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Anomalous transport in random superconducting composite systems

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We have studied anomalous diffusion in the random superconducting network (RSN) with a wide distribution of conductivity. We consider a composite medium of superconducting and normal conducting regions in which the normal conducting component obeys a transfer-rate distribution of the form $W^{-(1+\alpha)}$ $(0<\alpha<1)$. In the static (dc) case below the percolation threshold p_c, one finds that the dc conductivity varies as $(p_c-p)^{-s'}$ in the vicinity of p_c with $s'=1/\alpha$. Above the percolation threshold and in the RSN limit, the superconducting component is considered to possess a large but finite transfer rate W_s. In this limit, the dc conductivity follows the $p=p_c$ behavior for small W_s, crossing over to the behavior of ordinary percolation at a crossover value of the superconducting transfer rate $W_{s,\rm co}$, which is found to vary as $(p-p_c)^{-(1+\alpha)/\alpha}$. The results are in good accord with scaling relations. Right at the percolation threshold, the frequency-dependent conductivity is calculated in the RSN limit. The real part of the conductivity (σ_R) at low frequencies initially follows the dc behavior, crossing over to the behavior $\sigma_R\sim\omega^{1-\alpha}$ at high frequencies. The crossover frequencies are estimated for various relevant regions. The imaginary part of the conductivity (σ_I) has even more complex behaviors. At high frequencies, σ_I varies as $\omega^{1-\alpha}$, being the same as the real part. The results are in accord with the scaling relations generalized to finite frequencies. The model is also numerically solved in the effective medium approximation to compare with the analytic results. Good agreements are found.

I. INTRODUCTION

Diffusion and transport problems on percolating systems have been extensively studied recently, as percolation is a simple model for binary composite systems on one hand, and its fractal geometry is realized near the percolation threshold[1] on the other hand. In the case of composite systems, the main concern is how the laws of diffusion and transport are modified when the medium is a random mixture of good and poor conducting regions. The question has received substantial attention for two limiting cases, namely (i) the random resistor network (RRN) in which the poor-conducting component has zero conductivity, and (ii) the random superconducting network (RSN) in which the good-conducting components has infinite conductivity.

de Gennes[2] introduced the termite diffusion model to deal with the random superconducting network. The result for the effective conductivity was found to be noncritical. Stanley *et al.*[3] modified the model by introducing two time scales for the problem and critical behaviors were successfully obtained. As a consequence of the analysis, the termite (RSN) limit is shown to relate to the ant (RRN) limit by a simple change of time scale.[3] The terms "termite",[2] or "ant,"[4] or "butterfly"[5] have been introduced because one can replace the conductivity problem by the diffusion problem via the Einstein relation.[6] The termite limit has proved to be rather subtle in many respects, as it requires us to handle diffusion involving two time scales.

The problem is further complicated by the fact that in a real composite system the constituent components may have a wide range of conductivity. For instance, a rock is composed of tiny grains of different conductivities (to heat, to fluid flow, to electricity, etc.). Also, the newly discovered high-temperature oxide superconductors are in fact composites of at least two phases.[7] As a model, we may consider a random mixture of insulator and normal conductor in which the conducting component has a wide range of conductivity or the insulator has a wide range of dielectric constant. Or we may consider a composite of superconductor and good conductor in which the good conductor has a wide range of conductivity. This wide distribution of conductivity may lead to nonuniversal diffusion and transport behaviors. The problem of distribution-induced nonuniversality has been introduced a few years ago by several groups.[8–10] Recently it has also been shown to relate to continuum percolation.[11]

Different methods have been used to deal with diffusion problems on percolating network, both in the static (dc) and the dynamic cases. In particular, scaling analysis and numerical simulations have been extensively used by several groups.[3(a),11(b)] Here we shall use the effective-medium approximation (EMA)[12–14] to calculate the frequency-dependent (ac) conductivity and find complex crossover behaviors.

The present paper is a continuation of Ref. 13(c) (hereafter referred to as paper I). We study on a percolating network a continuous-time random walk described by the usual master equation

$$\frac{dP_n}{dt}=-\Big(\sum_\delta W_{n+\delta,n}\Big)P_n+\sum_\delta W_{n,n+\delta}P_{n+\delta}, \qquad (1)$$

where δ is a nearest neighbor of site n. In this equation, $P_n(t)$ is the probability of the random walker being on site n at time t, $W_{m,n}=W_{n,m}$ being the nearest-neighbor transfer rates. They are all independent random variables and obey a probability distribution $P(W)$. For an ordinary binary composite

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system, the transfer rates obey a binary distribution

$$P(W) = p\delta(W - W_s) + (1-p)\delta(W - W_0),$$

where $\delta(x)$ is the Dirac delta function. W_s and W_0 are the transfer rates associated with the good and poor conducting regions, respectively, and $W_0 < W_s$. In the ant model, we shall take $W_s = 1$ and $W_0 = 0$, the transfer rate being finite with probability p and zero with probability $1-p$. The distribution of transfer rates has the following form:

$$P_A(W) = p\delta(W-1) + (1-p)\delta(W). \quad (2a)$$

This case has been extensively studied as a model of diffusion on a random resistor network.

In the termite model, we take $W_s = \infty$ and $W_0 = 1$, the transfer rate being infinite with probability p and finite with probability $1-p$. The distribution of transfer rates has the following form:

$$P_T(W) = p\delta(W-\infty) + (1-p)\delta(W-1). \quad (2b)$$

This case has been proposed by Stanley *et al.*[3] to discuss a random superconducting network.

