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CONTINUOUS-MODE MULTIPHOTON FILTERING*

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Abstract. The purpose of this paper is to derive filters for an arbitrary open quantum system driven by a light wave packet prepared in a continuous-mode multiphoton state. A continuous-mode multiphoton state is a state of a traveling light wave packet that contains a definite number of photons and is characterized by a temporal (or, equivalently, spectral) profile. After the interaction with the system, the outgoing light can be monitored by means of homodyne detection or photodetection. Filters for both measurement schemes are derived in this paper. Unlike the vacuum or the coherent state case, the annihilation operator of the light field acting on a multiphoton state changes the state by annihilating a photon, and this makes the traditional filtering techniques inapplicable. To circumvent this difficulty, we adopt a non-Markovian embedding technique proposed in [J. E. Gough, M. R. James, and H. I. Nurdin, *Quantum Inf. Process.*, 12 (2013), pp. 1469–1499] for the study of the single-photon filtering problem. However, the multiphoton nature of the problem addressed in this paper makes the study much more mathematically involved. Moreover, as demonstrated by an example—a two-level system driven by a continuous-mode two-photon state—multiphoton filters can reveal interesting strong nonlinear optical phenomena absent in both the single-photon state case and the continuous-mode Fock state case.

Key words. open quantum systems, quantum filtering, multiphoton states

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1. Introduction.

When light impinges on a quantum system, e.g., an atom or a quantum-mechanical oscillator, partial system information may be carried away by the outgoing light. The outgoing light can be directed to another quantum system, thus serving as a (directional) link to facilitate cascade connection [2, 3, 4, 5, 6]. Alternatively, the outgoing light may be continuously monitored to produce photocurrent, on which the state of the quantum system can be conditioned. The stochastic evolution of the conditional system state is commonly called a *quantum trajectory*. A quantum filter can be designed to estimate quantum trajectories [2, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]; this is the basis of quantum-measurement-based feedback control [15, 16, 19, 20, 21, 22].

In quantum optics, the familiar formalism of quantum filtering considers incident lights in Gaussian states, including the vacuum state, coherent states, thermal states, and squeezed states [22, 23, 24]. This is natural as Gaussian states are commonly used in quantum optics laboratories and have been well studied. With the advent of modern experimental technology, nowadays non-Gaussian states, such as single-photon states, multiphoton states, and Schrödinger cat states, can be reliably generated and manipulated. Therefore, very recently, there is a growing interest in deriving quantum

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filters for non-Gaussian states. For example, filters have been derived for quantum systems driven by light fields prepared in single-photon states or cat states [1, 21, 25].

Single-photon states and multiphoton states are very useful resources in quantum computing, quantum communication, and quantum cryptography; see [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39] and references therein. Roughly speaking, a continuous-mode n-photon state is a state of a traveling light wave packet containing exactly n photons that share a common pulse shape superposed on a continuum of spectral modes. When $n = 0$, the wave packet is in the vacuum state, whose filtering equation is well known in the quantum optics community. When $n = 1$, the wave packet is in a single-photon state, whose filtering equations have recently been derived in [1, 25]. When $n > 1$, for convenience we call the state a *multiphoton state*.

In this paper, we study the problem of quantum filtering for arbitrary quantum systems driven by continuous-mode multiphoton states. Due to the multiphoton nature of the problem, it turns out that the derivation of multiphoton filters is very mathematically involved. For example, if the input is a wave packet containing n photons, we need a hierarchy of $\frac{2^n(2^n+1)}{2}$ differential equations to determine an n-photon filter. When $n = 0$, namely, the vacuum state case, a single differential equation is sufficient. When $n = 1$, namely, the single-photon state case, we need 3 coupled differential equations. When $n = 2$, a system of 10 equations is required. Similarly, a hierarchy of 36 differential equations is required for the case of a 3-photon state, and so on. Therefore, to present the main ideas clearly, we investigate the 2-photon case in detail before proceeding to the general n-photon case.

For the 2-photon case, master equations are given in Theorem 3.1 and Corollary 3.2, while quantum filters for homodyne detection are given in Theorem 3.6 and Corollary 3.7. These results contain those in [1] and [25] for the single-photon case as special cases. Numerical studies conducted in Examples 2 and 3 show that two-photon excitation of a two-level system has highly nonlinear optical phenomena, which are absent in both the single-photon state case and the continuous-mode Fock state case. For the general n-photon case, quantum filters are given in Theorem 4.4 for the homodyne detection case. For photodetection, the quantum filter is given in Theorem 4.6, which reduces to the single-photon filter for photodetection when $n = 1$ which is studied in [1, 25]. Finally, the multiphoton master equations are given in Theorem 4.1, which in the Fock state case is actually the master equation (20) in [40] for continuous-mode Fock states. Therefore, the results presented in this paper are indeed very general. Due to the multiphoton nature, the mathematical description of general multiphoton filtering equations are very messy; in fact, the lexicographical ordering [41] plays an essential role.

The rest of the paper is organized in the following way. Section 2 introduces open quantum systems and poses the filtering problem. Section 3 focuses on the two-photon case. Here, we first define two-photon states in subsection 3.1, then present the master equations in subsection 3.2. In order to derive the two-photon filtering equation, we define an extended system in subsection 3.3, and derive the filtering equation for this extended system in subsection 3.4, based on which in subsection 3.5 we derive the quantum filter for the original system driven by a two-photon state. After the study of the two-photon filtering problem in section 3, we proceed to the general multiphoton case in section 4, where we present the general filtering equations for both the homodyne detection case and the photon-counting case. Section 5 concludes the paper.

Notation. $|0\rangle$ is the vacuum state of the free field. $|\eta\rangle$ is the initial state of the quantum system of interest. $\mathbb{R}^+$ is the set of nonnegative real numbers, $L_2(\mathbb{R}^+, \mathbb{C})$ is

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the space of Lebesgue measurable and square integrable functions from $\mathbb{R}^+$ to $\mathbb{C}$. For ξ_1 and $\xi_2 \in L_2(\mathbb{R}^+, \mathbb{C})$, their inner product is $\langle \xi_1 | \xi_2 \rangle \triangleq \int_0^\infty \xi_1^*(t)\xi_2(t)\mathrm{d}t$. The norm of a function $\xi \in L_2(\mathbb{R}^+, \mathbb{C})$ is $\|\xi\| \triangleq \sqrt{\langle \xi | \xi \rangle}$. δ_{jk} is the Kronecker delta, namely, $\delta_{jk} = 1$ if $j = k$ or 0 otherwise. $\otimes$ stands for the tensor product. X^* denotes the complex conjugate of X if X is a complex number or the adjoint operator of X if X is an operator. The commutator of two operators A and B is defined to be $[A, B] \triangleq AB - BA$.

2. Preliminaries. In this section we briefly introduce quantum systems and pose the multiphoton filtering problem.

2.1. Quantum systems. This subsection gives a very brief introduction to quantum systems; more details can be found in, e.g., [4, 6, 16, 22, 42, 43, 44].

The model we study is an arbitrary quantum system G driven by a single-channel light field which can be effectively described by the so-called (S, L, H) language [5, 6]. Here, S is a unitary scattering operator, L is a coupling operator that describes how the system is coupled to the input field, and the self-adjoint operator H is the initial system Hamiltonian. S, L, and H are system operators on a separable Hilbert space H_S where the system states reside. The single-channel light field has an annihilation operator $b(t)$ and a creation operator $b^*(t)$, which are operators on a Fock space H_F (an infinite-dimensional Hilbert space). $B(t) \triangleq \int_0^t b(r)\mathrm{d}r$ and $B^*(t) \triangleq \int_0^t b^*(r)\mathrm{d}r$ are integrated annihilation and creation field operators, respectively. The gauge process, often called counting process, $\Lambda(t) \triangleq \int_0^t b^*(\tau)b(\tau)\mathrm{d}\tau$ is also an integrated operator on the Fock space H_F for the input field. In this paper, the input field is *canonical*, that is, the nonzero Ito products are

$$\mathrm{d}B(t)\mathrm{d}B^*(t) = \mathrm{d}t, \mathrm{d}B(t)\mathrm{d}\Lambda(t) = \mathrm{d}B(t), \mathrm{d}\Lambda(t)\mathrm{d}\Lambda(t) = \mathrm{d}\Lambda(t), \mathrm{d}\Lambda(t)\mathrm{d}B^*(t) = \mathrm{d}B^*(t).$$

The temporal evolution of the composite system composed of the system and the field can be described by a unitary operator $U(t)$ on the tensor product Hilbert space $\mathsf{H}_S \otimes \mathsf{H}_F$, and is given by the the following Hudson–Parthasarathy quantum stochastic differential equation (QSDE)

$$\mathrm{d}U(t) = \left\{ (S - I)\mathrm{d}\Lambda(t) + L\mathrm{d}B^*(t) - L^*S\mathrm{d}B(t) - (\frac{1}{2}L^*L + \mathrm{i}H)\mathrm{d}t \right\} U(t), \quad t > 0,$$

with the initial condition $U(0) = I$ (the identity operator) and $\mathrm{i} = \sqrt{-1}$.

In the Heisenberg picture, the system operator X at time t is given by $X(t) \equiv j_t(X) \triangleq U^*(t)(X \otimes I)U(t)$, which is an operator on $\mathsf{H}_S \otimes \mathsf{H}_F$, and whose temporal evolution is governed by the following Heisenberg equation of motion,

$$\mathrm{d}j_t(X) = j_t(\mathcal{L}_{00}(X))\mathrm{d}t + j_t(\mathcal{L}_{01}(X))\mathrm{d}B(t) + j_t(\mathcal{L}_{10}(X))\mathrm{d}B^*(t) + j_t(\mathcal{L}_{11}(X))\mathrm{d}\Lambda(t), \tag{2.1}$$

where the Evans–Parthasarathy superoperators are

$$\mathcal{L}_{00}(X) \triangleq \frac{1}{2}L^*[X, L] + \frac{1}{2}[L^*, X]L - \mathrm{i}[X, H], \quad \mathcal{L}_{01}(X) \triangleq [L^*, X]S, \tag{2.2}$$

$$\mathcal{L}_{10}(X) \triangleq S^*[X, L] = (\mathcal{L}_{01}(X^*))^*, \quad \mathcal{L}_{11}(X) \triangleq S^*XS - X. \tag{2.3}$$

After interaction, the quantum field becomes $B_{\mathrm{out}}(t) \triangleq U^*(t)(I \otimes B(t))U(t)$, an operator on $\mathsf{H}_S \otimes \mathsf{H}_F$, whose dynamics are given by the following QSDE,

$$\mathrm{d}B_{\mathrm{out}}(t) = j_t(L)\mathrm{d}t + j_t(S)\mathrm{d}B(t).$$

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The output field can be monitored. Homodyne detection and photodetection are the two most commonly used measurement methods in quantum optics. In homodyne detection, the noise quadrature

$$Y(t) \triangleq U^*(t)(I \otimes (B(t) + B^*(t)))U(t) = B_{\rm out}(t) + B^*_{\rm out}(t)$$

may be measured, while in photodetection (photon counting),

$$Y^\Lambda(t) \triangleq U^*(t)(I \otimes \Lambda(t))U(t)$$

is measured. By Ito rules, the observation processes $Y(t)$ and $Y^\Lambda(t)$ satisfy

$$\mathrm{d}Y(t) = j_t(S^*)\mathrm{d}B^*(t) + j_t(S)\mathrm{d}B(t) + j_t(L + L^*)\mathrm{d}t$$

and

$$\mathrm{d}Y^\Lambda(t) = \mathrm{d}\Lambda(t) + j_t(S^*L)\mathrm{d}B^*(t) + j_t(L^*S)\mathrm{d}B(t) + j_t(L^*L)\mathrm{d}t,$$

respectively. Moreover, $Y(t)$ and $Y^\Lambda(t)$ obey the so-called *self-nondemolition* property, i.e.,

$$[Y(t), Y(s)] = [Y^\Lambda(t), Y^\Lambda(s)] = 0, \quad 0 \le s \le t.$$

We denote by $\mathscr{Y}(t)$ and $\mathscr{Y}^\Lambda(t)$ the commutative von Neumann algebras generated by $\{Y(s);\ 0 \le s \le t\}$ and $\{Y^\Lambda(s);\ 0 \le s \le t\}$, respectively.

2.2. Quantum filtering. Simply speaking, the quantum filtering problem studied in this paper is about finding a least mean square estimate of system observables $j_t(X)$ based on the past measurement outcome information up to time t for a quantum system driven by a continuous-mode multiphoton state. In the homodyne detection case, it is about the computation of the quantum conditional expectation

$$\pi_t^{n;n}(X) \triangleq \mathbb{E}_{n;n}[j_t(X)|\mathscr{Y}(t)]. \tag{2.4}$$

Here, the subscript "$n; n$" in the expectation notation $\mathbb{E}$ is used to indicate that the input field is in an n-photon state. The exact form of n-photon states and the notation $\mathbb{E}_{n;n}$ will be made clear in due course. As introduced in subsection 2.1, $\mathscr{Y}(t)$ is the commutative von Neumann algebra generated by the observation processes $Y(s),\ 0 \le s \le t$. The quantum conditional expectation in (2.4) is well-defined due to the fact that $j_t(X)$ satisfies the *nondemolition* condition $[j_t(X), Y(s)] = 0$ for all $s \le t$. The quantum conditional expectation for the photodetection case can be defined in a similar manner, specifically,

$$\hat{\pi}_t^{n;n}(X) \triangleq \mathbb{E}_{n;n}[j_t(X)|\mathscr{Y}^\Lambda(t)]. \tag{2.5}$$

Due to the complexity of the multiphoton filtering problem, to better present the main ideas, we first focus on the two-photon case and conduct a detailed study of the two-photon filtering problem in section 3. After that we proceed to the general n-photon case in section 4.

Example 1. In this example we demonstrate the above-mentioned (S, L, H) language by means of a toy model: a two-level system in a one-way waveguide. Here, "one-way" means that photons can only propagate along one direction in the waveguide [45, 46]. The state space of the two-level system G is $\mathsf{H}_S = \mathbb{C}^2$ whose basis

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vectors are the ground state $|g\rangle = [0\ 1]^T$ and the excited state $|e\rangle = [1\ 0]^T$. System operators are 2-by-2 matrices of complex numbers, for example, $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma_+ = |e\rangle\langle g|$, and $\sigma_- = |g\rangle\langle e|$. In the interaction picture the total Hamiltonian of the composite system is ($\hbar = 1$)

$$H_{\text{total}} = \frac{\omega_c}{2}\sigma_z + \int_{-\infty}^{\infty} \omega b^*(\omega)b(\omega)\,\mathrm{d}\omega + \mathrm{i}\sqrt{\frac{\kappa}{2\pi}}\int_{-\infty}^{\infty} (\sigma_+ b(\omega) - \sigma_- b^*(\omega))\,\mathrm{d}\omega,$$

in which the first term and second term on the right-hand side are the free Hamiltonians of the system and the field, respectively, while the third one is the interaction Hamiltonian. The detuning $\omega_c = \Omega - \omega_0$, where Ω is the atomic transition frequency between the ground state $|g\rangle$ and the excited state $|e\rangle$ and ω_0 is the carrier frequency of the input light field, $b(\omega)$ and its adjoint $b^*(\omega)$ are the annihilation operator and the creation operator of the input field, respectively, and $\kappa > 0$ is related to the coupling constant between the two-level system G and the field. For this model, in the Heisenberg picture we have the following expressions at $t \geq t_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}b(\omega,t) = -\mathrm{i}[b(\omega,t), H_{\text{total}}(t)] = -\mathrm{i}\omega b(\omega,t) - \sqrt{\frac{\kappa}{2\pi}}\sigma_-(t), \tag{2.6}$$

$$\dot{\sigma}_-(t) = -\mathrm{i}[\sigma_-(t), H_{\text{total}}(t)] = -\mathrm{i}\omega_c\sigma_-(t) - \sqrt{\frac{\kappa}{2\pi}}\sigma_z(t)\int_{-\infty}^{\infty} b(\omega,t)\mathrm{d}\omega. \tag{2.7}$$

Here, t_0 is the initial time, namely, the time when the system and the field start to interact. Integrating (2.6) from t_0 to t yields

$$b(\omega,t) = \mathrm{e}^{-\mathrm{i}\omega(t-t_0)}b(\omega,t_0) - \sqrt{\frac{\kappa}{2\pi}}\int_{t_0}^{t} \mathrm{e}^{-\mathrm{i}\omega(t-r)}\sigma_-(r)\mathrm{d}r. \tag{2.8}$$

Putting (2.8) back into (2.7) we have

$$\dot{\sigma}_-(t) = -\left(\frac{\kappa}{2} + \mathrm{i}\omega_c\right)\sigma_-(t) + \sqrt{\kappa}\sigma_z(t)b(t),$$

where

$$b(t) \triangleq -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega(t-t_0)}b(\omega,t_0)\mathrm{d}\omega$$

is the annihilation operator introduced in subsection 2.1. On the other hand, let t_1 be the terminal time. Integrating (2.6) from t to t_1 we have

$$b(\omega,t) = \mathrm{e}^{-\mathrm{i}\omega(t-t_1)}b(\omega,t_1) - \sqrt{\frac{\kappa}{2\pi}}\int_{t_1}^{t} \mathrm{e}^{-\mathrm{i}\omega(t-r)}\sigma_-(r)\mathrm{d}r.$$

Define the output operator by

$$b_{\text{out}}(t) \triangleq -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega(t-t_1)}b(\omega,t_1)\mathrm{d}\omega.$$

It can be easily shown that the input operator $b(t)$ and the output operator $b_{\text{out}}(t)$ are related by

$$b_{\text{out}}(t) = \sqrt{\kappa}\sigma_-(t) + b(t).$$

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Finally, the following rotations $\sigma_-(t) \to e^{i\omega_c t}\sigma_-(t)$, $b(t) \to e^{i\omega_c t}b(t)$, $b_{\text{out}}(t) \to e^{i\omega_c t}b_{\text{out}}(t)$ yield the final model of interest

$$\dot{\sigma}_-(t) = -\frac{\kappa}{2}\sigma_-(t) + \sqrt{\kappa}\sigma_z(t)b(t), \tag{2.9}$$

$$b_{\text{out}}(t) = \sqrt{\kappa}\sigma_-(t) + b(t). \tag{2.10}$$

For the model (2.9)–(2.10) we have $S = I$, $L = \sqrt{\kappa}\sigma_-$, and $H = 0$.

