Chang et al. Journal of Inequalities and Applications 2012, 2012:37
http://www.journalofinequalitiesandapplications.com/content/2012/1/37

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A generalization and improvement of Chidume theorems for total asymptotically nonexpansive mappings in Banach spaces

Shih-sen Chang[1], Jong Kyu Kim[2*], HW Joseph Lee[3] and Chi Kin Chan[3]

* Correspondence: jongkyuk@kyungnam.ac.kr
[2]Department of Mathematics Education, Kyungnam University, Masan, Kyungnam 631-701, South Korea
Full list of author information is available at the end of the article

Abstract
The purpose of this article is to establish some new approximation theorems of common fixed points for a countable family of total asymptotically quasi-nonexpansive mappings in Banach spaces which generalize and improve the corresponding theorems of Chidume et al. and others.
2000 AMS Subject Classification: 47J05; 47H09; 49J25.

Keywords: total asymptotically nonexpansive mapping, total asymptotically quasi-nonexpansive mapping, asymptotically quasi-nonexpansive mapping, asymptotically nonexpansive mapping, asymptotically nonexpansive in the intermediate sense

1. Introduction

Throughout this article, we assume that E is a real Banach space, C is a nonempty closed convex subset of E. In the sequel, we use $F(T)$ to denote the set of fixed points of a mapping T, and use $\mathfrak{R}$ and $\mathfrak{R}^+$ to denote the set of all real numbers and the set of all nonnegative real numbers, respectively.

Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if,

$$||Tx - Ty|| \le ||x - y||, \quad \forall x, y \in C.$$

T is called *asymptotically nonexpansive* if, there exists a sequence $\{v_n\} \subset (0, \infty)$ with $\lim_{n\to\infty} v_n = 0$ such that for all $x, y \in C$

$$||T^n x - T^n y|| \le (1 + v_n)||x - y||, \forall n \ge 1.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] as a generalization of the class of nonexpansive mappings. They proved that if C is a nonempty closed and convex bounded subset of a real uniformly convex Banach space and $T : C \to C$ is an asymptotically nonexpansive mapping, then T has a fixed point.

A mapping $T : C \to C$ is said to be *asymptotically nonexpansive in the intermediate sense* [2], if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} (||T^n x - T^n y|| - ||x - y||) \le 0. \tag{1.1}$$

If $F(T) \neq \emptyset$ and (1.1) holds for all $x \in C$, $y \in F(T)$, then T is called *asymptotically quasi-nonexpansive in the intermediate sense*. Observe that if we define

$$a_n := \sup_{x,y \in C} \left(||T^n x - T^n y|| - ||x - y|| \right) \ \textit{and}\ \sigma_n = \max\{0,\ a_n\}, \tag{1.2}$$

then $\sigma_n \to 0$ as $n \to \infty$ and (1.1) reduces to

$$||T^n x - T^n y|| \ \leq \ ||x - y|| + \sigma_n, \forall x, y \in C, n \geq 1. \tag{1.3}$$

The class of asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al. [2]. It is known [3] that if C is a nonempty closed and convex bounded subset of a uniformly convex Banach space E and $T : C \to C$ is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

Recently, Alber et al. [4] introduced the concept of *total asymptotically nonexpansive mappings* which is more general than asymptotically nonexpansive mappings and studied the approximation methods of fixed points for this kind of mappings.

Definition 1.1 A mapping $T : C \to C$ is said to be *total asymptotically nonexpansive* if, there exist nonnegative real sequences $\{\nu_n\}$ and $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\zeta : \mathfrak{R}^+ \to \mathfrak{R}^+$ with $\zeta(0) = 0$ such that for all $x, y \in C$,

$$||T^n x - T^n y|| \ \leq \ ||x - y|| + \nu_n \zeta \left(||x - y|| \right) \ + \ \mu_n, \forall n \geq 1. \tag{1.4}$$

If $F(T) \neq \emptyset$ and (1.4) holds for all $x \in C$, $y \in F(T)$, then T is called *total asymptotically quasi-nonexpansive*.