We have considered the random resistor network with an anomalous distribution of conductivity (anomalous ant model) in paper I. The transfer-rate distribution has the following form:

$$P(W) = (1-p)\delta(W) + ph_A(W),$$

where

$$h_A(W) = (1-\alpha)W^{-\alpha}, \quad 0 < W < 1,$$
$$= 0, \quad \text{otherwise}, \quad (3)$$

$p(0<p<1)$ being the percolation probability and $0<\alpha<1$. The anomalous distribution $h_A(W)$ leads to nonuniversal modifications of the conductivity exponent.[13(c)]

In the present paper we study the random superconducting network with an anomalous distribution of conductivity (anomalous termite model) by considering the following distribution of transfer rates:

$$P(W) = p\delta(W - W_s) + (1-p)h_T(W),$$

where

$$h_T = (\alpha W_0^\alpha / W^{1+\alpha})\Theta(W - W_0), \quad (4)$$

again $p(0<p<1)$ being the percolation probability and $0<\alpha<1$; $\Theta(x)$ is the Heavyside step function. We shall set $W_0 = 1$ without loss of generality. The anomalous distribution $h_T(W)$ possesses a long tail[15]; it reflects the existence of a large weight of good conductors in addition to a finite concentration of superconductors ($W_s = \infty$). We shall also consider the termite (RSN) limit in which W_s is large but finite.

The paper is organized as follows: in Sec. II, we examine the exact scaling results[3] for the conductivity and establish a relation between the RSN and the RRN limits. Section III deals with the derivation of the self-consistent EMA equations applicable to the anomalous termite case. From this we shall study the static (dc) limit of both the ordinary RRN and RSN models, obtaining the dc conductivity as a function of the percolation probability. The conductivity exponents are obtained and results are compared to well-known scaling results. In Sec. IV, we use EMA to extract analytic results for the dc conductivity of the anomalous RRN model for $p > p_c$ and for the anomalous RSN model for $p < p_c$. The distribution-induced conductivity exponents (t' and s') are obtained and compared to Straley's predictions.[10] Then we discuss the problem in the RSN limit (large but finite W_s) for $p = p_c$ and $p > p_c$. Analysis has been done for a d-dimensional hypercubic network. Special attention has been paid to the complex crossover behaviors and the determination of the crossover transfer rate. In Sec. V, we calculate the frequency-dependent conductivity near the percolation threshold. We also discuss at finite frequencies the various crossover phenomena, which have proved to be very complex. The crossover frequencies are obtained. In Sec. VI, we solve the EMA equations numerically for a three-dimensional cubic network and present numerical solutions for the dc and ac conductivities. Comparison with the analytical results (Secs. IV and V) will be made.

II. EXACT SCALING RESULTS

In this section we present some general scaling results and establish a relation between the RSN and the RRN limits. Our main concern here is on the dc conductivity σ_{dc}, the diffusion coefficient D, and the ac conductivity $\sigma(\omega)$. The relations show that physical laws in the vicinity of the RSN limit are identical to those in the vicinity of the RRN limit.[16] Let us denote the dc conductivity of a composite system of two components W_s and W_0 by $\sigma_{dc}(W_s, W_0)$ and let $\eta = W_0/W_s$. Here the analysis is valid for any arbitrary concentration p. Since $W_0 < W_s$, $0 < \eta < 1$. If we consider a second composite system in which all transfer rates were *doubled*, then the conductivity would be *doubled* also. From this homogeneity requirement,

$$\sigma_{dc}(W_s, W_0) = W_s\sigma_{dc}(1, W_0/W_s) = W_0\sigma_{dc}(W_s/W_0, 1),$$

one immediately obtains

$$\sigma_{dc}(\eta^{-1}, 1) = \eta^{-1}\sigma_{dc}(1, \eta). \quad (5)$$

This relation, originally proposed by Straley,[16] gives a relation between the RSN and the RRN limits; it can readily be generalized to finite frequencies.

A similar relation holds for the mean-square displacement $\langle R^2(t)\rangle$, as a generalization of Eq. (5). If we consider a second system in which all transfer rates were *doubled*, then the mean-square displacement of the original system would be reached in *half* the time. One can define a time-dependent diffusion coefficient by the relation $D(W_s, W_0, t) = \langle R^2(t)\rangle/t$. One must obtain

$$D(\eta^{-1}, 1, t) = \eta^{-1}D(1, \eta, t\eta^{-1}). \quad (6)$$

Thus, the behavior of the random walker near the RSN limit can be obtained from that near the RRN limit by a simple change of time scale. From the Einstein relation, the ac conductivity $\sigma(W_s, W_0, \omega)$ obeys a similar relation:

$$\sigma(\eta^{-1}, 1, \omega) = \eta^{-1}\sigma(1, \eta, \eta\omega). \quad (7)$$

Thus, the ac conductivity at a frequency ω in the RSN limit can be determined from the knowledge of the conductivity at a frequency $\eta\omega$ in the RRN limit. Having established the exact relation between the RSN limit and the RRN limit, we are now in a position to present some general scaling results for the dc conductivity in the vicinity of the percolation threshold. Let us define a dimensionless parameter $\epsilon = (p - p_c)/p_c$ to measure how close we are to the percolation threshold. In the RRN limit, the dc conductivity approaches zero as ϵ goes to zero with a conductivity exponent t; the t exponent is defined as

$$\sigma_{dc}(1,\eta = 0,\epsilon) \sim \epsilon^t \quad \text{for} \quad \epsilon > 0. \tag{8}$$

