf{e}_t q_t] \\ s_t\mathbb{E}[q_t] & \mathbb{E}[q_t\mathbf{e}_t'] & \mathbb{E}[q_t^2] \end{pmatrix} \succ 0.$$

Define the excess return vector of risky assets as $\mathbf{P}_t = (P_t^1, \cdots, P_t^n)' = (e_t^1 - s_t, \cdots, e_t^n - s_t)'$. The following is then true for $t = 0, 1, \cdots, T-1$:

$$\begin{pmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{P}_t'] & s_t\mathbb{E}[q_t] \\ s_t\mathbb{E}[\mathbf{P}_t] & \mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] & \mathbb{E}[\mathbf{P}_t q_t] \\ s_t\mathbb{E}[q_t] & \mathbb{E}[q_t\mathbf{P}_t'] & \mathbb{E}[q_t^2] \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}' & 0 \\ -\mathbf{1} & I & \mathbf{0} \\ 0 & \mathbf{0}' & 1 \end{pmatrix}\begin{pmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{e}_t'] & s_t\mathbb{E}[q_t] \\ s_t\mathbb{E}[\mathbf{e}_t] & \mathbb{E}[\mathbf{e}_t\mathbf{e}_t'] & \mathbb{E}[\mathbf{e}_t q_t] \\ s_t\mathbb{E}[q_t] & \mathbb{E}[q_t\mathbf{e}_t'] & \mathbb{E}[q_t^2] \end{pmatrix}\begin{pmatrix} 1 & -\mathbf{1}' & 0 \\ \mathbf{0} & I & \mathbf{0} \\ 0 & \mathbf{0}' & 1 \end{pmatrix} \succ 0,$$

where $\mathbf{1}$ and $\mathbf{0}$ are the n-dimensional all-one and all-zero vectors, respectively, and I is the $n \times n$ identity matrix, which further implies,

$$\begin{aligned} \mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] \succ 0, &\quad t = 0, 1, \cdots, T-1, \\ s_t^2(1 - \mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]) > 0, &\quad t = 0, 1, \cdots, T-1, \\ \mathbb{E}[q_t^2] - \mathbb{E}[q_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t] > 0, &\quad t = 0, 1, \cdots, T-1. \end{aligned}$$

Denote

$$\begin{aligned} B_t &\triangleq \mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t], \\ \widehat{B}_t &\triangleq \mathbb{E}[q_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t], \\ \widetilde{B}_t &\triangleq \mathbb{E}[q_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t]. \end{aligned}$$

Thus, $0 \le B_t < 1,\ 0 \le \widetilde{B}_t < \mathbb{E}[q_t^2],\ t = 0, 1, \cdots, T-1$.

During the investment, the investor may be forced to leave the financial market at time τ on or before T by some uncontrollable reasons. The uncertain exit time τ is supposed to be an exogenous random variable with probability mass function $\tilde{p}_t = \Pr\{\tau = t\}, t = 1, 2, \cdots$. Therefore, the actual exit time of the investor is $T \wedge \tau = \min\{T, \tau\}$, and its probability mass function is

$$\alpha_t \triangleq \Pr\{T \wedge \tau = t\} = \begin{cases} 0, & t = 0, \\ \tilde{p}_t, & t = 1, 2, \cdots, T-1, \\ 1 - \sum_{j=1}^{T-1} \tilde{p}_j, & t = T. \end{cases}$$

Let x_t and l_t be the wealth and liability of the investor at the beginning of period t, respectively. Let $\pi_t^i,\ i = 1, 2, \cdots, n$, be the amount invested in the i-th risky asset at period t. Then,

$x_t - \sum_{i=1}^n \pi_t^i$ represents the amount invested in the risk-free asset at period t. The information set at the beginning of period t is denoted as

$$\mathcal{F}_t = \sigma(\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{t-1}, q_0, q_1, q_{t-1}),$$

and $\mathcal{F}_0$ is the trivial σ-algebra over Ω. Therefore, $\mathbb{E}[\cdot|\mathcal{F}_0]$ is just the unconditional expectation $\mathbb{E}[\cdot]$. We confine an admissible investment strategy to be $\mathcal{F}_t$-measurable Markov strategy. Then, $(\mathbf{P}_t', q_t)'$ and $\boldsymbol{\pi}_t = (\pi_t^1, \cdots, \pi_t^n)'$ are independent, $\{x_t, l_t\}$ is an adapted Markovian process and $\mathcal{F}_t = \sigma(x_t, l_t)$.

The multi-period asset-liability mean-variance portfolio selection problem with uncertain exit time is to seek the best strategy, $\boldsymbol{\pi}_t^* = (\pi_t^{1*}, \pi_t^{2*}, \cdots, \pi_t^{n*})'$, $t = 0, 1, \cdots, T-1$, which is the optimizer of the following stochastic optimal control problem with an uncertain exit time,

$$\left\{\begin{array}{ll} \min & \mathrm{Var}^{(\tau)}(x_{T\wedge\tau} - l_{T\wedge\tau}) - w\mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}], \\ s.t. & x_{t+1} = \sum_{i=1}^n e_t^i \pi_t^i + \left(x_t - \sum_{i=1}^n \pi_t^i\right) s_t \\ & \qquad = s_t x_t + \mathbf{P}_t' \boldsymbol{\pi}_t, \qquad t = 0, 1, \cdots, T-1, \\ & l_{t+1} = q_t l_t, \qquad t = 0, 1, \cdots, T-1, \end{array}\right. \tag{1}$$

where $w \geq 0$ is the trade-off parameter between the mean and the variance, and

$$\mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] \triangleq \sum_{t=0}^{T} \mathbb{E}\big[x_{T\wedge\tau} - l_{T\wedge\tau} \big| T\wedge\tau = t\big] \Pr\{T\wedge\tau = t\} = \sum_{t=0}^{T} \mathbb{E}[x_t - l_t]\alpha_t,$$

$$\mathrm{Var}^{(\tau)}(x_{T\wedge\tau} - l_{T\wedge\tau}) \triangleq \sum_{t=0}^{T} \mathrm{Var}\big(x_{T\wedge\tau} - l_{T\wedge\tau} \big| T\wedge\tau = t\big) \Pr\{T\wedge\tau = t\} = \sum_{t=0}^{T} \mathrm{Var}(x_t - l_t)\alpha_t,$$

respectively. Then the multi-period mean-variance problem (1) can be equivalently re-written into the following problem,

$$\left\{\begin{array}{ll} \min & \sum_{t=1}^T \alpha_t \Big\{\mathrm{Var}(x_t - l_t) - w\mathbb{E}[x_t - l_t]\Big\}, \\ \text{s.t.} & x_{t+1} = s_t x_t + \mathbf{P}_t' \boldsymbol{\pi}_t, \qquad t = 0, 1, \cdots, T-1, \\ & l_{t+1} = q_t l_t, \qquad t = 0, 1, \cdots, T-1. \end{array}\right. \tag{2}$$

Since the variance operation does not satisfy the smoothing property, the problem (2) is nonseparable in the sense of dynamic programming, i.e., it cannot be decomposed by a stage-wise backward recursion and then is no longer solved by dynamic programming. We try to tackle it by mean-field formulation. For $t = 0, 1, \cdots, T-1$, taking the expectation operator of the dynamic system specified in (2) and noticing that $\mathbf{P}_t$ and $\boldsymbol{\pi}_t$, q_t and l_t are independent, we deduce

$$\left\{\begin{array}{l} \mathbb{E}[x_{t+1}] = s_t \mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\boldsymbol{\pi}_t], \\ \mathbb{E}[l_{t+1}] = \mathbb{E}[q_t]\mathbb{E}[l_t], \\ \mathbb{E}[x_0] = x_0, \\ \mathbb{E}[l_0] = l_0. \end{array}\right. \tag{3}$$

Combining (2) and (3) into the following, for $t = 0, 1, \cdots, T-1$,

$$\begin{cases} x_{t+1} - \mathbb{E}[x_{t+1}] &= s_t\big(x_t - \mathbb{E}[x_t]\big) + \mathbf{P}_t'\boldsymbol{\pi}_t - \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\boldsymbol{\pi}_t] \\ &= s_t\big(x_t - \mathbb{E}[x_t]\big) + \mathbf{P}_t'\big(\boldsymbol{\pi}_t - \mathbb{E}[\boldsymbol{\pi}_t]\big) + \big(\mathbf{P}_t' - \mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\boldsymbol{\pi}_t], \\ l_{t+1} - \mathbb{E}[l_{t+1}] &= q_t l_t - \mathbb{E}[q_t]\mathbb{E}[l_t] \\ &= q_t\big(l_t - \mathbb{E}[l_t]\big) + \big(q_t - \mathbb{E}[q_t]\big)\mathbb{E}[l_t], \\ x_0 - \mathbb{E}[x_0] &= 0, \\ l_0 - \mathbb{E}[l_0] &= 0. \end{cases} \tag{4}$$

Then the state space (x_t, l_t) and the control space $(\boldsymbol{\pi}_t)$ are enlarged into $(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t])$ and $(\mathbb{E}[\boldsymbol{\pi}_t], \boldsymbol{\pi}_t - \mathbb{E}[\boldsymbol{\pi}_t])$, respectively. Although we can select the control vector $\mathbb{E}[\boldsymbol{\pi}_t]$ and $\boldsymbol{\pi}_t - \mathbb{E}[\boldsymbol{\pi}_t]$ independently at time t, they should be chosen such that

$$\mathbb{E}[\boldsymbol{\pi}_t - \mathbb{E}[\boldsymbol{\pi}_t]] = \mathbf{0}, \quad t = 0, 1, \cdots, T-1,$$

and thus

$$\mathbb{E}[x_t - \mathbb{E}[x_t]] = 0, \quad t = 0, 1, \cdots, T-1,$$

is satisfied. We also confine admissible investment strategies $(\mathbb{E}[\boldsymbol{\pi}_t], \boldsymbol{\pi}_t - \mathbb{E}[\boldsymbol{\pi}_t])$ to be $\mathcal{F}_t$-measurable Markov controls.

