https://doi.org/10.1016/j.orl.2016.09.004

Continuous-Time Markowitz's Model with Constraints on Wealth and Portfolio☆

Xun Li[1], Zuo Quan Xu[1,*]

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China

Abstract

We consider a continuous-time Markowitz's model with bankruptcy prohibition and convex cone portfolio constraints. The problem is long-standing and difficult not only because of its theoretical significance, but also for its practical importance. We first transform the problem into an equivalent one with bankruptcy prohibition but *without* portfolio constraints. The latter is then treated by martingale theory. This approach allows one to directly present the semi-analytical expressions of the pre-committed efficient mean-variance policy without using the viscosity solution technique but within the framework of cone portfolio constraints. The numerical simulation also sheds light on results established in this paper.

Keywords: Continuous-time, Markowitz's mean-variance model, bankruptcy prohibition, convex cone constraints, efficient frontier, stochastic LQ control, HJB equation

1. Introduction

Since Markowitz [14] published his seminal work on the mean-variance portfolio selection nearly sixty years ago, the mean-risk portfolio selection framework has become one of the most prominent topics in quantitative finance. Recently, there has been increasing interest in studying the dynamic mean-variance portfolio problem with various constraints, as well as addressing their financial applications. Typical contributions include [1], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and [18]. The dynamic mean-variance problem can be treated in a forward-looking way by starting with the initial state. In some financial engineering problems, however, one needs to study stochastic systems with constrained conditions, such as cone-constrained policies. This naturally results in a continuous-time mean-variance portfolio selection problem with *constraints for the wealth process* (see [1]), and/or *constraints for the policies* (see [10] and [13]). To the best of our knowledge, despite active research efforts put in this direction in recent years, there has barely any progress in the study of the continuous-time mean-variance problem with the *mixed* restriction of bankruptcy prohibition and convex cone portfolio constraints. In this paper, we aim to address this long-standing and notoriously difficult problem, not only for its theoretical significance, but also for its practical importance. Our new approach, significantly different from those developed in the existing literature, will establish a general theory for stochastic control problems with mixed constraints for both state and control variables.

Li, Zhou and Lim [13] considered a continuous-time mean-variance problem with no-shorting constraints, while Cui, Gao, Li and Li [3] developed its counterpart in discrete-time. Bielecki, Jin, Pliska and Zhou [1] paved the way for investigating continuous-time mean-variance with bankruptcy prohibition using the martingale approach. Labbe and Heunis [11] employed a duality method to analyze both the mean-variance portfolio selection and mean-variance hedging problems with general convex constraints. Czichowsky and Schweizer [5] further studied cone-constrained continuous-time mean-variance portfolio selection problem with the price processes being semi-martingales. Meanwhile, Pham and Touzi [15] showed that in a constrained market, no arbitrage opportunity is equivalent to the existence of a supermartingale measure, under which the discounted wealth process of any admissible policy is a supermartingale (see [2] for a situation with upper bounds on proportion positions). In particular, Xu and Shreve [16, 17] investigated a utility maximization problem with no-shorting constraints using the duality analysis. Recently, Heunis [8] considered the expected value of a general quadratic loss function of the wealth in a more general constraint setting.

The existing theories and methods cannot easily handle the continuous-time mean-variance problem with the mixed restriction of bankruptcy prohibition and convex cone portfolio constraints. Based on our analysis, we find out that the market price of risk in policy is actually independent of the wealth process. This important finding allows us to overcome the difficulty of the original problem and also makes the similar continuous-time financial investment problem both interesting and practical. In fact, we first show that the problem with the mixed restriction is equivalent to one only with bankruptcy prohibition via studying the Hamilton-Jacobi-Bellman (HJB) equations of

☆We thank the area editor and the anonymous referee for their comments and suggestions. This research work was partially supported by Research Grants Council of Hong Kong under grants 529711, 519913, 15209614 and 15224215, Hong Kong Early Career Scheme 533112, and National Natural Science Foundation of China under grants 11471276.

*Corresponding author.

Email addresses: malixun@polyu.edu.hk (Xun Li), maxu@polyu.edu.hk (Zuo Quan Xu)

the two problems. We then discuss the equivalent problem using the results obtained in [1].

The reminder of the paper is organized as follows. In Section 2, we formulate a mean-variance problem with bankruptcy prohibition and convex cone portfolio constraints. In Section 3, we transform our problem into an equivalent mean-variance problem with bankruptcy prohibition but *without* convex cone portfolio constraints. Then, in Section 4, we further derive the pre-committed policy for the problem using the results derived in [1]. In Section 5, we discuss properties of mean-variance problems with different constraints. In Section 6, we present a numerical simulation to illustrate results established in the previous sections. Finally, we summarize the paper in Section 7.

2. Problem Formulation and Preliminaries

2.1. Notation

We use the following notation throughout the paper:

- M' : the transpose of any matrix or vector M;
- $|a|$: $= \sqrt{\sum_i a_i^2}$ for any vector $a = (a_i)$;
- $\|M\|$: $= \sqrt{\sum_{i,j} m_{ij}^2}$ for any matrix $M = (m_{ij})$;
- $\mathbb{R}^m$: m dimensional real Euclidean space;
- $\mathbb{R}^m_+$: the subset of $\mathbb{R}^m$ consisting of elements with nonnegative components;
- $\mathbf{1}_A$: the indicator function for an event A that is equal to 1 if A happens, and 0 otherwise.

The underlying uncertainty is generated on a fixed filtered complete probability space $(\Omega, \mathbf{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\geqslant 0})$ on which is defined a standard $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted m-dimensional Brownian motion $W(\cdot) \equiv (W^1(\cdot), \cdots, W^m(\cdot))'$. Given a Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|_{\mathcal{H}}$, we can define a Banach space

$$L^2_{\mathcal{F}}(a, b; \mathcal{H}) = \left\{ \varphi(\cdot) \middle| \begin{array}{l} \varphi(\cdot) \text{ is an } \mathcal{F}_t\text{-adapted, } \mathcal{H}\text{-valued measurable} \\ \text{process on } [a, b] \text{ and } \|\varphi(\cdot)\|_{\mathcal{F}} < +\infty \end{array} \right\}$$

with the norm

$$\|\varphi(\cdot)\|_{\mathcal{F}} = \left(\mathbf{E}\left[\int_a^b \|\varphi(t, \omega)\|^2_{\mathcal{H}} \,\mathrm{d}t \right] \right)^{\frac{1}{2}}.$$

2.2. Problem Formulation

Consider an arbitrage-free financial market where $m+1$ assets are traded continuously on a finite horizon $[0, T]$. One asset is a *bond*, whose price $S_0(t)$ evolves according to the ordinary differential equation

$$\begin{cases} \mathrm{d}S_0(t) = r(t) S_0(t)\,\mathrm{d}t, & t \in [0, T], \\ S_0(0) = s_0 > 0, \end{cases}$$

where $r(t)$ is the interest rate of the bond at time t. The remaining m assets are *stocks*, and their prices are modeled by the system of stochastic differential equations

$$\begin{cases} \mathrm{d}S_i(t) = S_i(t)\{b_i(t)\,\mathrm{d}t + \sum_{j=1}^m \sigma_{ij}(t)\,\mathrm{d}W^j(t)\}, & t \in [0, T], \\ S_i(0) = s_i > 0, \end{cases}$$

where $b_i(t)$ is the appreciation rate of the i^{th} stock and $\sigma_{ij}(t)$ is the volatility coefficient at time t. Denote $b(t) := (b_1(t), \cdots, b_m(t))'$ and $\sigma(t) := (\sigma_{ij}(t))$. We assume throughout that $r(t)$, $b(t)$ and $\sigma(t)$ are given deterministic, measurable, and uniformly bounded functions on $[0, T]$. In addition, we assume that the non-degeneracy condition on $\sigma(\cdot)$, that is,

$$y'\sigma(t)\sigma(t)'y \geqslant \delta y'y, \quad \forall\ (t, y) \in [0, T] \times \mathbb{R}^m, \tag{1}$$

is satisfied for some scalar $\delta > 0$. Also, we define the excess return vector $B(t) = (b_1(t) - r(t), \cdots, b_m(t) - r(t))$.