In EMA $t = 1$, while in the mean field theory $t = 3$. Right at the percolation threshold, we define the u exponent

$$\sigma_{dc}(1,\eta,\epsilon = 0) \sim \eta^u. \tag{9}$$

The parameter $\eta = W_0/W_s$ also behaves as an external field variable. A general scaling relation can be obtained by combining Eqs. (8) and (9) for $\epsilon > 0$:

$$\sigma_{dc}(1,\eta,\epsilon) = \eta^u g(\epsilon\eta^{-\phi}). \tag{10}$$

Here ϕ is the crossover exponent between ϵ and η. In order to reproduce the limiting behaviors in Eqs. (8) and (9), one must have

$$\phi = u/t. \tag{11}$$

We next come to $\epsilon < 0$, i.e., we are below the percolation threshold. In the RSN limit, the dc conductivity diverges as ϵ goes to zero with an exponent s; the s exponent is defined as

$$\sigma_{dc}(\eta^{-1} = \infty,1,\epsilon) \sim |\epsilon|^{-s} \quad \text{for} \quad \epsilon < 0. \tag{12}$$

In EMA $s = 1$, while in the mean-field theory $s = 0$. Right at the percolation threshold, a similar equation for η in the RSN limit can be obtained. From Eqs. (5) and (6), one finds at $p = p_c$,

$$\sigma_{dc}(\eta^{-1},1,\epsilon = 0) \sim \eta^{-(1-u)}. \tag{13}$$

Again η behaves as an external field variable. Combining the results, one finds a general scaling relation for $\epsilon < 0$:

$$\sigma_{dc}(\eta^{-1},1,\epsilon) = \eta^{-(1-u)} g(\epsilon\eta^{-\phi}). \tag{14}$$

Again ϕ is the crossover exponent between ϵ and η. In order to reproduce the limiting behaviors of Eqs. (12) and (13), one must have $\phi = (1-u)/s$, and this expression must be identical to Eq. (11). One finds

$$\phi = 1/(s + t). \tag{15}$$

It is immediately clear that $\phi = \frac{1}{2}$ in EMA. We are now in a position to deal with the frequency-dependent (ac) conductivity $\sigma(W_s,W_0,\epsilon,\omega)$ and establish some analogous scaling relations. In the RRN limit and right at the percolation threshold, the ac conductivity obeys the following relation[1]:

$$\sigma(1,0,0,\omega) \sim \omega^{\theta/(2+\theta)}, \tag{16}$$

where the exponent θ describes the range dependence of the diffusion coefficient.[1] In EMA, $\theta = 2$, while in the mean field theory, $\theta = 4$. From this limit, one can establish analogous scaling ansatz for the ac conductivity:

$$\sigma(1,\eta,0,\omega) = \eta^u H(\omega\eta^{-\psi}) \tag{17}$$

and

$$\sigma(1,0,\epsilon,\omega) = |\epsilon|^t A_{\pm}(\omega|\epsilon|^{-z}), \tag{18}$$

where $+(-)$ denotes above (below) the percolation threshold. The crossover exponents ψ and z are readily determined to be $\psi = u(2+\theta)/\theta$ and $z = t(2+\theta)/\theta$, respectively. The RSN limit can be readily obtained from Eq. (7) by a simple change of frequency scale. The next section will deal with the effective medium approximate solutions of the anomalous RSN limit and comparison with these scaling relations will be made if possible.

III. EFFECTIVE-MEDIUM APPROXIMATION (EMA)

The EMA has been derived previously.[13] Here we extend it for the anomalous RSN case. In this approximation, we replace the random transfer rates in Eq. (1) by a homogeneous value, which is to be determined self-consistently:

$$\langle (W - \overline{W})/[1 + Q(W - \overline{W})] \rangle = 0, \tag{19a}$$

where W is the EMA transfer rate, Q being the EMA impedance. The brackets denote an average over the distribution of transfer rate $P(W)$:

$$Q = (1/d\overline{W})(1 - sP_0), \tag{19b}$$

where d is the dimensionality of the hypercubic network, s is the Laplace-transformed variable, and $P_0(s)$ is the zero-site probability. In the effective-medium approximation,

$$P_0(s) = \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int \frac{dq_1 \cdots dq_d}{s + 2\overline{W} \Sigma_{i=1}^d (1 - \cos q_i)}. \tag{20}$$

In one dimension, $P_0(s)$ can be evaluated exactly and one obtains

$$P_0(s) = (s^2 + 4\overline{W}s)^{-1/2}, \tag{21a}$$

while in three dimensions, one can approximate $P_0(s)$ by the following expression[1]:

$$P_0(s) = 2/\{s + 6\overline{W} + [s(s + 12\overline{W})]^{1/2}\}. \tag{21b}$$

We shall solve these self-consistent equations to obtain the EMA transfer rate as a small s expansion. The frequency-dependent conductivity is given by the generalized Einstein relation derived from the fluctuation-dissipation theorem[6]

$$\sigma(\omega) = (ne^2/k_B T)\langle D(i\omega)\rangle = (ne^2/k_B T)\overline{W}(s = i\omega). \tag{22}$$

In Eq. (22), n is the density of charge carriers of charge e, T being the temperaure. In order to develop an EMA applicable for the anomalous case, we substitute $y = W - \overline{W}$ into Eq. (19a). After considerable simplifications, we obtain $1 = \langle (1 + Qy)^{-1} \rangle$. We further let $f = Q\overline{W}$, so that

$$f = (1 - sP_0)/d, \tag{23}$$

and let

$$h = \overline{W}(1 - f)/f, \tag{24}$$

so that h is related to the effective medium by a trivial proportional factor. We obtain

$$f/\overline{W} = (1 - f)/h = \langle 1/(W + h) \rangle. \tag{25}$$

Again the brackets denote an average over the distribution of transfer rates $P(W)$.