Remark 1. It is worth noticing that the model (2.9)–(2.10) can also be used to describe a two-level atom in free space, as previously studied in [1, 25, 40, 47, 48, 49].

3. Two-photon filtering.

In this section we present a detailed study of the quantum filtering problem for an arbitrary quantum system driven by a two-photon state. Two-photon states are defined in subsection 3.1, the master equations are presented in subsection 3.2, and the filtering equations for the homodyne detection case are derived in subsections 3.3–3.5.

3.1. Two-photon states.

Given a function $\xi \in L_2(\mathbb{R}^+, \mathbb{C})$, define an operator

$$B(\xi) \triangleq \int_0^\infty \xi^*(t)b(t)\mathrm{d}t$$

whose adjoint operator is

$$B^*(\xi) \triangleq \int_0^\infty \xi(t)b^*(t)\mathrm{d}t. \tag{3.1}$$

Given two functions $\xi_1, \xi_2 \in L_2(\mathbb{R}^+, \mathbb{C})$ satisfying $\|\xi_1\| = \|\xi_2\| = 1$, define single-photon states $|\Phi_{10}\rangle$ and $|\Phi_{01}\rangle$ to be

$$|\Phi_{10}\rangle \triangleq B^*(\xi_1)|0\rangle \qquad \text{and} \qquad |\Phi_{01}\rangle \triangleq B^*(\xi_2)|0\rangle, \tag{3.2}$$

respectively. Then we define a two-photon state

$$|\Phi_{11}\rangle \triangleq \frac{1}{\sqrt{N_2}}B^*(\xi_1)B^*(\xi_2)|0\rangle, \tag{3.3}$$

where $N_2 = 1 + |\langle\xi_1|\xi_2\rangle|^2$ is a normalization coefficient. If $\xi_1 \equiv \xi_2$, then $|\Phi_{11}\rangle$ is a continuous-mode two-photon Fock state. Finally, for notational convention, denote $|\Phi_{00}\rangle \triangleq |0\rangle$.

For these states we have

$$\begin{aligned} &\mathrm{d}B(t)|\Phi_{00}\rangle = 0, \quad \mathrm{d}B(t)|\Phi_{10}\rangle = \xi_1(t)|\Phi_{00}\rangle\mathrm{d}t, \quad \mathrm{d}B(t)|\Phi_{01}\rangle = \xi_2(t)|\Phi_{00}\rangle\mathrm{d}t, \\ &\mathrm{d}B(t)|\Phi_{11}\rangle = \frac{\xi_1(t)}{\sqrt{N_2}}|\Phi_{01}\rangle\mathrm{d}t + \frac{\xi_2(t)}{\sqrt{N_2}}|\Phi_{10}\rangle\mathrm{d}t. \end{aligned} \tag{3.4}$$

3.2. Master equations.

In this subsection we present the master equations for a quantum system G driven by the two-photon state $|\Phi_{11}\rangle$ defined in (3.3).

For a given system operator X on H_S, define expectations

$$\omega_t^{jk;mn}(X) \triangleq \mathbb{E}_{jk;mn}[j_t(X)] \equiv \langle\eta\Phi_{jk}|j_t(X)|\eta\Phi_{mn}\rangle \qquad \forall\ j,k,m,n = 0,1, \tag{3.5}$$

where $|\eta\rangle$ is the initial state of the system. It can be easily verified that

$$\omega_t^{mn;jk}(X) = (\omega_t^{jk;mn}(X^*))^* \qquad \forall\ j,k,m,n = 0,1. \tag{3.6}$$

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In view of (2.1) and (3.4), if we differentiate $\omega_t^{11;11}(X)$, we will get such expressions as $\omega_t^{11;01}(X)$, $\omega_t^{11;10}(X)$, $\omega_t^{10;11}(X)$, and $\omega_t^{01;11}(X)$. Following this logic, in order to derive the master equation for $\omega_t^{11;11}(X)$, we have to find derivatives of $\omega_t^{jk;mn}(X)$ for all $j,k,m,n=0,1$.

Theorem 3.1. *The master equation in the Heisenberg picture for the quantum system G driven by the two-photon input field state $|\Phi_{11}\rangle$ is given by the system of differential equations*

$$\begin{aligned}
\dot{\omega}_t^{00;00}(X) &= \omega_t^{00;00}(\mathcal{L}_{00}(X)),\\
\dot{\omega}_t^{00;10}(X) &= \omega_t^{00;10}(\mathcal{L}_{00}(X)) + \xi_1(t)\omega_t^{00;00}(\mathcal{L}_{01}(X)),\\
\dot{\omega}_t^{00;01}(X) &= \omega_t^{00;01}(\mathcal{L}_{00}(X)) + \xi_2(t)\omega_t^{00;00}(\mathcal{L}_{01}(X)),\\
\dot{\omega}_t^{10;10}(X) &= \omega_t^{10;10}(\mathcal{L}_{00}(X)) + \xi_1(t)\omega_t^{10;00}(\mathcal{L}_{01}(X)) + \xi_1^*(t)\omega_t^{00;10}(\mathcal{L}_{10}(X))\\
&\quad + |\xi_1(t)|^2\omega_t^{00;00}(\mathcal{L}_{11}(X)),\\
\dot{\omega}_t^{10;01}(X) &= \omega_t^{10;01}(\mathcal{L}_{00}(X)) + \xi_2(t)\omega_t^{10;00}(\mathcal{L}_{01}(X)) + \xi_1^*(t)\omega_t^{00;01}(\mathcal{L}_{10}(X))\\
&\quad + \xi_1^*(t)\xi_2(t)\omega_t^{00;00}(\mathcal{L}_{11}(X)),\\
\dot{\omega}_t^{01;01}(X) &= \omega_t^{01;01}(\mathcal{L}_{00}(X)) + \xi_2(t)\omega_t^{01;00}(\mathcal{L}_{01}(X)) + \xi_2^*(t)\omega_t^{00;01}(\mathcal{L}_{10}(X))\\
&\quad + |\xi_2(t)|^2\omega_t^{00;00}(\mathcal{L}_{11}(X)),\\
\dot{\omega}_t^{00;11}(X) &= \omega_t^{00;11}(\mathcal{L}_{00}(X)) + \frac{1}{\sqrt{N_2}}\xi_1(t)\omega_t^{00;01}(\mathcal{L}_{01}(X)) + \frac{1}{\sqrt{N_2}}\xi_2(t)\omega_t^{00;10}(\mathcal{L}_{01}(X)),\\
\dot{\omega}_t^{10;11}(X) &= \omega_t^{10;11}(\mathcal{L}_{00}(X)) + \xi_1^*(t)\omega_t^{00;11}(\mathcal{L}_{10}(X))\\
&\quad + \frac{1}{\sqrt{N_2}}\xi_1(t)\omega_t^{10;01}(\mathcal{L}_{01}(X)) + \frac{1}{\sqrt{N_2}}\xi_2(t)\omega_t^{10;10}(\mathcal{L}_{01}(X))\\
&\quad + \frac{1}{\sqrt{N_2}}|\xi_1(t)|^2\omega_t^{00;01}(\mathcal{L}_{11}(X)) + \frac{1}{\sqrt{N_2}}\xi_1^*(t)\xi_2(t)\omega_t^{00;10}(\mathcal{L}_{11}(X)),\\
\dot{\omega}_t^{01;11}(X) &= \omega_t^{01;11}(\mathcal{L}_{00}(X)) + \xi_2^*(t)\omega_t^{00;11}(\mathcal{L}_{10}(X))\\
&\quad + \frac{1}{\sqrt{N_2}}\xi_1(t)\omega_t^{01;01}(\mathcal{L}_{01}(X)) + \frac{1}{\sqrt{N_2}}\xi_2(t)\omega_t^{01;10}(\mathcal{L}_{01}(X))\\
&\quad + \frac{1}{\sqrt{N_2}}|\xi_2(t)|^2\omega_t^{00;10}(\mathcal{L}_{11}(X)) + \frac{1}{\sqrt{N_2}}\xi_1(t)\xi_2^*(t)\omega_t^{00;01}(\mathcal{L}_{11}(X)),\\
\dot{\omega}_t^{11;11}(X) &= \omega_t^{11;11}(\mathcal{L}_{00}(X)) + \frac{1}{\sqrt{N_2}}\xi_1(t)\omega_t^{11;01}(\mathcal{L}_{01}(X)) + \frac{1}{\sqrt{N_2}}\xi_2(t)\omega_t^{11;10}(\mathcal{L}_{01}(X))\\
&\quad + \frac{1}{\sqrt{N_2}}\xi_1^*(t)\omega_t^{01;11}(\mathcal{L}_{10}(X)) + \frac{1}{\sqrt{N_2}}\xi_2^*(t)\omega_t^{10;11}(\mathcal{L}_{10}(X))\\
&\quad + \frac{1}{N_2}|\xi_1(t)|^2\omega_t^{01;01}(\mathcal{L}_{11}(X)) + \frac{1}{N_2}\xi_1^*(t)\xi_2(t)\omega_t^{01;10}(\mathcal{L}_{11}(X))\\
&\quad + \frac{1}{N_2}\xi_1(t)\xi_2^*(t)\omega_t^{10;01}(\mathcal{L}_{11}(X)) + \frac{1}{N_2}|\xi_2(t)|^2\omega_t^{10;10}(\mathcal{L}_{11}(X))
\end{aligned}$$

with the initial conditions $\omega_0^{jk;mn}(X) = \langle\eta|X|\eta\rangle\langle\Phi_{jk}|\Phi_{mn}\rangle$ for all $j,k,m,n=0,1$. Moreover, the differential equations for $\omega_t^{10;00}(X)$, $\omega_t^{01;00}(X)$, $\omega_t^{01;10}(X)$, $\omega_t^{11;00}(X)$, $\omega_t^{11;10}(X)$, and $\omega_t^{11;01}(X)$ can be obtained from the above differential equations by means of the property (3.6).

Remark 2. The system of equations in Theorem 3.1 can be established directly by means of (2.1) and (3.4). Alternatively, they can be obtained from the system of filtering equations by averaging over the environment (to be discussed in subsection

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3.5). Finally, as will be pointed out in Remark 5, Theorem 3.1 is an immediate consequence of Theorem 3.3 in subsection 3.3. Thus, the proof of Theorem 3.1 is omitted.

Remark 3. The first equation in Theorem 3.1 is nothing else but the master equation when the input state is the vacuum state $|0\rangle$. Moreover, the first, second, and fourth equations in the theorem are the system of master equations when the input state is the single-photon state $|\Phi_{10}\rangle$, as derived in [1, 25].

Next, we present the master equations in the Schrodinger picture. Define operators $\varrho_t^{jk;mn}$ on H_S via

$$\mathrm{Tr}[(\varrho_t^{jk;mn})^* X] = \omega_t^{jk;mn}(X) \quad \forall\, j,k,m,n = 0,1. \tag{3.7}$$

Clearly, $\varrho_t^{jk;mn}$ are reduced system density operators. Given a system operator ϱ on H_S, define superoperators

$$\begin{aligned}
&\mathcal{D}_{00}(\varrho) \triangleq \frac{1}{2}[L\varrho, L^*] + \frac{1}{2}[L, \varrho L^*] - \mathrm{i}[H, \varrho],\\
&\mathcal{D}_{01}(\varrho) \triangleq [S\varrho, L^*],\ \mathcal{D}_{10}(\varrho) \triangleq [L, \varrho S^*],\ \mathcal{D}_{11}(\varrho) \triangleq S\varrho S^* - \varrho.
\end{aligned}$$

The master equation in the Schrodinger picture is given in the following corollary, which is a direct consequence of Theorem 3.1 and (3.7).

Corollary 3.2. *The master equation in the Schrodinger picture for the quantum system G driven by the two-photon input field state $|\Phi_{11}\rangle$ is given by the system of differential equations*

$$\begin{aligned}
\dot{\varrho}_t^{00;00} &= \mathcal{D}_{00}(\varrho_t^{00;00}),\\
\dot{\varrho}_t^{00;10} &= \mathcal{D}_{00}(\varrho_t^{00;10}) + \xi_1^*(t)\mathcal{D}_{10}(\varrho_t^{00;00}),\\
\dot{\varrho}_t^{00;01} &= \mathcal{D}_{00}(\varrho_t^{00;01}) + \xi_2^*(t)\mathcal{D}_{10}(\varrho_t^{00;00}),\\
\dot{\varrho}_t^{10;10} &= \mathcal{D}_{00}(\varrho_t^{10;10}) + \xi_1^*(t)\mathcal{D}_{10}(\varrho_t^{10;00}) + \xi_1(t)\mathcal{D}_{01}(\varrho_t^{00;10}) + |\xi_1(t)|^2\mathcal{D}_{11}(\varrho_t^{00;00}),\\
\dot{\varrho}_t^{10;01} &= \mathcal{D}_{00}(\varrho_t^{10;01}) + \xi_2^*(t)\mathcal{D}_{10}(\varrho_t^{10;00}) + \xi_1(t)\mathcal{D}_{01}(\varrho_t^{00;01}) + \xi_1(t)\xi_2^*(t)\mathcal{D}_{11}(\varrho_t^{00;00}),\\
\dot{\varrho}_t^{01;01} &= \mathcal{D}_{00}(\varrho_t^{01;01}) + \xi_2^*(t)\mathcal{D}_{10}(\varrho_t^{01;00}) + \xi_2(t)\mathcal{D}_{01}(\varrho_t^{00;01}) + |\xi_2(t)|^2\mathcal{D}_{11}(\varrho_t^{00;00}),\\
\dot{\varrho}_t^{00;11} &= \mathcal{D}_{00}(\varrho_t^{00;11}) + \frac{1}{\sqrt{N_2}}\xi_1^*(t)\mathcal{D}_{10}(\varrho_t^{00;01}) + \frac{1}{\sqrt{N_2}}\xi_2^*(t)\mathcal{D}_{10}(\varrho_t^{00;10}),\\
\dot{\varrho}_t^{10;11} &= \mathcal{D}_{00}(\varrho_t^{10;11}) + \frac{1}{\sqrt{N_2}}\xi_1^*(t)\mathcal{D}_{10}(\varrho_t^{10;01}) + \frac{1}{\sqrt{N_2}}\xi_2^*(t)\mathcal{D}_{10}(\varrho_t^{10;10}) + \xi_1(t)\mathcal{D}_{01}(\varrho_t^{00;11})\\
&\quad + \frac{1}{\sqrt{N_2}}|\xi_1(t)|^2\mathcal{D}_{11}(\varrho_t^{00;01}) + \frac{1}{\sqrt{N_2}}\xi_1(t)\xi_2^*(t)\mathcal{D}_{11}(\varrho_t^{00;10}),\\
\dot{\varrho}_t^{01;11} &= \mathcal{D}_{00}(\varrho_t^{01;11}) + \frac{1}{\sqrt{N_2}}\xi_1^*(t)\mathcal{D}_{10}(\varrho_t^{01;01}) + \frac{1}{\sqrt{N_2}}\xi_2^*(t)\mathcal{D}_{10}(\varrho_t^{01;10}) + \xi_2(t)\mathcal{D}_{01}(\varrho_t^{00;11})\\
&\quad + \frac{1}{\sqrt{N_2}}|\xi_2(t)|^2\mathcal{D}_{11}(\varrho_t^{00;10}) + \frac{1}{\sqrt{N_2}}\xi_1^*(t)\xi_2(t)\mathcal{D}_{11}(\varrho_t^{00;01}),
\end{aligned}$$

$$\begin{aligned}
\dot{\varrho}_t^{11;11} &= \mathcal{D}_{00}(\varrho_t^{11;11}) + \frac{1}{\sqrt{N_2}}\xi_1^*(t)\mathcal{D}_{10}(\varrho_t^{11;01}) + \frac{1}{\sqrt{N_2}}\xi_2^*(t)\mathcal{D}_{10}(\varrho_t^{11;10})\\
&\quad + \frac{1}{\sqrt{N_2}}\xi_1(t)\mathcal{D}_{01}(\varrho_t^{01;11}) + \frac{1}{\sqrt{N_2}}\xi_2(t)\mathcal{D}_{01}(\varrho_t^{10;11}) + \frac{1}{N_2}|\xi_1(t)|^2\mathcal{D}_{11}(\varrho_t^{01;01})\\
&\quad + \frac{1}{N_2}\xi_1(t)\xi_2^*(t)\mathcal{D}_{11}(\varrho_t^{01;10}) + \frac{1}{N_2}\xi_1^*(t)\xi_2\mathcal{D}_{11}(\varrho_t^{10;01}) + \frac{1}{N_2}|\xi_2(t)|^2\mathcal{D}_{11}(\varrho_t^{10;10}),
\end{aligned}$$