Remark 1.2 If $\zeta(t) = t$, $t \geq 0$, then (1.4) reduces to

$$||T^n x - T^n y|| \ \leq (1 + \nu_n)||x - y|| + \mu_n, \ \forall n \geq 1.$$

In addition, if $\mu_n = 0$, $\forall n \geq 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. If $\nu_n = \mu_n = 0$, $\forall n \geq 1$, then total asymptotically nonexpansive mappings coincide with nonexpansive mappings. If $\nu_n = 0$ and $\mu_n = \sigma_n := \max\{0, a_n\}$, where a_n is defined by (1.2), then (1.4) reduces to (1.3) which has been studied as asymptotically nonexpansive mappings in the intermediate sense.

Within the past 30 years, research on iterative approximation of common fixed points of nonexpansive mappings, asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings have been considered by many authors (see, for example, [1-20] and the references therein).

Especially, recently Chidume and Ofoedu [13,14] introduced the following iterative scheme for approximation of a common fixed point of a finite family of total asymptotically nonexpansive mappings in Banach spaces which extend and generalize the corresponding results of Kirk [3], Alber et al. [4], Quan et al. [5], Shahzad et al. [6], Chang et al. [9], Jung [10], Shioji et al. [11], Suzuki [12], and Schu [19].

Theorem 1.3 [[13,14]] Let E be a real Banach space, C be a nonempty closed convex subset of E and $T_i : C \to C$, $i = 1, 2, ..., m$ be m total asymptotically nonexpansive mappings with sequences $\{\nu_{in}\}$, $\{\mu_{in}\}$, $i = 1, 2, ..., m$, such that $\mathfrak{F} := \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n x_n, \ if \ m = 1, \ n \geq 1 \end{cases}$$

$$\begin{cases} x_1 \in C; \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n y_{1n} \\ y_{1n} = (1-\alpha_n)x_n + \alpha_n T_2^n y_{2n} \\ \vdots \\ y_{(m-2)n} = (1-\alpha_n)x_n + \alpha_n T_{m-1}^n y_{(m-1)n} \\ y_{(m-1)n} = (1-\alpha_n)x_n + \alpha_n T_m^n x_n, \ if \ m \geq 2, \ n \geq 1. \end{cases} \tag{1.5}$$

Suppose $\sum_{n=1}^{\infty} \nu_{in} < \infty$, $\sum_{n=1}^{\infty} \mu_{in} < \infty$, i = 1, 2, ..., m and suppose that there exist M_i, $M_i^* > 0$ such that $\zeta_i(\lambda_i) \leq M_i^* \lambda_i$ for all $\lambda_i \geq M_i$, i = 1, 2, ..., m. Then the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} ||x_n - p||$ exists, $p \in \mathfrak{F}$. Moreover, the sequence $\{x_n\}$ converges strongly to a common fixed point of T_i, i = 1, 2, ..., m if and only if $\liminf_{n\to\infty} d(x_n, \mathfrak{F}) = 0$, where $d(x_n, \mathfrak{F}) = \inf_{y\in\mathfrak{F}} ||x_n - y||$, $n \geq 1$.

Theorem 1.4 [[13,14]] Let E be a uniformly convex real space, C be a nonempty closed convex subset of E, and $T_i : C \to C$, i = 1, 2, ..., m be m uniformly continuous total asymptotically nonexpansive mappings with sequences $\{\nu_{in}\}$, $\{\mu_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \nu_{in} < \infty$, $\sum_{n=1}^{\infty} \mu_{in} < \infty$, i = 1, 2, ..., m and $\mathfrak{F} := \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Let $\{\alpha_{in}\} \subset [\varepsilon, 1 - \varepsilon]$ for some $\varepsilon \in (0, 1)$. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.5). Suppose that there exist M_i, $M_i^* > 0$ such that $\zeta_i(\lambda_i) \leq M_i^* \lambda_i$ whenever $\lambda_i \geq M_i$, i = 1, 2, ..., m and that one of T_1, T_2, ..., T_m is compact, then $\{x_n\}$ converges strongly to some $p \in \mathfrak{F}$.