Therefore, the problem (2) can be reformulated as the following mean-filed type of linear-quadratic optimal stochastic control problem

$$\begin{cases} \min & \displaystyle\sum_{t=1}^{T} \alpha_t \Big\{ \mathbb{E}\big[\big(x_t - l_t - \mathbb{E}[x_t - l_t]\big)^2\big] - w\mathbb{E}\big[x_t - l_t\big] \Big\}, \\ \text{s.t.} & \{\mathbb{E}[x_t], \mathbb{E}[l_t], \mathbb{E}[\boldsymbol{\pi}_t]\} \text{ satisfy dynamic equation (3)}, \\ & \{x_t - \mathbb{E}[x_t], l_t - \mathbb{E}[l_t], \boldsymbol{\pi}_t - \mathbb{E}[\boldsymbol{\pi}_t]\} \text{ satisfy dynamic equation (4)}, \\ & \mathbb{E}[\boldsymbol{\pi}_t - \mathbb{E}[\boldsymbol{\pi}_t]] = \mathbf{0}, \quad t = 0, 1, \cdots, T-1. \end{cases} \tag{5}$$

Now, it is indeed a separable linear-quadratic optimal stochastic control problem, which can be solved by classic dynamic programming approach.

3 The Optimal Strategies and Mean-Variance Efficient Frontiers

Before deriving the main results, we present a useful lemma.

Lemma 1 (Sherman-Morrison formula) *Suppose that A is an invertible square matrix and μ and ν are two given vectors. If*

$$1 + \nu' A^{-1}\mu \neq 0,$$

then the following holds,

$$(A + \mu\nu')^{-1} = A^{-1} - \frac{A^{-1}\mu\nu' A^{-1}}{1 + \nu' A^{-1}\mu}.$$

Firstly, we derive the optimal strategy and efficient frontier for the general case, where the rates of return of assets and liability are correlated, and the exit time is uncertain. For simplicity, we define the following backward recursions for seven deterministic sequences of parameters, $\{\xi_t\}$, $\{\eta_t\}$, $\{\zeta_t\}$, $\{\theta_t\}$, $\{\psi_t\}$, $\{\epsilon_t\}$ and $\{\delta_t\}$, as

$$\begin{aligned}
&\xi_t = \xi_{t+1}(1-B_t)s_t^2 + \alpha_t, \quad \eta_t = \eta_{t+1}\big(\mathbb{E}[q_t] - \widehat{B}_t\big)s_t + \alpha_t,\\
&\zeta_t = \zeta_{t+1}s_t + \alpha_t, \quad \theta_t = \theta_{t+1}\mathbb{E}[q_t] - \frac{\zeta_{t+1}\eta_{t+1}}{\xi_{t+1}}\frac{\widehat{B}_t - \mathbb{E}[q_t]B_t}{1-B_t} + \alpha_t,\\
&\psi_t = \psi_{t+1} - \frac{\zeta_{t+1}^2}{4\xi_{t+1}}\frac{B_t}{1-B_t}, \quad \epsilon_t = \epsilon_{t+1}\mathbb{E}[q_t^2] - \eta_{t+1}^2\xi_{t+1}^{-1}\widetilde{B}_t + \alpha_t,\\
&\delta_t = \delta_{t+1}(\mathbb{E}[q_t])^2 + \epsilon_{t+1}(\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2) - \frac{\eta_{t+1}^2}{\xi_{t+1}}\left(\widetilde{B}_t - (\mathbb{E}[q_t])^2 + \frac{(\widehat{B}_t - \mathbb{E}[q_t])^2}{1-B_t}\right),
\end{aligned}$$

for $t = T-1, T-2, \cdots, 0$, with terminal conditions

$$\xi_T = \alpha_T,\ \eta_T = \alpha_T,\ \epsilon_T = \alpha_T,\ \zeta_T = \alpha_T,\ \theta_T = \alpha_T,\ \delta_T = 0,\ \psi_T = 0.$$

These parameters can also be expressed as follows,

$$\begin{aligned}
&\xi_t = \sum_{k=t}^{T}\alpha_k\prod_{j=t}^{k-1}(1-B_j)s_j^2, \quad \eta_t = \sum_{k=t}^{T}\alpha_k\prod_{j=t}^{k-1}\big(\mathbb{E}[q_j] - \widehat{B}_j\big)s_j,\\
&\zeta_t = \sum_{k=t}^{T}\alpha_k\prod_{j=t}^{k-1}s_j, \quad \theta_t = \sum_{k=t}^{T-1}\left(\alpha_k - \frac{\zeta_{k+1}\eta_{k+1}}{\xi_{k+1}}\frac{\widehat{B}_k - \mathbb{E}[q_k]B_k}{1-B_k}\right)\prod_{j=t}^{k-1}\mathbb{E}[q_j] + \alpha_T\prod_{j=t}^{T-1}\mathbb{E}[q_j],\\
&\psi_t = -\sum_{k=t}^{T-1}\frac{\zeta_{k+1}^2}{4\xi_{k+1}}\frac{B_k}{1-B_k}, \quad \epsilon_t = \sum_{k=t}^{T-1}(\alpha_k - \eta_{k+1}^2\xi_{k+1}^{-1}\widetilde{B}_k)\prod_{j=t}^{k-1}\mathbb{E}[q_j^2] + \alpha_T\prod_{j=t}^{T-1}\mathbb{E}[q_j^2],\\
&\delta_t = \sum_{k=t}^{T-1}\left[\epsilon_{k+1}(\mathbb{E}[q_k^2] - (\mathbb{E}[q_k])^2) - \frac{\eta_{k+1}^2}{\xi_{k+1}}\left(\widetilde{B}_k - (\mathbb{E}[q_k])^2 + \frac{(\widehat{B}_k - \mathbb{E}[q_k])^2}{1-B_k}\right)\right]\prod_{j=t}^{k-1}(\mathbb{E}[q_j])^2.
\end{aligned}$$

Using the above seven deterministic parameters, we can express the optimal strategy and efficient frontier of problem (2) in explicit forms as follows.

Theorem 1 *The optimal strategy of problem (2) is given by*

$$\begin{aligned}
\boldsymbol{\pi}_t^* = &-\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\left[x_t - \mathbb{E}[x_t] - \frac{w\zeta_{t+1} + 2\eta_{t+1}\big(\widehat{B}_t - \mathbb{E}[q_t]\big)\mathbb{E}[l_t]}{2s_t\xi_{t+1}(1-B_t)}\right]\\
&+\frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_tq_t]l_t,
\end{aligned} \tag{6}$$

where

$$\mathbb{E}[x_t] = x_0\prod_{j=0}^{t-1}s_j + \sum_{k=0}^{t-1}\left(\prod_{j=k+1}^{t-1}s_j\right)\left(\frac{w\zeta_{k+1}B_k}{2\xi_{t+1}(1-B_k)} + \frac{\eta_{k+1}}{\xi_{k+1}}\frac{\widehat{B}_k - \mathbb{E}[q_k]B_k}{1-B_k}\prod_{j=0}^{k-1}\mathbb{E}[q_j]l_0\right). \tag{7}$$

Moreover, the corresponding efficient frontier of problem (2) is given by

$$\begin{aligned}
Var^{(\tau)}(x_{T\wedge\tau} - l_{T\wedge\tau}) = -(4\psi_0)^{-1}\big(\mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] - \zeta_0x_0 + \theta_0l_0\big)^2 + \delta_0l_0^2,&\\
\text{for } \mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] \geq \zeta_0x_0 - \theta_0l_0.&
\end{aligned} \tag{8}$$

Proof. Please refer to Appendix A. □

Compared with Yi et al. [17], our results involve much simpler yet more efficient formulas for the optimal strategy and efficient frontier, and all the coefficients can be computed more accurately. In the embedding scheme, a class of auxiliary linear-quadratic optimal stochastic control problems should be solved explicitly first, and then the optimal auxiliary parameters are determined. This procedure may possibly involve some unnecessary and complicated expressions or computational errors, resulting in complicated or even inaccurate formulas. However, by adopting mean-filed formulation, the model can be solved directly and much neater expressions for the optimal strategy and efficient frontier can be derived.