Suppose an agent has an initial wealth $x_0 > 0$ and the total wealth of his position at time t is $X(t)$. Denote by $\pi_i(t)$, $i = 1, \cdots, m$, the total market value of the agent's wealth in the i^{th} stock at time t. We call $\pi(\cdot) := (\pi_1(\cdot), \cdots, \pi_m(\cdot))' \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ a portfolio. We will consider self-financing portfolios here. Then it is well-known that $X(\cdot)$ follows (see [18])

$$\begin{cases} \mathrm{d}X(t) = [r(t)X(t) + \pi(t)'B(t)]\,\mathrm{d}t + \pi(t)'\sigma(t)\,\mathrm{d}W(t), \\ X(0) = x_0. \end{cases} \tag{2}$$

An important restriction considered in this paper is the convex cone portfolio constraints, that is $\pi(\cdot) \in C$, where

$$C = \{\pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \colon C(t)'\pi(t) \in \mathbb{R}^k_+, \forall\ t \in [0, T]\},$$

and $C : [0, T] \mapsto \mathbb{R}^{m\times k}$ is a given deterministic and measurable function. Another important restriction considered in this paper is the prohibition of bankruptcy, namely

$$X(t) \geqslant 0, \quad \forall\ t \in [0, T]. \tag{3}$$

Meanwhile, borrowing from the money market (at the interest rate $r(\cdot)$) is still allowed; that is, the money invested in the bond $\pi_0(\cdot) = X(\cdot) - \sum_{i=1}^m \pi_i(\cdot)$ has no constraint.

Definition 1. *A portfolio $\pi(\cdot)$ is called an admissible control (or portfolio) if $\pi(\cdot) \in C$ and the corresponding wealth process $X(\cdot)$ defined in* (2) *satisfies* (3)*. In this case, the process $X(\cdot)$ is called an admissible wealth process, and $(X(\cdot), \pi(\cdot))$ is called an admissible pair.*

Remark 1. *In view of the boundedness of $\sigma(\cdot)$ and the non-degeneracy condition* (1)*, we have that $\pi(\cdot) \in L^2_{\mathcal{F}}(a, b; \mathbb{R}^m)$ if and only if $\sigma(\cdot)'\pi(\cdot) \in L^2_{\mathcal{F}}(a, b; \mathbb{R}^m)$. The latter is often used to define the admissible process in the literature, for instance, [1].*

Remark 2. *It is easy to show that both the set of all admissible controls and the set of all admissible wealth processes are convex. As a consequence, the set of all expected terminal wealths $\{\mathbf{E}[X(T)] : X(\cdot)$ is an admissible process$\}$ is an interval.*

Mean-variance portfolio selection refers to the problem of, given a favorable mean level d, finding an allowable investment policy (i.e., a dynamic portfolio satisfying all the constraints), such that the expected terminal wealth $\mathbf{E}[X(T)]$ is d while the risk measured by the variance of the terminal wealth

$$\mathbf{Var}(X(T)) = \mathbf{E}[X(T) - \mathbf{E}[X(T)]]^2 = \mathbf{E}[X(T) - d]^2$$

is minimized.

The following assumption is standard in the mean-risk portfolio selection literature (see, e.g., Assumption 2.1 in [13]).

Assumption 1. *The value of the expected terminal wealth d satisfies $d \geqslant x_0 e^{\int_0^T r(s)\,\mathrm{d}s}$.*

Definition 2. *The mean-variance portfolio selection problem is formulated as the following optimization problem parameterized by d:*

$$\begin{aligned} \min_{\pi(\cdot)} \quad & \mathbf{Var}(X(T)) = \mathbf{E}[X(T)-d]^2, \\ \text{subject to} \quad & \begin{cases} \mathbf{E}[X(T)] = d, \\ \pi(\cdot) \in C \text{ and } X(\cdot) \geqslant 0, \\ (X(\cdot), \pi(\cdot)) \text{ satisfies the equation (2).} \end{cases} \end{aligned} \tag{4}$$

An optimal control satisfying (4) *is called an efficient strategy, and* $(\sqrt{\mathbf{Var}(X(T))}, d)$*, where* $\mathbf{Var}(X(T))$ *is the optimal value of* (4) *corresponding to d, is called an efficient point. The set of all efficient points, when the parameter d runs over all possible values, is called the efficient frontier.*

In the current setting, the admissible controls belong to a convex cone, so the value of the expected terminal wealth may not be arbitrary. Denote by $V(d)$ the optimal value of the problem (4). Denote

$$\widehat{d} = \sup\{\mathbf{E}[X(T)] : X(\cdot) \text{ is an admissible process}\}.$$

Taking $\pi(\cdot) \equiv 0$, we see that $X(t) \equiv x_0 e^{\int_0^t r(s)\,\mathrm{d}s}$ is an admissible process, so $\widehat{d} \geqslant \mathbf{E}[X(T)] = x_0 e^{\int_0^T r(s)\,\mathrm{d}s}$. The following nontrivial example shows that it is possible that $\widehat{d} = x_0 e^{\int_0^T r(s)\,\mathrm{d}s}$.

Example 1. *Let $B(\cdot) = -C(\cdot)\chi$, where χ is any positive vector of appropriate dimension. Then for any admissible control $\pi(\cdot) \in C$, we have $\pi(\cdot)'B(\cdot) = -\pi(\cdot)'C(\cdot)\chi \leqslant 0$. Therefore, by* (2),

$$\mathrm{d}(\mathbf{E}[X(t)]) = (r(t)\,\mathbf{E}[X(t)] + \mathbf{E}[\pi(t)'B(t)])\,\mathrm{d}t \leqslant r(t)\,\mathbf{E}[X(t)]\,\mathrm{d}t,$$

which implies $\mathbf{E}[X(T)] \leqslant x_0 e^{\int_0^T r(s)\,\mathrm{d}s}$*. Hence* $\widehat{d} = x_0 e^{\int_0^T r(s)\,\mathrm{d}s}$*.*

Theorem 1. *Assume that $\widehat{d} = x_0 e^{\int_0^T r(s)\,\mathrm{d}s}$. Then the optimal value of the problem* (4) *is 0.*

Proof. From Assumption 1 and with $\widehat{d} = x_0 e^{\int_0^T r(s)\,\mathrm{d}s}$, we obtain that the only possible value of d is $x_0 e^{\int_0^T r(s)\,\mathrm{d}s}$. Note that $(X(t), \pi(t)) \equiv (x_0 e^{\int_0^t r(s)\,\mathrm{d}s}, 0)$ is an admissible pair satisfying the constraint of the problem (4), so $V(d) \leqslant \mathbf{E}[X(T)-d]^2 = \mathbf{E}[x_0 e^{\int_0^T r(s)\,\mathrm{d}s} - d]^2 = 0$. The claim follows immediately. □

From now on we assume $\widehat{d} > x_0 e^{\int_0^T r(s)\,\mathrm{d}s}$. Denote $\mathfrak{D} = (0, \widehat{d}\,)$ and $\mathfrak{D}^+ = [x_0 e^{\int_0^T r(s)\,\mathrm{d}s}, \widehat{d}\,)$.

