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OPTIMALITY CONDITIONS VIA EXACT PENALTY FUNCTIONS*

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Abstract. In this paper, we study KKT optimality conditions for constrained nonlinear programming problems and strong and Mordukhovich stationarities for mathematical programs with complementarity constraints using l_p penalty functions, with $0 \leq p \leq 1$. We introduce some optimality indication sets by using contingent derivatives of penalty function terms. Some characterizations of optimality indication sets are obtained by virtue of the original problem data. We show that the KKT optimality condition holds at a feasible point if this point is a local minimizer of some l_p penalty function with p belonging to the optimality indication set. Our result on constrained nonlinear programming includes some existing results from the literature as special cases.

Key words. KKT optimality condition, nonlinear programming problem, mathematical programs with complementarity constraints, strong stationarity, Mordukhovich stationarity, exact penalty function

AMS subject classifications. 90C30, 90C33, 90C46

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1. Introduction. Consider the following inequality and equality constrained optimization problem:

$$
\begin{array}{llll}
(P) & \min & f(x) & \\
& \text{s.t.} & g_i(x) \leq 0, & i \in I, \\
& & h_j(x) = 0, & j \in J,
\end{array}
$$

where $I = \{1, 2, \ldots, m\}$, $J = \{m+1, m+2, \ldots, m+q\}$, and $f, g_i, h_j : R^n \to R$ are all assumed to be continuously differentiable functions. The well-known KKT optimality condition is said to hold at a local minimizer $\bar{x}$ of (P) if there is a multiplier $\lambda = (\lambda^g, \lambda^h) \in R^{m+q}$ such that

$$\nabla f(\bar{x}) + \sum_{i \in I} \lambda_i^g \nabla g_i(\bar{x}) + \sum_{j \in J} \lambda_j^h \nabla h_j(\bar{x}) = 0, \quad \lambda_i^g \geq 0,\ \lambda_i^g g_i(\bar{x}) = 0 \quad \forall\, i \in I.$$

An important topic in the study of KKT optimality conditions concerns various constraint qualifications (CQs), under which KKT optimality conditions are valid at local minimizers of (P); see [3] and the references therein. Note that CQs are independent of the objective function and that the Guignard constraint qualification (GCQ) is the weakest in the sense that the GCQ holds at $\bar{x}$ if and only if the KKT optimality condition is valid at $\bar{x}$ for every (P), which has the same constraints and the same local minimizer $\bar{x}$; see [13].

When the GCQ is violated at $\bar{x}$, another type of regularity condition that depends not only on constraint functions but also on the objective function can be invoked to ensure that KKT optimality conditions at $\bar{x}$ are valid. With the aid of exact

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penalty functions, this type of regularity condition was first studied in [5]; see also [4] for a survey on this topic. More recently, KKT optimality conditions of (P) were studied via a lower order exact penalty function in [33]. In this paper, we consider the following l_p $(0 \le p \le 1)$ penalty function for (P):

$$\mathcal{F}_p(x) := f(x) + \mu S^p(x),$$

where $\mu > 0$ is the penalty parameter, the penalty function term $S^p(x)$ is defined by

$$S^p(x) := \left(\sum_{i \in I} (g_i(x))_+ + \sum_{j \in J} |h_j(x)| \right)^p,$$

with $t_+ := \max\{t, 0\}$ for all $t \in R$, and the convention $0^0 = 0$ is used when $p = 0$. The l_p penalty function $\mathcal{F}_p$ is said to be exact at a local minimizer $\bar{x}$ of (P) if there is some $\mu > 0$ such that $\mathcal{F}_p$ has an unconstrained local minimizer $\bar{x}$. It is well known that the KKT optimality condition is valid at $\bar{x}$ if the l_1 penalty function is exact at $\bar{x}$; see [6] and [4]. But, for $0 \le p < 1$, the KKT optimality condition at $\bar{x}$ cannot always be derived from an l_p exact penalty function unless some additional conditions are imposed on the constraints; see [33] for more details. For a comprehensive study of lower order penalty functions, we refer the reader to [18] and [26].

In this paper, we present a unified approach for the study of KKT optimality conditions. We define an optimality indication set of (P) with respect to S and a feasible point $\bar{x}$ as follows:

$$\Pi(S, \bar{x}) = \{p \in [0,1] \mid KerDS^p(\bar{x})^* \subset KerDS^1(\bar{x})^*\},$$

with $KerDS^p(\bar{x})$ being the kernel of the contingent derivative $DS^p(\bar{x})$ of $S^p(x)$ at $\bar{x}$, and $KerDS^p(\bar{x})^*$ being the polar cone of $KerDS^p(\bar{x})$. The definition of the contingent derivative will be given at the end of this section. We will show that the KKT optimality condition is valid at $\bar{x}$ if there exists some $p \in \Pi(S, \bar{x})$ such that the l_p penalty function is exact at $\bar{x}$. This result includes both various CQs and regularity conditions obtained in terms of exact penalty functions as special cases; see section 2.

In section 3, optimality conditions of the following mathematical program with complementarity constraints (MPCC) will be studied:

$$\begin{array}{llll} \text{(MPCC)} & \min & f(x) & \\ & \text{s.t.} & g_i(x) \le 0, & i \in I, \\ & & h_j(x) = 0, & j \in J, \\ & & G_k(x) \ge 0, H_k(x) \ge 0, G_k(x)H_k(x) = 0, & k \in K, \end{array}$$

where $f, g_i, i \in I, h_j, j \in J$ are given as in (P), and $G_k, H_k : R^n \to R, k \in K = \{m + q+1, m+q+2, \ldots, m+q+l\}$ are assumed to be continuously differentiable. Stationarity (or first-order optimality) conditions for (MPCC) have been the subject of many recent papers and books; see [27, 28, 18, 19, 34, 23, 9, 10, 12]. Since there are several different approaches for deriving optimality conditions, various stationarity conditions arise; see a very recent thesis [8] for their definitions and connections. In this paper, we will focus only on strong stationarity and Mordukhovich stationarity. Specifically, a local minimizer $\bar{x}$ of (MPCC) is said to be a strongly (resp., a Mordukhovich) stationary point of (MPCC) if there is $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{m+q+2l}$ such that

$$\nabla f(\bar{x}) + \sum_{i\in I} \lambda_i^g \nabla g_i(\bar{x}) + \sum_{j\in J} \lambda_j^h \nabla h_j(\bar{x}) - \sum_{k\in K} [\lambda_k^G \nabla G_k(\bar{x}) + \lambda_k^H \nabla H_k(\bar{x})] = 0,$$
$$\lambda_\gamma^G = 0, \quad \lambda_\alpha^H = 0,$$
$$\forall i \in I, \quad \lambda_i^g \geq 0, \quad \lambda_i^g g_i(\bar{x}) = 0,$$
$$\forall k \in \beta, \quad \lambda_k^G \geq 0, \quad \lambda_k^H \geq 0$$
$$(\text{resp.}, \forall k \in \beta, \text{ either } \lambda_k^G > 0, \ \lambda_k^H > 0, \text{ or } \lambda_k^G \lambda_k^H = 0),$$

where α, β, γ are very useful index sets in what follows:

$$\begin{aligned} \alpha &:= \alpha(\bar{x}) = \{k \in K \mid 0 = G_k(\bar{x}) < H_k(\bar{x})\}, \\ \beta &:= \beta(\bar{x}) = \{k \in K \mid G_k(\bar{x}) = H_k(\bar{x}) = 0\}, \\ \gamma &:= \gamma(\bar{x}) = \{k \in K \mid G_k(\bar{x}) > H_k(\bar{x}) = 0\}. \end{aligned}$$

Clearly, strong stationarity implies Mordukhovich stationarity. Note that $\bar{x}$ is a strongly stationary point if and only if $\bar{x}$ is a KKT point of (MPCC); see [11] for more details. Therefore, similarly as for (P), the GCQ is the weakest CQ for strong stationarity of (MPCC). Moreover, it is easy to see from [4] that $\bar{x}$ is a strongly stationary point if it is a local minimizer of the following penalty function $\mathcal{G}_p$ (with $p = 1$) of (MPCC):

$$\mathcal{G}_p(x) = f(x) + \mu U^p(x),$$

where $\mu > 0$, and

$$U^p(x) = \left(S(x) + \sum_{k\in K} [(-G_k(x))_+ + (-H_k(x))_+ + |G_k(x)H_k(x)|] \right)^p.$$

In contrast to the result involving the penalty function $\mathcal{G}_1$, it was shown in [10] that $\bar{x}$ is a Mordukhovich stationary point if it is a local minimizer of the following penalty function $\mathcal{H}_p$ (with $p = 1$) of (MPCC):

$$\mathcal{H}_p(x) = f(x) + \mu V^p(x),$$

where $\mu > 0$, and

$$V^p(x) = \left(S(x) + \sum_{k\in K} |\phi_{min}(G_k(x), H_k(x))| \right)^p,$$

with $\phi_{min}(a, b) := \min\{a, b\}$ being an NCP function. As for various CQs ensuring Mordukhovich stationarity, we refer the reader to [34, 10] and [8] for more details.