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and

$$\varrho_t^{10;00} = (\varrho_t^{00;10})^*, \quad \varrho_t^{01;00} = (\varrho_t^{00;01})^*, \quad \varrho_t^{01;10} = (\varrho_t^{10;01})^*,$$
$$\varrho_t^{11;00} = (\varrho_t^{00;11})^*, \quad \varrho_t^{11;10} = (\varrho_t^{10;11})^*, \quad \varrho_t^{11;01} = (\varrho_t^{01;11})^*$$

with the initial conditions

$$\varrho_0^{jk;mn} = \langle \Phi_{mn} | \Phi_{jk} \rangle |\eta\rangle\langle\eta| \qquad \forall\ j,k,m,n = 0,1. \tag{3.8}$$

Remark 4. Restricted to the 2-photon Fock state case, i.e., $\xi_1 \equiv \xi_2$, the above master equations reduce to the master equation (41) in [40], while the initial conditions (3.8) reduce to (42)–(43) in [40]. It is clear that the general 2-photon case is much more complicated than the 2-photon Fock state case.

Example 2. We illustrate the two-photon master equations derived above by means of the model (2.9)–(2.10) studied in Example 1. Let the two-level system G be driven by a wave packet prepared in the two-photon state $|\Phi_{11}\rangle$, as defined in (3.3). We show that, in this two-photon case, interesting phenomena can be observed which are absent from both the single-photon case and the two-photon Fock state case, as previously studied in [1, 25, 40, 47, 50]. Here, we assume $\kappa = 1$ and the two-level system is initialized in the ground state $|g\rangle$. For the two-photon input state $|\Phi_{11}\rangle$, we use Gaussian pulse shapes. Specifically, we choose

$$\xi_i(t) = \Big(\frac{\Omega_i^2}{2\pi}\Big)^{1/4} \exp\Big(-\frac{\Omega_i^2}{4}(t-t_i)^2\Big), \quad i = 1,2. \tag{3.9}$$

For the single-photon state $|\Phi_{10}\rangle$ or $|\Phi_{01}\rangle$ defined in (3.2), t_i can be interpreted as the peak arrival time of the photon, and Ω_i is the frequency bandwidth. More discussions on the physical aspect of the model can be found in, e.g., [40, 48, 49, 50]. Let $\varrho_t^{11;11}$ be the solution to the master equations in Corollary 3.2. Then the unconditional excitation probability (the probability of finding the two-level system in the excited state $|e\rangle$) is $\mathbb{P}_e(t) \triangleq \mathrm{Tr}[\varrho_t^{11;11}|e\rangle\langle e|]$.

The excitation of a two-level system by a light field in a single-photon state has been studied extensively; e.g., [40, 48, 49, 50]. If the two-level system G is driven by $|\Phi_{10}\rangle$ with the Gaussian pulse shape in (3.9), it is found in [48] that the largest value, denoted $\mathbb{P}_e^{\max}$, of the excitation probability $\mathbb{P}_e(t)$ is around 0.8, which is achieved when $t_1 = 3$ and $\Omega_1 = 1.46\kappa$ (the optimal bandwidth). This value has been confirmed in [25, 40, 49].

The situation of the excitation of a two-level system by a 2-photon state is much more complicated than the single-photon case. A two-level system is a nonlinear system, it can at most absorb or emit one photon at a given time; and the absorption of one photon by the two-level system may have a drastic effect on the system's response to the second coming photon. This nonlinear photon-photon interaction mediated by a two-level system gives rise to interesting physical phenomena. In what follows we study this by means of the master equations derived above. We have the following observations.

(1) If the two peak arrival times t_1 and t_2 are very far away from each other, then the two photons interact with the two-level system one by one, thus the mediated photon-photon interaction does not happen. This is similar to the single-photon case and the maximal excitation probability $\mathbb{P}_e(t) = 0.805$.

(2) When $t_1 = t_2 = 3$ and $\Omega_1 = \Omega_2 = 1.46\kappa$, it can be seen from the black solid line in Figure 1 that $\mathbb{P}_e^{\max} = 0.805$, which is consistent with Figure 2(a) in [40]. In

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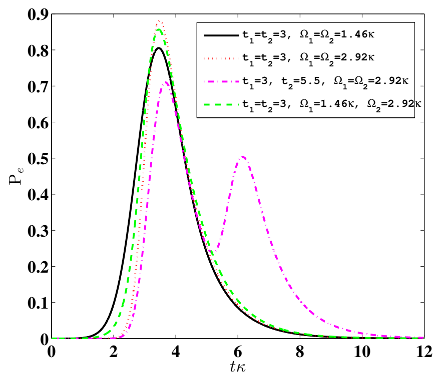

FIG. 1. *Excitation probability for the model in Example* 2.

this case, it appears that one photon is absorbed by the two-level system while the other one just goes through without interaction. Therefore, this case is similar to the one-photon case.

(3) When $t_1 = t_2 = 3$, and $\Omega_1 = \Omega_2 = 2.92\kappa$, it can be seen from the red dotted line in Figure 1 that $\mathbb{P}_e^{\max} = 0.8796$. This cannot occur in the single-photon state case. It is also interesting to notice that in this case the optimal bandwidth (equivalently, the ratio of the bandwidth and the decay rate) is *exactly* twice of that for the single-photon case. To the best knowledge of the author, this has never been reported in the literature.

(4) When $t_1 = t_2 = 3$, $\Omega_1 = 1.46\kappa$, and $\Omega_2 = 2.92\kappa$, as can be seen from the green dashed line in Figure 1, there is one peak whose value is approximately 0.8556, which is still bigger than 0.805 for the optimal single-photon case. Clearly, this is caused by the nonlinear photon-photon interaction mediated by the two-level system.

(5) When $\Omega_1 = \Omega_2 = 2.92\kappa$, and $t_1 = 3, t_2 = 5.5$, as shown by the magenta dash-dotted line in Figure 1, the value of the first peak is around 0.7102 which is even less than 0.805 for the optimal single-photon case (the black solid line in Figure 1), while the value of the second peak is only around 0.5. Moreover $\mathbb{P}_e(t)$ does not drop to zero after the first peak. This means that the excited two-level system is being affected by the other photon in the field in its decay process. The authors are not aware of existing literature that reports such interesting nonlinear atom-photon interaction phenomena to the mathematical rigor presented here.

We will return to this example later and study its two-photon filters. We will show that there are many quantum trajectories whose largest excitation probability can be very close to 1 in all the above scenarios.

3.3. Master equations for an extended system. In this subsection we define an ancillary, then derive the master equations for the extended system: system plus field plus ancillary.

Let

$$|e_{11}\rangle = |e\rangle \otimes |e\rangle,\ |e_{10}\rangle = |e\rangle \otimes |g\rangle,\ |e_{01}\rangle = |g\rangle \otimes |e\rangle,\ |e_{00}\rangle = |g\rangle \otimes |g\rangle \tag{3.10}$$

be an orthonormal basis for $\mathbb{C}^4$. Define a state $|\Sigma\rangle \in \mathbb{C}^4 \otimes \mathsf{H}_S \otimes \mathsf{H}_F$ to be

$$|\Sigma\rangle \triangleq \alpha_{11}|e_{11}\eta\Phi_{11}\rangle + \alpha_{10}|e_{10}\eta\Phi_{10}\rangle + \alpha_{01}|e_{01}\eta\Phi_{01}\rangle + \alpha_{00}|e_{00}\eta\Phi_{00}\rangle, \tag{3.11}$$

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where $\alpha_{11},\ \alpha_{10},\ \alpha_{01}$, and α_{00} are nonzero complex numbers satisfying the normalization condition $\sum_{j,k=0}^{1}|\alpha_{jk}|^2=1$.

Now we have an extended system defined on the tensor product space $\mathbb{C}^4\otimes\mathsf{H}_S\otimes\mathsf{H}_F$. We assume that operators defined on $\mathbb{C}^4$ do not evolve temporally. More specifically, for an arbitrary 4×4 complex matrix A on $\mathbb{C}^4$ and X on H_S, the temporal evolution of $A\otimes X\otimes I$ is governed by $(I\otimes U^*(t))(A\otimes X\otimes I)(I\otimes U(t))=A\otimes j_t(X)$. The adoption of the auxiliary space $\mathbb{C}^4$ allows us to define conditional expectations and derive their filtering equations on the extended space $\mathbb{C}^4\otimes\mathsf{H}_S\otimes\mathsf{H}_F$ with respect to the superposition state $|\Sigma\rangle$. Such conditional expectations in terms of the orthonormal vectors $|e_{jk}\rangle$ in equation (3.10) help us find the conditional expectations for the original quantum system G driven by the 2-photon state $|\Phi_{11}\rangle$; cf. (3.25) in the following. Therefore, careful manipulation on the quantum filters for the extended system enables us to derive 2-photon filters for the original system G. This is the so-called non-Markovian embedding method, which has already been used in [1] for the problem of single-photon filtering.

For an arbitrary 4×4 complex matrix A, define superoperators

$$\mathcal{K}_{00}(A)\triangleq A, \tag{3.12}$$

$$\begin{aligned}\mathcal{K}_{01}(A)\triangleq{}&\frac{\alpha_{11}}{\alpha_{01}\sqrt{N_2}}\xi_1(t)A|e_{11}\rangle\langle e_{01}|+\frac{\alpha_{11}}{\alpha_{10}\sqrt{N_2}}\xi_2(t)A|e_{11}\rangle\langle e_{10}|\\&+\frac{\alpha_{10}}{\alpha_{00}}\xi_1(t)A|e_{10}\rangle\langle e_{00}|+\frac{\alpha_{01}}{\alpha_{00}}\xi_2(t)A|e_{01}\rangle\langle e_{00}|,\end{aligned} \tag{3.13}$$

$$\mathcal{K}_{10}(A)\triangleq\mathcal{K}_{01}(A^*)^*, \tag{3.14}$$

$$\mathcal{K}_{11}(A)\triangleq\mathcal{K}_{10}(\mathcal{K}_{01}(A)). \tag{3.15}$$

For these superoperators the following relations hold:

$$\mathbb{E}_\Sigma[A\mathrm{d}t]=\mathbb{E}_\Sigma[\mathcal{K}_{00}(A)]\mathrm{d}t,\ \mathbb{E}_\Sigma[A\otimes\mathrm{d}B(t)]=\mathbb{E}_\Sigma[\mathcal{K}_{01}(A)]\mathrm{d}t, \tag{3.16}$$

$$\mathbb{E}_\Sigma[A\otimes\mathrm{d}B^*(t)]=\mathbb{E}_\Sigma[\mathcal{K}_{10}(A)]\mathrm{d}t,\ \mathbb{E}_\Sigma[A\otimes\mathrm{d}\Lambda(t)]=\mathbb{E}_\Sigma[\mathcal{K}_{11}(A)]\mathrm{d}t. \tag{3.17}$$

The expectation of $A\otimes j_t(X)$ with respect to the superposition state $|\Sigma\rangle$ is defined by $\tilde{\omega}_t(A\otimes X)\triangleq\mathbb{E}_\Sigma[A\otimes j_t(X)]\equiv\langle\Sigma|A\otimes j_t(X)|\Sigma\rangle$. This expectation is normalized, that is, $\tilde{\omega}_t(I\otimes I)=1$.

THEOREM 3.3. *The temporal evolution of the expectation $\tilde{\omega}_t(A\otimes X)$ is governed by the following master equation*

$$\dot{\tilde{\omega}}_t(A\otimes X)=\tilde{\omega}_t(\mathcal{G}(A\otimes X)), \tag{3.18}$$

where the superoperator $\mathcal{G}(A\otimes X)$ is defined as

$$\mathcal{G}(A\otimes X)\triangleq\sum_{j,k=0}^{1}\mathcal{K}_{jk}(A)\otimes\mathcal{L}_{jk}(X). \tag{3.19}$$

Proof. By (2.1) and (3.16)–(3.17), we have

$$\begin{aligned}\mathrm{d}\tilde{\omega}_t(A\otimes X)={}&\mathbb{E}_\Sigma\big[A\otimes j_t(\mathcal{L}_{00}(X))\mathrm{d}t\big]+\mathbb{E}_\Sigma\big[A\otimes j_t(\mathcal{L}_{01}(X))\mathrm{d}B(t)\big]\\&+\mathbb{E}_\Sigma\big[A\otimes j_t(\mathcal{L}_{10}(X))\mathrm{d}B^*(t)\big]+\mathbb{E}_\Sigma\big[A\otimes j_t(\mathcal{L}_{11}(X))\mathrm{d}\Lambda(t)\big]\\={}&\tilde{\omega}_t(\mathcal{G}(A\otimes X))\mathrm{d}t.\end{aligned}$$

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Remark 5. It can be easily verified that the expectations $\omega_t^{jk;mn}(X)$, defined in (3.5), are scaled components of $\tilde{\omega}_t(A\otimes X)$, that is,

$$I\otimes\omega_t^{jk;mn}(X)=\frac{\tilde{\omega}_t(|e_{jk}\rangle\langle e_{mn}|\otimes X)}{\alpha_{jk}^*\alpha_{mn}}\quad\forall\ j,k,m,n=0,1.$$

As a result, the system of master equations for $\omega_t^{jk;mn}(X)$ in Theorem 3.1 can be derived from (3.18) by setting $A=|e_{jk}\rangle\langle e_{mn}|$ with $j,k,m,n=0,1$.

Remark 6. Note that $\omega_t^{jk;mn}(X)$ can be alternatively rewritten as

$$(I\otimes\omega_t^{jk;mn}(X))\tilde{\omega}_t(|e_{11}\rangle\langle e_{11}|\otimes I)=\frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\omega}_t(|e_{jk}\rangle\langle e_{mn}|\otimes X)\ \ \forall\ j,k,m,n=0,1.$$

Interestingly, a similar relation also holds in the filtering problem; cf. (3.25).

3.4. Quantum filter for the extended system. In this subsection, we consider the homodyne detection and present the quantum filter for the extended system as introduced in the previous subsection.

Define the quantum conditional expectation by

$$\tilde{\pi}_t(A\otimes X)\triangleq\mathbb{E}_\Sigma[A\otimes j_t(X)|I\otimes\mathscr{Y}(t)]. \tag{3.20}$$

THEOREM 3.4. *In the case of homodyne detection, the filtering equation for the conditional expectation $\tilde{\pi}_t(A\otimes X)$ for the extended system is*

$$\mathrm{d}\tilde{\pi}_t(A\otimes X)=\tilde{\pi}_t(\mathcal{G}(A\otimes X))\mathrm{d}t+\tilde{\mathcal{H}}_t(A\otimes X)\mathrm{d}\tilde{W}(t), \tag{3.21}$$

in which the superoperator $\mathcal{G}(A\otimes X)$ is that defined in (3.19) and

$$\tilde{\mathcal{H}}_t(A\otimes X)=\tilde{M}_t(A\otimes X)-\tilde{\pi}_t(A\otimes X)\tilde{M}_t(I\otimes I)$$

with

$$\tilde{M}_t(A\otimes X)\triangleq\tilde{\pi}_t(\mathcal{K}_{00}(A)\otimes(XL+L^*X)+\mathcal{K}_{01}(A)\otimes XS+\mathcal{K}_{10}(A)\otimes S^*X). \tag{3.22}$$

Moreover, the stochastic process $\tilde{W}(t)$, which satisfies the following Ito equation

$$\mathrm{d}\tilde{W}(t)=I\otimes\mathrm{d}Y(t)-\tilde{M}_t(I\otimes I)\mathrm{d}t,$$

is a Wiener process with respect to the state $|\Sigma\rangle$.