Lemma 1. *The value function $V(\cdot)$ is convex on $\mathfrak{D}$ and strictly increasing on $\mathfrak{D}^+$.*

Proof. Let $(\overline{X}(\cdot), \overline{\pi}(\cdot))$ and $(\widetilde{X}(\cdot), \widetilde{\pi}(\cdot))$ be any two admissible pairs such that $d_1 = \mathbf{E}[\overline{X}(T)]$ and $d_2 = \mathbf{E}[\widetilde{X}(T)]$ are different and both in $\mathfrak{D}$. For any $0 < \alpha < 1$, define $(\widehat{X}(\cdot), \widehat{\pi}(\cdot)) = (\alpha\overline{X}(\cdot) + (1-\alpha)\widetilde{X}(\cdot), \alpha\overline{\pi}(\cdot) + (1-\alpha)\widetilde{\pi}(\cdot))$. Then $(\widehat{X}(\cdot), \widehat{\pi}(\cdot))$ satisfies (2). Moreover, $\widehat{\pi}(\cdot) \in C$, $\widehat{X}(\cdot) \geqslant 0$ and $\mathbf{E}[\widehat{X}(T)] = \alpha d_1 + (1-\alpha)d_2 \in \mathfrak{D}$, so $(\widehat{X}(\cdot), \widehat{\pi}(\cdot))$ is an admissible pair. Hence,

$$\begin{aligned} V(\alpha d_1 + (1-\alpha)d_2) &\leqslant \mathbf{Var}(\widehat{X}(T)) = \mathbf{Var}(\alpha\overline{X}(\cdot) + (1-\alpha)\widetilde{X}(\cdot)) \\ &\leqslant \alpha\,\mathbf{Var}(\overline{X}(T)) + (1-\alpha)\,\mathbf{Var}(\widetilde{X}(T)), \end{aligned}$$

by the convexity of square function. Because $(\overline{X}(\cdot), \overline{\pi}(\cdot))$ and $(\widetilde{X}(\cdot), \widetilde{\pi}(\cdot))$ are arbitrary chosen, we conclude that

$$V(\alpha d_1 + (1-\alpha)d_2) \leqslant \alpha V(d_1) + (1-\alpha)V(d_2).$$

This establishes the convexity of $V(\cdot)$.

If $\pi(\cdot) \equiv 0$, then $X(T) = x_0 e^{\int_0^T r(s)\,\mathrm{d}s}$. This clearly implies that $V\left(x_0 e^{\int_0^T r(s)\,\mathrm{d}s}\right) = 0$. It is known that if there are no portfolio constraints (i.e. $C(t) \equiv 0$), then the optimal value is positive on $\mathfrak{D}^+$ (see [1]), so $V(\cdot)$ must be positive in the interior of $\mathfrak{D}^+$. The convexity of $V(\cdot)$ implies that it is strictly increasing on $\mathfrak{D}^+$. □

Corollary 1. *The value function $V(\cdot)$ is finite and continuous on $\mathfrak{D}$.*

Since the problem (4) is a convex optimization problem, the mean constraint $\mathbf{E}[X(T)] = d$ can be dealt with by introducing a Lagrange multiplier. As is well-known, the mean-variance portfolio selection problem (4) is meaningful only when $d \in \mathfrak{D}^+$. We will focus on this case from now on.

Because $V(\cdot)$ is convex on $\mathfrak{D}$ and strictly increasing at any $d \in (x_0 e^{\int_0^T r(s)\,\mathrm{d}s}, \widehat{d}\,)$, there is a constant $\lambda > 0$ such that $V(x) - 2\lambda x \geqslant V(d) - 2\lambda d$ for all $x \in \mathfrak{D}$. In this way the portfolio selection problem (4) is equivalent to the following problem

$$\begin{aligned} \min_{\pi(\cdot)} \quad & \mathbf{E}[X(T)-d]^2 - 2\lambda(\mathbf{E}[X(T)] - d), \\ \text{subject to} \quad & \begin{cases} \pi(\cdot) \in C \text{ and } X(\cdot) \geqslant 0, \\ (X(\cdot), \pi(\cdot)) \text{ satisfies the equation (2),} \end{cases} \end{aligned}$$

or equivalently,

$$\begin{aligned} \min_{\pi(\cdot)} \quad & \mathbf{E}[X(T) - (d+\lambda)]^2, \\ \text{subject to} \quad & \begin{cases} \pi(\cdot) \in C \text{ and } X(\cdot) \geqslant 0, \\ (X(\cdot), \pi(\cdot)) \text{ satisfies the equation (2)} \end{cases} \end{aligned}$$

in the sense that these problems have exactly the same optimal pair if one of them admits one.

We plan to use dynamic programming to study the aforementioned problems, so we denote by $\hat{V}(t,x)$ the optimal value of problem

$$\begin{aligned} \min_{\pi(\cdot)} \quad & \mathbf{E}[(X(T) - (d+\lambda))^2 | \mathcal{F}_t, X(t) = x], \\ \text{subject to} \quad & \begin{cases} \pi(\cdot) \in C \text{ and } X(\cdot) \geqslant 0, \\ (X(\cdot), \pi(\cdot)) \text{ satisfies the equation (2).} \end{cases} \end{aligned} \tag{5}$$

Lemma 2. *The function $\widehat{V}(t,\cdot)$ is strictly decreasing and convex on $(0, (d+\lambda)e^{-\int_t^T r(s)\,\mathrm{d}s}]$ for every fixed $t \in [0,T]$.*

Proof. The proof is similar to that of Lemma 1. We leave the proof for the interested readers. □

Remark 3. *As is well-known, if the initial wealth $X(t) = x$ is too big compared to the target $d+\lambda$, then the mean-variance portfolio selection problem* (5) *is not meaningful. This makes us focus on the small initials in* $(0, (d+\lambda)e^{-\int_t^T r(s)\,\mathrm{d}s}]$.

Lemma 3. *If $X(\cdot)$ is a feasible wealth process with $X(t) = 0$ for some $t \in [0,T]$, then $X(s) = 0$ for all $s \in [t,T]$.*

PROOF. Since $X(\cdot)$ is a feasible wealth process, we have $X(s) \geqslant 0$, for all $s \in [t, T]$. If $\mathbf{P}(X(s) > 0)$ is positive for some $s \in [t, T]$, then this leads to an arbitrage opportunity. $\square$

Lemma 4. *We have that* $\widehat{V}(t, 0) = (d + \lambda)^2$ *and* $\widehat{V}\left(t, (d + \lambda)e^{-\int_t^T r(s)\,\mathrm{d}s}\right) = 0$ *for all* $t \in [0, T]$.

PROOF. If $X(t) = 0$, then $X(T) = 0$ by Lemma 3. Hence, $\widehat{V}(t, 0) = (d + \lambda)^2$.

Suppose $X(t) = (d + \lambda)e^{-\int_t^T r(s)\,\mathrm{d}s}$. Then taking $\pi(\cdot) \equiv 0$, we obtain that $X(T) = d+\lambda$, so $\widehat{V}\left(t, (d + \lambda)e^{-\int_t^T r(s)\,\mathrm{d}s}\right) \leqslant \mathbf{E}[X(T)-(d + \lambda)]^2 = 0$. The proof is complete. $\square$

3. An Equivalent Stochastic Problem

Since the Riccati equation approach to solve the problem (5) is not applicable in this case, we consider the corresponding Hamilton-Jacobi-Bellman (HJB) equation. This is the following partial differential equation:

$$\begin{cases} \mathcal{L}v = 0, & (t, x) \in \mathfrak{S}, \\ v\left(t, (d + \lambda)e^{-\int_t^T r(s)\,\mathrm{d}s}\right) = 0, v(t, 0) = (d + \lambda)^2, & 0 \leqslant t \leqslant T, \\ v(T, x) = (x - (d + \lambda))^2, & 0 < x < d + \lambda, \end{cases} \tag{6}$$

where

$$\mathcal{L}v = v_t(t, x)+\inf_{\pi\in C_t}\left\{v_x(t, x)[r(t)x + \pi' B(t)]+\frac{1}{2}v_{xx}(t, x)\pi'\sigma(t)\sigma(t)'\pi\right\},$$

$$\mathfrak{S} = \left\{(t, x) : 0 \leqslant t < T,\ 0 < x < (d + \lambda)e^{-\int_t^T r(s)\,\mathrm{d}s}\right\},$$

and $C_t = \{z \in \mathbb{R}^m : C(t)'z \in \mathbb{R}^k_+\}$.