It is our purpose in this article to construct a new iterative sequence much simpler than (1.5) for approximation of common fixed points of a countable family of total asymptotically nonexpansive mappings and give necessary and sufficient conditions for the convergence of the scheme to common fixed points of the mappings in arbitrary real Banach spaces. As well as a sufficient condition for convergence of the iteration process to a common fixed point of mappings under the setting of uniformly convex Banach space is also established. The results presented in the article not only generalize and improve the corresponding results of Chidume et al. [13-15] but also unify, extend and generalize the corresponding result of [3-7,9-12,19].

2. Preliminaries

For the sake of convenience we first give the following lemmas which will be needed in proving our main results.

Lemma 2.1 [[20]] Let E be a uniformly convex Banach space, r >0 be a positive number and $B_r(0)$ be a closed ball of E. Then, for any sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous, strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$, $g(0) = 0$ such that for any positive integers $i, j \geq 1$, $i \neq j$, the following holds:

$$||\sum_{n=1}^{\infty} \lambda_n x_n||^2 \leq \sum_{n=1}^{\infty} \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||). \tag{2.1}$$

Lemma 2.2 Let $\{a_n\}$, $\{b_n\}$, and $\{\lambda_n\}$ be sequences of nonnegative real numbers such that

$$a_{n+1} \le (1+\lambda_n)a_n + b_n, \forall n \ge n_0, \tag{2.2}$$

where n_0 is some positive integer. If $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\{a_n\}$ is bounded and $\lim_{n\to\infty} a_n$ exists. Moreover, if, in addition, $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Definition 3.1 Let E be a real Banach space, C be a nonempty closed convex subset of E.

(1) Let $\{T_i\}$ be a countable family of mappings from C into itself. $\{T_i\}$ is said to be *uniformly total asymptotically quasi-nonexpansive mappings* if, $\mathfrak{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, and there exist nonnegative real sequences $\{\nu_n\}$ and $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\zeta : \mathfrak{R}^+ \to \mathfrak{R}^+$ with $\zeta(0) = 0$ such that for all $x \in C$, $p \in \mathfrak{F}$,

$$||T_i^n x - p|| \le ||x-p|| + \nu_n \zeta(||x-p||) + \mu_n, \forall i \ge 1, n \ge 1. \tag{3.1}$$

(2) A mapping $T : C \to C$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$||T^n x - T^n y|| \le L||x-y||, \ \forall x, y \in C, \ \forall n \ge 1.$$

Let $\{T_i\}$ be a countable family of uniformly total asymptotically quasi-nonexpansive mappings from C into itself and for each $i \ge 1$, T_i is uniformly L_i-Lipschitz continuous. For any given $x_0 \in C$, we define an iterative sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) z_n, \\ z_n = \beta_{n,0} x_n + \sum_{i=1}^{\infty} \beta_{n,i} T_i^n x_n, \ \forall n \ge 0. \end{cases} \tag{3.2}$$

Theorem 3.2 Let E be a real Banach space, C be a nonempty closed convex subset of E. Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of uniformly total asymptotically quasi-nonexpansive mappings from C into itself with nonnegative real sequences $\{\nu_n\}$ and $\{\mu_n\}$ and a strictly increasing continuous function $\zeta : \mathfrak{R}^+ \to \mathfrak{R}^+$ such that $\zeta(0) = 0$, $\mathfrak{F} := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and $\sum_{n=1}^{\infty} (\nu_n + \mu_n) < \infty$. Let $\{x_n\}$ be the sequence defined by (3.2), where $\{\beta_{n,i}\}$, $i = 0, 1, 2, \ldots$ and $\{\alpha_n\}$ are sequences in [0, 1] satisfying the following conditions:

(a) for each $n \ge 0$, $\sum_{i=0}^{\infty} \beta_{n,i} = 1$; If $\{x_n\}$ and $\mathfrak{F}$ are bounded, then the following conclusions hold:

(1) for each $p \in \mathfrak{F}$, $\lim_{n\to\infty} ||x_n - p||$ exists;

(2) the sequence $\{x_n\}$ converges strongly to a common fixed point $x^* \in \mathfrak{F}$ if and only if $\liminf_{n\to\infty} d(x_n, \mathfrak{F}) = 0$, where $d(x_n, \mathfrak{F}) = \inf_{y\in\mathfrak{F}} ||x_n - y||$, $n \ge 1$.