Secondly, we provide the optimal strategy and efficient frontier for three particular cases, where the expressions of the optimal strategy and efficient frontier are much simpler yet neater. Although these results are direct corollaries of Theorem 1, they will help investors better understand influences of the liability and the uncertain exit time, and may be useful for real-life applications.

Particular case 1: the rates of return of assets and liability are correlated, and exit time is fixed to the terminal time T, i.e., $\alpha_T = 1$ and $\alpha_t = 0$ for $t = 0, 1, \cdots, T-1$. Then, the seven deterministic parameters are reduced into the following expressions,

$$\xi_t = \prod_{j=t}^{T-1}(1-B_j)s_j^2, \quad \eta_t = \prod_{j=t}^{T-1}\big(\mathbb{E}[q_j]-\widehat{B}_j\big)s_j, \quad \zeta_t = \prod_{j=t}^{T-1}s_j, \quad \theta_t = \prod_{j=t}^{T-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{1-B_j},$$

$$\psi_t = -\frac{1-\prod_{j=t}^{T-1}(1-B_j)}{4\prod_{j=t}^{T-1}(1-B_j)}, \quad \epsilon_t = -\sum_{k=t}^{T-1}\widetilde{B}_k\prod_{j=k+1}^{T-1}\frac{(\mathbb{E}[q_j]-\widehat{B}_j)^2}{1-B_j}\prod_{j=t}^{k-1}\mathbb{E}[q_j^2] + \prod_{j=t}^{T-1}\mathbb{E}[q_j^2],$$

$$\delta_t = -\prod_{j=t}^{T-1}\frac{(\mathbb{E}[q_j]-\widehat{B}_j)^2}{1-B_j} - \sum_{k=t}^{T-1}\widetilde{B}_k\Big(\prod_{j=k+1}^{T-1}\frac{\big(\mathbb{E}[q_j]-\widehat{B}_j\big)^2}{1-B_j}\Big)\Big(\prod_{j=t}^{k-1}\mathbb{E}[q_j^2]\Big) + \prod_{j=t}^{T-1}\mathbb{E}[q_j^2].$$

Furthermore, according to equation (7) in Theorem 1, it is easy to compute that

$$\begin{aligned}
\mathbb{E}[x_t] =\ & x_0\prod_{j=0}^{t-1}s_j + \frac{w}{2}\Big(\prod_{j=t}^{T-1}\frac{1}{(1-B_j)s_j}\Big)\sum_{k=0}^{t-1}\frac{B_k}{1-B_k}\Big(\prod_{j=k+1}^{t-1}\frac{1}{1-B_j}\Big) \\
&+\Big(\prod_{j=t}^{T-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{(1-B_j)s_j}\Big)\sum_{k=0}^{t-1}\frac{\widehat{B}_k-\mathbb{E}[q_k]B_k}{1-B_k}\Big(\prod_{j=k+1}^{t-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{1-B_j}\Big)\prod_{j=0}^{k-1}\mathbb{E}[q_j]l_0 \\
=\ & x_0\prod_{j=0}^{t-1}s_j + \frac{w}{2}\Big(\prod_{j=t}^{T-1}\frac{1}{(1-B_j)s_j}\Big)\frac{1-\prod_{j=0}^{t-1}(1-B_j)}{\prod_{j=0}^{t-1}(1-B_j)} \\
&+\Big(\prod_{j=t}^{T-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{(1-B_j)s_j}\Big)\Big(\prod_{j=0}^{t-1}\mathbb{E}[q_j] - \prod_{j=0}^{t-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{1-B_j}\Big)l_0,
\end{aligned}$$

and

$$\mathbb{E}[x_t] + \frac{w\zeta_{t+1}+2\eta_{t+1}\big(\widehat{B}_t-\mathbb{E}[q_t]\big)\mathbb{E}[l_t]}{2s_t\xi_{t+1}(1-B_t)} = \Big(x_0\prod_{j=0}^{T-1}s_j + \frac{w}{2}\prod_{j=0}^{T-1}\frac{1}{1-B_j} - l_0\prod_{j=0}^{T-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{1-B_j}\Big)\prod_{j=t}^{T-1}s_j^{-1}.$$

Hence, we get the following corollary for Particular case 1.

Corollary 1 *When the rates of return of assets and liability are correlated and exit time is fixed to the terminal time T, the optimal strategy of problem (2) is given by*

$$\boldsymbol{\pi}_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\Big(x_t - \gamma\prod_{j=t}^{T-1}s_j^{-1}\Big) + \Big(\prod_{j=t+1}^{T-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{s_j(1-B_j)}\Big)\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t]l_t,$$

where

$$\gamma = x_0\prod_{j=0}^{T-1}s_j + \frac{w}{2}\prod_{j=0}^{T-1}\frac{1}{1-B_j} - l_0\prod_{j=0}^{T-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{1-B_j}.$$

Moreover, the efficient frontier of problem (2) is given by

$$\mathrm{Var}(x_T - l_T) = \frac{\prod_{j=t}^{T-1}(1-B_j)}{1-\prod_{j=t}^{T-1}(1-B_j)}\Big(\mathbb{E}[x_T-l_T] - \prod_{j=0}^{T-1}s_jx_0 + \prod_{j=0}^{T-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{1-B_j}l_0\Big)^2 + \delta_0 l_0^2, \quad (9)$$

$$\textit{for } \mathbb{E}[x_T - l_T] \geq \prod_{j=0}^{T-1}s_jx_0 - \prod_{j=0}^{T-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{1-B_j}l_0.$$

Particular case 2: the rates of return of assets and liability are independent, and the exit time is uncertain. Then, we have

$$\widehat{B}_t = \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t] = \mathbb{E}[q_t]B_t, \quad t = 0,1,\cdots,T-1,$$
$$\widetilde{B}_t = \big(\mathbb{E}[q_t]\big)^2\mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t] = \big(\mathbb{E}[q_t]\big)^2B_t, \quad t = 0,1,\cdots,T-1,$$

which reduce the expressions of the seven deterministic parameters as follows,

$$\xi_t = \sum_{k=t}^{T}\alpha_k\prod_{j=t}^{k-1}(1-B_j)s_j^2, \quad \eta_t = \sum_{k=t}^{T}\alpha_k\prod_{j=t}^{k-1}(1-B_j)\mathbb{E}[q_j]s_j,$$
$$\zeta_t = \sum_{k=t}^{T}\alpha_k\prod_{j=t}^{k-1}s_j, \quad \theta_t = \sum_{k=t}^{T}\alpha_k\prod_{j=t}^{k-1}\mathbb{E}[q_j], \quad \psi_t = -\sum_{k=t}^{T-1}\frac{\zeta_{k+1}^2}{4\xi_{k+1}}\frac{B_k}{1-B_k},$$
$$\epsilon_t = \sum_{k=t}^{T-1}(\alpha_k - \eta_{k+1}^2\xi_{k+1}^{-1}B_k(\mathbb{E}[q_k])^2)\prod_{j=t}^{k-1}\mathbb{E}[q_j^2] + \alpha_T\prod_{j=t}^{T-1}\mathbb{E}[q_j^2],$$
$$\delta_t = \sum_{k=t}^{T-1}(\alpha_k - \eta_{k+1}^2\xi_{k+1}^{-1}B_k(\mathbb{E}[q_k])^2)\Big(\prod_{j=t}^{k-1}\mathbb{E}[q_j^2] - \prod_{j=t}^{k-1}(\mathbb{E}[q_j])^2\Big) + \alpha_T\Big(\prod_{j=t}^{T-1}\mathbb{E}[q_j^2] - \prod_{j=t}^{T-1}(\mathbb{E}[q_j])^2\Big).$$

Using the above notation, we get the following corollary for Particular case 2.