We need the following technical result.

Lemma 5. *Suppose the problem* (6) *admits a solution* $v \in C^{1,2}(\mathfrak{S})$ *which is convex in the second argument. Then* $v \leqslant (d + \lambda)^2$ *on* $\mathfrak{S}$.

PROOF. By the convexity of v in the second argument, we have, for each $(t, x) \in \mathfrak{S}$,

$$v(t, x) \leqslant \max\left\{v(t, 0), v\left(t, (d + \lambda)e^{-\int_t^T r(s)\,\mathrm{d}s}\right)\right\} = (d + \lambda)^2.$$

The proof is complete. $\square$

Now we are ready to establish the following verification theorem:

Theorem 2. *Suppose the problem* (6) *admits a solution* $v \in C^{1,2}(\mathfrak{S})$ *which is convex in the second argument. Then* $\widehat{V} = v$ *on* $\mathfrak{S}$.

PROOF. Without loss of generality, we shall show $\widehat{V}(0, x_0) = v(0, x_0)$. Let $(X(\cdot), \pi(\cdot))$ be an admissible pair. Define

$$\tau = \inf\left\{t \in [0, T] : X(t) = 0 \text{ or } X(t) = (d + \lambda)e^{-\int_t^T r(s)\,\mathrm{d}s}\right\} \wedge T,$$

$$\tau_N = \sup\left\{t \in [0, T] : \int_0^t \|v_x(s, X(s))\pi(s)'\sigma(s)\|^2\,\mathrm{d}s \leqslant N\right\} \wedge T.$$

Applying Itô's Lemma to $v(t, X_t)$ yields

$$\begin{aligned} & v(\tau \wedge \tau_N, X(\tau \wedge \tau_N)) \\ & = \int_0^{\tau\wedge\tau_N} \Big(v_t(t, X(t)) + v_x(t, X(t))[r(t)X(t) + \pi(t)'B(t)] \\ & \quad + \frac{1}{2}v_{xx}(t, X(t))\pi(t)'\sigma(t)\sigma(t)'\pi(t)\Big)\,\mathrm{d}t \\ & \quad + \int_0^{\tau\wedge\tau_N} v_x(t, X(t))\pi(t)'\sigma(t)\,\mathrm{d}W(t) + v(0, x_0) \\ & \geqslant \int_0^{\tau\wedge\tau_N} \mathcal{L}v(t, X(t))\,\mathrm{d}t + \int_0^{\tau\wedge\tau_N} v_x(t, X(t))\pi(t)'\sigma(t)\,\mathrm{d}W(t)+v(0, x_0) \\ & \geqslant \int_0^{\tau\wedge\tau_N} v_x(t, X(t))\pi(t)'\sigma(t)\,\mathrm{d}W(t) + v(0, x_0). \end{aligned}$$

By taking expectation on both sides, we obtain

$$\begin{aligned} & \mathbf{E}[v(\tau \wedge \tau_N, X(\tau \wedge \tau_N))] \\ & \geqslant \mathbf{E}\left[\int_0^{\tau\wedge\tau_N} v_x(t, X(t))\pi(t)'\sigma(t)\,\mathrm{d}W(t) + v(0, x_0)\right] = v(0, x_0). \end{aligned}$$

Because v is continuous, and $\tau \wedge \tau_N$ and $X(\tau \wedge \tau_N)$ are both uniformly bounded, by letting $N \to \infty$ and applying the dominated convergence theorem, we get

$$\mathbf{E}[v(\tau, X(\tau))] \geqslant v(0, x_0). \tag{7}$$

If $X(\tau) = 0$, then $X(T) = 0$ by Lemma 3. Applying Lemma 5 yields

$$\begin{aligned} \mathbf{E}[v(T, X(T))|\mathcal{F}_\tau]\mathbf{1}_{\{X(\tau)=0\}} &= (d + \lambda)^2\mathbf{1}_{\{X(\tau)=0\}} \\ & \geqslant v(\tau, X(\tau))\mathbf{1}_{\{X(\tau)=0\}}. \end{aligned} \tag{8}$$

If $X(\tau) = (d + \lambda)e^{-\int_\tau^T r(s)\,\mathrm{d}s}$, then $v(\tau, X(\tau)) = 0$. This trivially leads to

$$\begin{aligned} & \mathbf{E}[v(T, X(T))|\mathcal{F}_\tau]\mathbf{1}_{\{X(\tau)=(d+\lambda)e^{-\int_\tau^T r(s)\,\mathrm{d}s}\}} \\ & \qquad \geqslant v(\tau, X(\tau))\mathbf{1}_{\{X(\tau)=(d+\lambda)e^{-\int_\tau^T r(s)\,\mathrm{d}s}\}}. \end{aligned} \tag{9}$$

If $0 < X(\tau) < (d + \lambda)e^{-\int_\tau^T r(s)\,\mathrm{d}s}$, then $\tau = T$ by its definition. Hence,

$$\begin{aligned} & \mathbf{E}[v(T, X(T))|\mathcal{F}_\tau]\mathbf{1}_{\{0<X(\tau)<(d+\lambda)e^{-\int_\tau^T r(s)\,\mathrm{d}s}\}} \\ & = \mathbf{E}[v(\tau, X(\tau))|\mathcal{F}_\tau]\mathbf{1}_{\{0<X(\tau)<(d+\lambda)e^{-\int_\tau^T r(s)\,\mathrm{d}s}\}} \\ & = v(\tau, X(\tau))\mathbf{1}_{\{0<X(\tau)<(d+\lambda)e^{-\int_\tau^T r(s)\,\mathrm{d}s}\}}. \end{aligned} \tag{10}$$

From (8), (9) and (10), we obtain

$$\mathbf{E}[v(T, X(T))|\mathcal{F}_\tau] \geqslant v(\tau, X(\tau)).$$

Together with (7), it leads to

$$\mathbf{E}[v(T, X(T))] \geqslant \mathbf{E}[v(\tau, X(\tau))] \geqslant v(0, x_0).$$

Note $v(T, X(T)) = (X(T) - (d + \lambda))^2$, so $\widehat{V}(0, x_0) \geqslant v(0, x_0)$.

On the other hand, define a portfolio

$$\pi(t) = \begin{cases} 0, & \text{if } X(t) = 0; \\ 0, & \text{if } X(t) = (d+\lambda)e^{-\int_t^T r(s)\,\mathrm{d}s}; \\ \operatorname*{argmin}_{\pi \in C_t} \left\{ v_x(t, X(t))\pi' B(t) + \frac{1}{2} v_{xx}(t, X(t))\pi'\sigma(t)\sigma(t)'\pi \right\}, & \\ & \text{otherwise.} \end{cases}$$

It is not hard to see that $(X(\cdot), \pi(\cdot))$ is an admissible pair. Then we see that (7), (8), (9) and (10) become identities, so

$$\mathbf{E}[X(T) - (d+\lambda)]^2 = \mathbf{E}[v(T, X(T))] = \mathbf{E}[v(\tau, X(\tau))] = v(0, x_0).$$

This implies that $\widehat{V}(0, x_0) \leqslant v(0, x_0)$. The proof is complete. □

Before going further, we need the following key result.