Proof. (1) For any $n \ge 0$ and for any given $p \in \mathfrak{F}$ we have

$$||x_{n+1} - p|| \le \alpha_n ||x_n - p|| + (1-\alpha_n)||z_n - p||. \tag{3.3}$$

Denoting by $M = \sup_{n\geq 0, p\in\mathfrak{F}}\{||x_n|| + ||x_n - p||\} < \infty$, from (3.2) we have

$$\begin{aligned} ||z_n - p|| &\leq \beta_{n,0}||x_n - p|| + \sum_{i=1}^{\infty} \beta_{n,i}||T_i^n x_n - p|| \\ &\leq \beta_{n,0}||x_n - p|| + \sum_{i=1}^{\infty} \beta_{n,i}[||x_n - p|| + \nu_n \zeta(||x_n - p||) + \mu_n] \\ &\leq \beta_{n,0}||x_n - p|| + \sum_{i=1}^{\infty} \beta_{n,i}[||x_n - p|| + \nu_n \zeta(M) + \mu_n] \\ &\leq ||x_n - p|| + \gamma_n, \end{aligned} \tag{3.4}$$

where $\gamma_n = \nu_n\zeta(M) + \mu_n$. By the assumption, $\sum_{n=1}^{\infty}\gamma_n < \infty$. Substituting (3.4) into (3.3) and simplifying we have

$$||x_{n+1} - p|| \leq ||x_n - p|| + \gamma_n, \forall n \geq 1 \textit{ and for each } p \in \mathfrak{F}. \tag{3.5}$$

It follows from Lemma 2.2 that $\lim_{n\to\infty} ||x_n - p||$ exists. The conclusion (1) is proved.

(2) From (3.5) we have that

$$d(x_{n+1}, \mathfrak{F}) \leq d(x_n, \mathfrak{F}) + \gamma_n, \textit{for each } n \geq 1.$$

Then Lemma 2.2 implies that $\lim_{n\to\infty} d(x_n, \mathfrak{F})$ exists. By the assumption that $\liminf_{n\to\infty} d(x_n, \mathfrak{F}) = 0$, therefore we have $\lim_{n\to\infty} d(x_n, \mathfrak{F}) = 0$.

This completes the proof of Theorem 3.2.

Theorem 3.3 Let $E, C, \{T_n\}_{n=1}^{\infty}$, $\mathfrak{F}$ be the same as in Theorem 3.2. If there exist constants $K, K^* > 0$ such that $\zeta(t) \leq K^* t$, $\forall t \geq K$, then the sequence $\{x_n\}$ defined by (3.2) is bounded and so the conclusions of Theorem 3.2 still hold.

Proof In fact, from (3.2) for any given $p \in \mathfrak{F}$, we have that

$$\begin{aligned} ||z_n - p|| &\leq \beta_{n,0}||x_n - p|| + \sum_{i=1}^{\infty} \beta_{n,i}||T_i^n x_n - p|| \\ &\leq \beta_{n,0}||x_n - p|| + \sum_{i=1}^{\infty} \beta_{n,i}\{||x_n - p|| + \nu_n \zeta(||x_n - p||) + \mu_n\} \end{aligned}$$

By the assumption, it is easy to see that

$$\zeta(||x_n - p||) \leq \begin{cases} \zeta(K), & \text{if } ||x_n - p|| < K, \\ K^*||x_n - p||, & \text{if } ||x_n - p|| \geq K. \end{cases}$$