Corollary 2 *When the rates of return of assets and liability are independent, and the exit time is uncertain, the optimal strategy of problem (2) is given by*

$$\boldsymbol{\pi}_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\Big[x_t - \mathbb{E}[x_t] - \frac{w\zeta_{t+1}}{2s_t\xi_{t+1}(1-B_t)} - \frac{\eta_{t+1}\mathbb{E}[q_t](l_t - \mathbb{E}[l_t])}{s_t\xi_{t+1}}\Big],$$

where

$$\mathbb{E}[x_t] = x_0\prod_{j=0}^{t-1}s_j + \frac{w}{2}\sum_{k=0}^{t-1}\Big(\prod_{j=k+1}^{t-1}s_j\Big)\frac{\zeta_{k+1}}{\xi_{k+1}}\frac{B_k}{1-B_k}.$$

Moreover, the efficient frontier of problem (2) is given by

$$\begin{aligned}&\mathrm{Var}(x_T-l_T)\\ =&\Big(\sum_{k=0}^{T-1}\frac{\zeta_{k+1}^2}{\xi_{k+1}}\frac{B_k}{1-B_k}\Big)^{-1}\Big(\mathbb{E}[x_T-l_T]-\sum_{k=0}^{T}\alpha_k\prod_{j=0}^{k-1}s_jx_0+\sum_{k=0}^{T}\alpha_k\prod_{j=0}^{k-1}\mathbb{E}[q_j]l_0\Big)^2+\delta_0l_0^2, \qquad (10)\\ &\text{for } \mathbb{E}[x_T-l_T]\geq\sum_{k=0}^{T}\alpha_k\prod_{j=0}^{k-1}s_jx_0-\sum_{k=0}^{T}\alpha_k\prod_{j=0}^{k-1}\mathbb{E}[q_j]l_0.\end{aligned}$$

Particular case 3: the rates of return of assets and liability are independent, and exit time is fixed to the terminal time T. Then, the seven deterministic parameters are reduced to

$$\begin{aligned}&\xi_t=\prod_{j=t}^{T-1}(1-B_j)s_j^2,\quad \eta_t=\prod_{j=t}^{T-1}(1-B_j)\mathbb{E}[q_j]s_j,\quad \zeta_t=\prod_{j=t}^{T-1}s_j,\quad \theta_t=\prod_{j=t}^{T-1}\mathbb{E}[q_j],\\ &\psi_t=-\frac{1-\prod_{j=t}^{T-1}(1-B_j)}{4\prod_{j=t}^{T-1}(1-B_j)},\quad \epsilon_t=-\sum_{k=t}^{T-1}(\mathbb{E}[q_k])^2B_k\prod_{j=k+1}^{T-1}(\mathbb{E}[q_j])^2(1-B_j)\prod_{j=t}^{k-1}\mathbb{E}[q_j^2]+\prod_{j=t}^{T-1}\mathbb{E}[q_j^2],\\ &\delta_t=-\prod_{j=t}^{T-1}(\mathbb{E}[q_j])^2(1-B_j)-\sum_{k=t}^{T-1}(\mathbb{E}[q_k])^2B_k\Big(\prod_{j=k+1}^{T-1}(\mathbb{E}[q_j])^2(1-B_j)\Big)\Big(\prod_{j=t}^{k-1}\mathbb{E}[q_j^2]\Big)+\prod_{j=t}^{T-1}\mathbb{E}[q_j^2].\end{aligned}$$

We get the following corollary for Particular case 3.

Corollary 3 *When the rates of return of assets and liability are independent, and exit time is fixed to the terminal time T, the optimal strategy of problem (2) is given by*

$$\pi_t^*=-\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\Big(x_t-\gamma\prod_{j=t}^{T-1}s_j^{-1}-\prod_{j=t}^{T-1}\frac{\mathbb{E}[q_j]}{s_j}l_t\Big),$$

where

$$\gamma=x_0\prod_{j=0}^{T-1}s_j+\frac{w}{2}\prod_{j=0}^{T-1}\frac{1}{1-B_j}-l_0\prod_{j=0}^{T-1}\mathbb{E}[q_j].$$

Moreover, the efficient frontier of problem (2) is given by

$$\begin{aligned}\mathrm{Var}(x_T-l_T)=\frac{\prod_{j=t}^{T-1}(1-B_j)}{1-\prod_{j=0}^{T-1}(1-B_j)}\Big(\mathbb{E}[x_T-l_T]-\prod_{j=0}^{T-1}s_jx_0+\prod_{j=0}^{T-1}\mathbb{E}[q_j]l_0\Big)^2+\delta_0l_0^2, \qquad (11)\\ \text{for } \mathbb{E}[x_T-l_T]\geq\prod_{j=0}^{T-1}s_jx_0-\prod_{j=0}^{T-1}\mathbb{E}[q_j]l_0.\end{aligned}$$

4 Sensitivity Analysis and a Real-life Example

In this section, we study the impacts of different parameters through sensitivity analysis of an artificial example. Then, we analyze a real-life example to learn the practical consequences of the derived optimal strategy.

Table 1: Information of the risky asset and the liability

	risky asset	liability
Expected return	14%	10%
Standard deviation	18.5%	20%
Correlation coefficient	0.25	

First, let us consider an artificial market, in which there are one risk-free asset, one risky asset and one liability. The annual return of the risk-free asset is 5%. Other information of the annual rates of return of the risky asset and the liability are given in Table 1. We further assume that the investor has initial wealth $x_0 = 3$, initial liability $l_0 = 1$, a trade-off parameter between the mean and the variance $w = 1$ and an investment horizon $T = 5$ years. Then, according to equation (6) in Theorem 1, the optimal mean-variance strategy takes a linear feedback form as follows,

$$\pi_t^* = -k_t x_t + \tilde{k}_t l_t + k_t \hat{k}_t, \quad t = 0, 1, 2, 3, 4,$$

where

$$\begin{aligned} k_t &= \mathbb{E}^{-1}[P_t^2]\mathbb{E}[P_t]s_t = 2.2327, \\ \hat{k}_t &= \mathbb{E}[x_t] + \frac{w\zeta_{t+1} + 2\eta_{t+1}\big(\widehat{B}_t - \mathbb{E}[q_t]\big)\mathbb{E}[l_t]}{2s_t\xi_{t+1}(1 - B_t)}, \\ \tilde{k}_t &= \frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}^{-1}[P_t^2]\mathbb{E}[P_t q_t]. \end{aligned}$$

The optimal strategy consists of three components: a wealth-dependent component $-k_t x_t$, a liability-dependent component $\tilde{k}_t l_t$ and a time-varying component $k_t \hat{k}_t$. It is easy to see that the coefficient of the wealth-dependent component only relies on the investment opportunity of the market (i.e., the parameter P_t and s_t).

In the following three subsections, we analyze the impacts of the uncertain exit time τ, the liability l_t and the correlation coefficient ρ, respectively.

4.1 The impact of the uncertain exit time

In this subsection, we assume that the actual exit time $T \wedge \tau$ has four different probability mass functions, $\boldsymbol{\alpha}^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}, \alpha_4^{(i)}, \alpha_5^{(i)})'$, $(i = 1, 2, 3, 4)$, as follows,

$$\begin{aligned} \boldsymbol{\alpha}^{(1)} &= (0.1, 0.15, 0.2, 0.25, 0.3)', \quad \boldsymbol{\alpha}^{(2)} = (0, 0.1, 0.1, 0.3, 0.5)', \\ \boldsymbol{\alpha}^{(3)} &= (0, 0, 0.1, 0.2, 0.7)', \quad \boldsymbol{\alpha}^{(4)} = (0, 0, 0, 0, 1)', \end{aligned}$$

where $\boldsymbol{\alpha}^{(4)}$ means that the investor exits the market at the terminal time T. Table 2 shows the parameters $\hat{k}_t$, $\tilde{k}_t$ of the optimal strategy, the expected value and variance of the wealth at exit time, for different $\boldsymbol{\alpha}^{(i)}$s. We can see that under our setting, the later the investor exits the financial market, the bigger the values of $\hat{k}_t$ and $\tilde{k}_t$, i.e., the larger the liability-dependent component and the time-varying component. As a result, a larger expected value of the wealth can be achieved. Furthermore, when t increases, $\hat{k}_t$ increases and $\tilde{k}_t$ decreases. Thus, the investor would like to increase the time-varying component and decrease the liability-dependent component of the optimal strategy as time evolves.

Figure 1 describes the efficient frontiers with different $\boldsymbol{\alpha}^{(i)}$s by changing ω from 0 to $+\infty$. We can see that the efficient frontiers cross each other. When the investor can bear a relative

Table 2: The impact of the uncertain exit time

	α_1	α_2	α_3	α_4
$\hat{k}_0$	2.7946	2.8847	2.9425	3.0048
$\hat{k}_1$	2.9619	3.0289	3.0896	3.1550
$\hat{k}_2$	3.1393	3.2024	3.2441	3.3128
$\hat{k}_3$	3.3245	3.3754	3.4195	3.4784
$\hat{k}_4$	3.5164	3.5655	3.6034	3.6523
$\tilde{k}_0$	2.7047	2.7562	2.7868	2.8171
$\tilde{k}_1$	2.6619	2.6904	2.7203	2.7498
$\tilde{k}_2$	2.6243	2.6452	2.6553	2.6842
$\tilde{k}_3$	2.5899	2.5950	2.6050	2.6201
$\tilde{k}_4$	2.5576	2.5576	2.5576	2.5576
$\mathbb{E}^{(\tau)}(x_{5\wedge\tau})$	2.8442	3.0563	3.1894	3.3349
$\mathrm{Var}^{(\tau)}(x_{5\wedge\tau})$	0.4643	0.5870	0.6651	0.7516

large risk (i.e., ω is large), the later the investor exits the financial market, the better efficient frontier he can achieve. However, when the investor aims to minimize the risk only or can bear a relative small risk (i.e., ω is small), the sooner the investor exists the financial market, the better the achieved efficient frontier. The reason behind is the existence of the liability. When the investor exits the financial market early, the uncertainty of the liability is not large and can be largely reduced by the asset, which results in a portfolio with low risk. But when the investor exits the financial market late, the uncertainty of the liability is large and can only be partially reduced by the asset.