Motivated by the work reported in [12], section 3 starts with a summary of characterizations of strong and Mordukhovich stationarities. Then, we apply the results obtained in section 2 to (MPCC) to derive sufficient conditions respectively for Mordukhovich stationarity by means of lower order exact penalty functions $\mathcal{G}_p$ and $\mathcal{H}_p$, and for strong stationarity by means of lower order exact penalty functions $\mathcal{G}_p$ (when $0 \leq p < 1$). This is done by introducing some stationarity indication sets, which are defined by the polar cone of $KerDU^p(\bar{x})$ or $KerDV^p(\bar{x})$. Some properties of $KerDU^p(\bar{x})$ and $KerDV^p(\bar{x})$ are also given. Furthermore, by applying nonsmooth analysis tools, in particular, Dini derivatives and Taylor expansions, we obtain some characterizations of the stationarity indication sets by virtue of the original data of (MPCC). Section 3 ends with some discussion on relationships between penalty functions $\mathcal{G}_p$ and $\mathcal{H}_p$ and their stationarity indication sets.

We conclude this section by reviewing some concepts that are needed in what follows. For a function $\phi : R^n \to R$, we say that ϕ is $\mathcal{C}^{1,1}$ if ϕ is differentiable and its gradient is locally Lipschitz. The Dini upper directional derivative of a function $\phi : R^n \to R$ at $x \in R^n$ in the direction $u \in R^n$ is defined by

$$D_+\phi(x; u) = \limsup_{t\to 0+} \frac{\phi(x + tu) - \phi(x)}{t}.$$

Let

$$KerD_+\phi(x) = \{u \in R^n \mid D_+\phi(x; u) = 0\}.$$

The generalized lower and upper second-order directional derivatives of a continuously differentiable function $\phi : R^n \to R$ at $x \in R^n$ in the direction $u \in R^n$ are defined, respectively, by (see [7, 32])

$$\phi_{oo}(x; u) = \liminf_{y\to x,\, t\to 0+} \frac{\nabla\phi(y + tu)^T u - \nabla\phi(y)^T u}{t}$$

and

$$\phi^{oo}(x; u) = \limsup_{y\to x,\, t\to 0+} \frac{\nabla\phi(y + tu)^T u - \nabla\phi(y)^T u}{t}.$$

Let A be a nonempty subset of R^n and a point $\bar{x} \in A$. The polar cone of A is defined by

$$A^* = \{v \in R^n \mid \langle v, x\rangle \le 0,\ \forall\, x \in A\}.$$

A vector $w \in R^n$ is tangent to A at $\bar{x}$, written as $w \in T_A(\bar{x})$, if there are $\tau^\nu \to 0+$ and $w^\nu \to w$ such that $\bar{x} + \tau^\nu w^\nu \in A$ for all ν. The regular normal cone $\hat{N}_A(\bar{x})$ to A at $\bar{x}$ is the polar cone of $T_A(\bar{x})$, i.e., $\hat{N}_A(\bar{x}) = T_A(\bar{x})^*$. A vector $v \in R^n$ is normal to A at $\bar{x}$, written as $v \in N_A(\bar{x})$, if there are sequences $x^\nu \to \bar{x}$ and $v^\nu \to v$ with $x^\nu \in A$ and $v^\nu \in \hat{N}_A(x^\nu)$ for all ν. See Chapter 6 of [25] for more details on $T_A(\bar{x})$, $\hat{N}_A(\bar{x})$, and $N_A(\bar{x})$. Let $M : R^n \rightrightarrows R^s$ be a set-valued map and $(x, y) \in gphM$, where $gphM$ denotes the graph of M. The contingent derivative of M at (x, y) is defined by the set-valued map $DM(x, y) : R^n \rightrightarrows R^s$ such that

$$gph(DM(x, y)) = T_{gphM}(x, y).$$

In particular, when M is single-valued at x, i.e., $M(x) = \{y\}$, we use $DM(x)$ to denote $DM(x, y)$ for simplicity, and we define the kernel of $DM(x)$ by

$$KerDM(x) = \{u \in R^n \mid 0 \in DM(x)(u)\}.$$

2. Constrained optimization problems.

Throughout this section, let C be the feasible set of (P) and let $\bar{x} \in C$ be a local minimizer of (P). The basic properties of $KerDS^p(\bar{x})$ are summarized in the following lemma.

LEMMA 2.1. *Let $0 \le p \le 1$. Then, $KerDS^p(\bar{x})$ is a closed cone with the following properties:*

(i) *$u \in KerDS^p(\bar{x})$, with $p \neq 0$ if and only if there exist $t_k \to 0+$ and $u_k \to u$ such that* $\max\left\{\frac{g_i(\bar{x}+t_k u_k)}{t_k^{1/p}}, 0\right\} \to 0\ \forall i \in I_0$ *and* $\frac{h_j(\bar{x}+t_k u_k)}{t_k^{1/p}} \to 0\ \forall j \in J$, *where* $I_0 := \{i \in I \mid g_i(\bar{x}) = 0\}$ *denotes the active inequality index set of* (P) *at* $\bar{x}$.

(ii) $KerDS^p(\bar{x}) \subset KerDS^{p'}(\bar{x})\ \forall\, p' \in (p, 1]$.

(iii) $KerDS^0(\bar{x}) = T_C(\bar{x})$ *and* $KerDS^1(\bar{x}) = L_C(\bar{x})$, *where* $L_C(\bar{x})$ *is the linearized tangent cone to* C *at* $\bar{x}$ *defined by*

$$L_C(\bar{x}) = \left\{ u \in R^n \;\middle|\; \begin{array}{ll} \nabla g_i(\bar{x})^T u \leq 0, & i \in I_0, \\ \nabla h_j(\bar{x})^T u = 0, & j \in J. \end{array} \right\}$$

(iv) $KerDS^p(\bar{x}) = T_C(\bar{x})$ *if there are* $\delta > 0$ *and* $\tau > 0$ *such that, for all* $x \in B_\delta(\bar{x}) := \{x \in R^n \mid \|x - \bar{x}\| < \delta\}$,

$$d(x, C) \leq \tau S^p(x); \tag{1}$$

i.e., C has a local error bound at $\bar{x}$ with respect to S^p; see [21] *and* [31].

Proof. Since $KerDS^p(\bar{x}) \times \{0\} = (R^n \times \{0\}) \cap T(gphS^p, (\bar{x}, 0))$, $KerDS^p(\bar{x})$ is clearly a closed cone. Properties (i)–(iii) follow directly from the definitions of the contingent derivative and the contingent cone. By properties (ii) and (iii), $T_C(\bar{x}) \subset KerDS^p(\bar{x})$. On the other hand, it follows directly from the local error bound condition (1) that $KerDS^p(\bar{x}) \subset T_C(\bar{x})$. Thus, (iv) holds. The proof is complete. □

Now, define an optimality indication set of (P) with respect to S and $\bar{x}$ as follows:

$$\Pi(S, \bar{x}) := \{p \in [0, 1] \mid KerDS^p(\bar{x})^* \subset KerDS^1(\bar{x})^*\}.$$

Proposition 2.2. *The following statements are true:*

(i) $\Pi(S, \bar{x}) \neq \emptyset$, *since* $1 \in \Pi(S, \bar{x})$.

(ii) $[a, 1] \subset \Pi(S, \bar{x})$ *if* $a \in \Pi(S, \bar{x})$.

(iii) $\Pi(S, \bar{x}) = [0, 1]$ *if and only if the GCQ holds at* $\bar{x}$, *i.e.,* $T_C(\bar{x})^* = L_C(\bar{x})^*$.

Proof. All statements follow easily from Lemma 2.1(ii) and (iii). □

The following proposition sheds some light on how to identify a subset of $\Pi(S, \bar{x})$ by replacing $KerDS^p(\bar{x})$ with $KerD_+S^p(\bar{x})$, which is much easier to calculate.

Proposition 2.3. *Let* $0 \leq p \leq 1$. *Then,* $KerD_+S^p(\bar{x})$ *is a cone with the following statements holding true:*

(i) *When* $p \neq 0$,

$$KerD_+S^p(\bar{x}) = \left\{ u \in R^n \;\middle|\; \begin{array}{ll} \limsup\limits_{t \to 0+} \dfrac{g_i(\bar{x}+tu)}{t^{1/p}} \leq 0, & i \in I_0, \\ \lim\limits_{t \to 0+} \dfrac{h_j(\bar{x}+tu)}{t^{1/p}} = 0, & j \in J. \end{array} \right\}$$

(ii) $KerD_+S^1(\bar{x}) = L_C(\bar{x})$ *and* $KerD_+S^0(\bar{x}) = F_C(\bar{x}) \subset T_C(\bar{x})$, *where* $F_C(\bar{x})$ *is the feasible direction cone of* C *at* $\bar{x}$, *i.e.,* $u \in F_C(\bar{x})$ *if and only if there exists* $\delta > 0$ *such that* $\bar{x} + tu \in C$ *for all* $0 < t \leq \delta$. *Moreover, if* C *has a local error bound at* $\bar{x}$ *with respect to* S^p *(i.e., (1) holds), then*

$$F_C(\bar{x}) \subset KerD_+S^{p'}(\bar{x}) \subset T_C(\bar{x})\ \forall\, p' \in [0, p].$$

(iii) $KerD_+S^p(\bar{x}) \subset KerD_+S^{p'}(\bar{x})\ \forall\, p' \in [p, 1]$.

(iv) $KerD_+S^p(\bar{x}) \subset KerDS^p(\bar{x})$.

(v) $\{p \in [0, 1] \mid KerD_+S^p(\bar{x})^* \subset KerD_+S^1(\bar{x})^*\} \subset \Pi(S, \bar{x})$.