This implies that

$$\zeta(||x_n - p||) \leq \zeta(K) + K^*||x_n - p||.$$

Therefore, we have

$$\begin{aligned} ||z_n - p|| &\leq \beta_{n,0}||x_n - p|| + \sum_{i=1}^{\infty} \beta_{n,i}\{||x_n - p|| + \nu_n[\zeta(K) + K^*||x_n - p||] + \mu_n\} \\ &\leq (1 + \nu_n K^*)||x_n - p|| + \xi_n, \end{aligned} \tag{3.6}$$

where $\xi_n = \nu_n\zeta(K) + \mu_n$. Substituting (3.6) into (3.3) and simplifying, we have that

$$||x_{n+1} - p|| \leq (1 + \nu_n K^*)||x_n - p|| + \xi_n, \forall n \geq 1. \tag{3.7}$$

By Lemma 2.2, for each $p \in \mathfrak{F}$, $\lim_{n\to\infty} ||x_n - p||$ exists, and so $\{x_n\}$ is bounded. The conclusions of Theorem 3.3 can be obtained from Theorem 3.2 immediately.

From Theorem 3.3 we can obtain the following result:

Corollary 3.4 Let E, C be as in Theorem 3.2. Let $T_i : C \to C$, $i = 1, 2, ..., m$ be m total asymptotically quasi-nonexpansive mappings with nonnegative real sequences $\{\nu_{in}\}$, $\{\mu_{in}\}$ and a strictly increasing continuous function $\zeta_i : \mathfrak{R}^+ \to \mathfrak{R}^+$ with $\zeta_i(0) = 0$ such that $\mathfrak{F} := \bigcap_{i=1}^m F(T_i)$ is nonempty and bounded, $\sum_{n=1}^\infty (\nu_{in} + \mu_{in}) < \infty$, $i = 1, 2, ..., m$. Let $\{x_n\}$ be the sequence defined by:

$$\begin{cases} x_0 \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) z_n, \\ z_n = \beta_{n,0} x_n + \sum_{i=1}^m \beta_{n,i} T_i^n x_n, \ \forall n \geq 0. \end{cases} \tag{3.8}$$

where $\{\beta_{n,\ i}\}$, $i = 0, 1, 2, ..., m$, and $\{\alpha_n\}$ are sequences in [0, 1] satisfying the following conditions:

(a) for each $n \geq 0$, $\sum_{i=0}^m \beta_{n,i} = 1$;

If there exist constants K, $K^* > 0$ such that for each $i = 1, 2, ..., m$, $\zeta_i(t) \leq K^* t$, $\forall t \geq K$, then $\{x_n\}$ is bounded and the conclusions of Theorem 3.2 still hold.

Proof Let $\nu_n = \max_{1\leq i\leq m} \nu_{in}$, $\mu_n = \max_{1\leq i\leq m} \mu_{in}$ and $\zeta = \max_{1\leq i\leq m} \zeta_i$, then $\sum_{n=1}^\infty \{\nu_n + \mu_n\} < \infty$ and $\zeta : \mathfrak{R}^+ \to \mathfrak{R}^+$ is a strictly increasing continuous function with $\zeta(0) = 0$ and there exist constants K, $K^* > 0$ such that $\zeta(t) \leq K^* t$, $\forall t \geq K$. Therefore all conditions in Theorem 3.3 are satisfied. The conclusions of Corollary 3.4 can be obtained from Theorem 3.3 immediately.

If the space E is uniformly convex, and one of $\{T_i\}$ is compact, then we can obtain the following more better result.

Theorem 3.5 Let E be a uniformly convex real Banach space, C be a nonempty closed convex subset of E and $\{T_i\}_{i=1}^\infty$ be a countable family of uniformly total asymptotically quasi-nonexpansive mappings from C into itself with nonnegative real sequences $\{\nu_n\}$ and $\{\mu_n\}$ and a strictly increasing continuous function $\zeta : \mathfrak{R}^+ \to \mathfrak{R}^+$ such that $\zeta(0) = 0$, $\mathfrak{F} := \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ and $\sum_{n=1}^\infty (\nu_n + \mu_n) < \infty$, and for each $i \geq 1$, T_i is uniformly Li-Lipschitzian continuous. Let $\{x_n\}$ be the sequence defined by (3.2), where $\{\beta_{n,\ i}\}$, $i = 0, 1, 2, ...$ and $\{\alpha_n\}$ are sequences in [0, 1] satisfying the conditions (a), (b) in Theorem 3.2 and $\liminf \beta_{n,0}\beta_{n;\ i} > 0$ for any $i \geq 1$. If $\{x_n\}$ and $\mathfrak{F}$ both are bounded and one of $\{T_i\}$ is compact, then the following conclusions hold:

(1) $\lim_{n\to\infty}||x_n - T_j^n x_n|| = 0$ uniformly in $j \geq 1$,

(2) the sequence $\{x_n\}$ converges strongly to some point $p \in \mathfrak{F}$.