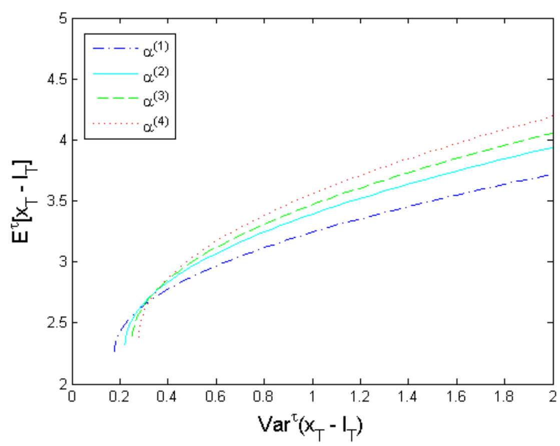

Figure 1: Efficient frontiers with different $\boldsymbol{\alpha}^{(i)}$s

4.2 The impact of the liability

In this subsection, we choose $\boldsymbol{\alpha} = (0.1, 0.15, 0.2, 0.25, 0.3)$. To see the impact of the liability, we consider the case without liability as benchmark and assume that the standard deviation of the rate of return of the liability has three choices, 0.2, 0.3, 0.4, (i.e., $\mathrm{Std}[q_t] = 0.2, 0.3, 0.4$).

The parameters $\hat{k}_t$, $\tilde{k}_t$ of the optimal strategy, the expected value and variance of the wealth at exit time are showed in Table 3 and the efficient frontiers are represented in Figure 2.

Table 3: The impact of the liability

	no liability	$\text{Std}[q_t] = 0.2$	$\text{Std}[q_t] = 0.3$	$\text{Std}[q_t] = 0.4$
$\hat{k}_0$	3.8780	2.7946	2.8348	2.8737
$\hat{k}_1$	4.1091	2.9619	3.0088	3.0541
$\hat{k}_2$	4.3564	3.1393	3.1945	3.2480
$\hat{k}_3$	4.6175	3.3245	3.3896	3.4529
$\hat{k}_4$	4.8913	3.5164	3.5929	3.6676
$\tilde{k}_0$	0.0000	2.7047	2.7466	2.7843
$\tilde{k}_1$	0.0000	2.6619	2.7237	2.7820
$\tilde{k}_2$	0.0000	2.6243	2.7033	2.7799
$\tilde{k}_3$	0.0000	2.5899	2.6846	2.7780
$\tilde{k}_4$	0.0000	2.5576	2.6669	2.7761
$\mathbb{E}^{(\tau)}(x_{5\wedge\tau})$	4.1414	2.8442	2.8965	2.9472
$\text{Var}^{(\tau)}(x_{5\wedge\tau})$	0.2877	0.4643	0.7067	1.1

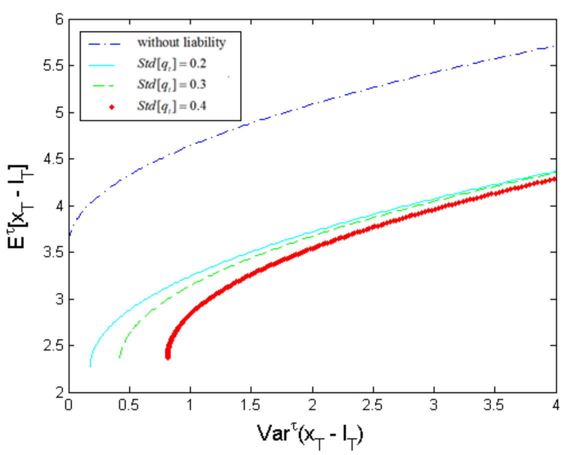

Figure 2: Efficient frontiers with different $\text{Std}[q_t]$s

From Table 3, we can see that both the liability-dependent component and the time-varying component increase as the standard deviation of the liability increases. Thus, the investor should maintain larger risky asset holding to offset the increasing uncertainty of the liability. Similar to the result of Subsection 4.1, the investor would like to increase the time-varying component and decrease the liability-dependent component of the optimal strategy as time evolves. From Figure 2, we can see that if there is no liability, the efficient frontier is much higher and the minimum variance portfolio can achieve zero risk level by only investing the risk-free asset. When there exists a liability in the market, the efficient frontier decreases as the standard deviation of the rate of return of the liability increases.

4.3 The impact of the correlation coefficient

In this subsection, we choose $\boldsymbol{\alpha} = (0.1, 0.15, 0.2, 0.25, 0.3)$. To see the impact of the correlation coefficient, we assume that $\rho = -1, -0.5, 0, 0.5, 1$. The parameters $\hat{k}_t$, $\tilde{k}_t$ of the optimal strategy, the expected value and variance of the wealth at exit time are showed in Table 4 and the efficient frontiers are represented in Figure 3.

Table 4: The impact of the correlation coefficient

	$\rho = -1$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1$
$\hat{k}_0$	2.3129	2.5240	2.7101	2.8737	3.0171
$\hat{k}_1$	2.4036	2.6477	2.8637	3.0541	3.2216
$\hat{k}_2$	2.4877	2.7714	3.0239	3.2480	3.4466
$\hat{k}_3$	2.5656	2.8940	3.1888	3.4529	3.6892
$\hat{k}_4$	2.6376	3.0152	3.3575	3.6676	3.9485
$\tilde{k}_0$	2.0105	2.3541	2.6075	2.7843	2.8967
$\tilde{k}_1$	1.8358	2.2146	2.5279	2.7820	2.9830
$\tilde{k}_2$	1.6927	2.0971	2.4588	2.7799	3.0624
$\tilde{k}_3$	1.5711	1.9943	2.3966	2.7780	3.1384
$\tilde{k}_4$	1.4649	1.9019	2.3390	2.7761	3.2132
$\mathbb{E}^{(\tau)}(x_{5\wedge\tau})$	2.2224	2.4939	2.7345	2.9472	3.1347
$\mathrm{Var}^{(\tau)}(x_{5\wedge\tau})$	0.2915	0.4548	0.4865	0.4219	0.2880

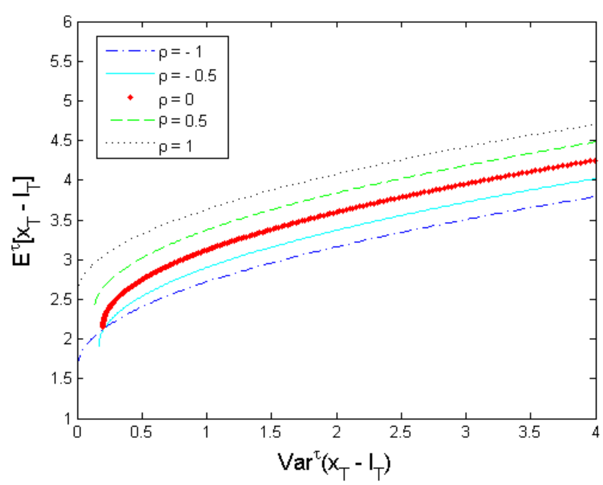

Figure 3: Efficient frontiers with different ρs

As shown in Table 4, when the correlation coefficient ρ increases, both the liability-dependent component and the time-varying component increases. Taking $\rho = -1$ (or $\rho = 1$) as an example, the investor would like to take short position (or long position) in the risky asset to hedge the liability. From the expression of the optimal strategy π^*, we can see that the only way to maintain a short position (or long position) in risky asset is decreasing (or increasing) the liability-dependent component and the time-varying component. Furthermore, the case of $\rho = 0$ gives the largest risk. The reason behind is that when the absolute value of the correlation coefficient is large, it is much easier for the investor to overcome the uncertainty of the liability through investing the risky asset. Figure 3 shows that the efficient frontier with higher ρ is better. It is also straightforward. To control the risk, the investor tends to take more and more long position in the risky asset as ρ increases. As the expect rate of excess return of the risky asset is positive, the larger long position in risky asset, the bigger the expected value of the wealth at the exit time, i.e., the better the efficient frontier.

4.4 A Real-life Example

In this subsection, we consider a real-life example and study the practical consequences of the derived optimal strategy.

The investor is a 60-year old worker, who will retire at age 65. He has 3 million Chinese yuan cash and 1 million Chinese yuan loan to repay. The loan is adjusted annually according to the inflation rate in the year. The worker wants to invest in a bank account and Chinese stock market index ETF (CSI 300 Index ETF) until his retirement. During the investment, he suffers from a possible death, which is characterised by the life table. Therefore, the worker's investment decision problem can be formulated as a 5-year asset-liability mean-variance portfolio selection with an uncertain exit time.