We first consider the RRN model by evaluating the average in Eq. (25) with respect to Eq. (2a) (with $W_s = 1$ and $W_0 = 0$). We obtain

$$(1 - f)/h = [p/(1 + h)] + [(1 - p)/h].$$

In the static limit $s = 0$ and $f = p_c$, we immediately obtain $h = (p - p_c)/p_c$, or $\overline{W} = (p - p_c)/(1 - p_c)$ for $p \geqslant p_c$. This gives immediately the dc conductivity via Eq. (22):

$$\sigma_{dc} = (p - p_c)/(1 - p_c) \sim (p - p_c)^t, \tag{26}$$

with the conductivity exponent $t = 1$ in EMA, in agreement with Kirkpatrick.[12(b)] Also, when $s = 0$ and $p = p_c$, if $\eta = W_0/W_s$, one finds that $h \sim \eta^{1/2}$. Comparing with Eq. (9), one finds $u = \frac{1}{2}$.

We next consider the RSN model by evaluating the average in Eq. (25) with respect to Eq. (2b) (with $W_s = \infty$ and $W_0 = 1$). We obtain in this limit

$$(1 - f)/h = (1 - p)/(1 + h).$$

In the static limit $s = 0$ and $f = p_c$, we immediately obtain

$$h = [(p_c - p)/(1 - p_c)]^{-1} \quad \text{or} \quad \overline{W} = p_c(p_c - p)^{-1}$$

for $p < p_c$. Again the dc conductivity is obtained by Eq. (22):

$$\sigma_{dc} = p_c(p_c - p)^{-1} \sim (p_c - p)^{-s}, \tag{27}$$

with the superconductivity exponent $s = 1$ in EMA. This is consistent with the duality relation[16(a)] $s = t$ in two dimensions.

When $p \geqslant p_c$, we consider the RSN limit in which W_s is large but finite. Equation (25) becomes

$$(1 - p_c)/h = [p/(W_s + h)] + [(1 - p)/(1 + h)], \tag{28}$$

in the static case $s = 0$. We obtain the dc conductivity via the Einstein relation [Eq. (22)]

$$\sigma_{dc} = (p - p_c)W_s/(1 - p_c), \quad p > p_c, \tag{29a}$$

$$\sigma_{dc} = [p_c W_s/(1 - p_c)]^{1/2}, \quad p = p_c. \tag{29b}$$

Following Eq. (13) $\sigma_{dc} \sim W_s(W_0/W_s)^u$ when W_s approaches infinity at $p = p_c$. We find that $u = \frac{1}{2}$, in accord with the scaling predictions.

Now we consider the average with respect to the anomalous ant limit in Eq. (3); we get

$$(1 - f)/h = [(1 - p)/h] + p\langle 1/(W + h)\rangle_\alpha.$$

Now the average $\langle \ldots \rangle$ is performed with respect to $h_A(W)$ only. Upon a change of variables $W = hx$, we obtain the following expression:

$$(1 - f)/h = [(1 - p)/h] + p(1 - \alpha)h^{-\alpha}K_\alpha^A(h), \tag{30}$$

where

$$K_\alpha^A(h) = \int_0^{h^{-1}} \frac{x^{-\alpha}\,dx}{x + 1} \tag{31}$$

is a function of h.

The anomalous termite limit is rather tricky. By evaluating the average with respect to Eq. (4), we obtain

$$(1 - f)/h = [p/(W_s + h)] + (1 - p)\langle 1/(W + h)\rangle_\alpha.$$

Here W_s is large but finite. The average $\langle \ldots \rangle$ is performed with respect to $h_T(W)$ only; it diverges because $0 < \alpha < 1$. We need a regularization

$$\left\langle \frac{1}{W + h} \right\rangle_\alpha = \frac{\alpha}{h}\left(\int_1^\infty W^{-(1+\alpha)}\,dW - \int_1^\infty \frac{W^{-\alpha}\,dW}{W + h} \right).$$

Finally we obtain

$$(1 - f)/h = [p/(W_s + h)] + [(1 - p)/h] - \alpha(1 - p)h^{-(1+\alpha)}K_\alpha^T(h), \tag{32}$$

where

$$K_\alpha^T(h) = \int_{1/h}^\infty \frac{x^{-\alpha}\,dx}{x + 1}$$

is a function of h. A simple approximation can be obtained by setting the upper limit of Eq. (31) to infinity and the lower limit of Eq. (32) to zero. The inclusion of the finite upper limit in Eq. (31) leads to nonanalytic corrections[17] which are important near $\alpha = 0$ and 1.

IV. ANOMALOUS RRN AND RSN MODELS IN THE STATIC LIMIT

In the static case, $s = 0$ and $f = p_c$ by virtue of Eq. (23). When $p > p_c$, we are above the percolation threshold. For the anomalous RRN model, let us consider Eq. (30) in the static limit. Solving for h in this limit, we obtain

$$h = [(p - p_c)/pA_\alpha]^{1/(1-\alpha)},$$

where

$$A_\alpha = (1 - \alpha)K_\alpha^A(0) = (1 - \alpha)\pi/\sin(\pi\alpha).$$

The dc conductivity can be obtained from Eq. (22):

$$\sigma_{dc} \sim (p - p_c)^{t'}. \tag{33}$$

The anomalous distribution has modified the critical exponent t. This is consistent with Straley[9] that $t' = 1/(1 - \alpha)$.

We next consider the anomalous RSN limit. When $p < p_c$, we are below the percolation threshold. For simplicity, we set $W_s = \infty$ and obtain from Eq. (32) the solution

$$h = [(1 - p)A'_\alpha/(p_c - p)]^{1/\alpha},$$

where

$$A'_\alpha = \alpha K_\alpha^T(\infty) = \pi\alpha/\sin(\pi\alpha).$$

The dc conductivity can be found from Eq. (22):

$$\sigma_{dc} \sim (p_c - p)^{-s'}. \tag{34}$$

Again the anomalous distribution has modified the critical exponent. This is consistent with Straley[9] that $s' = 1/\alpha$.