Proof (1) Since $\{x_n\}$ and $\mathfrak{F}$ both are bounded and the norm $||\cdot||^2$ is convex, for any given $p \in \mathfrak{F}$, it follows from (3.2) that

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||\alpha_n x_n + (1 - \alpha_n) z_n - p||^2 \\ &\leq \alpha_n ||x_n - p||^2 + (1 - \alpha_n)||z_n - p||^2. \end{aligned} \tag{3.9}$$

By Lemma 2.1, for any positive integer $j \geq 1$ we have

$$\begin{aligned} ||z_n - p||^2 &= ||\beta_{n,0}x_n + \sum_{i=1}^{\infty} \beta_{n,i}T_i^n x_n - p||^2 \\ &\leq \beta_{n,0}||x_n - p||^2 + \sum_{i=1}^{\infty} \beta_{n,i}||T_i^n x_n - p||^2 \\ &\quad - \beta_{n,0}\beta_{n,j}g(||x_n - T_j^n x_n||). \end{aligned} \tag{3.10}$$

Since

$$\begin{aligned} ||T_i^n x_n - p||^2 &\leq (||x_n - p|| + \nu_n \zeta(||x_n - p||) + \mu_n)^2 \\ &\leq (||x_n - p|| + \nu_n \zeta(M) + \mu_n)^2 \\ &\leq ||x_n - p||^2 + \gamma_n, \end{aligned} \tag{3.11}$$

where

$$M = \sup_{n \geq 1, p \in \mathfrak{F}} ||x_n - p||,$$

$$\gamma_n = (\nu_n \zeta(M) + \mu_n)(\nu_n \zeta(M) + \mu_n + 2M) \to 0 \ (as\ n \to \infty)$$

First substituting (3.11) into (3.10), then substituting (3.10) into (3.9) and simplifying we have that

$$\begin{aligned} ||x_{n+1} - p||^2 &\leq \alpha_n ||x_n - p||^2 + (1 - \alpha_n) \Big\{ \beta_{n,0}||x_n - p||^2 \\ &\quad + \sum_{i=1}^{\infty} \beta_{n,i}(||x_n - p||^2 + \gamma_n) - \beta_{n,0}\beta_{n,j}g(||x_n - T_j^n x_n||) \Big\} \\ &\leq \alpha_n||x_n - p||^2 + (1 - \alpha_n)\{||x_n - p||^2 + \gamma_n - \beta_{n,0}\beta_{n,j}g(||x_n - T_j^n x_n||)\} \\ &\leq ||x_n - p||^2 + \gamma_n - (1 - \alpha_n)\beta_{n,0}\beta_{n,j}g(||x_n - T_j^n x_n||)\}. \end{aligned}$$

This together with Theorem 3.2 (1) shows that for each $j \geq 1$

$$(1 - \alpha_n)\beta_{n,0}\beta_{n,j}g(||x_n - T_j^n x_n||) \leq ||x_n - p||^2 + \gamma_n - ||x_{n+1} - p||^2 \to 0(as\ n \to \infty).$$

By conditions (b) and lim inf $\beta_{n,0}\beta_{n,\ i} > 0$ for any $i \geq 1$, this implies that

$$\lim_{n \to \infty} g(||x_n - T_j^n x_n||) = 0 \ uniformly\ in\ j \geq 1.$$

By the property of g, we have that

$$\lim_{n \to \infty} ||x_n - T_j^n x_n|| = 0 \ uniformly\ in\ j \geq 1. \tag{3.12}$$

The conclusion (1) is proved.