In details, the expected value and variance of the annual rate of return of CSI 300 Index ETF is computed using the monthly data of CSI 300 Index from January 2001 to February 2016. The expected value and variance of the annual rate of return of the loan are computed using the monthly data of CPI (Consumer Price Index) from January 2001 to February 2016. Then, the correlation coefficient between the annual rates of return of CSI 300 Index ETF and the load is also estimated. The annual rate of return of the bank account is set 1.75%, which is the one-year fixed deposit rate. The probability mass function of the uncertain exit time is computed using 2010 Life Table for Chinese Males. More specifically, we have

$$\begin{aligned}
&x_0 = 3 \text{ million}, \quad l_0 = 1 \text{ million}, \quad s_t = 1.0175, \quad \mathbb{E}[P_t] = 0.0411, \quad \mathrm{Var}[P_t] = 0.0822, \\
&\mathbb{E}[q_t] = 1.0268, \quad \mathrm{Var}[q_t] = 4.84 \times 10^{-4}, \quad \rho = 0.0751, \\
&\alpha_1 = 0.0085, \quad \alpha_2 = 0.00934, \quad \alpha_3 = 0.01033 \quad \alpha_4 = 0.01106, \quad \alpha_5 = 0.96077.
\end{aligned}$$

As the worker may want to achieve a remarkable wealth for his retirement, it is reasonable to assume that he has a moderate risk-averse attitude. Thus, we choose $\omega = 10$. Table 5 shows the optimal strategy at time 0, the parameters $\hat{k}_t$, $\tilde{k}_t$ of the optimal strategy, the expected value and variance of the wealth at exit time for four cases. We can see that once the uncertain exit time is considered, the worker tends to take smaller positions in the risky asset. However, as the probability of death before retirement is small, the uncertain exit time has limited impact on the optimal strategy and the expected value and variance of the wealth. When the liability is considered, the worker tends to take larger positions in the risky asset at time 0, and achieves a larger risk. As the correlation coefficient and the variance of the liability are small, the liability has limited impact on the optimal strategy and the variance of the wealth. But it does have a large impact on the expected value of the wealth, for the liability should be repaid at the exit time.

The main difference between multi-period asset-liability mean-variance portfolio selection with an uncertain exit time and a classical multi-period mean-variance portfolio selection is the inclusion of the liability and the uncertain exit time. The probability mass function of the exit time, the variance of the liability and the correlation coefficients are key parameters, which need to be estimated accurately.

Table 5: The parameters $\hat{k}_t$, $\tilde{k}_t$ and investment performances

	general case	fixed exit time	no liability	no liability and fixed exit time
π_0^*	2.5355	2.5362	2.5295	2.5302
$\hat{k}_0$	7.0295	7.0301	8.0740	8.0754
$\hat{k}_1$	7.1527	7.1531	8.2158	8.2167
$\hat{k}_2$	7.2781	7.2783	8.3600	8.3605
$\hat{k}_3$	7.4055	7.4056	8.5066	8.5068
$\hat{k}_4$	7.5352	7.5352	8.6556	8.6557
$\tilde{k}_0$	0.5267	0.5271	0.0000	0.0000
$\tilde{k}_1$	0.5222	0.5224	0.0000	0.0000
$\tilde{k}_2$	0.5177	0.5178	0.0000	0.0000
$\tilde{k}_3$	0.5132	0.5133	0.0000	0.0000
$\tilde{k}_4$	0.5087	0.5087	0.0000	0.0000
$\mathbb{E}^{(\tau)}(x_{5\wedge\tau})$	2.6544	2.6671	3.7917	3.8072
$\mathrm{Var}^{(\tau)}(x_{5\wedge\tau})$	2.6282	2.6795	2.6254	2.6766

5 Conclusion

In financial practices, the investor may encounter multi-period asset-liability mean-variance portfolio selection problem with an uncertain exit time. Different from other approaches in the current literature, we propose a mean-field reformulation for dealing with this problem, and derive its analytical optimal strategy and efficient frontier. The corresponding sensitivity analysis and a real-life example show that the probability mass function of the exit time, the variance of the liability and the correlation coefficients are key parameters for the model. The investor tends to enlarge the risky assets' holding during the investment, when i) he exits the market late, ii) the variance of the liability increases, or iii) the correlation coefficient increases.

Appendix A: Proof of Theorem 1

We prove the main results by dynamic programming approach. For the information set $\mathcal{F}_t$, the cost-to-go functional at period t is computed by

$$\begin{aligned}
&J_t(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\
=&\min_{\{\boldsymbol{\pi}_t - \mathbb{E}[\boldsymbol{\pi}_t], \mathbb{E}[\boldsymbol{\pi}_t]\}} \mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]\big)\big|\mathcal{F}_t\big] \\
&+ \alpha_t\big(x_t - l_t - \mathbb{E}[x_t - l_t]\big)^2 - w\alpha_t\mathbb{E}[x_t - l_t].
\end{aligned}$$

The cost-to-go functional at terminal time T is

$$\begin{aligned}
&J_T\big(\mathbb{E}[x_T], x_T - \mathbb{E}[x_T], \mathbb{E}[l_T], l_T - \mathbb{E}[l_T]\big) \\
=&\alpha_T\big(x_T - l_T - \mathbb{E}[x_T - l_T]\big)^2 - w\alpha_T\mathbb{E}\big[x_T - l_T\big] \\
=&\xi_T\big(x_T - \mathbb{E}[x_T]\big)^2 - 2\eta_T\big(l_T - \mathbb{E}[l_T]\big)\big(x_T - \mathbb{E}[x_T]\big) + \epsilon_T\big(l_T - \mathbb{E}[l_T]\big)^2 \\
&- w\zeta_T\mathbb{E}[x_T] + w\theta_T\mathbb{E}[l_T] + \delta_T(\mathbb{E}[l_T])^2 + w^2\psi_T.
\end{aligned}$$

Assume that the cost-to-go functional at time $t+1$ is the following expression

$$\begin{aligned}
&J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1}-\mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1}-\mathbb{E}[l_{t+1}]\big)\\
=&\,\xi_{t+1}\big(x_{t+1}-\mathbb{E}[x_{t+1}]\big)^2 - 2\eta_{t+1}\big(l_{t+1}-\mathbb{E}[l_{t+1}]\big)\big(x_{t+1}-\mathbb{E}[x_{t+1}]\big) + \epsilon_{t+1}\big(l_{t+1}-\mathbb{E}[l_{t+1}]\big)^2\\
&-w\zeta_{t+1}\mathbb{E}[x_{t+1}] + \omega\theta_{t+1}\mathbb{E}[l_{t+1}] + \delta_{t+1}(\mathbb{E}[l_{t+1}])^2 + w^2\psi_{t+1}.
\end{aligned}$$

We prove that the above statement still holds at time t. For given information set $\mathcal{F}_t$, i.e., knowing $x_t-\mathbb{E}[x_t]$, $\mathbb{E}[x_t]$, $l_t-\mathbb{E}[l_t]$ and $\mathbb{E}[l_t]$, we have

$$\begin{aligned}
&\mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1}-\mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1}-\mathbb{E}[l_{t+1}]\big)\big|\mathcal{F}_t\big]\\
=&\mathbb{E}\Big[\xi_{t+1}\Big[s_t\big(x_t-\mathbb{E}[x_t]\big) + \mathbf{P}_t'\big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big) + \big(\mathbf{P}_t'-\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\boldsymbol{\pi}_t]\Big]^2\\
&-2\eta_{t+1}\Big[q_t\big(l_t-\mathbb{E}[l_t]\big) + \big(q_t-\mathbb{E}[q_t]\big)\mathbb{E}[l_t]\Big]\Big[s_t\big(x_t-\mathbb{E}[x_t]\big) + \mathbf{P}_t'\big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big) + \big(\mathbf{P}_t'-\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\boldsymbol{\pi}_t]\Big]\\
&+\epsilon_{t+1}\big[q_t(l_t-\mathbb{E}[l_t]) + (q_t-\mathbb{E}[q_t])\mathbb{E}[l_t]\big]^2 - w\zeta_{t+1}\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\boldsymbol{\pi}_t]\big)\\
&+w\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + w^2\psi_{t+1}\Big|\mathcal{F}_t\Big]\\
=&\xi_{t+1}\Big[s_t^2\big(x_t-\mathbb{E}[x_t]\big)^2 + \big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big) + 2s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}[\mathbf{P}_t']\big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big)\\
&+\mathbb{E}[\boldsymbol{\pi}_t']\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\boldsymbol{\pi}_t] + 2\big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big)'\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\boldsymbol{\pi}_t]\Big]\\
&-2\eta_{t+1}\Big[s_t\mathbb{E}[q_t]\big(l_t-\mathbb{E}[l_t]\big)\big(x_t-\mathbb{E}[x_t]\big) + \mathbb{E}[q_t\mathbf{P}_t']\big(l_t-\mathbb{E}[l_t]\big)\big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big)\\
&+\big(\mathbb{E}[q_t\mathbf{P}_t']-\mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\Big(\mathbb{E}[l_t]\big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big) + \big(l_t-\mathbb{E}[l_t]\big)\mathbb{E}[\boldsymbol{\pi}_t] + \mathbb{E}[l_t]\mathbb{E}[\boldsymbol{\pi}_t]\Big)\Big]\\
&+\epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t-\mathbb{E}[l_t])^2 + 2(\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(l_t-\mathbb{E}[l_t])\mathbb{E}[l_t] + (\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big]\\
&-w\zeta_{t+1}\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\boldsymbol{\pi}_t]\big) + w\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + w^2\psi_{t+1}.
\end{aligned}$$