We next examine $p = p_c$; one is right at the percolation threshold. From Eq. (32), we expect $h \ll W_s$ in the RSN limit (large but finite W_s). In this limit, we obtain

$$h = [(1 - p_c)A'_\alpha W_s/p_c]^{1/(1+\alpha)},$$

and the dc conductivity is

$$\sigma_{dc} \sim W_s^{1/(1+\alpha)}. \tag{35}$$

Here $u' = s'/(s' + t) = 1/(1 + \alpha)$. This finding suggests that the general scaling result [Eq. (14)] continues to hold in the anomalous RSN limit, as we shall numerically confirm in Sec. VI.

Now we come to the case $p > p_c$, where one is above the percolation threshold. From Eq. (32), one obtains

$$[(p - p_c)/h] + (1 - p)A'_\alpha h^{-(1+\alpha)} = p/(W_s + h), \tag{36}$$

where $A'_\alpha = \pi\alpha/\sin(\pi\alpha)$.

We should compare the magnitude of the two terms on the left-hand side of Eq. (36). When $(p - p_c)/h$ is much greater than the second term, one finds

$$h = (p - p_c) W_s / p_c,$$

or the dc conductivity is

$$\sigma_{dc} \sim (p - p_c) W_s, \quad (37)$$

which is exactly the same as the result of the ordinary RSN problem; the singular distribution is irrelevant.

When the second term is much greater than $(p - p_c)/h$, we find

$$h = [(1-p) A'_\alpha W_s / p]^{1/(1+\alpha)},$$

and the dc conductivity is

$$\sigma_{dc} \sim W_s^{1/(1+\alpha)}, \quad (38)$$

which is exactly the same as the $p = p_c$ case [Eq. (35)]. As a result, one expects a crossover between the two behaviors described by Eqs. (37) and (38). The crossover value of W_s can be obtained by equating the two behaviors of interest. One finds

$$W_{s,co} = [(1-p) A'_\alpha / p]^{1/\alpha} [(p - p_c)/p_c]^{-(1+\alpha)/\alpha}. \quad (39)$$

This crossover value has the following meaning: When $W_s \ll W_{s,co}$, the second behavior dominates, i.e., $\sigma_{dc} \sim W_s^{1/(1+\alpha)}$ crossing over to the behavior $\sigma_{dc} \sim W_s$ when W_s increases to be much greater than $W_{s,co}$, where $W_{s,co} \sim (p - p_c)^{-(1+\alpha)/\alpha}$, as we shall numerically confirm in Sec. VI.

V. FREQUENCY-DEPENDENT CONDUCTIVITY AT THE PERCOLATION THRESHOLD

Here we shall calculate the frequency-dependent conductivity in the effective medium approximation. In one dimension, we consider the case $p = 0$ for the anomalous RSN model. The distribution of transfer rates is purely singular:

$$P(W) = (\alpha / W^{1+\alpha}) \Theta(W - 1). \quad (40)$$

One can obtain a solution from Eq. (32) by putting $p = 0$,

$$h = (A'_\alpha / f)^{1/\alpha}. \quad (41)$$

Let us consider the low-frequency limit. As s approaches zero, we approximate $P_0(s)$ by $(4\overline{W}s)^{-1/2}$ and $f \sim 1$. With this approximation,

$$\overline{W} = 4 (A'_\alpha)^{2/\alpha} s^{-1}, \quad (42)$$

which diverges as s approaches zero.[18]

We now consider the high-frequency limit $s \gg \overline{W}$. One can approximate f by $2\overline{W}/s$. From Eq. (41), one obtains

$$\overline{W} = (s/2)^{1-\alpha} A'_\alpha, \quad (43)$$

being exactly the same as the 3D results as we next show.

In higher dimensions $(d > 1)$, we need to consider low- and high-frequency expansions of $P_0(s)$ in Eq. (21). For low frequencies, $s \ll \overline{W}$,

$$P_0(s) = B / p_c \overline{W}, \quad d > 2,$$
$$= (B'/p_c s)(s/\overline{W})^{d/2}, \quad 1 < d < 2. \quad (44a)$$

As a result,

$$f = p_c - Bs/\overline{W}, \quad d > 2,$$
$$= p_c - B'(s/\overline{W})^{d/2}, \quad 1 < d < 2, \quad (44b)$$

where B and B' are dimension-dependent coefficients.

In the opposite limit $s \gg \overline{W}$, we have

$$P_0(s) = 1/s - 2d\overline{W}/s^2, \quad (45a)$$

and therefore

$$f = 2\overline{W}/s, \quad (45b)$$

which is true for any dimensions.

Let us consider the case $p = p_c$, where we are right at the percolation threshold. From Eq. (32), one obtains

$$[(p_c - f)/h] + (1 - p_c) A'_\alpha h^{-(1+\alpha)} = p_c / (W_s + h), \quad (46)$$

where $A'_\alpha = \pi\alpha / \sin(\pi\alpha)$.

Let us consider $s \ll \overline{W}$, and let $f = p_c - B(s, \overline{W})$, where $B(s, \overline{W})$ is given by Eq. (44b). In $d > 2$, we obtain

$$h = h_0 \{1 - [B^* s W_s / (1+\alpha) p_c h_0^2]\}, \quad (47)$$

where B^* is a dimension-dependent coefficient, and h_0 is given by

$$h_0 = [(1 - p_c) A'_\alpha W_s / p_c]^{1/(1+\alpha)}. \quad (48)$$

The conductivity can be obtained from Eq. (22); we find

$$\sigma_R(\omega) \sim h_0 \sim W_s^{1/(1+\alpha)}, \quad (49a)$$

$$\sigma_I(\omega) \sim W_s^{\alpha/(1+\alpha)}. \quad (49b)$$

The imaginary part of the conductivity $\sigma_I(\omega)$ is linear in ω and varies as $W_s^{\alpha/(1+\alpha)}$.