Proof. We use the characteristic function method by postulating the filter to be of the form

$$\mathrm{d}\tilde{\pi}_t(A\otimes X)=\tilde{\mathcal{F}}_t(A\otimes X)\mathrm{d}t+\tilde{\mathcal{H}}_t(A\otimes X)I\otimes\mathrm{d}Y(t), \tag{3.23}$$

where $\tilde{\mathcal{F}}_t(A\otimes X)$ and $\tilde{\mathcal{H}}_t(A\otimes X)$ are to be determined. For an arbitrary function $f\in L_2(\mathbb{R}^+,\mathbb{C})$, define a random process $c_f(t)$ via $c_f(t)\triangleq \mathrm{e}^{\int_0^t f(s)\mathrm{d}Y(s)-\frac{1}{2}\int_0^t f^2(s)\mathrm{d}s}$. Clearly $c_f(0)=1$. Moreover, $c_f(t)$ satisfies $\mathrm{d}c_f(t)=f(t)c_f(t)\mathrm{d}Y(t)$. Thus $I\otimes c_f(t)$ is adapted to $I\otimes\mathscr{Y}(t)$. By a property of conditional expectations, we have

$$\mathbb{E}_\Sigma[(A\otimes j_t(X))(I\otimes c_f(t))]=\mathbb{E}_\Sigma[\tilde{\pi}_t(A\otimes X)(I\otimes c_f(t))]. \tag{3.24}$$

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Differentiating both sides of (3.24) and by means of properties of conditional expectations, we find

$$\mathrm{d}\mathbb{E}_\Sigma[A\otimes j_t(X)c_f(t)]=\mathbb{E}_\Sigma\Big[\big(I\otimes c_f(t)\big)\tilde{\pi}_t(\mathcal{G}(A\otimes X))+\big(I\otimes f(t)c_f(t)\big)\tilde{M}_t(A\otimes X)\Big]\mathrm{d}t$$

and

$$\begin{aligned}\mathrm{d}\mathbb{E}_\Sigma[\tilde{\pi}_t(A\otimes X)(I\otimes c_f(t))]=\mathbb{E}_\Sigma\Big[&\big(I\otimes c_f(t)\big)\big\{\tilde{\mathcal{F}}_t(A\otimes X)+\tilde{\mathcal{H}}_t(A\otimes X)\tilde{M}_t(I\otimes I)\big\}\\&+\big(I\otimes f(t)c_f(t)\big)\big\{\tilde{\pi}_t(A\otimes X)\tilde{M}_t(I\otimes I)+\tilde{\mathcal{H}}_t(A\otimes X)\big\}\Big]\mathrm{d}t.\end{aligned}$$

Comparing the coefficients of $c_f(t)$ and $f(t)c_f(t)$, respectively, we find the exact forms of $\tilde{\mathcal{F}}_t(A\otimes X)$ and $\tilde{\mathcal{H}}_t(A\otimes X)$. Putting them back into (3.23) yields the filter (3.21).

We now prove the martingale property $\mathbb{E}_\Sigma[\tilde{W}(t)-\tilde{W}(s)|I\otimes\mathscr{Y}(s)]=0$ for all $0\le s\le t$. This is equivalent to proving that $\mathbb{E}_\Sigma[(\tilde{W}(t)-\tilde{W}(s))(I\otimes K)]=0$ for all $K\in\mathscr{Y}(s), 0\le s\le t$. Obviously,

$$\begin{aligned}&\mathbb{E}_\Sigma[(\tilde{W}(t)-\tilde{W}(s))(I\otimes K)]\\&=\mathbb{E}_\Sigma[I\otimes(Y(t)-Y(s))(I\otimes K)]-\mathbb{E}_\Sigma\Big[\int_s^t\tilde{M}_r(I\otimes I)\mathrm{d}r(I\otimes K)\Big]\\&=\mathbb{E}_\Sigma[I\otimes(Y(t)-Y(s))(I\otimes K)]\\&\quad-\mathbb{E}_\Sigma\Big[\int_s^t\tilde{\pi}_r\big(\mathcal{K}_{00}(I)\otimes(L+L^*)+\mathcal{K}_{01}(I)\otimes S+\mathcal{K}_{10}(I)\otimes S^*\big)\mathrm{d}r(I\otimes K)\Big]\\&=0.\end{aligned}$$

Finally, since $\mathrm{d}\tilde{W}(t)\mathrm{d}\tilde{W}(t)=\mathrm{d}t$, Levy's theorem implies that $\tilde{W}(t)$ is a Wiener process. ∎

Remark 7. Due to the martingale property of the innovations process $\tilde{W}(t)$, if we take the expected value of (3.21), we can recover the master equation (3.18).

3.5. Two-photon quantum filter. In this subsection, we derive the quantum filter for the original quantum system G driven by the two-photon state $|\Phi_{11}\rangle$ defined in (3.3).

Define implicitly the conditional expectations $\pi_t^{jk;mn}(X)$, $j,k,m,n=0,1$, for the original system G via

$$(I\otimes\pi_t^{jk;mn}(X))\tilde{\pi}_t(|e_{11}\rangle\langle e_{11}|\otimes I)=\frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(|e_{jk}\rangle\langle e_{mn}|\otimes X),\tag{3.25}$$

where $\tilde{\pi}_t(A\otimes X)$ is the conditional expectation for the extended system, as defined in (3.20). Clearly, $\pi_0^{jk;mn}(X)=\langle\eta|X|\eta\rangle\langle\Phi_{jk}|\Phi_{mn}\rangle$, and

$$\pi_t^{mn;jk}(X)=(\pi_t^{jk;mn}(X^*))^*\quad\forall\ j,k,m,n=0,1.$$

Equation (3.25) is very important in the derivation of the two-photon quantum filter, since it establishes a relationship between the conditional expectations of the extended system and the original system. And $\pi_t^{11;11}(X)$ defined in this way is exactly the quantum conditional expectation for the two-photon field state $|\Phi_{11}\rangle$ as shown by the following lemma. Therefore we can get the desired two-photon quantum filter by means of the filtering equations for the extended system, Theorem 3.6.

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LEMMA 3.5. *For all $K \in \mathscr{Y}(t)$,*

$$\mathbb{E}_{11;11}[\pi_t^{jk;mn}(X)K] = \mathbb{E}_{jk;mn}[j_t(X)K] \quad \forall\ j,k,m,n = 0,1. \tag{3.26}$$

In particular,

$$\mathbb{E}_{11;11}[\pi_t^{11;11}(X)K] = \mathbb{E}_{11;11}[j_t(X)K].$$

That is, $\pi_t^{11;11}(X)$ is exactly the quantum conditional expectation for the two-photon field state $|\Phi_{11}\rangle$, namely, $\pi_t^{11;11}(X) = \mathbb{E}_{11;11}[j_t(X)|\mathscr{Y}(t)]$.

In what follows we derive the quantum filtering equations for the quantum conditional expectation $\pi_t^{11;11}(X)$. To present the results clearly, we define the superoperators $M_t^{jk;mn}(X)$ $(j,k,m,n = 0,1)$ for an arbitrary system operator X as follows:

$$\begin{aligned} &M_t^{jk;mn}(X) \\ &\triangleq \pi_t^{jk;mn}(XL + L^*X) + \delta_{m1}\delta_{n0}\xi_1(t)\pi_t^{jk;00}(XS) + \delta_{m0}\delta_{n1}\xi_2(t)\pi_t^{jk;00}(XS) \\ &\quad + \delta_{j1}\delta_{k0}\xi_1^*(t)\pi_t^{00;mn}(S^*X) + \delta_{j0}\delta_{k1}\xi_2^*(t)\pi_t^{00;mn}(S^*X) + \frac{\delta_{m1}\delta_{n1}}{\sqrt{N_2}}\xi_1(t)\pi_t^{jk;01}(XS) \\ &\quad + \frac{\delta_{m1}\delta_{n1}}{\sqrt{N_2}}\xi_2(t)\pi_t^{jk;10}(XS) + \frac{\delta_{j1}\delta_{k1}}{\sqrt{N_2}}\xi_1^*(t)\pi_t^{01;mn}(S^*X) + \frac{\delta_{j1}\delta_{k1}}{\sqrt{N_2}}\xi_2^*(t)\pi_t^{10;mn}(S^*X). \end{aligned} \tag{3.27}$$

THEOREM 3.6. *In the case of homodyne detection, the quantum filter for the quantum system G driven by the 2-photon state $|\Phi_{11}\rangle$ is given by the following system of Ito differential equations:*

$$\begin{aligned} \mathrm{d}\pi_t^{11;11}(X) = \Bigg[&\pi_t^{11;11}(\mathcal{L}_{00}(X)) + \frac{1}{\sqrt{N_2}}\xi_1(t)\pi_t^{11;01}(\mathcal{L}_{01}(X)) + \frac{1}{\sqrt{N_2}}\xi_2(t)\pi_t^{11;10}(\mathcal{L}_{01}(X)) \\ &+ \frac{1}{\sqrt{N_2}}\xi_1^*(t)\pi_t^{01;11}(\mathcal{L}_{10}(X)) + \frac{1}{\sqrt{N_2}}\xi_2^*(t)\pi_t^{10;11}(\mathcal{L}_{10}(X)) \\ &+ \frac{1}{N_2}|\xi_1(t)|^2\pi_t^{01;01}(\mathcal{L}_{11}(X)) + \frac{1}{N_2}|\xi_2(t)|^2\pi_t^{10;10}(\mathcal{L}_{11}(X)) \\ &+ \frac{1}{N_2}\xi_1(t)\xi_2^*(t)\pi_t^{10;01}(\mathcal{L}_{11}(X)) + \frac{1}{N_2}\xi_1^*(t)\xi_2(t)\pi_t^{01;10}(\mathcal{L}_{11}(X))\Bigg]\mathrm{d}t \\ &+ \Bigg[M_t^{11;11}(X) - \pi_t^{11;11}(X)M_t^{11;11}(I)\Bigg]\mathrm{d}W(t), \end{aligned}$$

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where

$$
\begin{aligned}
\mathrm{d}\pi_t^{00;00}(X) &= \pi_t^{00;00}(\mathcal{L}_{00}(X))\mathrm{d}t + \left[M_t^{00;00}(X) - \pi_t^{00;00}(X)M_t^{11;11}(I)\right]\mathrm{d}W(t),\\
\mathrm{d}\pi_t^{00;10}(X) &= \left[\pi_t^{00;10}(\mathcal{L}_{00}(X)) + \xi_1(t)\pi_t^{00;00}(\mathcal{L}_{01}(X))\right]\mathrm{d}t\\
&\quad + \left[M_t^{00;10}(X) - \pi_t^{00;10}(X)M_t^{11;11}(I)\right]\mathrm{d}W(t),\\
\mathrm{d}\pi_t^{00;01}(X) &= \left[\pi_t^{00;01}(\mathcal{L}_{00}(X)) + \xi_2(t)\pi_t^{00;00}(\mathcal{L}_{01}(X))\right]\mathrm{d}t\\
&\quad + \left[M_t^{00;01}(X) - \pi_t^{00;01}(X)M_t^{11;11}(I)\right]\mathrm{d}W(t),\\
\mathrm{d}\pi_t^{10;10}(X) &= \Big[\xi_1(t)\pi_t^{10;00}(\mathcal{L}_{01}(X)) + \xi_1^*(t)\pi_t^{00;10}(\mathcal{L}_{10}(X)) + |\xi_1(t)|^2\pi_t^{00;00}(\mathcal{L}_{11}(X))\\
&\quad + \pi_t^{10;10}(\mathcal{L}_{00}(X))\Big]\mathrm{d}t + \left[M_t^{10;10}(X)) - \pi_t^{10;10}(X)M_t^{11;11}(I)\right]\mathrm{d}W(t),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}\pi_t^{10;01}(X) &= \Big[\xi_2(t)\pi_t^{10;00}(\mathcal{L}_{01}(X)) + \xi_1^*(t)\pi_t^{00;01}(\mathcal{L}_{10}(X)) + \xi_1^*(t)\xi_2(t)\pi_t^{00;00}(\mathcal{L}_{11}(X))\\
&\quad + \pi_t^{10;01}(\mathcal{L}_{00}(X))\Big]\mathrm{d}t + \left[M_t^{10;01}(X) - \pi_t^{10;01}(X)M_t^{11;11}(I)\right]\mathrm{d}W(t),\\
\mathrm{d}\pi_t^{01;01}(X) &= \Big[\xi_2(t)\pi_t^{01;00}(\mathcal{L}_{01}(X)) + \xi_2^*(t)\pi_t^{00;01}(\mathcal{L}_{10}(X)) + |\xi_2(t)|^2\pi_t^{00;00}(\mathcal{L}_{11}(X))\\
&\quad + \pi_t^{01;01}(\mathcal{L}_{00}(X))\Big]\mathrm{d}t + \left[M_t^{01;01}(X) - \pi_t^{01;01}(X)M_t^{11;11}(I)\right]\mathrm{d}W(t),\\
\mathrm{d}\pi_t^{00;11}(X) &= \Big[\frac{1}{\sqrt{N_2}}\xi_1(t)\pi_t^{00;01}(\mathcal{L}_{01}(X)) + \frac{1}{\sqrt{N_2}}\xi_2(t)\pi_t^{00;10}(\mathcal{L}_{01}(X))\\
&\quad + \pi_t^{00;11}(\mathcal{L}_{00}(X))\Big]\mathrm{d}t + \left[M_t^{00;11}(X) - \pi_t^{00;11}(X)M_t^{11;11}(I)\right]\mathrm{d}W(t),\\
\mathrm{d}\pi_t^{10;11}(X) &= \Big[\frac{1}{\sqrt{N_2}}\xi_1(t)\pi_t^{10;01}(\mathcal{L}_{01}(X)) + \frac{1}{\sqrt{N_2}}\xi_2(t)\pi_t^{10;10}(\mathcal{L}_{01}(X))\\
&\quad + \xi_1^*(t)\pi_t^{00;11}(\mathcal{L}_{10}(X)) + \frac{1}{\sqrt{N_2}}|\xi_1(t)|^2\pi_t^{00;01}(\mathcal{L}_{11}(X))\\
&\quad + \frac{1}{\sqrt{N_2}}\xi_1^*(t)\xi_2(t)\pi_t^{00;10}(\mathcal{L}_{11}(X)) + \pi_t^{10;11}(\mathcal{L}_{00}(X))\Big]\mathrm{d}t\\
&\quad + \left[M_t^{10;11}(X) - \pi_t^{10;11}(X)M_t^{11;11}(I)\right]\mathrm{d}W(t),\\
\mathrm{d}\pi_t^{01;11}(X) &= \Big[\frac{1}{\sqrt{N_2}}\xi_1(t)\pi_t^{01;01}(\mathcal{L}_{01}(X)) + \frac{1}{\sqrt{N_2}}\xi_2(t)\pi_t^{01;10}(\mathcal{L}_{01}(X))\\
&\quad + \xi_2^*(t)\pi_t^{00;11}(\mathcal{L}_{10}(X)) + \frac{1}{\sqrt{N_2}}\xi_1(t)\xi_2^*(t)\pi_t^{00;01}(\mathcal{L}_{11}(X))\\
&\quad + \frac{1}{\sqrt{N_2}}|\xi_2(t)|^2\pi_t^{00;10}(\mathcal{L}_{11}(X)) + \pi_t^{01;11}(\mathcal{L}_{00}(X))\Big]\mathrm{d}t\\
&\quad + \left[M_t^{01;11}(X) - \pi_t^{01;11}(X)M_t^{11;11}(I)\right]\mathrm{d}W(t),
\end{aligned}
$$

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and

$$\pi_t^{10;00}(X) = (\pi_t^{00;10}(X^*))^*,\ \pi_t^{01;00}(X) = (\pi_t^{00;01}(X^*))^*,\ \pi_t^{11;00}(X) = (\pi_t^{00;11}(X^*))^*,$$
$$\pi_t^{01;10}(X) = (\pi_t^{10;01}(X^*))^*,\ \pi_t^{11;01}(X) = (\pi_t^{01;11}(X^*))^*,\ \pi_t^{11;10}(X) = (\pi_t^{10;11}(X^*))^*,$$

with the initial conditions $\pi_0^{jk;mn}(X) = \langle\eta|X|\eta\rangle\langle\Phi_{jk}|\Phi_{mn}\rangle$ *for all* $j,k,m,n = 0,1$. *Moreover, the innovation process* $W(t)$*, defined by* $\mathrm{d}W(t) = \mathrm{d}Y(t) - M_t^{11;11}(I)\mathrm{d}t$*, is a Wiener process with respect to the two-photon state* $|\Phi_{11}\rangle$.