Since any admissible strategy of $(\mathbb{E}[\boldsymbol{\pi}_t], \boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t])$ satisfies $\mathbb{E}[\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]] = \mathbf{0}$ and $\mathbb{E}[l_t-\mathbb{E}[l_t]] = 0$ holds, we have

$$\begin{aligned}
&\mathbb{E}\Big[\big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big)'\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\boldsymbol{\pi}_t]\Big|\mathcal{F}_0\Big] = 0,\\
&\mathbb{E}\Big[\big(\mathbb{E}[q_t\mathbf{P}_t']-\mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[l_t]\big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big)\Big|\mathcal{F}_0\Big] = 0,\\
&\mathbb{E}\Big[\big(\mathbb{E}[q_t\mathbf{P}_t']-\mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\big(l_t-\mathbb{E}[l_t]\big)\mathbb{E}[\boldsymbol{\pi}_t]\Big|\mathcal{F}_0\Big] = 0,\\
&\mathbb{E}\Big[(\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(l_t-\mathbb{E}[l_t])\mathbb{E}[l_t]\Big|\mathcal{F}_0\Big] = 0.
\end{aligned}$$

We first identify optimal $(\mathbb{E}[\boldsymbol{\pi}_t^*], \boldsymbol{\pi}_t^*-\mathbb{E}[\boldsymbol{\pi}_t^*])$ by minmimizing the following equivalent cost functional,

$$
\begin{aligned}
&\mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1}-\mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1}-\mathbb{E}[l_{t+1}]\big)\big|\mathcal{F}_t\big]\\
=&\xi_{t+1}\Big[s_t^2\big(x_t-\mathbb{E}[x_t]\big)^2+\big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big)\\
&+2s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}[\mathbf{P}_t']\big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big)+\mathbb{E}[\boldsymbol{\pi}_t']\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\boldsymbol{\pi}_t]\Big]\\
&-2\eta_{t+1}\Big[s_t\mathbb{E}[q_t]\big(l_t-\mathbb{E}[l_t]\big)\big(x_t-\mathbb{E}[x_t]\big)+\mathbb{E}[q_t\mathbf{P}_t']\big(l_t-\mathbb{E}[l_t]\big)\big(\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]\big)\\
&+\big(\mathbb{E}[q_t\mathbf{P}_t']-\mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[l_t]\mathbb{E}[\boldsymbol{\pi}_t]\Big]+\epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t-\mathbb{E}[l_t])^2+(\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big]\\
&-w\zeta_{t+1}\big(s_t\mathbb{E}[x_t]+\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\boldsymbol{\pi}_t]\big)+w\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t]+\delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2+w^2\psi_{t+1},
\end{aligned}
$$

without considering the linear constraint $\mathbb{E}[\boldsymbol{\pi}_t-\mathbb{E}[\boldsymbol{\pi}_t]]=\mathbf{0}$, and verify then the derived optimal strategy satisfies this constraint automatically.

It is easy to see that $\boldsymbol{\pi}_t^*-\mathbb{E}[\boldsymbol{\pi}_t^*]$ can be expressed by the linear form of states and their expected states, and $\mathbb{E}[\boldsymbol{\pi}_t^*]$ can be constructed by the linear form of the expected states, i.e.,

$$
\boldsymbol{\pi}_t^*-\mathbb{E}[\boldsymbol{\pi}_t^*]=-\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\big(x_t-\mathbb{E}[x_t]\big)+\eta_{t+1}\xi_{t+1}^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_tq_t]\big(l_t-\mathbb{E}[l_t]\big), \tag{12}
$$

$$
\begin{aligned}
\mathbb{E}[\boldsymbol{\pi}_t^*]=&\ \big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)^{-1}\left[\frac{w\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[\mathbf{P}_t]+\frac{\eta_{t+1}}{\xi_{t+1}}\big(\mathbb{E}[\mathbf{P}_tq_t]-\mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t]\big)\mathbb{E}[l_t]\right]\\
=&\ \frac{w\zeta_{t+1}+2\eta_{t+1}\big(\widehat{B}_t-\mathbb{E}[q_t]\big)\mathbb{E}[l_t]}{2\xi_{t+1}(1-B_t)}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]+\frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_tq_t]\mathbb{E}[l_t]. \qquad (13)
\end{aligned}
$$

In order to get the explicit expression of the cost-to-go functional at time t, we substitute

$\boldsymbol{\pi}_t^* - \mathbb{E}[\boldsymbol{\pi}_t^*]$ and $\mathbb{E}[\boldsymbol{\pi}_t^*]$ back and derive

$$
\begin{aligned}
&J_t\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]\big)\\
=&\min_{\{\boldsymbol{\pi}_t - \mathbb{E}[\boldsymbol{\pi}_t], \mathbb{E}[\boldsymbol{\pi}_t]\}} \mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]\big)\big|\mathcal{F}_t\big]\\
&+ \alpha_t\big(x_t - l_t - \mathbb{E}[x_t - l_t]\big)^2 - w\alpha_t\mathbb{E}[x_t - l_t]\\
=&\xi_{t+1}s_t^2\big(x_t - \mathbb{E}[x_t]\big)^2 - 2\eta_{t+1}s_t\mathbb{E}[q_t]\big(l_t - \mathbb{E}[l_t]\big)\big(x_t - \mathbb{E}[x_t]\big) - w\zeta_{t+1}s_t\mathbb{E}[x_t] + w\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t]\\
&+ \epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t - \mathbb{E}[l_t])^2 + (\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + w^2\psi_{t+1}\\
&- \xi_{t+1}\Big[-\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t']s_t\big(x_t - \mathbb{E}[x_t]\big) + \eta_{t+1}\xi_{t+1}^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[q_t\mathbf{P}_t']\big(l_t - \mathbb{E}[l_t]\big)\Big]\\
&\cdot \mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\Big[-\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\big(x_t - \mathbb{E}[x_t]\big) + \eta_{t+1}\xi_{t+1}^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_tq_t]\big(l_t - \mathbb{E}[l_t]\big)\Big]\\
&- \xi_{t+1}\left[\frac{w\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[\mathbf{P}_t'] + \frac{\eta_{t+1}}{\xi_{t+1}}\big(\mathbb{E}[q_t\mathbf{P}_t'] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[l_t]\right]\\
&\cdot \big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)^{-1}\left[\frac{w\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[\mathbf{P}_t] + \frac{\eta_{t+1}}{\xi_{t+1}}\big(\mathbb{E}[\mathbf{P}_tq_t] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t]\big)\mathbb{E}[l_t]\right]\\
&+ \alpha_t\big[(x_t - \mathbb{E}[x_t]) - (l_t - \mathbb{E}[l_t])\big]^2 - w\alpha_t\mathbb{E}[x_t - l_t]\\
=&\xi_{t+1}s_t^2(1 - B_t)\big(x_t - \mathbb{E}[x_t]\big)^2 - 2\eta_{t+1}s_t\big(\mathbb{E}[q_t] - \widehat{B}_t\big)\big(l_t - \mathbb{E}[l_t]\big)\big(x_t - \mathbb{E}[x_t]\big)\\
&+ \big(\epsilon_{t+1}\mathbb{E}[q_t^2] - \eta_{t+1}^2\xi_{t+1}^{-1}\widetilde{B}_t\big)\big(l_t - \mathbb{E}[l_t]\big)^2 - w\zeta_{t+1}s_t\mathbb{E}[x_t]\\
&+ w\left[\theta_{t+1}\mathbb{E}[q_t] - \frac{\zeta_{t+1}\eta_{t+1}}{\xi_{t+1}}\frac{\widehat{B}_t - \mathbb{E}[q_t]B_t}{1 - B_t}\right]\mathbb{E}[l_t] + w^2\left[\psi_{t+1} - \frac{\zeta_{t+1}^2}{4\xi_{t+1}}\frac{B_t}{1 - B_t}\right]\\
&+ \left[\delta_{t+1}(\mathbb{E}[q_t])^2 + \epsilon_{t+1}(\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2) - \frac{\eta_{t+1}^2}{\xi_{t+1}}\left(\widetilde{B}_t - (\mathbb{E}[q_t])^2 + \frac{(\widehat{B}_t - \mathbb{E}[q_t])^2}{1 - B_t}\right)\right]\big(\mathbb{E}[l_t]\big)^2\\
&+ \alpha_t(x_t - \mathbb{E}[x_t])^2 - 2\alpha_t(l_t - \mathbb{E}[l_t])(x_t - \mathbb{E}[x_t]) + \alpha_t(l_t - \mathbb{E}[l_t])^2 - w\alpha_t\mathbb{E}[x_t] + w\alpha_t\mathbb{E}[l_t]\\
=&\xi_t\big(x_t - \mathbb{E}[x_t]\big)^2 - 2\eta_t\big(l_t - \mathbb{E}[l_t]\big)\big(x_t - \mathbb{E}[x_t]\big) + \epsilon_t\big(l_t - \mathbb{E}[l_t]\big)^2\\
&- w\zeta_t\mathbb{E}[x_t] + w\theta_t\mathbb{E}[l_t] + \delta_t(\mathbb{E}[l_t])^2 + w^2\psi_t.
\end{aligned}
$$