A rough estimate of the crossover frequency can be found by equating the two terms

$$B(s, \overline{W})/h_0 \quad \text{and} \quad (1 - p_c) A'_\alpha / h_0^{1+\alpha}.$$

For $d > 2$, we find

$$\omega_{co} \sim W_s^{(1-\alpha)/(1+\alpha)}, \quad (50a)$$

and for $1 < d < 2$, we find

$$\omega_{co} \sim W_s^{(d-2\alpha)/d(1+\alpha)}. \quad (50b)$$

At high frequencies $s \gg \overline{W}$, we have the approximate limiting behavior Eq. (45b): $f = 2\overline{W}/s$, so that

$$f/h = [(p_c/h) - (p_c / W_s + h)]$$
$$- (1 - p_c) A'_\alpha h^{-(1+\alpha)}. \quad (51)$$

We drop the first term compared to the second term on the right-hand side of (51) and obtain

$$h = [(1 - p_c) A'_\alpha / f]^{1/\alpha}. \quad (52)$$

One can obtain the conductivity from Eq. (22):

$$\sigma_R(\omega) \sim \omega^{1-\alpha}, \quad (53a)$$

$$\sigma_I(\omega) \sim \omega^{1-\alpha}, \quad (53b)$$

which is exactly the same as the 1D results.

A rough crossover frequency can be estimated by equating the two terms:

$$f/h \quad \text{and} \quad p_c \{(1/h) - [1/(W_s + h)]\}.$$

We find the crossover frequency,

$$\omega_{co} \sim W_s^{1/(1-\alpha)}. \quad (54)$$

This section concludes the analytic solution of the RSN problems in various limits. We shall study the numerical solutions to verify these analytic results in the next section.

VI. NUMERICAL SOLUTIONS OF THE EFFECTIVE MEDIUM EQUATIONS

Here we wish to solve the EMA self-consistent equations numerically so that a comparison to the analytic solutions presented in previous sections can be made. On the other hand, one can bridge the analytic solutions in various limits by continuously varying the parameters of interest in the numerical work.

We first solve the ant model [Eq. (30)] numerically in the dc limit. The percolation probability is chosen to be greater than the percolation threshold. We include the first correction term h/α to improve the previous calculations.[13(c)] The ordinary ant model [Eq. (2b)] is also solved for comparison. In Fig. 1, the dc conductivity is plotted as a function of $p-p_c$ for the ant model. As p approaches p_c from above p_c, the curves become straight, showing the behavior $\sigma_{dc} \sim (p-p_c)^{1/(1-\alpha)}$. The magnitude of the slope varies as $1/(1-\alpha)$, which increases as α increases. For p away from p_c, results approach the usual ant model.

We then solve the termite model [Eq. (32)] in the static case. The percolation probability is chosen to be smaller than the percolation threshold. We include the first correction term $h^{-(1-\alpha)}/(1-\alpha)$ to improve the accuracy. The ordinary termite model [Eq. (2b)] is also solved for comparison. In Fig. 2, the dc conductivity as a function of p_c-p is plotted for the termite model. As p approaches p_c from below, the curves become straight, showing the behavior $\sigma_{dc} \sim (p_c-p)^{-1/\alpha}$. The magnitude of the slope varies as $1/\alpha$, which increases as α decreases. For p away from p_c, the curves bend over, approaching the results for the usual termite model. The straight line at the lower left corner is shown for usual percolation.

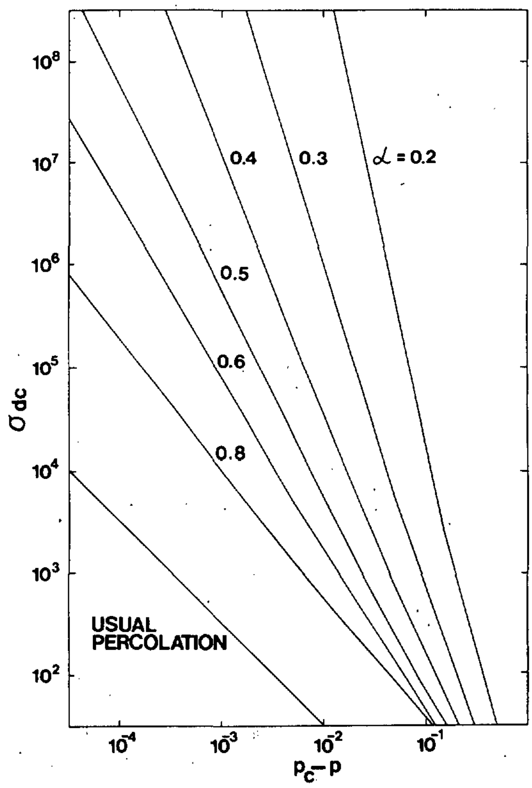


FIG. 2. The EMA dc conductivity plotted as a function of the percolation probability below the percolation threshold for several anomalous termite model parameterized by α.