The proof is given in Appendix A.

In what follows we present the stochastic master equations in the Schrodinger picture. Define conditional density operators $\rho_t^{jk;mn}$ on $\mathsf{H}_S \otimes \mathsf{H}_F$ in terms of

$$\pi_t^{jk;mn}(X) = \mathrm{Tr}[(\rho_t^{jk;mn})^*(I\otimes X)] \quad \forall\ j,k,m,n = 0,1. \tag{3.28}$$

Moreover, define superoperators $\mathcal{S}_t^{jk;mn}(\rho)$, $j,k,m,n = 0,1$, as follows:

$$\begin{aligned}\mathcal{S}_t^{jk;mn}(\rho) \triangleq{}& L\rho_t^{jk;mn} + \rho_t^{jk;mn}L^* + \delta_{m1}\delta_{n0}\xi_1^*(t)\rho_t^{jk;00}S^* + \delta_{m0}\delta_{n1}\xi_2^*(t)\rho_t^{jk;00}S^* \\ &+\delta_{j1}\delta_{k0}\xi_1(t)S\rho_t^{00;mn} + \delta_{j0}\delta_{k1}\xi_2(t)S\rho_t^{00;mn} + \frac{\delta_{m1}\delta_{n1}}{\sqrt{N_2}}\xi_1^*(t)\rho_t^{jk;01}S^* \\ &+\frac{\delta_{m1}\delta_{n1}}{\sqrt{N_2}}\xi_2^*(t)\rho_t^{jk;10}S^* + \frac{\delta_{j1}\delta_{k1}}{\sqrt{N_2}}\xi_1(t)S\rho_t^{01;mn} + \frac{\delta_{j1}\delta_{k1}}{\sqrt{N_2}}\xi_2(t)S\rho_t^{10;mn}.\end{aligned}$$

COROLLARY 3.7. *In the case of homodyne detection, the stochastic master equations for conditional densities* $\rho_t^{jk;mn}$ *of the quantum system* G *driven by the two-photon state* $|\Phi_{11}\rangle$ *are*

$$\begin{aligned}\mathrm{d}\rho_t^{11;11} ={}& \Bigg[\mathcal{D}_{00}(\rho_t^{11;11}) + \frac{1}{\sqrt{N_2}}\xi_1^*(t)\mathcal{D}_{10}(\rho_t^{11;01}) + \frac{1}{\sqrt{N_2}}\xi_2^*(t)\mathcal{D}_{10}(\rho_t^{11;10}) \\ &+\frac{1}{\sqrt{N_2}}\xi_1(t)\mathcal{D}_{01}(\rho_t^{01;11}) + \frac{1}{\sqrt{N_2}}\xi_2(t)\mathcal{D}_{01}(\rho_t^{10;11}) + \frac{1}{N_2}|\xi_1(t)|^2\mathcal{D}_{11}(\rho_t^{01;01}) \\ &+\frac{1}{N_2}\xi_1(t)\xi_2^*(t)\mathcal{D}_{11}(\rho_t^{01;10}) + \frac{1}{N_2}\xi_1^*(t)\xi_2(t)\mathcal{D}_{11}(\rho_t^{10;01}) \\ &+\frac{1}{N_2}|\xi_2(t)|^2\mathcal{D}_{11}(\rho_t^{10;10})\Bigg]\mathrm{d}t + \Big[\mathcal{S}_t^{11;11}(\rho) - \rho_t^{11;11}\mathrm{Tr}[\mathcal{S}_t^{11;11}(\rho)]\Big]\mathrm{d}\bar{W}_t,\end{aligned}$$

where

$$\mathrm{d}\rho_t^{00;00} = \mathcal{D}_{00}(\rho_t^{00;00})\mathrm{d}t + \Big[\mathcal{S}_t^{00;00}(\rho) - \rho_t^{00;00}\mathrm{Tr}[\mathcal{S}_t^{11;11}(\rho)]\Big]\mathrm{d}\bar{W}_t,$$

$$\mathrm{d}\rho_t^{00;10} = \Big[\mathcal{D}_{00}(\rho_t^{00;10}) + \xi_1^*(t)\mathcal{D}_{10}(\rho_t^{00;00})\Big]\mathrm{d}t + \Big[\mathcal{S}_t^{00;10}(\rho) - \rho_t^{00;10}\mathrm{Tr}[\mathcal{S}_t^{11;11}(\rho)]\Big]\mathrm{d}\bar{W}_t,$$

$$\mathrm{d}\rho_t^{00;01} = \Big[\mathcal{D}_{00}(\rho_t^{00;01}) + \xi_2^*(t)\mathcal{D}_{10}(\rho_t^{00;00})\Big]\mathrm{d}t + \Big[\mathcal{S}_t^{00;01}(\rho) - \rho_t^{00;01}\mathrm{Tr}[\mathcal{S}_t^{11;11}(\rho)]\Big]\mathrm{d}\bar{W}_t,$$

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$$
\begin{aligned}
\mathrm{d}\rho_t^{10;10} &= \left[\mathcal{D}_{00}(\rho_t^{10;10}) + \xi_1^*(t)\mathcal{D}_{10}(\rho_t^{10;00}) + \xi_1(t)\mathcal{D}_{01}(\rho_t^{00;10}) + |\xi_1(t)|^2\mathcal{D}_{11}(\rho_t^{00;00})\right]\mathrm{d}t \\
&\quad + \left[\mathcal{S}_t^{10;10}(\rho) - \rho_t^{10;10}\mathrm{Tr}[\mathcal{S}_t^{11;11}(\rho)]\right]\mathrm{d}\bar{W}_t, \\
\mathrm{d}\rho_t^{10;01} &= \left[\mathcal{D}_{00}(\rho_t^{10;01}) + \xi_2^*(t)\mathcal{D}_{10}(\rho_t^{10;00}) + \xi_1(t)\mathcal{D}_{01}(\rho_t^{00;01}) + \xi_1(t)\xi_2^*(t)\mathcal{D}_{11}(\rho_t^{00;00})\right]\mathrm{d}t \\
&\quad + \left[\mathcal{S}_t^{10;01}(\rho) - \rho_t^{10;01}\mathrm{Tr}[\mathcal{S}_t^{11;11}(\rho)]\right]\mathrm{d}\bar{W}_t, \\
\mathrm{d}\rho_t^{01;01} &= \left[\mathcal{D}_{00}(\rho_t^{01;01}) + \xi_2^*(t)\mathcal{D}_{10}(\rho_t^{01;00}) + \xi_2(t)\mathcal{D}_{01}(\rho_t^{00;01}) + |\xi_2(t)|^2\mathcal{D}_{11}(\rho_t^{00;00})\right]\mathrm{d}t \\
&\quad + \left[\mathcal{S}_t^{01;01}(\rho) - \rho_t^{01;01}\mathrm{Tr}[\mathcal{S}_t^{11;11}(\rho)]\right]\mathrm{d}\bar{W}_t, \\
\mathrm{d}\rho_t^{00;11} &= \left[\mathcal{D}_{00}(\rho_t^{00;11}) + \frac{1}{\sqrt{N_2}}\xi_1^*(t)\mathcal{D}_{10}(\rho_t^{00;01}) + \frac{1}{\sqrt{N_2}}\xi_2^*(t)\mathcal{D}_{10}(\rho_t^{00;10})\right]\mathrm{d}t \\
&\quad + \left[\mathcal{S}_t^{00;11}(\rho) - \rho_t^{00;11}\mathrm{Tr}[\mathcal{S}_t^{11;11}(\rho)]\right]\mathrm{d}\bar{W}_t,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}\rho_t^{10;11} &= \Bigg[\mathcal{D}_{00}(\rho_t^{10;11}) + \frac{1}{\sqrt{N_2}}\xi_1^*(t)\mathcal{D}_{10}(\rho_t^{10;01}) + \frac{1}{\sqrt{N_2}}\xi_2^*(t)\mathcal{D}_{10}(\rho_t^{10;10}) \\
&\quad + \xi_1(t)\mathcal{D}_{01}(\rho_t^{00;11}) + \frac{1}{\sqrt{N_2}}|\xi_1(t)|^2\mathcal{D}_{11}(\rho_t^{00;01}) + \frac{1}{\sqrt{N_2}}\xi_1(t)\xi_2^*(t)\mathcal{D}_{11}(\rho_t^{00;10})\Bigg]\mathrm{d}t \\
&\quad + \left[\mathcal{S}_t^{10;11}(\rho)) - \rho_t^{10;11}\mathrm{Tr}[\mathcal{S}_t^{11;11}(\rho)]\right]\mathrm{d}\bar{W}_t, \\
\mathrm{d}\rho_t^{01;11} &= \Bigg[\mathcal{D}_{00}(\rho_t^{01;11}) + \frac{1}{\sqrt{N_2}}\xi_1^*(t)\mathcal{D}_{10}(\rho_t^{01;01}) + \frac{1}{\sqrt{N_2}}\xi_2^*(t)\mathcal{D}_{10}(\rho_t^{01;10}) \\
&\quad + \xi_2(t)\mathcal{D}_{01}(\rho_t^{00;11}) + \frac{1}{\sqrt{N_2}}\xi_1^*(t)\xi_2(t)\mathcal{D}_{11}(\rho_t^{00;01}) + \frac{1}{\sqrt{N_2}}|\xi_2(t)|^2\mathcal{D}_{11}(\rho_t^{00;10})\Bigg]\mathrm{d}t \\
&\quad + \left[\mathcal{S}_t^{01;11}(\rho) - \rho_t^{01;11}\mathrm{Tr}[\mathcal{S}_t^{11;11}(\rho)]\right]\mathrm{d}\bar{W}_t,
\end{aligned}
$$

and

$$
\begin{aligned}
&\rho_t^{10;00} = (\rho_t^{00;10})^*, \quad \rho_t^{01;00} = (\rho_t^{00;01})^*, \quad \rho_t^{01;10} = (\rho_t^{10;01})^*, \\
&\rho_t^{11;00} = (\rho_t^{00;11})^*, \quad \rho_t^{11;10} = (\rho_t^{10;11})^*, \quad \rho_t^{11;01} = (\rho_t^{01;11})^*,
\end{aligned}
$$

where the innovation process $\bar{W}_t$ *is defined as* $\mathrm{d}\bar{W}_t = \mathrm{d}Y(t) - \mathrm{Tr}[\mathcal{S}_t^{11;11}(\rho)]\mathrm{d}t$*. The initial conditions are* $\rho_t^{jk;mn}(0) = \langle\Phi_{mn}|\Phi_{jk}\rangle|\eta\rangle\langle\eta|$ *for all* $j,k,m,n = 0,1$*.*

Remark 8. Corollary 3.7 is an immediate consequence of Theorem 3.6 and (3.28).

Example 3. We continue to study the system in Examples 1 and 2 by looking at its 2-photon filter. Here, we wish to calculate the conditional excitation probability (namely, the conditional excited state population) under homodyne detection, which can be expressed as $\mathbb{P}_e^c(t) \triangleq \mathrm{Tr}[\rho_t^{11;11}|e\rangle\langle e|]$, where $\rho_t^{11;11}$ is the solution to the filtering equations in Corollary 3.7. Individual trajectories $\mathbb{P}_e^c(t)$ are plotted in Figures 2–4. For comparison, we also plotted $\mathbb{P}_e(t)$ for the master equations in red solid lines. It can be seen clearly that in all three cases, many quantum trajectories can have maximal excitation probability very close to 1, namely, the unit probability.

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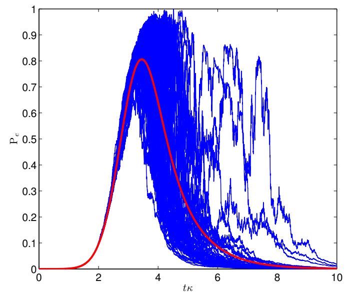

FIG. 2. *The conditional excitation probability of* 2*-photon filtering for the case when* $t_1 = t_2 = 3$ *and* $\Omega_1 = \Omega_2 = 1.46\kappa$*. The horizontal axis is time, while the vertical axis is excitation probability. The red solid line is the unconditional excitation probability* $\mathbb{P}_e(t)$ *as calculated by the master equation in Corollary* 3.2*. The blue lines are individual trajectories of conditional excitation probabilities* $\mathbb{P}_e^c(t)$*.*

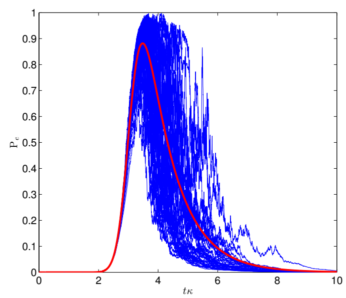

FIG. 3. *The conditional excitation probability of* 2*-photon filtering for the case when* $t_1 = t_2 = 3$ *and* $\Omega_1 = \Omega_2 = 2.92\kappa$*. The axes and lines are the same as in Figure* 2*.*

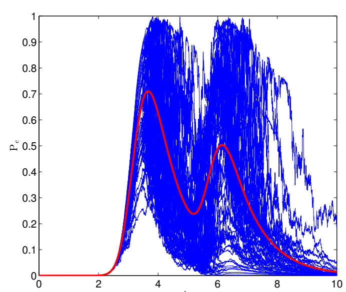

FIG. 4. *The conditional excitation probability of* 2*-photon filtering for the case when* $t_1 = 3$*,* $t_2 = 5.5$*, and* $\Omega_1 = \Omega_2 = 2.92\kappa$*. The axes and lines are the same as in Figure* 2*.*

4. Multiphoton filtering. In this section, we derive filtering equations for an arbitrary quantum system driven by a wave packet in an n-photon state. We follow the logic as carried out in section 3. That is, we first derive a filtering equation for an extended system, then present filtering equations for the original system. Multiphoton states are defined in subsection 4.1. The master equation is presented in subsection 4.2. Filters in the homodyne detection case are given in subsection 4.3,

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and filters in the photon counting case are presented in subsection 4.4.

It is worth noting that the notations used in the multiphoton context are slightly different from the 2-photon case, and the following notations turn out very convenient and useful in the derivation of the multiphoton filter. Define a set $\bar{n} \triangleq \{1, 2, \ldots, n\}$. It is implicitly assumed that the elements in each subset of $\bar{n}$ are ordered from the smallest to the largest. Moreover, given a set $R \subset \bar{n}$ and an integer $\mu \in \bar{n}$ but not in R, namely, $\mu \in \bar{n} \setminus R$, define a new (*ordered*) subset $R\mu \triangleq R \cup \{\mu\}$ of $\bar{n}$.

4.1. Multiphoton states. The continuous-mode n-photon state is defined as

$$|\Phi^n\rangle \triangleq \frac{1}{\sqrt{N_n}} \Pi_{j=1}^n B^*(\xi_j)|0\rangle, \tag{4.1}$$

where the superscript n indicates the number of photons, N_n is the normalization coefficient, and $B^*(\xi_j) = \int_0^\infty \xi_j(t) b^*(t) \mathrm{d}t$ is defined in (3.1). This state is completely determined by the set $M_n \triangleq \{\xi_1, \xi_2, \ldots, \xi_n\}$ of n functions in $L_2(\mathbb{R}^+, \mathbb{C})$. It is worth noting that we distinguish functions in terms of their subscript indices; thus, two (possibly identical) functions with *different* subscript indices are regarded as different functions. For simplicity, we assume all the functions ξ_k are normalized, that is, $\|\xi_k\| = 1$ for all $k = 1, \ldots, n$. However, these functions are not necessarily orthogonal to each other. If all the ξ_i $(i = 1, \ldots, n)$ are equal to ξ, the n-photon state defined in (4.1) reduces to a continuous-mode n-photon Fock state:

$$|F^n\rangle \triangleq \frac{1}{\sqrt{n!}} (B^*(\xi))^n |0\rangle. \tag{4.2}$$

4.2. Multiphoton master equation. In this subsection, we present the master equation of the quantum system G driven by an input field initialized in an n-photon state as defined in (4.1).

For an arbitrary system operator X on the Hilbert space H_S, define its expectation with respect to the n-photon state $|\Phi^n\rangle$ by $\omega_t^{n;n}(X) \triangleq \langle \eta\Phi^n | j_t(X) | \eta\Phi^n\rangle \equiv \mathbb{E}_{n;n}[j_t(X)]$. Geometrically, $\omega_t^{n;n}(X)$ indicates that, at each time instant t, how much information of $j_t(X)$ is contained in the projection space $|\eta\Phi^n\rangle\langle\eta\Phi^n|$.