Proof. Statements (i)–(iii) follow easily from the definition of the Dini upper directional derivative. Statement (iv) holds since, by the definitions of the Dini upper directional derivative and the contingent derivative,

$$D_+S^p(\bar{x}; d) = 0 \Longrightarrow 0 \in DS^p(\bar{x})(d).$$

Statement (v) follows from (ii) and (iv). The proof is complete. □

Remark 2.1. $KerD_+S^1(\bar{x})$ is closed and convex, but $KerD_+S^p(\bar{x})$ with $0 \leq p < 1$ is not necessarily closed or convex; see Examples 2.1 and 2.2.

Example 2.1. In (P), let $n = 2$, $m = 1$, $q = 0$, $\bar{x} = 0$, and $g_1(x) = x_1^2 - x_2$. Then,

$$KerD_+S^p(\bar{x}) = \begin{cases} \{d \in R^2 \mid d_2 > 0\} \cup \{(0,0)^T\} & \text{if } 0 \leq p \leq 0.5, \\ \{d \in R^2 \mid d_2 \geq 0\} & \text{if } 0.5 < p \leq 1. \end{cases}$$

Clearly, $KerD_+S^p(\bar{x})$ is convex but not closed when $0 \leq p \leq 0.5$, though

$$KerDS^p(\bar{x}) = T_C(\bar{x}) = L_C(\bar{x}) = \{d \in R^2 \mid d_2 \geq 0\}$$

is convex and closed for every $p \in (0.5, 1]$.

Example 2.2. In (P), let $n = 2$, $m = 2$, $q = 1$, $\bar{x} = 0$, $g_1(x) = x_1$, $g_2(x) = x_2$, and $h_3(x) = x_1x_2$. Then,

$$KerD_+S^p(\bar{x}) = KerDS^p(\bar{x}) = \begin{cases} \{d \in R^2 \mid d_1 \leq 0, d_2 \leq 0, d_1d_2 = 0\} & \text{if } 0 \leq p \leq 0.5, \\ \{d \in R^2 \mid d_1 \leq 0, d_2 \leq 0\} & \text{if } 0.5 < p \leq 1. \end{cases}$$

Clearly, both $KerD_+S^p(\bar{x})$ and $KerDS^p(\bar{x})$ are closed but not convex when $0 \leq p \leq 0.5$.

In view of Propositions 2.2 and 2.3, the following proposition, originally due to [33], is stated in terms of the optimality indication set $\Pi(S, \bar{x})$.

Proposition 2.4. *The following statements are true:*

(i) *If the constraint functions* g_i, $i \in I_0$ *and* h_j, $j \in J$ *are* $\mathcal{C}^{1,1}$, *then* $(0.5, 1] \subset \Pi(S, \bar{x})$.

(ii) *If* $g_i^{oo}(\bar{x}; u) \leq 0$ *for each* $i \in I_0$ *and* $u \in L_C(\bar{x}) \cap \nabla g_i(\bar{x})^\perp$, *and* $h_j^{oo}(\bar{x}; u) = h_{j\,oo}(\bar{x}; u) = 0$ *for each* $j \in J$ *and* $u \in L_C(\bar{x})$, *then* $[0.5, 1] \subset \Pi(S, \bar{x})$.

(iii) *Assume that* g_i, $i \in I_0$ *and* h_j, $j \in J$ *are* $\mathcal{C}^2$. *If* $u^T\nabla^2 g_i(\bar{x})u \leq 0$ *for each* $i \in I_0$ *and* $u \in L_C(\bar{x}) \cap \nabla g_i(\bar{x})^\perp$, *and* $u^T\nabla^2 h_j(\bar{x})u = 0$ *for each* $j \in J$ *and* $u \in L_C(\bar{x})$, *then* $[0.5, 1] \subset \Pi(S, \bar{x})$.

(iv) *Assume that* $q = 0$. *If* $g_i^{oo}(\bar{x}; u) < 0$ *for each* $i \in I_0$ *and* $u \in (L_C(\bar{x}) \cap \nabla g_i(\bar{x})^\perp)\backslash\{0\}$, *then* $\Pi(S, \bar{x}) = [0, 1]$.

Remark 2.2. When g_i, $i \in I_0$ and h_j, $j \in J$ are $\mathcal{C}^2$, conditions in statement (iii) are satisfied if and only if

$$\{p \in [0, 1] \mid KerD_+S^1(\bar{x}) \subset KerD_+S^p(\bar{x})\} = [0.5, 1]; \tag{2}$$

see Lemma 2.3 of [33]. But in some cases, the left-hand side of (2) is merely a proper subset of $\Pi(S, \bar{x})$; see Example 2.3 where neither the conditions in statement (iii) nor the GCQ at $\bar{x}$ is satisfied.

Example 2.3. In (P), let $n = m = 2$, $q = 0$, $g_1(x) = x_1^2x_2$, $g_2(x) = x_2^2 - x_1$, and $\bar{x} = 0$. Clearly, g_1 and g_2 are $\mathcal{C}^2$. By direct calculation, we have $T_C(\bar{x}) = R_+ \times (-R_+)$, $L_C(\bar{x}) = R_+ \times R$,

$$KerD_+S^p(\bar{x}) = \begin{cases} R_+ \times (-R_+)\backslash\{0\} \times (-R_+) & \text{if } 0 \leq p \leq \frac{1}{2}, \\ R_+ \times R & \text{if } \frac{1}{2} < p \leq 1, \end{cases}$$

and

$$KerDS^p(\bar{x}) = \begin{cases} R_+ \times (-R_+) & \text{if } 0 \le p \le \frac{1}{5}, \\ R_+ \times (-R_+) \cup \{0\} \times R_+ & \text{if } \frac{1}{5} < p \le \frac{1}{3}, \\ R_+ \times R & \text{if } \frac{1}{3} < p \le 1. \end{cases}$$

Thus, the GCQ is invalid at $\bar{x}$ since $T_C(\bar{x})^* = (-R_+) \times R_+ \neq L_C(\bar{x})^* = (-R_+) \times \{0\}$. Moreover, the conditions in statement (iii) of Proposition 2.4 are not satisfied since $u^T \nabla^2 g_2(\bar{x}) u = 2u_2^2 > 0$ for each $u = (0, u_2)^T \in L_C(\bar{x}) \cap \nabla g_2(\bar{x})^\perp$ with $u_2 \neq 0$. On the other hand, we have, by definition,

$$\{p \in [0,1] \mid KerD_+S^1(\bar{x}) \subset KerD_+S^p(\bar{x})\} = (0.5, 1] \subsetneq \Pi(S, \bar{x}) = \left(\frac{1}{5}, 1\right].$$

THEOREM 2.5. *If there exists $p \in \Pi(S, \bar{x})$ such that the l_p penalty function $\mathcal{F}_p$ is exact at $\bar{x}$, then the KKT optimality condition holds at $\bar{x}$.*

Proof. Let $p \in \Pi(S, \bar{x})$ be given. Since the l_p penalty function $\mathcal{F}_p$ is exact at $\bar{x}$, there is $\mu > 0$ such that $\bar{x}$ is a local minimum of $\mathcal{F}_p(x) = f(x) + \mu S^p(x)$ over R^n. By the definition of the contingent derivative, we thus have

$$D\mathcal{F}_p(\bar{x})(d) \subset R_+ \quad \forall\, d \in R^n.$$

Noting that the objective function f is assumed to be continuously differentiable, it follows easily from the sum rule of the contingent derivative (see [1] and [16]) that

$$D\mathcal{F}_p(\bar{x})(d) = \nabla f(\bar{x})^T d + \mu DS^p(\bar{x})(d).$$

Therefore, we have

$$-\nabla f(\bar{x})^T d \le 0 \quad \forall\, d \in KerDS^p(\bar{x}),$$

i.e., $-\nabla f(\bar{x}) \in KerDS^p(\bar{x})^*$. It then follows from $p \in \Pi(S, \bar{x})$ and Lemma 2.1(iii) that $-\nabla f(\bar{x}) \in L_C(\bar{x})^*$, which is equivalent to the validation of the KKT optimality condition at $\bar{x}$ by Farkas' lemma. This completes the proof. ∎

Remark 2.3.

(i) Theorem 2.5 can be applied when one of the following conditions holds:
  (a) GCQ. This is because the GCQ holds at $\bar{x}$ amounts to $0 \in \Pi(S, \bar{x})$, and the l_0 penalty function $\mathcal{F}_0$ is always exact at $\bar{x}$.
  (b) The l_1 penalty function $\mathcal{F}_1$ is exact. This is because $1 \in \Pi(S, \bar{x})$.
  (c) The l_p penalty function $\mathcal{F}_p$ with $0 < p < 1$ is exact, in addition to other conditions specified in Proposition 2.4.

(ii) More importantly, Example 2.4 shows that Theorem 2.5 can also be applied when none of the three conditions in (i) is satisfied.

(iii) In Example 2.5, a class of problems is given to illustrate the further application of Theorem 2.5, in which different cases on the parameters of the problem are considered, i.e., when the KKT optimality condition can be verified using one of the existing CQs, and when this condition can be verified only by Theorem 2.5.