Lemma 6. *Suppose* $\mathbf{A} \in \mathbb{R}^{m\times k}$, $\mathbf{B} \in \mathbb{R}^m$, $\mathbf{C} = \{z \in \mathbb{R}^m : \mathbf{A}' z \in \mathbb{R}^k_+\}$, *and* $\mathbf{D} \in \mathbb{R}^{m\times m}$ *is invertible. Then, for* $\alpha > 0$, *the following two convex optimization problems*

$$\min_{z\in\mathbf{C}} \tfrac{1}{2} z' \mathbf{D}\mathbf{D}' z - \alpha \mathbf{B}' z$$

and

$$\min_{z\in\mathbb{R}^m} \tfrac{1}{2} z' \mathbf{D}\mathbf{D}' z - \alpha \bar{z}' \mathbf{D}\mathbf{D}' z \tag{11}$$

have the same optimal solution $\alpha\bar{z}$ *and the same optimal value* $-\frac{1}{2}\alpha^2 \bar{z}\,\mathbf{D}'\,\mathbf{D}\,\bar{z}$, *where*

$$\bar{z} = \operatorname*{argmin}_{z\in\mathbf{C}} \| \mathbf{D}' z - \mathbf{D}^{-1}\mathbf{B} \|.$$

Proof. Because $\mathbf{C}$ is a cone, it is sufficient to study the case $\alpha = 1$. From the definition, $\bar{z}$ solves

$$\min_{z\in\mathbf{C}} \tfrac{1}{2} z' \mathbf{D}\mathbf{D}' z - \mathbf{B}' z = \min_{z\in\mathbb{R}^m,\ \mathbf{A}'z\in\mathbb{R}^k_+} \tfrac{1}{2} z' \mathbf{D}\mathbf{D}' z - \mathbf{B}' z,$$

which is equivalent to

$$\min_{z\in\mathbb{R}^m} \tfrac{1}{2} z' \mathbf{D}\mathbf{D}' z - \mathbf{B}' z - \nu' \mathbf{A}' z, \tag{12}$$

for some Lagrangian multiplier $\nu \in \mathbb{R}^k$. This problem clearly admits another solution $(\mathbf{D}\mathbf{D}')^{-1}(\mathbf{B} + \mathbf{A}\,\nu)$. Due to the uniqueness of its solution, we conclude $\bar{z} = (\mathbf{D}\mathbf{D}')^{-1}(\mathbf{B} + \mathbf{A}\,\nu)$, and hence $\mathbf{D}\mathbf{D}'\,\bar{z} = \mathbf{B} + \mathbf{A}\,\nu$. This implies that

$$\tfrac{1}{2} z' \mathbf{D}\mathbf{D}' z - \mathbf{B}' z - \nu' \mathbf{A}' z = \tfrac{1}{2} z' \mathbf{D}\mathbf{D}' z - \bar{z}' \mathbf{D}\mathbf{D}' z.$$

Hence the problem (12) is just the problem (11) with $\alpha = 1$. This completes proof. □

Remark 4. *By Lemma 2, we know that the solution* $\widehat{V}(t,\cdot)$ *to the problem (6) is strictly decreasing and convex on* $(0, (d+\lambda)e^{-\int_t^T r(s)\,\mathrm{d}s}]$ *for every* $t \in [0,T]$. *Therefore,* $v_x(t,x) < 0$ *and* $v_{xx}(t,x) > 0$ *for every* $t \in [0,T]$.

We now return to the HJB equation (6). Let

$$\bar{z}(t) := \operatorname*{argmin}_{z\in C_t} \|\sigma(t)' z - \sigma(t)^{-1} B(t)\|. \tag{13}$$

By Lemma 6 with $\alpha = -\frac{v_x(t,x)}{v_{xx}(t,x)} > 0$, the infimum in the HJB equation (6) is attained at

$$\pi = -\frac{v_x(t,x)}{v_{xx}(t,x)} \bar{z}(t) \in C_t\,.$$

Again by Lemma 6, the HJB equation (6) is equivalent to

$$\begin{cases} v_t(t,x) + \inf_{\pi\in\mathbb{R}^m} \Big\{ v_x(t,x)[r(t)x + \pi'\widehat{B}(t)] \\ \qquad\qquad + \frac{1}{2} v_{xx}(t,x)\pi'\sigma(t)\sigma(t)'\pi \Big\} = 0, \quad (t,x) \in \mathfrak{S}, \\ v\Big(t, (d+\lambda)e^{-\int_t^T r(s)\,\mathrm{d}s}\Big) = 0, v(t,0) = (d+\lambda)^2, \quad 0 \leqslant t \leqslant T, \\ v(T,x) = \big(x - (d+\lambda)\big)^2, \qquad 0 < x < d+\lambda, \end{cases}$$

where $\widehat{B}(t) = \sigma(t)\sigma(t)'\bar{z}(t)$.

On the other hand, the above equation is the HJB equation associated with the following problem

$$\begin{aligned} &\min_{\pi(\cdot)} && \mathbf{E}[(X(T) - (d+\lambda))^2], \\ &\text{subject to} && \begin{cases} \pi(\cdot) \in \widehat{C} \text{ and } X(\cdot) \geqslant 0, \\ (X(\cdot), \pi(\cdot)) \text{ satisfies the equation (14),} \end{cases} \end{aligned}$$

where $\widehat{C} = \{\pi(\cdot) : \pi(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)\}$, and

$$\begin{cases} \mathrm{d}X(t) = [r(t)X(t) + \pi(t)'\widehat{B}(t)]\,\mathrm{d}t + \pi(t)'\sigma(t)\,\mathrm{d}W(t), \\ X(0) = x_0. \end{cases} \tag{14}$$

Removing the Lagrange multiplier, the above problem has the same optimal control as the following mean-variance problem without constraints on the portfolio:

$$\begin{aligned} &\min_{\pi(\cdot)} && \mathbf{Var}(X(T)) = \mathbf{E}[X(T) - \tilde{d}]^2, \\ &\text{subject to} && \begin{cases} \mathbf{E}[X(T)] = \tilde{d}, \\ \pi(\cdot) \in \widehat{C} \text{ and } X(\cdot) \geqslant 0, \\ (X(\cdot), \pi(\cdot)) \text{ satisfies the equation (14),} \end{cases} \end{aligned}$$

for some $\tilde{d}$. Because the optimal solution to the above problem is also optimal to the problem (4), their means of the optimal terminal wealths should be the same, namely $\tilde{d} = d$. Therefore, we conclude that the problem (4) and the following problem

$$\begin{aligned} &\min_{\pi(\cdot)} && \mathbf{Var}(X(T)) = \mathbf{E}[X(T) - d]^2, \\ &\text{subject to} && \begin{cases} \mathbf{E}[X(T)] = d, \\ \pi(\cdot) \in \widehat{C} \text{ and } X(\cdot) \geqslant 0, \\ (X(\cdot), \pi(\cdot)) \text{ satisfies the equation (14),} \end{cases} \end{aligned} \tag{15}$$

have the same optimal solution.

The above mean-variance with bankruptcy prohibition problem was fully solved in [1], so is our problem (4). Moreover, these two problems have the same efficient frontier.