In what follows we derive the master equation for $\omega_t^{n;n}(X)$. In analogy to (3.4) for the 2-photon state case, the field operator $\mathrm{d}B(t)$ acting on the n-photon state $|\Phi^n\rangle$ generates n states, each having $n-1$ photons. Similarly, $\mathrm{d}B(t)$ acting on an $(n-1)$-photon state produces $n-1$ states, each of which has $n-2$ photons, and so on; cf. Figure 5. As a result, to derive the master equation for $\omega_t^{n;n}(X)$, we have to define the general $(n-k)$-photon states, $k = 1, 2, \ldots, n$. Moreover, for each k, due to the different choices of the functions in the set M_n, there are C_n^k different $(n-k)$-photon states. To efficiently distinguish between them, we adopt the symbol $|\Phi_{r_k}^{n-k}\rangle$, where the superscript $n-k$ indicates the number of photons, while the subscript $r_k \triangleq \{r^{(1)}, r^{(2)}, \ldots, r^{(k)}\} \subset \bar{n}$ indicates the set of functions $M_n \setminus \{\xi_{r^{(1)}}, \xi_{r^{(2)}}, \ldots, \xi_{r^{(k)}}\}$. Explicitly, the state $|\Phi_{r_k}^{n-k}\rangle$ is defined as $|\Phi_{r_k}^{n-k}\rangle \triangleq \frac{1}{\sqrt{N_{r_k}^{n-k}}} \Pi_{l \notin r_k} B^*(\xi_l)|0\rangle$, where $N_{r_k}^{n-k}$ is the corresponding normalization coefficient. In particular, if $k = n$, then $r_k = \bar{n}$. That is, $|\Phi_{r_n}^0\rangle = |0\rangle$ is the vacuum state; cf. the bottom level of the diagram in Figure 5. When $k = n-1$, $|\Phi_{r_{n-1}}^1\rangle$ is a single-photon state. Because all ξ_k are assumed to be normalized, $N_{r_{n-1}}^1 = 1$. There are $C_n^{n-1} = n$ such single-photon states which occupy the top-to-bottom level of the diagram in Figure 5. Finally, for notation's convenience, when $k = 0$, we denote $r_0 = \emptyset$ (the empty set), and correspondingly, $|\Phi_{r_0}^n\rangle = |\Phi^n\rangle$, which resides on the top level of the diagram in Figure 5.

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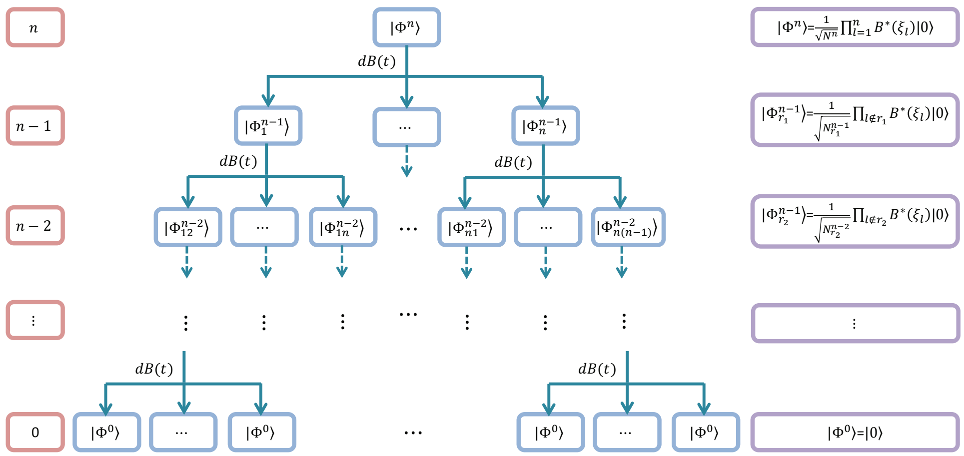

FIG. 5. *The hierarchical structure of the operation of* $\mathrm{d}B(t)$ *on photon states. The left column indicates the number of photons contained in each state on each level. The right column is the general expression of photon states on each level. The column in the middle shows how* $\mathrm{d}B(t)$ *acts on various photon states downward, whose "zoom-in" version is given in Figure* 6.

As illustrated in Figure 6, for the general state $|\Phi_{r_k}^{n-k}\rangle$, we find

$$\mathrm{d}B(t)|\Phi_{r_k}^{n-k}\rangle = \sum_{\mu\notin r_k} \frac{\sqrt{N_{r_k\mu}^{n-k-1}}}{\sqrt{N_{r_k}^{n-k}}}\xi_\mu(t)|\Phi_{r_k\mu}^{n-k-1}\rangle \mathrm{d}t, \quad k=0,\ldots,n-1, \tag{4.3}$$

with $\{j^{(1)}, j^{(2)}, \ldots, j^{(n-k)}\} = \bar{n}\setminus r_k$. Finally, when $k=n$, $r_k=\bar{n}$, $|\Phi_{r_n}^0\rangle = |0\rangle$, and thus $\mathrm{d}B(t)|\Phi_{r_n}^0\rangle = \mathrm{d}B(t)|0\rangle = 0$ which serves as the terminal condition.

As discussed above, to derive the master equation for the quantity $\omega_t^{n;n}(X)$, temporal evolutions of the following quantities

$$\omega_t^{n-j,l_j;n-k,r_k}(X) \triangleq \mathbb{E}_{n-j,l_j;n-k,r_k}[j_t(X)] \equiv \langle \eta\Phi_{l_j}^{n-j}|j_t(X)|\eta\Phi_{r_k}^{n-k}\rangle \quad \forall\, l_j, r_k \subset \bar{n}$$

have to be derived. Once $j=0$ or $k=0$, the notations $\omega_t^{n-j,l_j;n-k,r_k}$ can be simplified as $\omega_t^{n;n-k,r_k}$ or $\omega_t^{n-j,l_j;n}$, respectively. Finally, to simplify notation, we make use of $\omega_t^{n-j,l_j;n-k,r_k}(X) \equiv 0$ if either $j>n$ or $k>n$. This notational convention is very handy in our study of multiphoton filtering problem.

From (2.1) and (4.3), we can derive the master equation of the system G driven by the n-photon state $|\Phi^n\rangle$ as shown by the following theorem, which is the counterpart of Theorem 3.1 for the 2-photon case.

THEOREM 4.1. *The master equation in the Heisenberg picture for the system* G *driven by an input field in the* n*-photon state* $|\Phi^n\rangle$ *is given by the system of differential equations*

$$\begin{aligned}\dot{\omega}_t^{n;n}(X) &= \sum_{\mu=1}^{n} \frac{\sqrt{N_\mu^{n-1}}}{\sqrt{N_n}}\xi_\mu(t)\omega_t^{n;n-1,\mu}(\mathcal{L}_{01}(X)) + \sum_{\nu=1}^{n}\frac{\sqrt{N_\nu^{n-1}}}{\sqrt{N_n}}\xi_\nu^*(t)\omega_t^{n-1,\nu;n}(\mathcal{L}_{10}(X))\\ &\quad+\sum_{\mu=1}^{n}\sum_{\nu=1}^{n}\frac{\sqrt{N_\mu^{n-1}}\sqrt{N_\nu^{n-1}}}{N_n}\xi_\mu(t)\xi_\nu^*(t)\omega_t^{n-1,\nu;n-1,\mu}(\mathcal{L}_{11}(X)) + \omega_t^{n;n}(\mathcal{L}_{00}(X)),\end{aligned}$$

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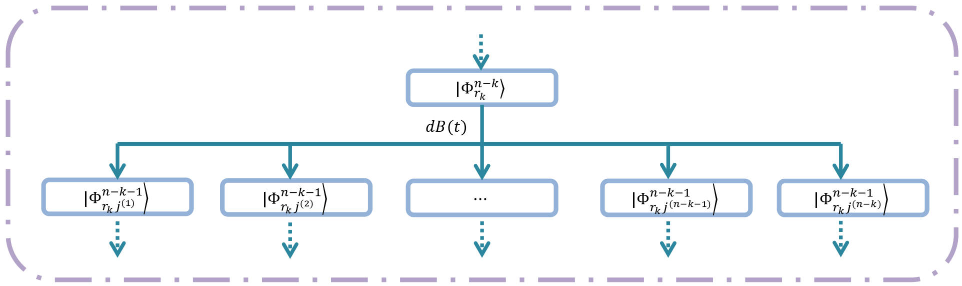

FIG. 6. *The zoom-in version of Figure* 5. *The action of* $\mathrm{d}B(t)$ *on the state* $|\Phi^{n-k}_{r_k}\rangle$ *produces* $n-k$ *states, each of which describes a wave packet containing* $n-k-1$ *photons. The subscripts* $r_k j^{(i)}$ *(i=1,... n-k) are introduced at the end of the second paragraph of section* 4.

where, for subsets $l_j, r_k \subset \bar{n}$ $\quad(\forall\ j,k=0,\dots,n)$,

$$
\begin{aligned}
&\dot{\omega}_t^{n-j,l_j;n-k,r_k}(X)\\
&= \omega_t^{n-j,l_j;n-k,r_k}(\mathcal{L}_{00}(X)) + \sum_{\mu\notin r_k}\frac{\sqrt{N^{n-k-1}_{r_k\mu}}}{\sqrt{N^{n-k}_{r_k}}}\xi_\mu(t)\omega_t^{n-j,l_j;n-k-1,r_k\mu}(\mathcal{L}_{01}(X))\\
&\quad + \sum_{\nu\notin l_j}\frac{\sqrt{N^{n-j-1}_{l_j\nu}}}{\sqrt{N^{n-j}_{l_j}}}\xi^*_\nu(t)\omega_t^{n-j-1,l_j\nu;n-k,r_k}(\mathcal{L}_{10}(X))\\
&\quad + \sum_{\mu\notin r_k}\sum_{\nu\notin l_j}\frac{\sqrt{N^{n-k-1}_{r_k\mu}}}{\sqrt{N^{n-k}_{r_k}}}\frac{\sqrt{N^{n-j-1}_{l_j\nu}}}{\sqrt{N^{n-j}_{l_j}}}\xi_\mu(t)\xi^*_\nu(t)\omega_t^{n-j-1,l_j\nu;n-k-1,r_k\mu}(\mathcal{L}_{11}(X))
\end{aligned}
$$

with initial conditions $\omega_0^{n-j,l_j;n-k,r_k}(X) = \langle\eta|X|\eta\rangle\langle\Phi^{n-j}_{l_j}|\Phi^{n-k}_{r_k}\rangle$.

Remark 9. It is clear that the above equations couple downward to the master equation for the vacuum state. This means that for the n-photon state case, we should totally consider 2^{2^n} equations. Luckily, with the help of the conjugation property $\omega_t^{n-j,l_j;n-k,r_k}(X) = (\omega_t^{n-k,r_k;n-j,l_j}(X^*))^*$, the number of differential equations can be reduced to $\frac{2^n(2^n+1)}{2}$. For example, for the 2-photon case, there are 10 differential equations as shown in Theorem 3.1.

Remark 10. Restricted to the Fock state $|F^n\rangle$ defined in (4.2), the master equation (20) in [40] can be derived from Theorem 4.1.

4.3. Multiphoton filter: The homodyne detection case. In this subsection, we derive the quantum filter for the homodyne detection case. Following the development in section 3 for the 2-photon case, we first derive a filter for the extended system, based on which we derive the filter for the original system.

In analogy to subsection 3.3, we construct a 2^n-level ancillary for the n-photon state case. Specifically, we choose an orthonormal basis $\{|e^{n-k}_{r_k}\rangle,\ r_k \subset \bar{n},\ k=0,\dots n\}$ for the vector space $\mathbb{C}^{2^n}$ which is defined in the following way. Each $|e^{n-k}_{r_k}\rangle$ has one and only one nonzero entry (which is 1) at the mth location (counted from the top to the bottom). More precisely, if $k=0$, then $m=1$, the vector $|e^n_{r_0}\rangle = [1,0,\dots,0]^T$. For $k\geq 1$, $m = C^0_n + C^1_n + \cdots + C^{k-1}_n + \Gamma(r_k)$, where $\Gamma(r_k)$ represents the location

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of the set $r_k \subset \bar{n}$ in the ordered collection of all subsets of $\bar{n}$ having k elements. Here the word "ordered" means the lexicographical order [41]. For example, $\{1,2,3\} \prec \{1,2,4\} \prec \{1,3,4\} \prec \{2,3,4\}$.

The extended system is initialized in the superposition state $|\Sigma^n\rangle \in \mathbb{C}^{2^n} \otimes \mathsf{H}_S \otimes \mathsf{F}$:

$$|\Sigma^n\rangle \triangleq \sum_{k=0}^{n} \sum_{r_k \subset \bar{n}} \alpha_{r_k}^{n-k} |e_{r_k}^{n-k} \eta \Phi_{r_k}^{n-k}\rangle, \tag{4.4}$$

where $\alpha_{r_k}^{n-k}$ ($k = 0, \ldots, n, r_k \subset \bar{n}$) are arbitrary nonzero numbers that satisfy the normalization condition $\sum_{k=0}^{n} \sum_{r_k \subset \bar{n}} |\alpha_{r_k}^{n-k}|^2 = 1$.

For an arbitrary $2^n \times 2^n$ complex matrix A on $\mathbb{C}^{2^n}$ and an arbitrary operator X on H_S, the expectation with respect to the superposition state $|\Sigma^n\rangle$ is $\tilde{\omega}_t^n(A \otimes X) \triangleq \mathbb{E}_{\Sigma^n}[A \otimes j_t(X)]$. Define superoperators $\mathcal{K}_{00}^n(A)$, $\mathcal{K}_{01}^n(A)$, $\mathcal{K}_{10}^n(A)$, and $\mathcal{K}_{11}^n(A)$ in the similar way as in the 2-photon case (cf. (3.12)–(3.15)), i.e.,

$$\begin{aligned} &\mathcal{K}_{00}^n(A) = A, \quad \mathcal{K}_{01}^n(A)|\Sigma^n\rangle \mathrm{d}t = A\mathrm{d}B(t)|\Sigma^n\rangle, \\ &\mathcal{K}_{10}^n(A) = (\mathcal{K}_{01}^n(A^*))^*, \quad \mathcal{K}_{11}^n(A) = \mathcal{K}_{10}^n(\mathcal{K}_{01}^n(A)). \end{aligned} \tag{4.5}$$

Then the master equations for $\tilde{\omega}_t^n(A \otimes X)$ are given by the following result.

THEOREM 4.2. *The expectation $\tilde{\omega}_t^n(A \otimes X)$ for the extended system evolves according to $\dot{\tilde{\omega}}_t^n(A \otimes X) = \tilde{\omega}_t^n(\mathcal{G}^n(A \otimes X))$, where*

$$\mathcal{G}^n(A \otimes X) \triangleq \sum_{j,k=0}^{1} \mathcal{K}_{jk}^n(A) \otimes \mathcal{L}_{jk}(X) \tag{4.6}$$

with $\mathcal{L}_{jk}(X)$ defined in (2.2)*–*(2.3)*.*

In the homodyne detection case, define the quantum conditional expectation for the extended system to be $\tilde{\pi}_t^n(A \otimes X) \triangleq \mathbb{E}_{\Sigma^n}[A \otimes j_t(X) | I \otimes \mathscr{Y}(t)]$. The following result presents the quantum filtering equation for the extended system, which is the counterpart of Theorem 3.4 for the 2-photon case.

THEOREM 4.3. *In the case of homodyne detection, the conditional expectation $\tilde{\pi}_t^n(A \otimes X)$ for the extended system satisfies*

$$\mathrm{d}\tilde{\pi}_t^n(A \otimes X) = \tilde{\pi}_t^n(\mathcal{G}^n(A \otimes X))\mathrm{d}t + \tilde{\mathcal{H}}_t^n(A \otimes X)\mathrm{d}\tilde{W}^n(t),$$

where the operator $\mathcal{G}^n(A \otimes X)$ is defined in (4.6), and

$$\tilde{\mathcal{H}}_t^n(A \otimes X) = \tilde{M}_t^n(A \otimes X) - \tilde{\pi}_t^n(A \otimes X)\tilde{M}_t^n(I \otimes I)$$

with

$$\tilde{M}_t^n(A \otimes X) \triangleq \tilde{\pi}_t^n(\mathcal{K}_{00}^n(A) \otimes (XL + L^*X)) + \tilde{\pi}_t^n(\mathcal{K}_{01}^n(A) \otimes XS) + \tilde{\pi}_t^n(\mathcal{K}_{10}^n(A) \otimes S^*X). \tag{4.7}$$

The innovation process $\tilde{W}^n(t)$, defined via $\mathrm{d}\tilde{W}^n(t) = I \otimes \mathrm{d}Y(t) - \tilde{M}_t^n(I \otimes I)\mathrm{d}t$, is a Wiener process with respect to the state $|\Sigma^n\rangle$.