(2) From (3.2) we have

$$||x_{n+1} - x_n|| = (1 - \alpha_n)||z_n - x_n|| \leq \sum_{i=1}^{\infty} \beta_{n,i}||T_i^n x_n - x_n||. \tag{3.13}$$

For any given $\varepsilon > 0$, from (3.12) there exists a positive integer n_0 such that

$$||x_n - T_i^n x_n|| < \varepsilon \ for\ all\ n \geq n_0,\ and\ any\ i \geq 1.$$

This together with (3.13) yields that

$$||x_{n+1} - x_n|| \ \to 0(as\ n \to \infty). \tag{3.14}$$

By the assumption that, there exists a mapping in $\{T_i\}$ which is compact. Without loss of generality, we can assume that T_1 is compact. Thus, there exists a subsequence $\{T_1^{n_k} x_{n_k}\}$ of $\{T_1^n x_n\}$ such that $T_1^{n_k} x_{n_k} \to x^*$ (as $k \to \infty$) for some point $x^* \in C$. Since T_1 is L_1-Lipschitzian, it is continuous. Thus we have $T_1 T_1^{n_k} x_{n_k} \to T_1 x^*$ (as $k \to \infty$). From (3.12), we have that $\lim_{k\to\infty} x_{n_k} = x^*$ Also from (3.12) for each $i \geq 1$, $\lim_{k\to\infty} T_i^{n_k} x_{n_k} = x^*$. Thus for each $i \geq 1$, $\lim_{k\to\infty} T_i T_i^{n_k} x_{n_k} = T_i x^*$. By (3.14), $\lim_{k\to\infty} ||x_{n_k+1} - x_{n_k}|| \ = 0$, it follows that $\lim_{k\to\infty} x_{n_k+1} = x^*$. Next, we prove that $x^* \in \mathfrak{F}$. In fact, for each $i \geq 1$, since T_i is uniformly L_i-Lipschitz continuous, we have

$$\begin{aligned} ||x^* - T_i x^*|| &\leq ||x^* - x_{n_k+1}|| + ||x_{n_k+1} - T_i^{n_k+1} x_{n_k+1}|| \\ &+ ||T_i^{n_k+1} x_{n_k+1} - T_i^{n_k+1} x_{n_k}|| + ||T_i^{n_k+1} x_{n_k} - T_i x^*|| \\ &\leq \ ||x^* - x_{n_k+1}|| + ||x_{n_k+1} - T_i^{n_k+1} x_{n_k+1}|| \\ &+ L_i ||x_{n_k+1} - x_{n_k}|| + ||T_i^{n_k+1} x_{n_k} - T_i x^*|| \ \to 0 \ (as\ k \to \infty). \end{aligned} \tag{3.15}$$

Therefore, we have $x^* = T_i x^*$, for each $i \geq 1$. This implies that $x^* \in \mathfrak{F}$. But by Theorem 3.2, for each $p \in \mathfrak{F}$, $\lim_{n\to\infty} ||x_n - p||$ exists. Hence $\{x_n\}$ converges strongly to $x^* \in \mathfrak{F}$. This completes the proof of Theorem 3.5.

Remark 3.6 By the same way as given in the proof of Theorem 3.3, we can prove that if the condition " $\{x_n\}$ is bounded" in Theorem 3.5 is replaced by the condition "if there exist constants K, $K^* > 0$ such that $\zeta(t) \leq K^* t$, $\forall t \geq K$", then the conclusions of Theorem 3.5 still hold.

Definition 3.7 Let $\{T_i\}$ be a family of mappings from C into itself.

(1) $\{T_i\}$ is said to be *a family of uniformly asymptotically nonexpansive mappings* if, there exists a sequence of nonnegative real numbers $\{\nu_n\}$ with $\nu_n \to 0$ (*as* $n \to \infty$) such that for any $x, y \in C$ and for any $i \geq 1$

$$||T_i^n x - T_i^n y|| \ \leq (1 + \nu_n)||x - y||, \quad \forall n \geq 1. \tag{3.16}$$