Substituting $\mathbb{E}[\boldsymbol{\pi}_t^*]$ to dynamics of $\mathbb{E}[x_t]$ in (3) yields

$$\mathbb{E}[x_{t+1}] = s_t\mathbb{E}[x_t] + \frac{w}{2}\frac{\zeta_{t+1}}{\xi_{t+1}}\frac{B_t}{1 - B_t} + \frac{\eta_{t+1}}{\xi_{t+1}}\frac{\widehat{B}_t - \mathbb{E}[q_t]B_t}{1 - B_t}\mathbb{E}[l_t],$$

which implies

$$\mathbb{E}[x_t] = x_0\prod_{j=0}^{t-1}s_j + \sum_{k=0}^{t-1}\left(\prod_{j=k+1}^{t-1}s_j\right)\left(\frac{w\zeta_{k+1}B_k}{2\xi_{k+1}(1 - B_k)} + \frac{\eta_{k+1}}{\xi_{k+1}}\frac{\widehat{B}_k - \mathbb{E}[q_k]B_k}{1 - B_k}\prod_{j=0}^{k-1}\mathbb{E}[q_j]l_0\right). \quad (14)$$

Hence, following from (12), (13) and (14), we derive the desired result, i.e., $\boldsymbol{\pi}_t^* = \big(\boldsymbol{\pi}_t^* - \mathbb{E}[\boldsymbol{\pi}_t^*]\big) + \mathbb{E}[\boldsymbol{\pi}_t^*]$ in (6).

Next, we show that this optimal strategy satisfies the linear constraints. At time 0, $\mathbb{E}[\boldsymbol{\pi}_0^* - \mathbb{E}[\boldsymbol{\pi}_0^*]] = \mathbf{0}$ is obvious due to $x_0 = \mathbb{E}[x_0]$ and $l_0 = \mathbb{E}[l_0]$. Then, according to the dynamic system of (4), we have $\mathbb{E}[x_1 - \mathbb{E}[x_1]] = 0$ and $\mathbb{E}[l_1 - \mathbb{E}[l_1]] = 0$, which further implies $\mathbb{E}[\boldsymbol{\pi}_1^* - \mathbb{E}[\boldsymbol{\pi}_1^*]] = \mathbf{0}$. Repeating this argument, we have $\mathbb{E}[\boldsymbol{\pi}_t^* - \mathbb{E}[\boldsymbol{\pi}_t^*]] = \mathbf{0}$ holds for all t.

Finally, we derive the efficient frontier. The optimal objective of problem (5) is as follows,

$$J_0\big(\mathbb{E}[x_0], 0, \mathbb{E}[l_0], 0\big) = -w\zeta_0x_0 + w\theta_0l_0 + \delta_0l_0^2 + w^2\psi_0. \quad (15)$$

In addition, from (14), we have

$$
\begin{aligned}
& \mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] \\
=\ & \sum_{t=0}^{T} \mathbb{E}[x_t]\alpha_t - \sum_{t=1}^{T} \mathbb{E}[l_t]\alpha_t \\
=\ & \sum_{t=0}^{T} \Bigg(x_0 \prod_{j=0}^{t-1} s_j + \sum_{k=0}^{t-1} \Bigg(\prod_{j=k+1}^{t-1} s_j \Bigg) \Bigg(\frac{w\zeta_{k+1}B_k}{2\xi_{t+1}(1-B_k)} + \frac{\eta_{k+1}}{\xi_{k+1}} \frac{\widehat{B}_k - \mathbb{E}[q_k]B_k}{1-B_k} \prod_{j=0}^{k-1} \mathbb{E}[q_j] l_0 \Bigg) - \mathbb{E}[l_t] \Bigg) \alpha_t \\
=\ & \zeta_0 x_0 - 2w\psi_0 - \theta_0 l_0,
\end{aligned}
$$

i.e.,

$$
w = -(2\psi_0)^{-1}\big(\mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] - \zeta_0 x_0 + \theta_0 l_0\big).
$$

Hence, according to (15), we can derive the variance term as

$$
\begin{aligned}
& \mathrm{Var}^{(\tau)}(x_{T\wedge\tau} - l_{T\wedge\tau}) \\
=& w\mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] + J_0\big(x_0, 0, l_0, 0\big) \\
=& -w^2\psi_0 + \delta_0 l_0^2 \\
=& -(4\psi_0)^{-1}\big(\mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] - \zeta_0 x_0 + \theta_0 l_0\big)^2 + \delta_0 l_0^2,
\end{aligned}
$$

which implies the efficient frontier.

References

[1] M. C. Chiu and D. Li, Asset and liability management under a continuous-time mean-variance optimization framework, *Insurance: Mathematics and Economics*, 39 (2006), 330-355.

[2] X. Y. Cui, J. J. Gao, X. Li, and D. Li, Optimal multiperiod mean-variance policy under no-shorting constraint, *European Journal of Operational Research*, 234 (2014), 459-468.

[3] X. Y. Cui, X. Li and D. Li, Unified framework of mean-field formulations for optimal multi-period mean-variance portfolio selection, *IEEE Transactions on Automatic Control*, 59 (2014), 1833-1844.

[4] C.P. Fu, A. Lari-Lavassani and X. Li, Dynamic mean variance portfolio selection with borrowing constraint, *European Journal of Operational Research*, 200 (2010), 312-319.

[5] A. Keel and H. Muller, Efficient portfolios in the asset liability context,*Astin Bulletin,* 25(1995), 33-48.

[6] C. G. Krouse, Portfolio Balancing Corporate Assets and Liabilities with Special Application to InsuranceManagement, *The Journal of Financial and Quantitative Analysis*, 5 (1970), 77-104.

[7] M. Leippold, F. Trojani and P. Vanini, A geometric approach to multiperiod mean variance optimization of assets and liabilities, *Journal of Economic Dynamics and Control*, 28 (2004), 1079-1113.

[8] C.J. Li, Z.F. Li, K. Fu and H.Q. Song, Time-consistent optimal portfolio strategy for asset-liability management under mean-variance criterion, *Accounting and Finance Research*, 2 (2013), 89-104.

[9] D. Li and W. L. Ng, Optimal dynamic portfolio selection: Multi-period mean-variance formulation, *Mathematical Finance*, 10 (2000), 387-406.

[10] X. Li, X.Y. Zhou and A.E.B. Lim, Dynamic mean-variance portfolio selection with no-shorting constraints, *SIAM Journal on Control and Optimization*, 40 (2002), 1540-1555.

[11] Z. F. Li and S. X. Xie, Mean-variance portfolio optimization under stochastic income and uncertain exit time, *Dynamics of Continuous, Discrete and Impulsive Systems*, 17 (2010), 131-147.

[12] H.M. Markowitz, Portfolio selection, *Journal of Finance*, 7 (1952), 77-91.

[13] W. F. Sharpe and L. G. Tint, Liabilities - a new approach, *Journal of Portfolio Management*, 16 (1990), 5-10.

[14] M. E. Yaari, Uncertain lifetime, life insurance, and the theory of the consumer, *The Review of Economic Studies*, 32 (1965), 137-150.

[15] H. X. Yao, Y. Z. Lai, and Z. F. Hao, Uncertain exit time multi-period mean-variance portfolio selection with endogenous liabilities and Markov jumps , *Automatica*, 49 (2013), 3258-3269.

[16] L. Yi, X. P. Wu, X. Li and X. Y. Cui, Mean-field formulation for optimal multi-period mean-variance portfolio selection with uncertain exit time, *Operations Research Letters*, 42 (2014), 489-494.

[17] L. Yi, Z. F. Li and D. Li, Multi-period portfolio selection for asset-liability management with uncertain investment horizon, *Journal of Industrial and Management Optimization*, 4 (2008), 535-552.

[18] L. Zhang and Z. F. Li, Multi-period mean-varinace portfolio selection with uncertain time horizon when returns are serially correlated, *Mathematical Probelems in Engineering*, Volume 2012 (2012), Article ID 216891, 17 pages.

[19] X. Y. Zhou and D. Li, Continuous-time mean-variance portfolio selection: A stochastic LQ framework, *Applied Mathematics and Optimization*, 42 (2000), 19-33.