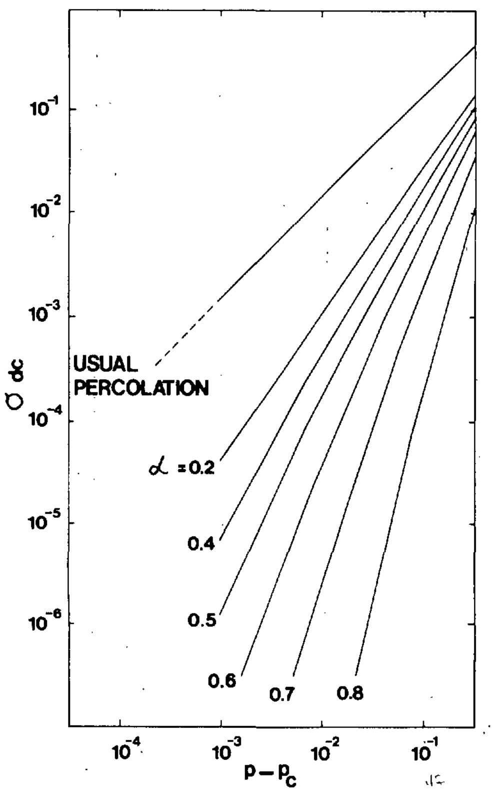
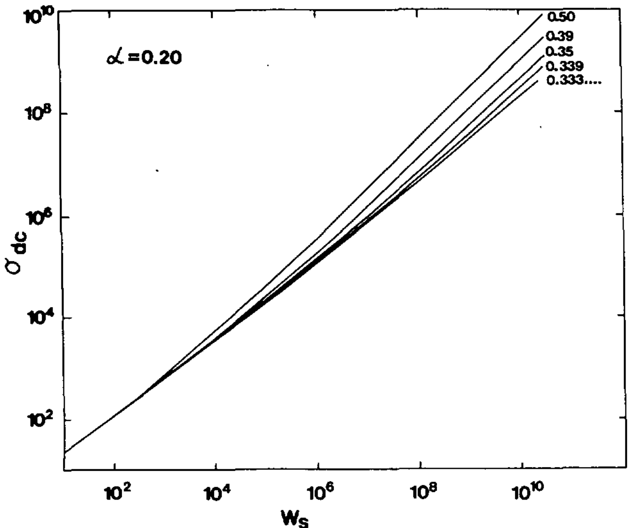


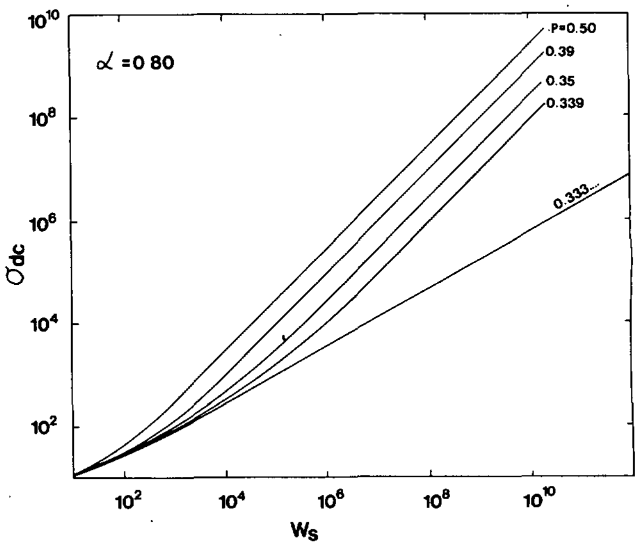
FIG. 1. The EMA dc conductivity plotted as a function of the percolation probability above the percolation threshold for several anomalous distributions of the ant model parametrized by α.

In the RSN limit and dc case, we solve Eq. (36) for several p above p_c. In Fig. 3, the dc conductivity as a function of the superconducting transfer rate W_s is plotted for several $p \gg p_c$. As W_s increases at fixed p_c, we see that the dc conductivity approaches a straight line with slope unity, showing the behavior $\sigma_{dc} \sim W_s$ for $p > p_c$. At $p = p_c$, the increase of the conductivity with W_s is slower, showing the behavior $\sigma_{dc} \sim W_s^{1/(1+\alpha)}$. For small $p > p_c$, the dc conductivity initially follows the $p = p_c$ behavior, crossing over to the behavior $\sigma_{dc} \sim W_s$. The crossover value of W_s varies as $(p-p_c)^{-(1+\alpha)/\alpha}$.

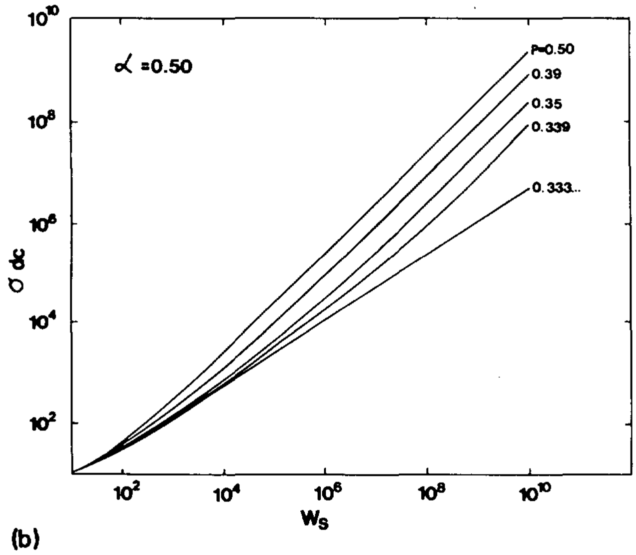
Finally, we solve the RSN model [Eq. (22)] in a three-dimensional cubic network. Several large values of the superconducting transfer rate ($10 \leqslant W_s \leqslant 10^8$) are chosen for the calculations. In Fig. 4, the real and imaginary parts of the ac conductivity are plotted as a function of frequency for several values of the superconducting transfer rate at fixed α right at the percolation threshold. The low-frequency $\sigma_R(\omega)$ follows the dc behavior, crossing over to the behavior $\sigma_R(\omega) \sim \omega^{1-\alpha}$ for frequencies above a crossover frequency that increases as W_s. The imaginary part of the conductivity



(a)



(c)



(b)

FIG. 3. The EMA dc conductivity plotted as a function of the superconducting transfer rate for several percolation probability above the percolation threshold at $\alpha =$ (a) 0.20, (b) 0.50, and (c)0.80.

has even more complex behaviors. For small W_s and large frequencies, one sees that $\sigma_I(\omega)$ varies as $\omega^{1-\alpha}$, being the same as the real part in this limit. For large W_s and small frequencies, one finds that $\sigma_I(\omega)$ varies linearly as the frequency. A complex crossover behavior occurs between these two limits.