(2) $\{T_i\}$ is said to be *a family of uniformly asymptotically nonexpansive in the intermediate sense* if, for each $i \geq 1$, T_i is continuous and there exists a sequence $\{\sigma_n\}$ of nonnegative real numbers with $\sigma_n \to 0$ (*as* $n \to \infty$) such that for any $x, y \in C$ and for any $i \geq 1$,

$$||T_i^n x - T_i^n y|| \ \leq \ ||x - y|| + \ \sigma_n, \forall n \geq 1. \tag{3.17}$$

From Theorem 3.5 and Remark 3.6 we can obtain the following

Theorem 3.8 Let E, C be the same as in Theorem 3.5. Let $\{T_i\}$ be a countable family of uniformly asymptotically nonexpansive mappings from C into itself with nonnegative real sequences $\{\nu_n\}$ such that $\mathfrak{F} := \bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and bounded and $\sum_{n=1}^{\infty} \nu_n < \infty$. Let $\{x_n\}$ be the sequence defined by (3.2), where $\{\beta_{n,\ i}\}$, $i = 0, 1, 2, \ldots$ and $\{\alpha_n\}$ are sequences in [0, 1] satisfying the conditions (a), (b) in Theorem 3.2 and lim inf $\beta_{n,\ 0}\beta_{n,\ i} > 0$ for any $i \geq 1$. If one of $\{T_i\}$ is compact, then the conclusions in Theorem 3.5 still hold:

Proof. Letting $\mu_n = 0$, $\forall n \geq 1$, $\zeta(t) = t$, $t \geq 0$, $K = 0$, and $K^* = 1$, therefore we have $||T_i^n x - T_i^n y|| \ \leq L||x - y||$, $\forall n \geq 1$, and $\zeta(t) = K^* t$, $\forall t \geq 0$. Again since $\nu_n \to 0$, $\{\nu_n\}$ is bounded. Setting $L = 1 + \sup_{n\geq 1} \nu_n$, it follows from (3.16) that

$$||T_i^n x - T_i^n y|| \leq L||x - y||, \ \forall n \geq 1,$$

i.e., for each $i \geq 1$, T_i is uniformly L-Lipschitz continuous. Therefore all conditions in Theorem 3.5 and Remark 3.6 are satisfied. The conclusions of Theorem 3.8 can be obtained from Theorem 3.5 and Remark 3.6 immediately.

Theorem 3.9 Let E, C be the same as in Theorem 3.5. Let $\{T_i\}$ be a countable family of uniformly asymptotically nonexpansive in the intermediate sense from C into itself with a nonnegative real sequence $\{\sigma_n\}$ such that $\mathfrak{F} := \bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and bounded and $\sum_{n=1}^{\infty} \sigma_n < \infty$. Let $\{x_n\}$ be the sequence defined by (3.2), where $\{\beta_{n,\ i}\}$, i = 0, 1, 2, ... and $\{\alpha_n\}$ are sequences in [0, 1] satisfying the conditions (a), (b) in Theorem 3.2 and lim inf $\beta_{n,\ 0}\beta_{n,\ i} > 0$ for any $i \geq 1$. If for each $i \geq 1$, T_i is uniformly L_i-Lipschitzian continuous and one of $\{T_i\}$ is compact, then the conclusions in Theorem 3.5 still hold.

Acknowledgements
The authors would like to express their thanks to the referees for their helpful comments and suggestions. This work was supported by the National Research Foundation of Korean Grant funded by the Korean Government (2011-0002581).

Author details
[1]Department of Mathematics, College of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China [2]Department of Mathematics Education, Kyungnam University, Masan, Kyungnam 631-701, South Korea [3]Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong

Authors' contributions
S-sC and JKK conceived the study and participated in its design and coordination. JKK and HWJL suggested many good ideas that are useful for achievement this paper and made the revision. JKK and CKC prepared the manuscript initially and performed all the steps of proof in this research. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 23 May 2011 Accepted: 16 February 2012 Published: 16 February 2012

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doi:10.1186/1029-242X-2012-37
Cite this article as: Chang *et al.*: **A generalization and improvement of Chidume theorems for total asymptotically nonexpansive mappings in Banach spaces.** *Journal of Inequalities and Applications* 2012 **2012**:37.