4. Optimal Portfolio

The result of the martingale pricing theory states that the set of random terminal payoffs that can be generated by the admissible trading strategies corresponds to the set of nonnegative $\mathcal{F}_T$-measurable random payoffs $X(T)$ which satisfy the budget constraint $\mathbf{E}[\phi(T)X(T)] \leqslant x_0$. Therefore, the dynamic problem (15), of choosing an optimal trading strategy $\pi(\cdot)$, is equivalent to the static problem of choosing an optimal payoff $X(T)$:

$$\begin{aligned} \min \quad & \mathbf{Var}(X(T)) = \mathbf{E}[X(T) - d]^2, \\ \text{subject to} \quad & \begin{cases} \mathbf{E}[X(T)] = d, \\ \mathbf{E}[\phi(T)X(T)] = x_0, \\ X(T) \geqslant 0, \end{cases} \end{aligned} \tag{16}$$

where $\phi(\cdot)$ is the *state price density*, or *stochastic discount factor*, defined by

$$\begin{cases} \mathrm{d}\phi(t) = \phi(t)\{-r(t)\,\mathrm{d}t - \widehat{\theta}(t)'\,\mathrm{d}W(t)\}, \\ \phi(0) = 1, \end{cases}$$

and $\widehat{\theta}(t) = \sigma(t)^{-1}\widehat{B}(t) = \sigma(t)'\bar{z}(t)$.

The above static optimization problem (16) was solved in [1]. The optimal random terminal payoff is $X^*(T) = (\mu - \gamma\phi(T))^+$, where $x^+ = \max\{x, 0\}$, and $(\mu, \gamma) \in \mathbb{R}^2$ solves the system of equations $\mathbf{E}[(\mu - \gamma\phi(T))^+] = d$, $\mathbf{E}[\phi(T)(\mu - \gamma\phi(T))^+] = x_0$. That is,

$$\begin{cases} \mu N\Big(\frac{\ln\left(\frac{\mu}{\gamma}\right) + \int_0^T [r(s) + \frac{1}{2}|\widehat{\theta}(s)|^2]\,\mathrm{d}s}{\sqrt{\int_0^T |\widehat{\theta}(s)|^2 ds}}\Big) \\ \qquad -\gamma e^{-\int_0^T r(s)\,\mathrm{d}s} N\Big(\frac{\ln\left(\frac{\mu}{\gamma}\right) + \int_0^T [r(s) - \frac{1}{2}|\widehat{\theta}(s)|^2]\,\mathrm{d}s}{\sqrt{\int_0^T |\widehat{\theta}(s)|^2 ds}}\Big) = d, \\ \mu N\Big(\frac{\ln\left(\frac{\mu}{\gamma}\right) + \int_0^T [r(s) - \frac{1}{2}|\widehat{\theta}(s)|^2]\,\mathrm{d}s}{\sqrt{\int_0^T |\widehat{\theta}(s)|^2 ds}}\Big) \\ -\gamma e^{-\int_0^T [r(s) - |\widehat{\theta}(s)|^2]\,\mathrm{d}s} N\Big(\frac{\ln\left(\frac{\mu}{\gamma}\right) + \int_0^T [r(s) - \frac{3}{2}|\widehat{\theta}(s)|^2]\,\mathrm{d}s}{\sqrt{\int_0^T |\widehat{\theta}(s)|^2 ds}}\Big) = x_0 e^{\int_0^T r(s)\,\mathrm{d}s}, \end{cases} \tag{17}$$

where $N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}}\,\mathrm{d}t$ is the cumulative distribution function of the standard normal distribution.

The investor's optimal wealth is given by

$$X^*(t) = \mathbf{E}\left[\frac{\phi(T)}{\phi(t)} X^*(T) \middle| \mathcal{F}_t\right] = f(t, \phi(t)), \tag{18}$$

where

$$f(t, y) = \mu N(-d_2(t, y)) e^{-\int_t^T r(s)\,\mathrm{d}s} - \gamma N(-d_1(t, y)) y e^{-\int_t^T [2r(s) - |\widehat{\theta}(s)|^2]\,\mathrm{d}s},$$

and

$$d_1(t, y) := \frac{\ln\left(\frac{\gamma}{\mu} y\right) + \int_t^T [-r(s) + \frac{3}{2}|\widehat{\theta}(s)|^2]\,\mathrm{d}s}{\sqrt{\int_t^T |\widehat{\theta}(s)|^2 ds}},$$

$$d_2(t, y) := d_1(t, y) - \sqrt{\int_t^T |\widehat{\theta}(s)|^2\,\mathrm{d}s}.$$

Applying Itô's lemma to $f(\cdot, \phi(\cdot))$ yields

$$\begin{aligned} &\mathrm{d}X^*(t) = \mathrm{d}f(t, \phi(t)) \\ &= \{\cdots\}\,\mathrm{d}t + \gamma\hat{\theta}(t) N(-d_1(t, \phi(t)))\phi(t) e^{-\int_t^T [2r(s) - |\widehat{\theta}(s)|^2] ds}\,\mathrm{d}W(t). \end{aligned}$$

Comparing this to the wealth evolution equation (14), we obtain the efficient portfolio

$$\pi^*(t) = \gamma(\sigma(t)\sigma(t)')^{-1}\widehat{B}(t) N(-d_1(t, \phi(t)))\phi(t) e^{-\int_t^T [2r(s) - |\widehat{\theta}(s)|^2]\,\mathrm{d}s}. \tag{19}$$

Remark 5. *The above results for the efficient portfolio and the associated wealth process were first derived in [1].*

Based on the above analysis, we have the following result.

Theorem 3. *Assume that* $\int_0^T |\widehat{\theta}(s)|^2\,\mathrm{d}s > 0$. *Then there exists a unique efficient portfolio for (4) corresponding to any given* $d \geqslant x_0 e^{\int_0^T r(s)\,\mathrm{d}s}$. *Moreover, the efficient portfolio is given by (19) and the associated wealth process is expressed by (18).*

5. Special Models

The mean-variance portfolio selection model, like many other stochastic optimization models, is based on averaging over all the possible random scenarios. We now discuss how the model (with different constraints) could guide the real investment in practice.

5.1. Bankruptcy Prohibition with Unconstrained Portfolio

The mean-variance unconstrained portfolio problem with bankruptcy prohibition is an interesting but practically relevant model. In this case, $k = m$ and $\pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$. It follows from (13) that

$$\bar{z}(t) = \operatorname*{argmin}_{z \in \mathbb{R}^m} \|\sigma(t)'z - \sigma(t)^{-1}B(t)'\| = (\sigma(t)\sigma(t)')^{-1}B(t)'.$$

Therefore, $\widehat{B}(t) = \sigma(t)\sigma(t)'\bar{z}(t) = B(t)$.

Proposition 1. *Assume that* $\int_0^T |\widehat{\theta}(s)|^2\,\mathrm{d}s > 0$. *Then there exists a unique efficient portfolio for this mean-variance model corresponding to any given* $d \geqslant x_0 e^{\int_0^T r(s)\,\mathrm{d}s}$. *Moreover, the efficient portfolio is given by (19) and the associated wealth process is expressed by (18), where* $\widehat{B}(t) = B(t)$ *and* $\widehat{\theta}(t) = \sigma(t)^{-1}B(t)$.

The proof of Proposition 1 can be found in [1].

5.2. Bankruptcy Prohibition with No-shorting Constraint

The mean-variance portfolio problem with mixed no-bankruptcy and no-shorting constraints is another interesting and challenging model. In this case, $k = m$ and $\pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m_+)$. It follows from (13) that

$$\bar{z}(t) = \operatorname*{argmin}_{z \in \mathbb{R}^m_+} \|\sigma(t)'z - \sigma(t)^{-1}B(t)'\| = (\sigma(t)\sigma(t)')^{-1}(B(t) + \lambda(t))',$$

where

$$\lambda(t) := \operatorname*{argmin}_{y \in \mathbb{R}^m_+} \|\sigma(t)^{-1}y + \sigma(t)^{-1}B(t)'\|. \tag{20}$$

Therefore, $\widehat{B}(t) = \sigma(t)\sigma(t)'\bar{z}(t) = B(t) + \lambda(t)$.