Example 2.4. In (P), let $n = 2$, $m = 3$, $q = 0$, $f(x) = -x_1^3 + x_2$, $g_1(x) = -x_2$, $g_2(x) = x_1^6 + x_2^3$, $g_3(x) = -x_1^2 + x_2^2$, and $\bar{x} = (0,0)^T$. By direct calculation, we have $T_C(\bar{x}) = C = \{\bar{x}\}$,

$$L_C(\bar{x}) = \{u = (u_1, u_2)^T \in R^2 \mid u_2 \ge 0\},$$

and

$$KerDS^{\frac{1}{2}}(\bar{x}) = \{u = (u_1, u_2)^T \in R^2 \mid u_2 \geq 0,\ u_1^2 \geq u_2^2\}.$$

Then, we have

$$KerDS^{\frac{1}{2}}(\bar{x})^* = KerDS^1(\bar{x})^* = L_C(\bar{x})^* = \{(0, v_2)^T \in R^2 \mid v_2 \leq 0\} \subsetneq T_C(\bar{x})^* = R^2.$$

Therefore, the GCQ does not hold at $\bar{x}$, and, by definition, $\frac{1}{2} \in \Pi(S, \bar{x})$. However, we cannot apply Proposition 2.4(iii) to obtain that $\frac{1}{2} \in \Pi(S, \bar{x})$ because $u^T \nabla^2 g_3(\bar{x}) u = 2u_2^2 > 0$ holds for every $u = (0, u_2)^T \in (L_C(\bar{x}) \cap \nabla g_3(\bar{x})^\perp) \backslash \{(0,0)^T\}$. In what follows, we will show that the l_p penalty function is exact at $\bar{x}$ for $p = \frac{1}{2}$ but not for $p > \frac{1}{2}$. Let $\delta \in (0,1)$ and $\tilde{\mu} = \frac{2}{1-\delta^2}$. Clearly, $\tilde{\mu} > 2$. Let $\mu \geq \tilde{\mu}$ and $x \in R^2$ be such that $|x_1| \leq \delta$ and $|x_2| \leq \delta$. We consider the following five cases:

Case 1. $x_1 \leq 0$ and $x_2 \geq 0$. We have

$$\begin{aligned} \mathcal{F}_{0.5}(x) &= -x_1^3 + x_2 + \mu\left((-x_2)_+ + (x_1^6 + x_2^3)_+ + (-x_1^2 + x_2^2)_+\right)^{0.5} \\ &\geq -x_1^3 + x_2 \\ &\geq 0. \end{aligned} \tag{3}$$

Case 2. $x_1 > 0$ and $x_2 \geq 0$. We have from (3)

$$\begin{aligned} \mathcal{F}_{0.5}(x) &\geq -x_1^3 + x_2 + \mu(x_1^6 + x_2^3)^{0.5} \\ &\geq -x_1^3 + \mu(x_1^6)^{0.5} \\ &= (\mu - 1)x_1^3 \\ &\geq 0. \end{aligned}$$

Case 3. $x_1 \leq 0$ and $x_2 < 0$. We have from (3)

$$\begin{aligned} \mathcal{F}_{0.5}(x) &\geq -x_1^3 + x_2 + \mu(-x_2)^{0.5} \\ &\geq x_2 + \mu(-x_2) \\ &= (\mu - 1)(-x_2) \\ &\geq 0. \end{aligned}$$

Case 4. $x_1 > 0$ and $x_2 \leq -x_1^2$. Since $|x_1| \leq \delta < 1$, we have $x_1^3 \leq x_1^2$, and from (3)

$$\begin{aligned} \mathcal{F}_{0.5}(x) &\geq -x_1^3 + x_2 + \mu(-x_2)^{0.5} \\ &\geq -x_1^2 + x_2 + \mu(-x_2)^{0.5} \\ &\geq 2x_2 + \mu(-x_2) \\ &= (\mu - 2)(-x_2) \\ &\geq 0. \end{aligned}$$

Case 5. $x_1 > 0$ and $-x_1^2 < x_2 < 0$. We have $x_1^6 + x_2^3 > 0$, $-x_2 + x_2^3 > 0$, and from (3)

$$\begin{aligned} \mathcal{F}_{0.5}(x) &\geq -x_1^3 + x_2 + \mu(-x_2 + x_1^6 + x_2^3)^{0.5} \\ &\geq -x_1^3 + x_2 + \tfrac{1}{2}\mu((-x_2 + x_2^3)^{0.5} + x_1^3) \\ &= (\tfrac{1}{2}\mu - 1)x_1^3 + (-x_2)(\tfrac{1}{2}\mu(1 - x_2^2) - 1) \\ &\geq (\tfrac{1}{2}\mu - 1)x_1^3 + (-x_2)(\tfrac{1}{2}\mu(1 - \delta^2) - 1) \\ &\geq 0, \end{aligned}$$

where the second inequality follows from Lemma 4.1 in [15].

To show that the l_p penalty function is not exact at $\bar{x}$ when $p > \frac{1}{2}$, we consider a sequence $x_\nu := (x_{1\nu}, 0) \in R^2$ such that $x_{1\nu} \to 0+$. It is easy to check that for any $\mu > 0$ and $p > \frac{1}{2}$, there exists ν_0 such that the condition

$$\mathcal{F}_p(x_\nu) = -x_{1\nu}^3 + \mu x_{1\nu}^{6p} < \mathcal{F}_p(\bar{x}) = 0$$

holds for all $\nu \geq \nu_0$. Thus, Theorem 2.5 can be applied to derive the KKT optimality condition at $\bar{x}$ only with $p = \frac{1}{2}$.

Example 2.5. Let $\bar{x} = 0 \in R^3$ be a local minimizer of the following inequality constrained optimization problem:

$$\begin{aligned} \text{(4)} \qquad & \min && f(x) \\ & \text{s.t.} && g_1(x) := a^T x + a_4 x_3^4 \leq 0, \\ & && g_2(x) := b^T x + b_4 x_3^4 \leq 0, \\ & && g_3(x) := c^T x + c_4 x_3^4 \leq 0, \end{aligned}$$

where $a = (a_1, a_2, a_3)^T$, $b = (b_1, b_2, b_3)^T$, $c = (c_1, c_2, c_3)^T \in R^3$, and a_4, b_4, $c_4 \in R$. Assume that vectors a and b are linearly independent, and that vector c has the following unique representation, for $k_1, k_2 \in R$:

$$c = -k_1 a - k_2 b.$$

If $k_1 < 0$ or $k_2 < 0$, Motzkin's theorem of the alternative ensures that the inequality system

$$a^T x < 0, \quad b^T x < 0, \quad \text{and} \quad c^T x < 0,$$

has a solution. Therefore, the MFCQ (see [20]) holds at $\bar{x}$. If $k_1 \geq 0$, $k_2 \geq 0$, and $k_1 a_4 + k_2 b_4 + c_4 \leq 0$, we will show that the GCQ holds at $\bar{x}$. In fact, by applying Theorem 1.17 of [2] or Theorem 2.5 of [22], it can be shown that the feasible set C of problem (4) has a local error bound at $\bar{x}$ with respect to

$$S(x) := \sum_{i=1}^{3} (g_i(x))_+$$

(see Lemma 2.1(iv)). Then, by Lemma 2.1(iii) and (iv), we have

$$T_C(\bar{x}) = Ker DS^1(\bar{x}) = L_C(\bar{x}),$$

which implies that the GCQ holds at $\bar{x}$.

In what follows, we assume that $k_1 \geq 0$, $k_2 \geq 0$, and $k_1 a_4 + k_2 b_4 + c_4 > 0$. Note that the inequality

$$k_1 g_1(x) + k_2 g_2(x) + g_3(x) \leq 0$$

implies that $x_3 = 0$. Thus, it is easy to check that $x \in C$ if and only if

$$a^T x \leq 0, \quad b^T x \leq 0, \quad c^T x \leq 0, \quad \text{and} \quad x_3 = 0.$$

By definition, we have

$$T_C(\bar{x}) = \left\{ x = (x_1, x_2, 0)^T \in R^3 \,\middle|\, a^T x \leq 0,\ b^T x \leq 0,\ c^T x \leq 0 \right\}$$

and

$$L_C(\bar{x}) = \left\{ x \in R^3 \,\middle|\, a^T x \le 0,\ b^T x \le 0,\ c^T x \le 0 \right\}.$$

By Farkas' lemma, we have

$$T_C(\bar{x})^* = \left\{ v = (v_1, v_2, v_3)^T \in R^3 \,\middle|\, \begin{array}{l} v_1 = y_1 a_1 + y_2 b_1 + y_3 c_1, \\ v_2 = y_1 a_2 + y_2 b_2 + y_3 c_2, \\ y_1, y_2, y_3 \ge 0,\ v_3 \in R, \end{array} \right\}$$

and

$$L_C(\bar{x})^* = \left\{ v = (v_1, v_2, v_3)^T \in R^3 \,\middle|\, \begin{array}{l} v_1 = y_1 a_1 + y_2 b_1 + y_3 c_1, \\ v_2 = y_1 a_2 + y_2 b_2 + y_3 c_2, \\ v_3 = y_1 a_3 + y_2 b_3 + y_3 c_3, \\ y_1, y_2, y_3 \ge 0. \end{array} \right\}$$

Therefore, $T_C(\bar{x})^* \ne L_C(\bar{x})^*$ and the GCQ does not hold at $\bar{x}$. So, in this case, the classical CQs cannot be used to verify the KKT optimality condition.

We will show that Theorem 2.5 is applicable to detect the validity of the KKT optimality condition when $k_1 \ge 0$, $k_2 \ge 0$, and $k_1 a_4 + k_2 b_4 + c_4 > 0$, and the objective function f takes the form

$$f(x) = w^T x + w_4 x_3^2, \tag{5}$$

where $w = (w_1, w_2, w_3)^T = -\rho_1 a - \rho_2 b$ with $\rho_1, \rho_2 \ge 0$, and $w_4 < 0$.