In analogy to (3.25), for all $l_j, r_k \subset \bar{n}$, define the conditional quantities $\pi_t^{n-j,l_j;n-k,r_k}(X)$ by means of

$$(I \otimes \pi^{n-j,l_j;n-k,r_k}(X))\tilde{\pi}_t^n(|e^n\rangle\langle e^n| \otimes I) \triangleq \frac{|\alpha^n|^2}{(\alpha_{l_j}^{n-j})^* \alpha_{r_k}^{n-k}} \tilde{\pi}_t^n(|e_{l_j}^{n-j}\rangle\langle e_{r_k}^{n-k}| \otimes X). \tag{4.8}$$

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Similarly to (3.26), it can be shown that

$$\mathbb{E}_{n;n}[\pi_t^{n-j,l_j;n-k,r_k}(X)K] = \mathbb{E}_{n-j,l_j;n-k,r_k}[j_t(X)K] \quad \forall\ K \in \mathscr{Y}(t).$$

In particular, when $j = k = 0$, the above equation reduces to

$$\mathbb{E}_{n;n}[\pi_t^{n;n}(X)K] = \mathbb{E}_{n;n}[j_t(X)K] \qquad \forall\ K \in \mathscr{Y}(t).$$

Since K is arbitrary, $\pi_t^{n;n}(X)$ is exactly the conditional system operator of the original system G driven by the n-photon state $|\Phi^n\rangle$, namely, (2.4).

Theorem 4.4. *In the case of homodyne detection, the quantum filter for the conditional expectation $\pi_t^{n;n}(X)$ is given by the following Ito differential equation,*

$$\mathrm{d}\pi_t^{n;n}(X) = L_t^{n;n}(X)\mathrm{d}t + \left[M_t^{n;n}(X) - \pi_t^{n;n}(X)M_t^{n;n}(I)\right]\mathrm{d}W^n(t).$$

And, more generally, for subsets $l_j, r_k \subset \bar{n} \quad (\forall\ j, l = 0, \ldots, n)$,

$$\begin{aligned}\mathrm{d}\pi_t^{n-j,l_j;n-k,r_k}(X) = L_t^{n-j,l_j;n-k,r_k}(X)\mathrm{d}t \\ + \left[M_t^{n-j,l_j;n-k,r_k}(X) - \pi_t^{n-j,l_j;n-k,r_k}(X)M_t^{n;n}(I)\right]\mathrm{d}W^n(t).\end{aligned}$$

Here,

$$\begin{aligned}&L_t^{n-j,l_j;n-k,r_k}(X)\\ &\triangleq \pi_t^{n-j,l_j;n-k,r_k}(\mathcal{L}_{00}(X)) + \sum_{\mu\notin r_k}\frac{\sqrt{N_{r_k\mu}^{n-k-1}}}{\sqrt{N_{r_k}^{n-k}}}\xi_\mu(t)\pi_t^{n-j,l_j;n-k-1,r_k\mu}(\mathcal{L}_{01}(X))\\ &\quad + \sum_{\nu\notin l_j}\frac{\sqrt{N_{l_j\nu}^{n-j-1}}}{\sqrt{N_{l_j}^{n-j}}}\xi_\nu^*(t)\pi_t^{n-j-1,l_j\nu;n-k,r_k}(\mathcal{L}_{10}(X))\\ &\quad + \sum_{\mu\notin r_k}\sum_{\nu\notin l_j}\frac{\sqrt{N_{r_k\mu}^{n-k-1}}}{\sqrt{N_{r_k}^{n-k}}}\frac{\sqrt{N_{l_j\nu}^{n-j-1}}}{\sqrt{N_{l_j}^{n-j}}}\xi_\mu(t)\xi_\nu^*(t)\pi_t^{n-j-1,l_j\nu;n-k-1,r_k\mu}(\mathcal{L}_{11}(X))\Big]\end{aligned}$$

and

$$\begin{aligned}&M_t^{n-j,l_j;n-k,r_k}(X)\\ &\triangleq \pi_t^{n-j,l_j;n-k,r_k}(XL + L^*X) + \sum_{\mu\notin r_k}\frac{\sqrt{N_{r_k\mu}^{n-k-1}}}{\sqrt{N_{r_k}^{n-k}}}\xi_\mu(t)\pi_t^{n-j,l_j;n-k-1,r_k\mu}(XS)\\ &\quad + \sum_{\nu\notin l_j}\frac{\sqrt{N_{l_j\nu}^{n-j-1}}}{\sqrt{N_{l_j}^{n-j}}}\xi_\nu^*(t)\pi_t^{n-j-1,l_j\nu;n-k,r_k}(S^*X).\end{aligned}$$

The innovation process $W^n(t)$ defined by $\mathrm{d}W^n(t) = \mathrm{d}Y(t) - M_t^{n;n}(I)\mathrm{d}t$, is a $\mathscr{Y}(t)$ Wiener process with respect to the n-photon state $|\Phi^n\rangle$. The initial conditions are $\pi_0^{n-j,l_j;n-k,r_k}(X) = \langle\Phi_{l_j}^{n-j}|\Phi_{r_k}^{n-k}\rangle\langle\eta|X|\eta\rangle$.

Remark 11. It can be verified that Theorem 4.4 reduces to Theorem 3.6 when $n = 2$, namely, the 2-photon case.

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4.4. Multiphoton filter: The photon-counting case. In this subsection, we present the multiphoton filter for the photon-counting case by deriving the evolution of the quantum conditional expectation $\hat{\pi}_t^{n;n}(X) \triangleq \mathbb{E}_{n;n}[j_t(X)|\mathscr{Y}^\Lambda(t)]$.

Similarly to the development in subsection 4.3, we first need to extend the system and initialize the state as $|\Sigma^n\rangle$ in (4.4), then we calculate the filtering equation for the extended system, i.e., the evolution of $\tilde{\pi}_t^\Lambda(A \otimes X) \triangleq \mathbb{E}_{\Sigma^n}[A \otimes j_t(X)|I \otimes \mathscr{Y}^\Lambda(t)]$. Finally, with the relationship between $\hat{\pi}_t^{n;n}(X)$ and $\tilde{\pi}_t^\Lambda(A\otimes X)$, we derive the quantum filter for $\hat{\pi}_t^{n;n}(X)$.

The following theorem is the filtering equation for the extended system, which is the counterpart of Theorem 4.3.

Theorem 4.5. *In the case of photon-counting monitoring, the conditional expectation $\tilde{\pi}_t^\Lambda(A \otimes X)$ for the extended system satisfies*

$$\mathrm{d}\tilde{\pi}_t^\Lambda(A \otimes X) = \tilde{\pi}_t^\Lambda(\mathcal{G}^n(A \otimes X))\mathrm{d}t + \tilde{H}_t^\Lambda(A \otimes X)\mathrm{d}\tilde{N}^\Lambda(t),$$

where $\mathcal{G}^n(A \otimes X)$ is defined in (4.6) and

$$\tilde{H}_t^\Lambda(A \otimes X) = (\tilde{Q}_t^\Lambda(I \otimes I))^{-1}\tilde{Q}_t^\Lambda(A \otimes X) - \tilde{\pi}_t^\Lambda(A \otimes X)$$

with

$$\begin{aligned}\tilde{Q}_t^\Lambda(A \otimes X) \triangleq\ & \tilde{\pi}_t^\Lambda(\mathcal{K}_{00}^n(A) \otimes L^*XL) + \tilde{\pi}_t^\Lambda(\mathcal{K}_{01}^n(A) \otimes L^*XS) + \tilde{\pi}_t^\Lambda(\mathcal{K}_{10}^n(A) \otimes S^*XL) \\ & + \tilde{\pi}_t^\Lambda(\mathcal{K}_{11}^n(A) \otimes S^*XS).\end{aligned}$$

The innovation process is given as $\mathrm{d}\tilde{N}^\Lambda(t) = \mathrm{d}Y^\Lambda(t) - \tilde{Q}_t^\Lambda(I \otimes I)\mathrm{d}t$.

Defining

$$(I \otimes \hat{\pi}_t^{n-j,l_j;n-k,r_k}(X))\tilde{\pi}_t^\Lambda(|e^n\rangle\langle e^n| \otimes I) = \frac{|\alpha^n|^2}{(\alpha_{l_j}^{n-j})^*\alpha_{r_k}^{n-k}}\tilde{\pi}_t^\Lambda(|e_{l_j}^{n-j}\rangle\langle e_{r_k}^{n-k}| \otimes X), \tag{4.9}$$

we can directly verify the following equation

$$\mathbb{E}_{n;n}[\hat{\pi}_t^{n-j,l_j;n-k,r_k}(X)K]{=}\mathbb{E}_{n-j,l_j;n-k,r_k}[j_t(X)K] \quad \forall\ K \in \mathscr{Y}^\Lambda(t).$$

If we set $j = k = 0$, and note that $K \in \mathscr{Y}^\Lambda(t)$ is arbitrary, we can deduce that $\hat{\pi}_t^{n;n}(X)$ defined in (4.9) is exactly the conditional expectation with respect to the n-photon field state $|\Phi^n\rangle$ for the photon detection, namely, (2.5).

The following result presents the quantum filter for photodetection, the counterpart of Theorem 4.4.

Theorem 4.6. *For photon-counting measurement, the quantum filter for the conditional expectation $\hat{\pi}_t^{n;n}(X)$ is given by the following Ito differential equation,*

$$\mathrm{d}\hat{\pi}_t^{n;n}(X) = \hat{P}_t^{n;n}(X)\mathrm{d}t + \left[(\Delta_t^{n;n}(I))^{-1}\,\Delta_t^{n;n}(X) - \hat{\pi}_t^{n;n}(X)\right]\mathrm{d}N_t.$$

And, more generally, for subsets $l_j, r_k \subset \bar{n}$ $\quad(\forall\ j,k = 0,\dots,n)$,

$$\begin{aligned}&\mathrm{d}\hat{\pi}_t^{n-j,l_j;n-k,r_k}(X) \\ &= \hat{P}_t^{n-j,l_j;n-k,r_k}(X)\mathrm{d}t + \left[(\Delta_t^{n;n}(I))^{-1}\,\Delta_t^{n-j,l_j;n-k,r_k}(X) - \hat{\pi}_t^{n-j,l_j;n-k,r_k}(X)\right]\mathrm{d}N_t,\end{aligned}$$

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where

$$\begin{aligned}
&\hat{P}_t^{n-j,l_j;n-k,r_k}(X)\\
&\triangleq \hat{\pi}_t^{n-j,l_j;n-k,r_k}(\mathcal{L}_{00}(X)) + \sum_{\mu\notin r_k}\frac{\sqrt{N_{r_k\mu}^{n-k-1}}}{\sqrt{N_{r_k}^{n-k}}}\xi_\mu(t)\hat{\pi}_t^{n-j,l_j;n-k-1,r_k\mu}(\mathcal{L}_{01}(X))\\
&\quad+\sum_{\nu\notin l_j}\frac{\sqrt{N_{l_j\nu}^{n-j-1}}}{\sqrt{N_{l_j}^{n-j}}}\xi_\nu^*(t)\hat{\pi}_t^{n-j-1,l_j\nu;n-k,r_k}(\mathcal{L}_{10}(X))\\
&\quad+\sum_{\substack{\mu\notin r_k\\ \nu\notin l_j}}\frac{\sqrt{N_{r_k\mu}^{n-k-1}}}{\sqrt{N_{r_k}^{n-k}}}\frac{\sqrt{N_{l_j\nu}^{n-j-1}}}{\sqrt{N_{l_j}^{n-j}}}\xi_\nu^*(t)\xi_\mu(t)\hat{\pi}_t^{n-j-1,l_j\nu;n-k-1,r_k\mu}(\mathcal{L}_{11}(X)),
\end{aligned}$$

and

$$\begin{aligned}
&\Delta_t^{n-j,l_j;n-k,r_k}(X)\\
&\triangleq \hat{\pi}_t^{n-j,l_j;n-k,r_k}(L^*XL) + \sum_{\mu\notin r_k}\frac{\sqrt{N_{r_k\mu}^{n-k-1}}}{\sqrt{N_{r_k}^{n-k}}}\xi_\mu(t)\hat{\pi}_t^{n-j,l_j;n-k-1,r_k\mu}(L^*XS)\\
&\quad+\sum_{\nu\notin l_j}\frac{\sqrt{N_{l_j\nu}^{n-j-1}}}{\sqrt{N_{l_j}^{n-j}}}\xi_\nu^*(t)\hat{\pi}_t^{n-j-1,l_j\nu;n-k,r_k}(S^*XL)\\
&\quad+\sum_{\substack{\mu\notin r_k\\ \nu\notin l_j}}\xi_\nu^*(t)\xi_\mu(t)\frac{\sqrt{N_{r_k\mu}^{n-k-1}}}{\sqrt{N_{r_k}^{n-k}}}\frac{\sqrt{N_{l_j\nu}^{n-j-1}}}{\sqrt{N_{l_j}^{n-j}}}\hat{\pi}_t^{n-j-1,l_j\nu;n-k-1,r_k\mu}(S^*XS).
\end{aligned}$$

The innovation process N_t *is defined by* $\mathrm{d}N_t = \mathrm{d}Y^\Lambda(t) - \Delta_t^{n;n}(I)\mathrm{d}t$*, and the initial conditions are* $\hat{\pi}_0^{n-j,l_j;n-k,r_k}(X) = \langle\Phi_{l_j}^{n-j}|\Phi_{r_k}^{n-k}\rangle\langle\eta|X|\eta\rangle$.

Remark 12. The single-photon filter for photodetection studied in [25, section 3-F] is a special case of of Theorem 4.6 when $n = 1$.

5. Conclusion. In this paper we have investigated the filtering problem for an arbitrary open quantum system driven by an incident wave packet prepared in a continuous-mode multiphoton state. A model of a two-level system driven by a two-photon wave packet has been used to demonstrate some of the results in the paper. This example reveals physical features of the two-photon case absent in the single-photon case and the Fock state case. Such interesting optical phenomena are due to the photon-photon interaction mediated by the two-level system. Encouraged by the success of two-photon filters, the general multiphoton filtering framework has also been proposed. Single-photon filters are probably no good for feedback control as the information is available only for a very short time period. In contrast, the multiphoton filtering framework developed here allows for any number of photons in a wave packet. As a result, by using homodyne detection to continuously measure the quadratures of the output field, one might be able to do state estimation and so feedback control. This is somehow similar to the photon box experiment done in Laboratoire Kastler Brossel, where a flow of atoms are measured and the information obtained is used to

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stabilize the number of photons in the cavity [7]. Nevertheless, whether or not the multiphoton filters developed here are applicable in feedback control depends crucially on the time scales of the underlying physical system, the photons' temporal envelope, and how fast homodyne measurement can be done. This is one of the things we would like to study more. Finally, we mention that at present most multiphoton states are generated in a probabilistic fashion. Nevertheless, deterministic generation of photon states is currently being investigated in the quantum optics community too; please see the review paper [39] and references therein.

Appendix A. The proof of Theorem 3.6 proceeds along the following three steps.

Step 1. Express the filtering equations in Theorem 3.6 in a unified manner in terms of the conditional expectations $\pi_t^{jk;mn}(X)$ defined in (3.25).

Step 2. Postulate the general form of the filtering equation of $\pi_t^{jk;mn}(X)$.

Step 3. Derive the exact expression of the quantum filter postulated in Step 2.

In what follows we work out the detail for each step.