Proposition 2. *Assume that* $\int_0^T |\widehat{\theta}(s)|^2\,\mathrm{d}s > 0$. *Then there exists a unique efficient portfolio for this mean-variance model corresponding to any given* $d \geqslant x_0 e^{\int_0^T r(s)\,\mathrm{d}s}$. *Moreover, the efficient portfolio is given by (19) and the associated wealth process is expressed by (18), where* $\widehat{B}(t) = B(t) + \lambda(t)$ *and* $\widehat{\theta}(t) = \sigma(t)^{-1}(B(t) + \lambda(t))$.

5.3. *No-shorting Constraint without Bankruptcy Prohibition*

The mean-variance portfolio problem with no-shorting constraints is also an important model in financial investment. In this case, $k = m$ and $\pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m_+)$. We again have $\widehat{B}(t) = B(t) + \lambda(t)$, where $\lambda(t)$ is determined by (20).

In particular, $d_1(t, \phi(t)) = -\infty$ and $d_2(t, \phi(t)) = -\infty$, that is, $N(-d_1(t, \phi(t))) = N(-d_2(t, \phi(t))) = 1$. The investor's optimal wealth is the stochastic process

$$X^*(t) = \mu e^{-\int_t^T r(s)\,ds} - \gamma\phi(t)e^{-\int_t^T [2r(s)-|\widehat{\theta}(s)|^2]\,ds} \tag{21}$$

and

$$\pi^*(t) = \gamma(\sigma(t)\sigma(t)')^{-1}\widehat{B}(t)\phi(t)e^{-\int_t^T [2r(s)-|\widehat{\theta}(s)|^2]\,ds} \tag{22}$$

where $\mu = \frac{\mathbf{E}[\phi(T)^2]d - x_0\,\mathbf{E}[\phi(T)]}{\mathbf{Var}(\phi(T))} = \frac{d - x_0 e^{\int_0^T [r(s)-|\widehat{\theta}(s)|^2]\,ds}}{1-e^{-\int_0^T |\widehat{\theta}(s)|^2\,ds}}$, $\gamma = \frac{\mathbf{E}[\phi(T)]d - x_0}{\mathbf{Var}(\phi(T))} = \frac{\left(d - x_0 e^{\int_0^T r(s)\,ds}\right)e^{\int_0^T [r(s)-|\widehat{\theta}(s)|^2]\,ds}}{1-e^{-\int_0^T |\widehat{\theta}(s)|^2\,ds}}$ and $\widehat{\theta}(t) = \sigma(t)^{-1}(B(t) + \lambda(t))$.

Proposition 3. *Assume that* $\int_0^T |\widehat{\theta}(s)|^2\,ds > 0$. *Then there exists a unique efficient portfolio for this mean-variance model corresponding to any given* $d \geqslant x_0 e^{\int_0^T r(s)\,ds}$. *Moreover, the efficient portfolio is given by (22) and the associated wealth process is expressed by (21).*

The same result can be found in [13].

6. A Numerical Example

In this section, a numerical example with constant coefficients is presented to demonstrate the results in the previous sections. Let $m = 3$. The interest rate of the bond and the appreciation rate of the m stocks are $r = 0.03$ and $(b_1, b_2, b_3)' = (0.12,\ 0.15,\ 0.18)'$, respectively, and the volatility matrix is

$$\sigma = \begin{bmatrix} 0.2500 & 0 & 0 \\ 0.1500 & 0.2598 & 0 \\ -0.2500 & 0.2887 & 0.3227 \end{bmatrix}.$$

Then $\theta = \sigma^{-1}B = (0.3600, 0.2540, 0.5164)'$. In addition, we suppose that the initial prices of stocks are $(S_1(0), S_2(0), S_3(0)) = (1, 1, 1)'$ and the initial wealth is $X(0) = 1$.

6.1. *Bankruptcy Prohibition with Unconstrained Portfolio*

In this subsection, we determine the optimal portfolio and the corresponding wealth process in Subsection 5.1 for the above market data. According to (17), we obtain the numerical results $\mu = 1.5046$ and $\gamma = 0.3154$. Hence, the wealth process (18) can be expressed by

$$X^*(t) = \mu N(-d_2(t, \phi(t)))e^{-r(T-t)} - \gamma N(-d_1(t, \phi(t)))\phi(t)e^{-[2r-|\hat{\theta}|^2](T-t)}, \tag{23}$$

where $d_1(t, \phi(t)) = \frac{\ln\left(\frac{\gamma}{\mu}\phi(t)\right) + [-r + \frac{3}{2}|\hat{\theta}|^2](T-t)}{\sqrt{|\hat{\theta}|^2(T-t)}}$, $d_2(t, \phi(t)) = d_1(t, \phi(t)) - \sqrt{|\hat{\theta}|^2(T-t)}$, $\phi(t) = e^{-[r+\frac{1}{2}|\hat{\theta}|^2](T-t) - \hat{\theta}(W(T)-W(t))}$, $\hat{\theta} = \theta = \sigma^{-1}B = (0.3600, 0.2540, 0.5164)'$. The efficient portfolio is given by

$$\pi^*(t) = \gamma(\sigma\sigma')^{-1}\widehat{B}N(-d_1(t, \phi(t)))\phi(t)e^{-[2r-|\hat{\theta}|^2](T-t)}. \tag{24}$$

where $(\sigma\sigma')^{-1}\widehat{B} = (\sigma\sigma')^{-1}B = (3.5200,\ -0.8000,\ 1.6000)'$.

In particular, the policy of investing in the second stock $\pi_2^*(t)$ is shorting.

6.2. *Bankruptcy Prohibition with No-shorting Constraint*

From Subsection 6.1, we see that there exists a shorting case in policy (24). Using (20), we obtain

$$\lambda = \underset{y \in \mathbb{R}^m_+}{\operatorname{argmin}} \|\sigma^{-1}y + \sigma^{-1}B'\| = (0,\ 0.03,\ 0)'.$$

Hence,

$$\begin{cases} \hat{\theta} = \sigma^{-1}\widehat{B} = \sigma^{-1}(B + \lambda) = (0.3600,\ 0.3695,\ 0.4131)', \\ (\sigma\sigma')^{-1}\widehat{B} = (\sigma\sigma')^{-1}(B + \lambda) = (2.72,\ 0,\ 1.28)'. \end{cases} \tag{25}$$

According to (17), we obtain the numerical results $\mu = 1.5253$ and $\gamma = 0.3368$. Hence, the wealth process (18) can be expressed by

$$X^*(t) = \mu N(-d_2(t, \phi(t)))e^{-r(T-t)} - \gamma N(-d_1(t, \phi(t)))\phi(t)e^{-[2r-|\hat{\theta}|^2](T-t)}. \tag{26}$$

The efficient portfolio is presented by

$$\pi^*(t) = \gamma(\sigma\sigma')^{-1}\widehat{B}N(-d_1(t, \phi(t)))\phi(t)e^{-[2r-|\hat{\theta}|^2](T-t)}. \tag{27}$$

Note that the policy in (27) is always non-negative, namely, a no-shorting policy.

6.3. *No-shorting Constraint without Bankruptcy Prohibition*

In this subsection, we present the optimal no-shorting policy without bankruptcy prohibition of Subsection 5.3 and its corresponding wealth process. According to (22), we find the numerical results $\mu = 1.5095$ and $\gamma = 0.3190$. Hence, the wealth process (18) can be expressed by

$$X^*(t) = \mu e^{-r(T-t)} - \gamma\phi(t)e^{-[2r-|\hat{\theta}|^2](T-t)} \tag{28}$$

and the portfolio is

$$\pi^*(t) = \gamma(\sigma\sigma')^{-1}\widehat{B}\phi(t)e^{-[2r-|\hat{\theta}|^2](T-t)}, \tag{29}$$

where $\hat{\theta}$ and $(\sigma\sigma')^{-1}\widehat{B}$ are given by (25).