First, we show that the l_p penalty function for problem (4),

$$\mathcal{F}_p(x) = f(x) + \mu (S(x))^p,$$

cannot be exact at $\bar{x}$ when $p > 0.5$. Consider a sequence $x_\nu := (x_{1\nu}, x_{2\nu}, x_{3\nu})^T \in R^3$ such that $x_\nu \to \bar{x}$, $x_{3\nu} \not\equiv 0$, $a^T x_\nu = 0$, and $b^T x_\nu = 0$. It is easy to check that for any $\mu > 0$ and $p > 0.5$, there exists ν_0 such that the inequality

$$\mathcal{F}_p(x_\nu) = w_4 x_{3\nu}^2 + \mu \left[(a_4 x_{3\nu}^4)_+ + (b_4 x_{3\nu}^4)_+ + (c_4 x_{3\nu}^4)_+ \right]^p < 0$$

holds for all $\nu \ge \nu_0$. Thus, the l_p ($p > 0.5$) penalty function for problem (4) is not exact at $\bar{x}$.

Next, we show that the $l_{0.5}$ penalty function is exact at $\bar{x}$. It is easy to see that

$$\begin{aligned} \mathcal{F}_{0.5}(x) :=& [w_4 + (\rho_1 a_4 + \rho_2 b_4) x_3^2] x_3^2 - \rho_1 (a^T x + a_4 x_3^4) - \rho_2 (b^T x + b_4 x_3^4) \\ &+ \mu \sqrt{(a^T x + a_4 x_3^4)_+ + (b^T x + b_4 x_3^4)_+ + (c^T x + c_4 x_3^4)_+}. \end{aligned}$$

Let $\delta = \min\{1, \frac{1}{\|a\| + |a_4|}, \frac{1}{\|b\| + |b_4|}\}$ and

$$\tilde{\mu} = 2 \max \left\{ 2\rho_1, 2\rho_2, \frac{|\rho_1 a_4 + \rho_2 b_4| - w_4}{\sqrt{\min\{\frac{1}{k_1}, \frac{1}{k_2}, 1\} (k_1 a_4 + k_2 b_4 + c_4)}} \right\},$$

where the convention $\frac{1}{0} := \infty$ is used when $k_1 = 0$ or $k_2 = 0$. Let $\mu \ge \tilde{\mu}$ and $\|x\| \le \delta$. By the definition of δ, we have

$$(\rho_1 a_4 + \rho_2 b_4) x_3^2 \ge -|\rho_1 a_4 + \rho_2 b_4| \delta^2 \ge -|\rho_1 a_4 + \rho_2 b_4| \tag{6}$$

and

$$|a^T x + a_4 x_3^4| \leq \|a\|\|x\| + |a_4| x_3^4 \leq \|a\|\delta + |a_4|\delta^4 \leq (\|a\| + |a_4|)\delta \leq 1, \tag{7}$$

and similarly,

$$|b^T x + b_4 x_3^4| \leq 1. \tag{8}$$

Thus, we obtain from (7) and (8)

$$\begin{aligned} &\sqrt{(a^T x + a_4 x_3^4)_+ + (b^T x + b_4 x_3^4)_+ + (c^T x + c_4 x_3^4)_+} \\ &\geq \sqrt{(a^T x + a_4 x_3^4)_+ + (b^T x + b_4 x_3^4)_+} \\ &\geq \tfrac{1}{2}\sqrt{(a^T x + a_4 x_3^4)_+} + \tfrac{1}{2}\sqrt{(b^T x + b_4 x_3^4)_+} \\ &\geq \tfrac{1}{2}(a^T x + a_4 x_3^4)_+ + \tfrac{1}{2}(b^T x + b_4 x_3^4)_+, \end{aligned} \tag{9}$$

where the second inequality follows from Lemma 4.1 in [15]. Since $k_1 a_4 + k_2 b_4 + c_4 > 0$, we have

$$\begin{aligned} &\sqrt{(a^T x + a_4 x_3^4)_+ + (b^T x + b_4 x_3^4)_+ + (c^T x + c_4 x_3^4)_+} \\ &\geq \sqrt{\min\{\tfrac{1}{k_1}, \tfrac{1}{k_2}, 1\}}\sqrt{k_1(a^T x + a_4 x_3^4)_+ + k_2(b^T x + b_4 x_3^4)_+ + (c^T x + c_4 x_3^4)_+} \\ &\geq \sqrt{\min\{\tfrac{1}{k_1}, \tfrac{1}{k_2}, 1\}}\sqrt{[k_1(a^T x + a_4 x_3^4) + k_2(b^T x + b_4 x_3^4) + (c^T x + c_4 x_3^4)]_+} \\ &= \left(\sqrt{\min\{\tfrac{1}{k_1}, \tfrac{1}{k_2}, 1\}}\sqrt{k_1 a_4 + k_2 b_4 + c_4}\right) x_3^2. \end{aligned} \tag{10}$$

In view of the definition of $\tilde{\mu}$, it follows from (6), (9), and (10) that

$$\begin{aligned} \mathcal{F}_{0.5}(x) &\geq (w_4 - |\rho_1 a_4 + \rho_2 b_4|)x_3^2 - \rho_1(a^T x + a_4 x_3^4) - \rho_2(b^T x + b_4 x_3^4) \\ &\quad + \tfrac{\mu}{4}(a^T x + a_4 x_3^4)_+ + \tfrac{\mu}{4}(b^T x + b_4 x_3^4)_+ \\ &\quad + \tfrac{\mu}{2}\left(\sqrt{\min\{\tfrac{1}{k_1}, \tfrac{1}{k_2}, 1\}}\sqrt{k_1 a_4 + k_2 b_4 + c_4}\right) x_3^2 \\ &\geq 0, \end{aligned}$$

which implies that the $l_{0.5}$ penalty function for problem (4) is exact at $\bar{x}$. Therefore, Theorem 2.5 is applicable because $0.5 \in \Pi(S, \bar{x})$, which follows readily from Proposition 2.4(iii).

3. Mathematical programs with complementarity constraints. Throughout this section, let $\bar{x} \in E$ be a local minimizer of (MPCC) where E denotes the feasible set of (MPCC). Moreover, let the index sets α, β, and γ be given as in the introduction and let the active index set I_0 be given as in Lemma 2.1(i). In what follows, we will study strong stationarity and Mordukhovich stationarity of (MPCC).

Before proceeding, we need to present a useful lemma. This lemma has been proved in [34] and [10] using Proposition 1 of [24], where polyhedral multifunctions are shown to be locally upper Lipschitz continuous. An alternative proof will be given in the appendix, with the affine structure of a mathematical program with affine complementarity constraints (MPACC) being carefully exploited.

LEMMA 3.1. *Let $\bar{x} \in E$, where E is now the feasible set of an (*MPACC*) defined by*

$$E = \left\{x \in R^n \mid Ax + a \leq 0, Bx + b \geq 0, Cx + c \geq 0, (Bx + b)^T(Cx + c) = 0\right\},$$

with $A \in R^{r\times n}$, $B, C \in R^{s\times n}$, $a \in R^r$, $b, c \in R^s$. *Then, there exist* $\delta > 0$ *and* $\tau > 0$ *such that, for all* $x \in B_\delta(\bar{x})$,

$$d(x, E) \leq \tau(\|(Ax + a)_+\| + \|\min\{Bx + b, Cx + c\}\|). \tag{11}$$

Let the linearized tangent cone of (MPCC) at $\bar{x}$ be defined by

$$T^{lin}(\bar{x}) = \left\{ u \in R^n \;\middle|\; \begin{array}{ll} \nabla g_i(\bar{x})^T u \leq 0, & i \in I_0, \\ \nabla h_j(\bar{x})^T u = 0, & j \in J, \\ \nabla G_k(\bar{x})^T u = 0, & k \in \alpha, \\ \nabla H_k(\bar{x})^T u = 0, & k \in \gamma, \\ \nabla G_k(\bar{x})^T u \geq 0, & k \in \beta, \\ \nabla H_k(\bar{x})^T u \geq 0, & k \in \beta, \end{array} \right\}$$

and let the (MPCC)-linearized tangent cone of (MPCC) at $\bar{x}$ be defined by

$$T^{lin}_{MPCC}(\bar{x}) = \left\{ u \in R^n \;\middle|\; \begin{array}{ll} \nabla g_i(\bar{x})^T u \leq 0, & i \in I_0, \\ \nabla h_j(\bar{x})^T u = 0, & j \in J, \\ \nabla G_k(\bar{x})^T u = 0, & k \in \alpha, \\ \nabla H_k(\bar{x})^T u = 0, & k \in \gamma, \\ \nabla G_k(\bar{x})^T u \geq 0, & k \in \beta, \\ \nabla H_k(\bar{x})^T u \geq 0, & k \in \beta, \\ (\nabla G_k(\bar{x})^T u)(\nabla H_k(\bar{x})^T u) = 0, & k \in \beta. \end{array} \right\}$$

Clearly, $T^{lin}_{MPCC}(\bar{x}) \subset T^{lin}(\bar{x})$. Following [12], we introduce two cones:

$$\Omega_1 = \left\{ (u, \xi_\beta, \eta_\beta) \in R^{n+2|\beta|} \mid \xi_k \geq 0, \eta_k \geq 0, \xi_k \eta_k = 0 \; \forall k \in \beta \right\}$$

and

$$\Omega_2 = \left\{ (u, \xi_\beta, \eta_\beta) \in R^{n+2|\beta|} \;\middle|\; \begin{array}{ll} \nabla g_i(\bar{x})^T u \leq 0, & i \in I_0, \\ \nabla h_j(\bar{x})^T u = 0, & j \in J, \\ \nabla G_k(\bar{x})^T u = 0, & k \in \alpha, \\ \nabla H_k(\bar{x})^T u = 0, & k \in \gamma, \\ \nabla G_k(\bar{x})^T u - \xi_k = 0, & k \in \beta, \\ \nabla H_k(\bar{x})^T u - \eta_k = 0, & k \in \beta. \end{array} \right\}$$

By virtue of Ω_1 and Ω_2, the following proposition gives some characterizations of strong stationarity and Mordukhovich stationarity.