In conclusion, we have studied anomalous diffusion and conductivity in the random superconducting network with an anomalous distribution of transfer rates. Exact scaling relations are obtained for the RSN limit and the results show that the behaviors near the RSN limit can be obtained from those near the RRN limit. The model is solved in the effective medium approximation both analytically and numerically. Various complex crossover behaviors are obtained. The results are also compared with the exact scaling relations and good agreements are found. This study may have some implications on recent high-temperature oxide superconductors. The x-ray diffraction patterns of the specimens indicate that they contain at least two phases. Also, the specimens that have equal compositions upon preparation exhibit quite different resistivity behaviors.[7] This could be explained by the fact that they have a wide range of conductivities. We hope that this analysis may shed light on the physics of high-temperature oxide superconductors.

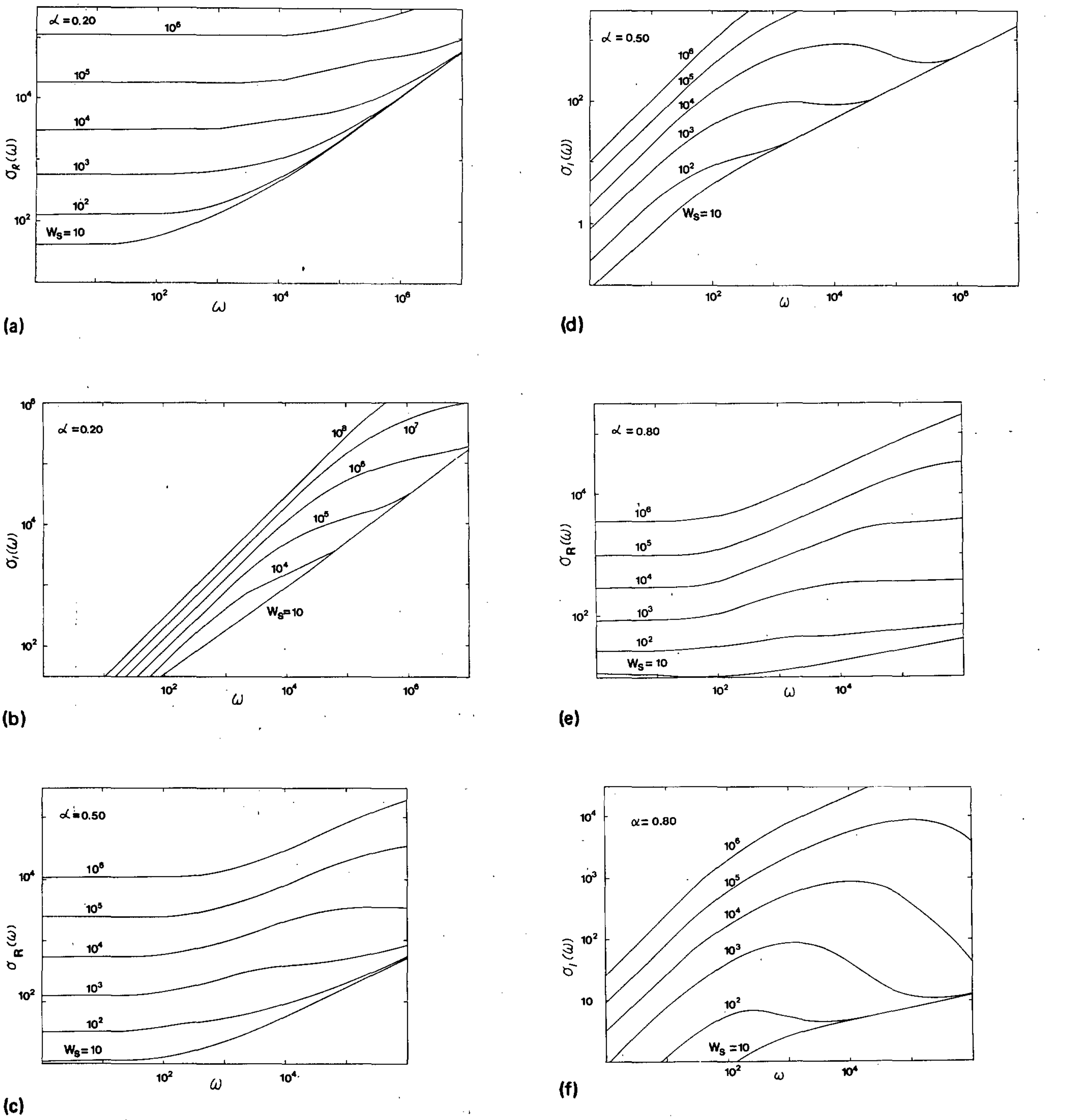


FIG. 4. The real and imaginary parts of the frequency-dependent conductivity in EMA plotted as functions of frequency for several values of the superconducting transfer rate in the anomalous termite limit. (a) Real part of the conductivity plotted as a function of frequency for $\alpha = 0.20$. (b) Imaginary part of the conductivity plotted as a function of frequency for $\alpha = 0.20$. (c) and (e) are the same as (a), but with different α. (d) and (f) are the same as (b), but with different α.

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