Note that the policy in (29) is non-negative, namely, a no-shorting policy. However, its corresponding wealth (28) is possibly negative. We shall further discuss this point by simulation results in the following subsection.

6.4. *Simulation*

In this subsection, we further analyze via simulation how the properties of the optimal portfolio strategies (24), (27) and (29) change, and compare their wealth processes (23), (26) and (28). We set the target wealth $d = 1.2X(0)$.

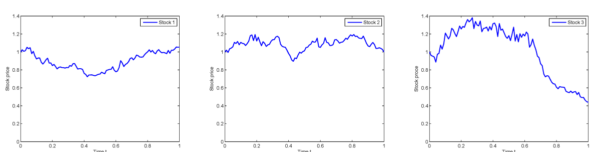

Figure 1: Prices of stocks

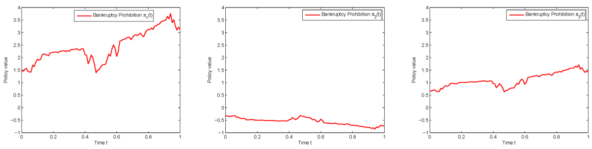

Figure 2: Policy: Bankruptcy Prohibition & Shorting Allowed

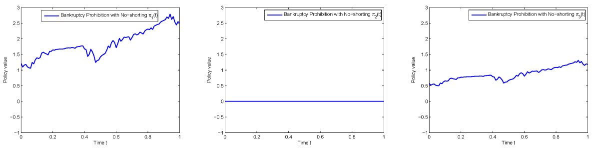

Figure 3: Policy: Bankruptcy Prohibition & No-shorting

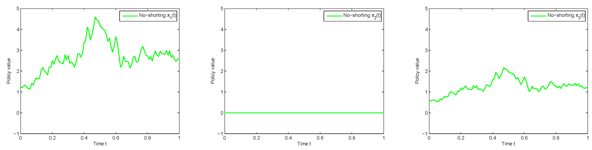

Figure 4: Policy: Bankruptcy Allowed & No-shorting

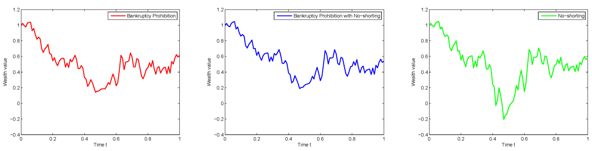

Figure 5: Optimal Wealth Processes

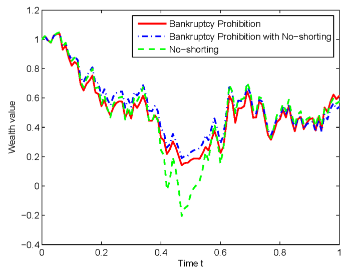

Figure 6: Comparison of Optimal Wealth Processes

According to the three scenarios in Figure 1, we plot the optimal portfolio strategies (24), (27) and (29) under different restrictions in Figure 2, Figure 3 and Figure 4. If shorting is allowed, then $(\sigma\sigma')^{-1}\widehat{B} = (3.5200, \ -0.8000, \ 1.6000)'$ and the optimal policy is a (positive) multiple of this vector. Hence, we see the shorting policy of the second stock in Figure 2. If shorting is not allowed, then $(\sigma\sigma')^{-1}\widehat{B} = (2.72, \ 0, \ 1.28)'$. Therefore, compared to the scenarios 1 and 3, there is no trading of the second stock over the whole trading horizon in Figure 3 and Figure 4. In addition, we plot their corresponding wealth processes under different restrictions in Figure 5, and we finally combine all wealth processes into one picture in Figure 6.

7. Conclusion

We studied the continuous-time mean-variance portfolio selection with mixed restrictions of bankruptcy prohibition and convex cone portfolio constraints. The main contribution is that we developed the semi-analytical expression for the pre-committed efficient mean-variance policy without the viscosity solution technique. A natural extension of our result to continuous-time linear-quadratic cone constrained controls with constrained states is straightforward, at least conceptually. On the other hand, if some of the market coefficients are random, the problem becomes more complicated. As pointed out by many researchers, trading costs are a major concern in active portfolio management, so it is a practically important and challenging problem to incorporate trading costs into our model. As is well-known, the existence of trading costs will fundamentally change the investment policy (see, e.g., [6]), we will address this fact in future works.

[1] T.R. Bielecki, H. Jin, S.R. Pliska and X.Y. Zhou, Continuous-time mean-variance portfolio selection with bankruptcy prohibition, *Mathematical Finance*, 15 (2005) 213–244.

[2] L. Carassus, H. Pham and N. Touzi, No arbitrage in discrete time under portfolio constraints, *Mathematical Finance*, 11 (2001) 315–329.

[3] X.Y. Cui, J.J. Gao, X. Li and D. Li, Optimal multiperiod mean-variance policy under no-shorting constraint, *European Journal of Operational Research*, 234 (2014) 459–468.

[4] X.Y. Cui, D. Li and X. Li, Mean-variance policy for discrete-time cone constrained markets: Time consistency in efficiency and minimum-variance signed supermartingale measure, to appear in *Mathematical Finance*, 2016.

[5] C. Czichowsky and M. Schweizer, Cone-constrained continuous-time Markowitz problems, *The Annals of Applied Probability*, 23 (2013) 427–857.

[6] M. Dai, Z. Q. Xu and X. Y. Zhou, Continuous-time Markowitz's model with transaction costs, *SIAM Journal on Financial Mathematics*, 1 (2009) 96–125.

[7] D. Duffie and H. Richardson, Mean-variance hedging in continuous time, *The Annals of Applied Probability*, 14 (1991) 1–15.

[8] A. Heunis, Quadratic minimization with portfolio and terminal wealth constraints, *Annals of Finance*, 11 (2015) 243–282.

[9] D. Hou and Z. Q. Xu, A robust Markowitz mean-variance portfolio selection model with an intractable claim, *SIAM Journal on Financial Mathematics*, 7 (2016) 124–151.

[10] Y. Hu and X.Y. Zhou, Constrained stochastic LQ control with random coefficients, and application to mean–variance portfolio selection, *SIAM Journal on Control and Optimization*, 44 (2005) 444–466.

[11] C. Labbé and A.J. Heunis, Convex duality in constrained mean-variance portfolio optimization, *Advances in Applied Probability*, 39 (2007) 77–104.

[12] D. Li and W.L. Ng, Optimal dynamic portfolio selection: Multi-period mean-variance formulation, *Mathematical Finance*, 10 (2000) 387–406.

[13] X. Li, X.Y. Zhou and A.E.B. Lim, Dynamic mean-variance portfolio selection with no-shorting constraints, *SIAM Journal on Control and Optimization*, 40 (2002) 1540–1555.

[14] H. Markowitz, Portfolio selection, *Journal of Finance*, 7 (1959) 77–91.

[15] H. Pham and N. Touzi, The fundamental theorem of asset pricing with cone constraints, *Journal of Mathematical Economics*, 31 (1999) 265–279.

[16] G.L. Xu and S.E. Shreve, A duality method for optimal consumption and investment under short-selling prohibition: I. general market coefficients, *The Annals of Applied Probability*, 2 (1992) 87–112.

[17] G.L. Xu and S.E. Shreve, A duality method for optimal consumption and investment under short-selling prohibition: II. constant market coefficients, *The Annals of Applied Probability*, 2 (1992) 314–328.

[18] X.Y. Zhou and D. Li, Continuous time mean-variance portfolio selection: A stochastic LQ framework, *Applied Mathematics and Optimization*, 42 (2000) 19–33.