PROPOSITION 3.2. *The following statements are true:*

(i) $\bar{x}$ *is a strongly stationary point of (MPCC) if and only if*

$$-\nabla f(\bar{x}) \in \{v \in R^n \mid (v, 0, 0) \in \hat{N}_{\Omega_1}(0, 0, 0) + \hat{N}_{\Omega_2}(0, 0, 0)\}.$$

(ii) $\bar{x}$ *is a Mordukhovich stationary point of (MPCC) if and only if*

$$-\nabla f(\bar{x}) \in \{v \in R^n \mid (v, 0, 0) \in N_{\Omega_1}(0, 0, 0) + N_{\Omega_2}(0, 0, 0)\}.$$

(iii) $T^{lin}(\bar{x})^* = \{v \in R^n \mid (v, 0, 0) \in \hat{N}_{\Omega_1}(0, 0, 0) + \hat{N}_{\Omega_2}(0, 0, 0)\}$.

(iv) $T^{lin}_{MPCC}(\bar{x})^* = \{v \in R^n \mid (v, 0, 0) \in \hat{N}_{\Omega_1 \cap \Omega_2}(0, 0, 0)\}$.

(v) $N_{\Omega_1 \cap \Omega_2}(0, 0, 0) \subset N_{\Omega_1}(0, 0, 0) + N_{\Omega_2}(0, 0, 0)$.

Proof. By the formulas of $\hat{N}_{\Omega_1}(0,0,0)$, $\hat{N}_{\Omega_2}(0,0,0)$, $N_{\Omega_1}(0,0,0)$, $N_{\Omega_2}(0,0,0)$ given in Proposition 2.2 of [17], statements (i)–(iii) follow directly from Farkas' lemma and the definitions of strong stationarity and Mordukhovich stationarity. Note that $(u,\xi_\beta,\eta_\beta)\in\Omega_1\cap\Omega_2$ if and only if $u\in T^{lin}_{MPCC}(\bar{x})$ and for every $k\in\beta$, $\xi_k=\nabla G_k(\bar{x})^Tu$, $\eta_k=\nabla H_k(\bar{x})^Tu$, and that $\hat{N}_{\Omega_1\cap\Omega_2}(0,0,0)=T_{\Omega_1\cap\Omega_2}(0,0,0)^*=(\Omega_1\cap\Omega_2)^*$. Thus, $(v,0,0)\in\hat{N}_{\Omega_1\cap\Omega_2}(0,0,0)$ if and only if

$$\langle(v,0,0),(u,\nabla G_\beta(\bar{x})^Tu,H_\beta(\bar{x})^Tu)\rangle=\langle v,u\rangle\le 0\;\forall u\in T^{lin}_{MPCC}(\bar{x}),$$

which amounts to $v\in T^{lin}_{MPCC}(\bar{x})^*$. Thus, statement (iv) holds.

Finally, to show (v), it suffices according to Corollary 4.2 of [14] to show that the set-valued map $M:R^{n+2|\beta|}\rightrightarrows R^{n+2|\beta|}$ defined by

$$M(y)=(\Omega_1-y)\cap\Omega_2 \tag{12}$$

is calm at $(0,0)\in R^{n+2|\beta|}\times R^{n+2|\beta|}$. By Theorem 3.1 of [29], calmness of M at $(0,0)$ amounts to having $\delta>0$ and $\tau>0$ such that, for all $(u,\xi_\beta,\eta_\beta)\in B_\delta(0)\cap\Omega_2$,

$$d((u,\xi_\beta,\eta_\beta),\Omega_1\cap\Omega_2)\le\tau d((u,\xi_\beta,\eta_\beta),\Omega_1), \tag{13}$$

where the distance function d is defined via the l_1 norm. Then, it follows from Lemma 3.1 that there are $\delta>0$ and $\tau>0$ such that, for all $(u,\xi_\beta,\eta_\beta)\in B_\delta(0)\cap\Omega_2$,

$$\begin{aligned}
&d((u,\xi_\beta,\eta_\beta),\Omega_1\cap\Omega_2)\\
&\le\tau\sum_{k\in\beta}|\min\{\xi_k,\eta_k\}|\\
&\le\tau\sum_{k\in\beta}d((\xi_k,\eta_k),\{(a,b)\in R^2\mid a\ge0,b\ge0,ab=0\})\\
&=\tau d((\xi_\beta,\eta_\beta),\{(a,b)\in R^{2|\beta|}\mid a_k\ge0,b_k\ge0,a_kb_k=0\;\forall k\in\beta\})\\
&=\tau d((u,\xi_\beta,\eta_\beta),\Omega_1),
\end{aligned}$$

which gives (13). This completes the proof. ◻

Remark 3.1. According to [13], any $v\in T_E(\bar{x})^*$ corresponds to a smooth objective function f such that $\bar{x}$ is a local minimizer of (MPCC) and $v=-\nabla f(\bar{x})$. Therefore, by statement (ii), the CQ

$$T_E(\bar{x})^*\times\{0\}\times\{0\}\subset N_{\Omega_1}(0,0,0)+N_{\Omega_2}(0,0,0) \tag{14}$$

is the weakest for Mordukhovich stationarity in the sense that it holds if and only if $\bar{x}$ is a Mordukhovich stationary point for every (MPCC) that has the same constraints and the same local minimizer $\bar{x}$ but different objective functions. It follows directly from statements (iv) and (v) and from $\hat{N}_{\Omega_1\cap\Omega_2}(0,0,0)\subset N_{\Omega_1\cap\Omega_2}(0,0,0)$ that

$$T^{lin}_{MPCC}(\bar{x})^*\times\{0\}\times\{0\}\subset N_{\Omega_1}(0,0,0)+N_{\Omega_2}(0,0,0). \tag{15}$$

Inclusions (14) and (15) together imply a well-known result (see [34] and [8]) that $\bar{x}$ is a Mordukhovich stationary point if MPCC − GCQ holds at $\bar{x}$, i.e., $T_E(\bar{x})^*=T^{lin}_{MPCC}(\bar{x})^*$. Since (14) can be true even if $T_E(\bar{x})^*=T^{lin}_{MPCC}(\bar{x})^*$ does not hold, MPCC−GCQ is not the weakest CQ for Mordukhovich stationarity; see Example 3.1.

Remark 3.2. It is interesting to note that the characterizations of strongly stationary and Mordukhovich stationary points can be obtained in such a unified way only by invoking the cones Ω_1 and Ω_2. This fundamental idea is borrowed from [12], where a rather direct proof was given to show that any local minimizer of (MPCC) is a Mordukhovich stationary point under MPCC – GCQ. In [12], statement (v) was obtained by invoking Proposition 1 of [24] to show that the polyhedral set-valued map M defined by (12) is calm at $(0, 0)$, while in our proof, statement (v) follows from Lemma 3.1, which can be proved without using Proposition 1 of [24]; see the appendix for a detailed proof of Lemma 3.1.

Example 3.1. In (MPCC), let $n = 1$, $m = q = 0$, $l = 1$, $G(x) = x$, $H(x) = x^2$, and $\bar{x} = 0$. Then, $T_E(\bar{x}) = \{0\}$, $T^{lin}_{MPCC}(\bar{x}) = T^{lin}(\bar{x}) = R_+$, and

$$\{v \in R \mid (v, 0, 0) \in N_{\Omega_1}(0, 0, 0) + N_{\Omega_2}(0, 0, 0)\} = R.$$

Thus, MPCC – GCQ does not hold at $\bar{x}$, but (14) holds.

Let $\mu > 0$ and $0 \le p \le 1$. Consider the following l_p penalty function $\mathcal{G}_p$ of (MPCC):

$$\mathcal{G}_p(x) = f(x) + \mu U^p(x),$$

where $\mu > 0$ and

$$U^p(x) := \left(S(x) + \sum_{k \in K} [(-G_k(x))_+ + (-H_k(x))_+ + |G_k(x)H_k(x)|] \right)^p.$$

The basic properties of $KerDU^p(\bar{x})$ are summarized in the following lemma.

LEMMA 3.3. *Let* $0 \le p \le 1$. *Then,* $KerDU^p(\bar{x})$ *is a closed cone with the following properties:*

(i) $KerDU^p(\bar{x}) \subset KerDU^{p'}(\bar{x}) \ \ \forall\, p' \in (p, 1]$.
(ii) $KerDU^0(\bar{x}) = T_E(\bar{x})$ *and* $KerDU^1(\bar{x}) = T^{lin}(\bar{x})$.
(iii) $KerDU^{0.5}(\bar{x}) \subset T^{lin}_{MPCC}(\bar{x})$.
(iv) $KerDU^p(\bar{x}) = T_E(\bar{x})$ *if* E *has a local error bound at* $\bar{x}$ *with respect to* U^p.