Step 1. For given $j, k, m, n = 0, 1$, define superoperators $\mathcal{T}_t^{jk;mn}(X)$ to be

$$
\begin{aligned}
\text{(A.1)}\quad & \mathcal{T}_t^{jk;mn}(X) \\
\triangleq\ & \pi_t^{jk;mn}(\mathcal{L}_{00}(X)) \\
& + [\delta_{j0}\delta_{k1}\ \xi_2^*(t) + \delta_{j1}\delta_{k0}\ \xi_1^*(t)]\pi_t^{00;mn}(\mathcal{L}_{10}(X)) \\
& + [\delta_{m0}\delta_{n1}\ \xi_2(t) + \delta_{m1}\delta_{n0}\ \xi_1(t)]\pi_t^{jk;00}(\mathcal{L}_{01}(X)) \\
& + \delta_{j1}\delta_{k1}\frac{1}{\sqrt{N_2}}[\xi_1^*(t)\pi_t^{01;mn}(\mathcal{L}_{10}(X)) + \xi_2^*(t)\pi_t^{10;mn}(\mathcal{L}_{10}(X))] \\
& + \delta_{m1}\delta_{n1}\frac{1}{\sqrt{N_2}}[\xi_1(t)\pi_t^{jk;01}(\mathcal{L}_{01}(X)) + \xi_2(t)\pi_t^{jk;10}(\mathcal{L}_{01}(X))] \\
& + \delta_{j1}\delta_{k1}\delta_{m1}\delta_{n1}\frac{1}{N_2}[|\xi_1(t)|^2\pi_t^{01;01}(\mathcal{L}_{11}(X)) + \xi_1(t)\xi_2^*(t)\pi_t^{10;01}(\mathcal{L}_{11}(X))] \\
& + \delta_{j1}\delta_{k1}\delta_{m1}\delta_{n1}\frac{1}{N_2}[\xi_1^*(t)\xi_2(t)\pi_t^{01;10}(\mathcal{L}_{11}(X)) + |\xi_2(t)|^2\pi_t^{10;10}(\mathcal{L}_{11}(X))] \\
& + \delta_{j0}\delta_{k1}\delta_{m1}\delta_{n1}\frac{1}{\sqrt{N_2}}[\xi_1(t)\xi_2^*(t)\pi_t^{00;01}(\mathcal{L}_{11}(X)) + |\xi_2(t)|^2\pi_t^{00;10}(\mathcal{L}_{11}(X))] \\
& + \delta_{j1}\delta_{k0}\delta_{m1}\delta_{n1}\frac{1}{\sqrt{N2}}[|\xi_1(t)|^2\pi_t^{00;01}(\mathcal{L}_{11}(X))) + \xi_1^*(t)\xi_2(t)\pi_t^{00;10}(\mathcal{L}_{11}(X))] \\
& + \delta_{j1}\delta_{k1}\delta_{m0}\delta_{n1}\frac{1}{\sqrt{N_2}}[\xi_1^*(t)\xi_2(t)\pi_t^{01;00}(\mathcal{L}_{11}(X)) + |\xi_2(t)|^2\pi_t^{10;00}(\mathcal{L}_{11}(X))] \\
& + \delta_{j1}\delta_{k1}\delta_{m1}\delta_{n0}\frac{1}{\sqrt{N_2}}[|\xi_1(t)|^2\pi_t^{01;00}(\mathcal{L}_{11}(X)) + \xi_1(t)\xi_2^*(t)\pi_t^{10;00}(\mathcal{L}_{11}(X))] \\
& + \delta_{j0}\delta_{k1}[\delta_{m0}\delta_{n1}\,|\xi_2(t)|^2 + \delta_{m1}\delta_{n0}\xi_1(t)\xi_2^*(t)]\pi_t^{00;00}(\mathcal{L}_{11}(X)) \\
& + \delta_{j1}\delta_{k0}[\delta_{m0}\delta_{n1}\xi_1^*(t)\xi_2(t) + \delta_{m1}\delta_{n0}|\xi_1(t)|^2]\pi_t^{00;00}(\mathcal{L}_{11}(X)).
\end{aligned}
$$

Then it can be verified that for all $j, k, m, n = 0, 1$, the equations in Theorem 3.6 can be rewritten in a unified way as

$$
\text{(A.2)}\quad d\pi_t^{jk;mn}(X) = \mathcal{T}_t^{jk;mn}(X)dt + \left\{M_t^{jk;mn}(X) - \pi_t^{jk;mn}(X)M_t^{11;11}(I)\right\}dW(t),
$$

where the superoperators $M_t^{jk;mn}(X)$ are given in (3.27). As a result, it suffices to establish (A.2) to prove Theorem 3.6.

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Step 2. We postulate the filtering equation of $\pi_t^{jk;mn}(X)$ to be

$$\mathrm{d}\pi_t^{jk;mn}(X) = F_t^{jk;mn}(X)\mathrm{d}t + H_t^{jk;mn}(X)\mathrm{d}Y(t) \quad \forall\, j,k,m,n = 0,1. \tag{A.3}$$

The expressions for $F_t^{jk;mn}(X)$ and $H_t^{jk;mn}(X)$ in (A.3) are to be determined in the next step.

Step 3. For the sake of clarity, we rewrite (3.25) as below,

$$\frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(|e_{jk}\rangle\langle e_{mn}| \otimes X) = (I \otimes \pi_t^{jk;mn}(X))\tilde{\pi}_t(|e_{11}\rangle\langle e_{11}| \otimes I). \tag{A.4}$$

Differentiating both sides of (A.4) and comparing corresponding terms we have

$$\begin{aligned}&\frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{M}_t(|e_{jk}\rangle\langle e_{mn}| \otimes X)\\ &\quad= I \otimes H_t^{jk;mn}(X)\tilde{\pi}_t(|e_{11}\rangle\langle e_{11}| \otimes I) + (I \otimes \pi_t^{jk;mn}(X))\tilde{M}_t(|e_{11}\rangle\langle e_{11}| \otimes I)\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}&\frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(\mathcal{G}(|e_{jk}\rangle\langle e_{mn}| \otimes X)) - \frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{M}_t(|e_{jk}\rangle\langle e_{mn}| \otimes X)\tilde{M}_t(I \otimes I)\\ &= I \otimes F_t^{jk;mn}(X)\tilde{\pi}_t(|e_{11}\rangle\langle e_{11}| \otimes I) - (I \otimes \pi_t^{jk;mn}(X))\tilde{M}_t(|e_{11}\rangle\langle e_{11}| \otimes I)\tilde{M}_t(I \otimes I)\\ &\quad+ I \otimes H_t^{jk;mn}(X)\left[\tilde{M}_t(|e_{11}\rangle\langle e_{11}| \otimes I) - \tilde{\pi}_t(|e_{11}\rangle\langle e_{11}| \otimes I)\tilde{M}_t(I \otimes I)\right].\end{aligned} \tag{A.6}$$

By means of (A.4), the definitions of $\tilde{M}_t(A \otimes X)$ in (3.22), and $M_t^{jk;mn}(X)$ in (3.27), we are able to derive

$$\frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{M}_t(|e_{jk}\rangle\langle e_{mn}| \otimes X) = \left(I \otimes M_t^{jk;mn}(X)\right)\tilde{\pi}_t(|e_{11}\rangle\langle e_{11}| \otimes I). \tag{A.7}$$

Putting (A.7) back into (A.5) yields

$$H_t^{jk;mn}(X) = M_t^{jk;mn}(X) - \pi_t^{jk;mn}(X)M_t^{11;11}(I). \tag{A.8}$$

That is, we have derived the expression for $H_t^{jk;mn}(X)$. Next we derive the expression for $F_t^{jk;mn}(X)$. Substituting (A.7)–(A.8) into (A.6) yields

$$\begin{aligned}&I \otimes F_t^{jk;mn}(X)\tilde{\pi}_t(|e_{11}\rangle\langle e_{11}| \otimes I)\\ &= \frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(\mathcal{G}(|e_{jk}\rangle\langle e_{mn}| \otimes X)) - \left(I \otimes H_t^{jk;mn}(X)M_t^{11;11}(I)\right)\tilde{\pi}_t(|e_{11}\rangle\langle e_{11}| \otimes I).\end{aligned} \tag{A.9}$$

Thus we have to find the expression for $\tilde{\pi}_t(\mathcal{G}(|e_{jk}\rangle\langle e_{mn}|\otimes X))$. Observe that by (3.19),

$$\begin{aligned}&\frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(\mathcal{G}(|e_{jk}\rangle\langle e_{mn}| \otimes X))\\ &= \frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(|e_{jk}\rangle\langle e_{mn}| \otimes \mathcal{L}_{00}(X)) + \frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(\mathcal{K}_{01}(|e_{jk}\rangle\langle e_{mn}|) \otimes \mathcal{L}_{01}(X))\\ &\quad+ \frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(\mathcal{K}_{10}(|e_{jk}\rangle\langle e_{mn}|) \otimes \mathcal{L}_{10}(X)) + \frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(\mathcal{K}_{11}(|e_{jk}\rangle\langle e_{mn}|) \otimes \mathcal{L}_{11}(X));\end{aligned} \tag{A.10}$$

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we need to calculate each term on the right-hand side of (A.10). First,

$$\text{(A.11)}\quad \frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(|e_{jk}\rangle\langle e_{mn}|\otimes\mathcal{L}_{00}(X))\tilde{\pi}_t(|e_{11}\rangle\langle e_{11}|\otimes I)^{-1} = I\otimes\pi_t^{jk;mn}(\mathcal{L}_{00}(X)).$$

Second, based on the definition of the superoperator $\mathcal{K}_{01}(A)$ in (3.13) we have

$$\text{(A.12)}\quad \begin{aligned} &\frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(\mathcal{K}_{01}(|e_{jk}\rangle\langle e_{mn}|)\otimes\mathcal{L}_{01}(X))\tilde{\pi}_t(|e_{11}\rangle\langle e_{11}|\otimes I)^{-1} \\ &= \delta_{m1}\delta_{n1}\frac{\xi_1(t)}{\sqrt{N_2}}(I\otimes\pi_t^{jk;01}(\mathcal{L}_{01}(X)) + \delta_{m1}\delta_{n1}\frac{\xi_2(t)}{\sqrt{N_2}}(I\otimes\pi_t^{jk;10}(\mathcal{L}_{01}(X)) \\ &\quad +\delta_{m1}\delta_{n0}\xi_1(t)(I\otimes\pi_t^{jk;00}(\mathcal{L}_{01}(X)) + \delta_{m0}\delta_{n1}\xi_2(t)(I\otimes\pi_t^{jk;00}(\mathcal{L}_{01}(X)). \end{aligned}$$

Third, according to the definition of the superoperator $\mathcal{K}_{10}(A)$ in (3.14), we have

$$\text{(A.13)}\quad \begin{aligned} &\frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(\mathcal{K}_{10}(|e_{jk}\rangle\langle e_{mn}|)\otimes\mathcal{L}_{10}(X))\tilde{\pi}_t(|e_{11}\rangle\langle e_{11}|\otimes I)^{-1} \\ &= \delta_{j1}\delta_{k1}\frac{\xi_1^*(t)}{\sqrt{N_2}}(I\otimes\pi_t^{01;mn}(\mathcal{L}_{10}(X)) + \delta_{j1}\delta_{k1}\frac{\xi_2^*(t)}{\sqrt{N_2}}(I\otimes\pi_t^{10;mn}(\mathcal{L}_{10}(X)) \\ &\quad +\delta_{j1}\delta_{k0}\xi_1^*(t)(I\otimes\pi_t^{00;mn}(\mathcal{L}_{10}(X)) + \delta_{j0}\delta_{k1}\xi_2^*(t)(I\otimes\pi_t^{00;mn}(\mathcal{L}_{10}(X)). \end{aligned}$$

Fourth, by the definition of the superoperator $\mathcal{K}_{11}(A)$ in (3.15), it can be shown that

$$\text{(A.14)}\quad \begin{aligned} &\frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(\mathcal{K}_{11}(|e_{jk}\rangle\langle e_{mn}|)\otimes\mathcal{L}_{11}(X))\tilde{\pi}_t(|e_{11}\rangle\langle e_{11}|\otimes I)^{-1} \\ &= \delta_{j1}\delta_{k1}\delta_{m1}\delta_{n1}\frac{|\xi_1(t)|^2}{N_2}I\otimes\pi_t^{01;01}(\mathcal{L}_{11}(X)) + \delta_{j1}\delta_{k1}\delta_{m1}\delta_{n1}\frac{\xi_1(t)\xi_2^*(t)}{N_2}I\otimes\pi_t^{10;01}(\mathcal{L}_{11}(X)) \\ &+\delta_{j1}\delta_{k0}\delta_{m1}\delta_{n1}\frac{|\xi_1(t)|^2}{\sqrt{N_2}}I\otimes\pi_t^{00;01}(\mathcal{L}_{11}(X)) + \delta_{j0}\delta_{k1}\delta_{m1}\delta_{n1}\frac{\xi_1(t)\xi_2^*(t)}{\sqrt{N_2}}I\otimes\pi_t^{00;01}(\mathcal{L}_{11}(X)) \\ &+\delta_{j1}\delta_{k1}\delta_{m1}\delta_{n1}\frac{\xi_1^*(t)\xi_2(t)}{N_2}I\otimes\pi_t^{01;10}(\mathcal{L}_{11}(X)) + \delta_{j1}\delta_{k1}\delta_{m1}\delta_{n1}\frac{|\xi_2(t)|^2}{N_2}I\otimes\pi_t^{10;10}(\mathcal{L}_{11}(X)) \\ &+\delta_{j1}\delta_{k0}\delta_{m1}\delta_{n1}\frac{\xi_1^*(t)\xi_2(t)}{\sqrt{N_2}}I\otimes\pi_t^{00;10}(\mathcal{L}_{11}(X)) + \delta_{j0}\delta_{k1}\delta_{m1}\delta_{n1}\frac{|\xi_2(t)|^2}{\sqrt{N_2}}I\otimes\pi_t^{00;10}(\mathcal{L}_{11}(X)) \\ &+\delta_{j1}\delta_{k1}\delta_{m1}\delta_{n0}\frac{|\xi_1(t)|^2}{\sqrt{N_2}}I\otimes\pi_t^{01;00}(\mathcal{L}_{11}(X)) + \delta_{j1}\delta_{k1}\delta_{m1}\delta_{n0}\frac{\xi_1(t)\xi_2^*(t)}{\sqrt{N_2}}I\otimes\pi_t^{10;00}(\mathcal{L}_{11}(X)) \\ &+\delta_{j1}\delta_{k0}\delta_{m1}\delta_{n0}|\xi_1(t)|^2I\otimes\pi_t^{00;00}(\mathcal{L}_{11}(X)) + \delta_{j0}\delta_{k1}\delta_{m1}\delta_{n0}\xi_1(t)\xi_2^*(t)I\otimes\pi_t^{00;00}(\mathcal{L}_{11}(X)) \\ &+\delta_{j1}\delta_{k1}\delta_{m0}\delta_{n1}\frac{\xi_1^*(t)\xi_2(t)}{\sqrt{N_2}}I\otimes\pi_t^{01;00}(\mathcal{L}_{11}(X)) + \delta_{j1}\delta_{k1}\delta_{m0}\delta_{n1}\frac{|\xi_2(t)|^2}{\sqrt{N_2}}I\otimes\pi_t^{10;00}(\mathcal{L}_{11}(X)) \\ &+\delta_{j1}\delta_{k0}\delta_{m0}\delta_{n1}\xi_1^*(t)\xi_2(t)I\otimes\pi_t^{00;00}(\mathcal{L}_{11}(X)) + \delta_{j0}\delta_{k1}\delta_{m0}\delta_{n1}|\xi_2(t)|^2I\otimes\pi_t^{00;00}(\mathcal{L}_{11}(X)). \end{aligned}$$

Finally, on substitution of (A.11)–(A.14) into (A.10), we have

$$\text{(A.15)}\quad \frac{|\alpha_{11}|^2}{\alpha_{jk}^*\alpha_{mn}}\tilde{\pi}_t(\mathcal{G}(|e_{jk}\rangle\langle e_{mn}|\otimes X)) = (I\otimes\mathcal{T}_t^{jk;mn}(X))\tilde{\pi}_t(|e_{11}\rangle\langle e_{11}|\otimes I),$$

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where $\mathcal{T}_t^{jk;mn}(X)$ is that defined in (A.1). Substituting (A.15) into (A.9) gives

$$F_t^{jk;mn}(X) = \mathcal{T}_t^{jk;mn}(X) - H_t^{jk;mn}(I)M_t^{11;11}(I). \tag{A.16}$$

That is, we have derived the expression for $F_t^{jk;mn}(X)$. Putting $H_t^{jk;mn}(X)$ in (A.8) and $F_t^{jk;mn}(X)$ in (A.16) back into (A.3) yields for all $j,k,m,n = 0,1$,

$$\mathrm{d}\pi_t^{jk;mn}(X) = \mathcal{T}_t^{jk;mn}(X)\mathrm{d}t + \left[M_t^{jk;mn}(X) - \pi_t^{jk;mn}(X)M_t^{11;11}(I)\right]\mathrm{d}W(t),$$

which is exactly (A.2). The proof is completed.

Remark 13. It is worth noting that the coefficients α_{jk} $(j,k = 0,1)$ in the superposition state $|\Sigma\rangle$ in (3.11) for the extended system studied in subsection 3.3 do not appear in the filtering equations in Theorem 3.6. This can be seen clearly from the above proof. Specifically, the unifying filtering equation (A.3) depends on two operators $F_t^{jk;mn}(X)$ and $H_t^{jk;mn}(X)$, which satisfy two coupled algebraic equations (A.5)–(A.6). Both these equations contain the coefficients α_{jk}. However, by (A.7), α_{jk} on the left-hand side of (A.5) disappear, so $H_t^{jk;mn}(X)$ does not depend on α_{jk}; cf. (A.8). Similarly, by (A.15) as well as (A.7), α_{jk} on the left-hand side of (A.6) disappear too. As a result, $F_t^{jk;mn}(X)$ does not depend on α_{jk} either; cf. (A.16). Therefore, the coefficients α_{jk} do not appear in the unifying filtering equation (A.3), or equivalently the filtering equations in Theorem 3.6.

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