Proof. Similarly as for Lemma 2.1, we get (i), (ii), and (iv) easily. Now, it remains for us to show (iii). By (i) and (ii), we have

$$KerDU^{0.5}(\bar{x}) \subset KerDU^1(\bar{x}) = T^{lin}(\bar{x}).$$

Therefore, to prove $KerDU^{0.5}(\bar{x}) \subset T^{lin}_{MPCC}(\bar{x})$, it suffices to show that for every $u \in KerDU^{0.5}(\bar{x})$ and $k \in \beta$, $(\nabla G_k(\bar{x})^T u)(\nabla H_k(\bar{x})^T u) = 0$. By the definition of the contingent derivative, for every $u \in KerDU^{0.5}(\bar{x})$, there exist $t^\nu \to 0+$ and $u^\nu \to u$ such that

$$\frac{G_k(\bar{x} + t^\nu u^\nu) H_k(\bar{x} + t^\nu u^\nu)}{(t^\nu)^2} \to 0 \quad \forall\, k \in \beta.$$

Since G_k and H_k are continuously differentiable, we have by the Taylor expansion rule that for all $k \in \beta$,

$$\frac{G_k(\bar{x} + t^\nu u^\nu) H_k(\bar{x} + t^\nu u^\nu)}{(t^\nu)^2} = \left(\nabla G_k(\bar{x})^T u^\nu + \frac{o(\|t^\nu u^\nu\|)}{t^\nu} \right) \left(\nabla H_k(\bar{x})^T u^\nu + \frac{o(\|t^\nu u^\nu\|)}{t^\nu} \right).$$

Thus, we have

$$(\nabla G_k(\bar{x})^T u)(\nabla H_k(\bar{x})^T u) = 0 \quad \forall u \in KerDU^{0.5}(\bar{x}) \quad \forall k \in \beta,$$

which completes the proof. ∎

We now define the strong stationarity indication set of (MPCC) with respect to U and $\bar{x}$, as follows:

$$\Pi_s(U,\bar{x}) := \{p \in [0,1] \mid KerDU^p(\bar{x})^* \subset KerDU^1(\bar{x})^*\}.$$

It follows from Proposition 3.2(iii) and Lemma 3.3(ii) that

$$KerDU^1(\bar{x})^* = T^{lin}(\bar{x})^* = \{v \in R^n \mid (v,0,0) \in \hat{N}_{\Omega_1}(0,0,0) + \hat{N}_{\Omega_2}(0,0,0)\}.$$

Thus,

$$\begin{aligned}&\Pi_s(U,\bar{x})\\ &= \{p \in [0,1] \mid KerDU^p(\bar{x})^* \subset T^{lin}(\bar{x})^*\}\\ &= \{p \in [0,1] \mid KerDU^p(\bar{x})^* \subset \{v \in R^n \mid (v,0,0) \in \hat{N}_{\Omega_1}(0,0,0) + \hat{N}_{\Omega_2}(0,0,0)\}\}.\end{aligned}$$

Furthermore, we define the Mordukhovich stationarity indication set of (MPCC) with respect to U and $\bar{x}$ as follows:

$$\Pi_m(U,\bar{x}) = \{p \in [0,1] \mid KerDU^p(\bar{x})^* \subset \{v \in R^n \mid (v,0,0) \in N_{\Omega_1}(0,0,0)+N_{\Omega_2}(0,0,0)\}\}.$$

According to (15) and the inclusion $T^{lin}(\bar{x})^* \subset T^{lin}_{MPCC}(\bar{x})^*$ due to $T^{lin}_{MPCC}(\bar{x}) \subset T^{lin}(\bar{x})$, we have

$$\Pi_s(U,\bar{x}) \subset \{p \in [0,1] \mid KerDU^p(\bar{x})^* \subset T^{lin}_{MPCC}(\bar{x})^*\} \subset \Pi_m(U,\bar{x}). \tag{16}$$

The second inclusion in (16) may not be held as an equality. For instance, using the data in Example 3.1, we have

$$KerDU^p(\bar{x}) = \begin{cases} R_+ & \text{if } \frac{1}{3} < p \leq 1, \\ \{0\} & \text{if } 0 \leq p \leq \frac{1}{3}, \end{cases}$$

and

$$\Pi_s(U,\bar{x}) = \{p \in [0,1] \mid KerDU^p(\bar{x})^* \subset T^{lin}_{MPCC}(\bar{x})^*\} = \left(\frac{1}{3},1\right] \subsetneq \Pi_m(U,\bar{x}) = [0,1].$$

THEOREM 3.4. *The following statements are true:*

(i) *If there exists some $p \in \Pi_s(U,\bar{x})$ such that the l_p penalty function $\mathcal{G}_p$ is exact at $\bar{x}$, then $\bar{x}$ is a strongly stationary point.*

(ii) *If there exists some $p \in \Pi_m(U,\bar{x})$ such that the l_p penalty function $\mathcal{G}_p$ is exact at $\bar{x}$, then $\bar{x}$ is a Mordukhovich stationary point.*

Proof. Similarly as in Theorem 2.5, we have $-\nabla f(\bar{x}) \in KerDU^p(\bar{x})^*$ whenever the l_p penalty function of $\mathcal{G}_p$ is exact at $\bar{x}$. Now all results follow easily from Proposition 3.2 and the definitions of $\Pi_s(U,\bar{x})$ and $\Pi_m(U,\bar{x})$. ∎

The following proposition characterizes the stationarity indication sets $\Pi_s(U,\bar{x})$ and $\Pi_m(U,\bar{x})$ by virtue of the original data of (MPCC).

PROPOSITION 3.5. *The following statements are true:*

(i) *If the functions $g_i, i \in I_0$, $h_j, j \in J$, G_k and H_k, $k \in K$ are $\mathcal{C}^{1,1}$, then $(0.5,1] \subset \Pi_s(U,\bar{x}) \subset \Pi_m(U,\bar{x})$.*

(ii) *If the functions g_i, $i \in I_0$, h_j, $j \in J$, G_k and H_k, $k \in K$ are $\mathcal{C}^2$, and the conditions*

  (a) $u^T\nabla^2 g_i(\bar{x})u \leq 0$ $\forall i \in I_0$, $\forall u \in T^{lin}_{MPCC}(\bar{x}) \cap \nabla g_i(\bar{x})^{\perp}$,

(b) $u^T\nabla^2 h_j(\bar{x})u = 0$ $\forall j \in J$, $\forall u \in T^{lin}_{MPCC}(\bar{x})$,

(c) $u^T\nabla^2 G_k(\bar{x})u = 0$ $\forall k \in \alpha$, $\forall u \in T^{lin}_{MPCC}(\bar{x})$,

(d) $u^T\nabla^2 H_k(\bar{x})u = 0$ $\forall k \in \gamma$, $\forall u \in T^{lin}_{MPCC}(\bar{x})$,

(e) $u^T\nabla^2 G_k(\bar{x})u \geq 0$ $\forall k \in \beta$, $\forall u \in (T^{lin}_{MPCC}(\bar{x}) \cap \nabla G_k(\bar{x})^\perp)\backslash\nabla H_k(\bar{x})^\perp$,

(f) $u^T\nabla^2 H_k(\bar{x})u \geq 0$ $\forall k \in \beta$, $\forall u \in (T^{lin}_{MPCC}(\bar{x}) \cap \nabla H_k(\bar{x})^\perp)\backslash\nabla G_k(\bar{x})^\perp$,

(g) $u^T\nabla^2 G_k(\bar{x})u \geq 0$ *and* $u^T\nabla^2 H_k(\bar{x})u \geq 0$ $\forall k \in \beta$, $\forall u \in T^{lin}_{MPCC}(\bar{x}) \cap \nabla G_k(\bar{x})^\perp \cap \nabla H_k(\bar{x})^\perp$

are satisfied, then $[0.5, 1] \subset \Pi_m(U, \bar{x})$.

Proof. Since U is defined in the same way as S in the sense that the complementarity constraints $G_k(x) \geq 0$, $H_k(x) \geq 0$, and $G_k(x)H_k(x) = 0$ are regarded as general inequality and equality constraints, statement (i) follows readily from Proposition 2.4(i), the definition of $\Pi_s(U, \bar{x})$, and (16). Now, we will show statement (ii) by proving that conditions (a)–(g) are satisfied if and only if $KerD_+U^{0.5}(\bar{x}) = KerDU^{0.5}(\bar{x}) = T^{lin}_{MPCC}(\bar{x})$, which implies that $0.5 \in \{p \in [0,1] \mid kerDU^p(\bar{x})^* \subset T^{lin}_{MPCC}(\bar{x})^*\} \subset \Pi_m(U, \bar{x})$ and hence $[0.5, 1] \subset \Pi_m(U, \bar{x})$. To start, it is easy to check that for each $0 < p \leq 1$,

$$KerD_+U^p(\bar{x}) = \left\{ u \in R^n \;\middle|\; \begin{array}{ll} \limsup\limits_{t\to 0+} \dfrac{g_i(\bar{x}+tu)}{t^{1/p}} \leq 0, & i \in I_0, \\ \lim\limits_{t\to 0+} \dfrac{h_j(\bar{x}+tu)}{t^{1/p}} = 0, & j \in J, \\ \lim\limits_{t\to 0+} \dfrac{G_k(\bar{x}+tu)}{t^{1/p}} = 0, & k \in \alpha, \\ \lim\limits_{t\to 0+} \dfrac{H_k(\bar{x}+tu)}{t^{1/p}} = 0, & k \in \gamma, \\ \liminf\limits_{t\to 0+} \dfrac{G_k(\bar{x}+tu)}{t^{1/p}} \geq 0, & k \in \beta, \\ \liminf\limits_{t\to 0+} \dfrac{H_k(\bar{x}+tu)}{t^{1/p}} \geq 0, & k \in \beta, \\ \lim\limits_{t\to 0+} \dfrac{G_k(\bar{x}+tu)H_k(\bar{x}+tu)}{t^{1/p}} = 0, & k \in \beta. \end{array} \right\} \tag{17}$$

Since $KerDU^{0.5}(\bar{x}) \subset T^{lin}_{MPCC}(\bar{x})$ according to Lemma 3.3(iii), and $KerD_+U^{0.5}(\bar{x}) \subset KerDU^{0.5}(\bar{x})$ due to the definitions of the Dini upper directional derivative and the contingent derivative, we have

$$KerD_+U^{0.5}(\bar{x}) = KerDU^{0.5}(\bar{x}) = T^{lin}_{MPCC}(\bar{x}) \Longleftrightarrow T^{lin}_{MPCC}(\bar{x}) \subset KerD_+U^{0.5}(\bar{x}).$$

As a result, it remains for us to show the equivalence of $T^{lin}_{MPCC}(\bar{x}) \subset KerD_+U^{0.5}(\bar{x})$ with conditions (a)–(g). This can be done by noticing (17) when $p = 0.5$ and the following limits for $\mathcal{C}^2$ functions $\varphi : R^n \to R$ and $\psi : R^n \to R$:

$$\frac{\varphi(x+tu)}{t^2} = \frac{\nabla\varphi(x)^T u}{t} + \frac{1}{2}u^T\nabla^2\varphi(x)u + \frac{o(t^2)}{t^2} \to -\infty \text{ as } t \to 0+, \tag{18}$$

whenever $\varphi(x) = 0$ and $\nabla\varphi(x)^T u < 0$;

$$\frac{\varphi(x+tu)}{t^2} = \frac{1}{2}u^T\nabla^2\varphi(x)u + \frac{o(t^2)}{t^2} \to \frac{1}{2}u^T\nabla^2\varphi(x)u \;\text{ as } t \to 0+, \tag{19}$$

whenever $\varphi(x) = 0$ and $\nabla\varphi(x)^T u = 0$;

$$\frac{\varphi(x+tu)\psi(x+tu)}{t^2} = \left(\frac{1}{2}u^T\nabla^2\varphi(x)u + \frac{o(t^2)}{t^2}\right)\psi(x+tu) \to 0 \text{ as } t \to 0+, \tag{20}$$

whenever $\varphi(x) = \psi(x) = 0$ and $\nabla\varphi(x)^T u = 0$. This completes the proof. ∎

Let $\mu > 0$ and $0 \le p \le 1$. Consider the following l_p penalty function $\mathcal{H}_p$ of (MPCC):

$$\mathcal{H}_p(x) = f(x) + \mu V^p(x),$$

where $\mu > 0$ and

$$V^p(x) := \left(S(x) + \sum_{k\in K} |\phi_{min}(G_k(x), H_k(x))|\right)^p,$$

with $\phi_{min}(a,b) := \min\{a,b\}$ being an NCP function. The basic properties of $KerDV^p(\bar{x})$ are summarized in the following lemma.

LEMMA 3.6. *Let $0 \le p \le 1$. Then, $KerDV^p(\bar{x})$ is a closed cone with the following properties:*

(i) $KerDV^p(\bar{x}) \subset KerDV^{p'}(\bar{x})\ \forall\, p' \in (p,1]$.

(ii) $KerDV^0(\bar{x}) = T_E(\bar{x})$ *and* $KerDV^1(\bar{x}) = T^{lin}_{MPCC}(\bar{x})$.

(iii) $KerDV^p(\bar{x}) = T_E(\bar{x})$ *if E has a local error bound at $\bar{x}$ with respect to V^p.*

Proof. All results except for $KerDV^1(\bar{x}) = T^{lin}_{MPCC}(\bar{x})$ can be obtained similarly as in Lemmas 2.1 or 3.3. Now it remains for us to show $KerDV^1(\bar{x}) = T^{lin}_{MPCC}(\bar{x})$. For every $u \in KerDV^1(\bar{x})$, by the definition of the contingent derivative, there exist $t^\nu \to 0+$ and $u^\nu \to u$ such that

$$\frac{\max\{g_i(\bar{x}+t^\nu u^\nu), 0\}}{t^\nu} \to 0 \quad \forall i \in I_0, \tag{21}$$

$$\frac{h_j(\bar{x}+t^\nu u^\nu)}{t^\nu} \to 0 \quad \forall j \in J, \tag{22}$$

$$\frac{G_k(\bar{x}+t^\nu u^\nu)}{t^\nu} \to 0 \quad \forall k \in \alpha, \tag{23}$$

$$\frac{H_k(\bar{x}+t^\nu u^\nu)}{t^\nu} \to 0 \quad \forall k \in \gamma, \tag{24}$$

and

$$\frac{\min\{G_k(\bar{x}+t^\nu u^\nu), H_k(\bar{x}+t^\nu u^\nu)\}}{t^\nu} \to 0 \quad \forall k \in \beta. \tag{25}$$

By applying the Taylor expansion rule of continuously differentiable functions to (21)–(25) and noticing that $\min\{\nabla G_k(\bar{x})^T u, \nabla H_k(\bar{x})^T u\} = 0$ if and only if

$$\nabla G_k(\bar{x})^T u \ge 0 \quad \nabla H_k(\bar{x})^T u \ge 0 \quad (G_k(\bar{x})^T u)(H_k(\bar{x})^T u) = 0,$$

we can easily get $KerDV^1(\bar{x}) \subset T^{lin}_{MPCC}(\bar{x})$. Letting $t^\nu \to 0+$ and $u^\nu \equiv u$ with $u \in T^{lin}_{MPCC}(\bar{x})$, and again applying the Taylor expansion rule for continuously differentiable functions, we will get (21)–(25), which together imply that $T^{lin}_{MPCC}(\bar{x}) \subset KerDV^1(\bar{x})$. This completes the proof. ∎

Now we define the Mordukhovich stationarity indication set of (MPCC) with respect to V and $\bar{x}$ by

$$\Pi_m(V,\bar{x}) := \{p \in [0,1] \mid KerDV^p(\bar{x})^* \subset \{v \in R^n \mid (v,0,0) \in N_{\Omega_1}(0,0,0)+N_{\Omega_2}(0,0,0)\}\}.$$

By (15) and Lemma 3.6(ii), we have the following relation:

$$1 \in \{p \in [0,1] \mid KerDV^p(\bar{x})^* \subset T^{lin}_{MPCC}(\bar{x})^*\} \subset \Pi_m(V,\bar{x}).$$

THEOREM 3.7. *If there exists $p \in \Pi_m(V,\bar{x})$ such that the l_p penalty function $\mathcal{H}_p$ is exact at $\bar{x}$, then $\bar{x}$ is a Mordukhovich stationary point.*

Proof. The proof is similar to that of Theorem 3.4. ∎

Remark 3.3. Theorem 3.7 is still valid if the NCP function ϕ_{min} in V is replaced by the Fischer–Burmeister function

$$\phi_{FB}(a,b) = a + b - \sqrt{a^2+b^2},$$

since it follows from Lemma 3.1 of [30] that for all $a, b \in R$,

$$(2-\sqrt{2})|\phi_{min}(a,b)| \le |\phi_{FB}(a,b)| \le (2+\sqrt{2})|\phi_{min}(a,b)|.$$

The following proposition characterizes $\Pi_m(V,\bar{x})$ by virtue of the original data of (MPCC).

PROPOSITION 3.8. *The following statements are true:*

(i) *If the functions $g_i, i \in I_0$, $h_j, j \in J$, G_k and H_k, $k \in K$ are $\mathcal{C}^{1,1}$, then*

$$(0.5,1] \subset \{p \in [0,1] \mid KerDV^p(\bar{x}) = T^{lin}_{MPCC}(\bar{x})\} \subset \Pi_m(V,\bar{x}).$$

(ii) *If the functions g_i, $i \in I_0$, h_j, $j \in J$, G_k and H_k, $k \in K$ are $\mathcal{C}^2$, and conditions* (a)–(d) *in Proposition* 3.5 *and the conditions*

(e′) $u^T\nabla^2 G_k(\bar{x})u = 0$ $\forall k \in \beta$, $\forall u \in (T^{lin}_{MPCC}(\bar{x}) \cap \nabla G_k(\bar{x})^\perp)\backslash \nabla H_k(\bar{x})^\perp$,

(f′) $u^T\nabla^2 H_k(\bar{x})u = 0$ $\forall k \in \beta$, $\forall u \in (T^{lin}_{MPCC}(\bar{x}) \cap \nabla H_k(\bar{x})^\perp)\backslash \nabla G_k(\bar{x})^\perp$,

(g′) $\phi_{min}(u^T\nabla^2 G_k(\bar{x})u, u^T\nabla^2 H_k(\bar{x})u) = 0$ $\forall k \in \beta$, $\forall u \in T^{lin}_{MPCC}(\bar{x}) \cap \nabla G_k(\bar{x})^\perp \cap \nabla H_k(\bar{x})^\perp$

are satisfied, then

$$[0.5,1] \subset \{p \in [0,1] \mid KerDV^p(\bar{x}) = T^{lin}_{MPCC}(\bar{x})\} \subset \Pi_m(V,\bar{x}).$$

Proof. By (15), the inclusion

$$\{p \in [0,1] \mid KerDV^p(\bar{x}) = T^{lin}_{MPCC}(\bar{x})\} \subset \Pi_m(V,\bar{x})$$

holds automatically. Now we will prove the rest of the results. To start, it is easy to check that for every $0 < p \le 1$,

(26)

$$KerD_+V^p(\bar{x}) = \left\{ u \in R^n \;\middle|\; \begin{array}{ll} \limsup\limits_{t\to 0+} \dfrac{g_i(\bar{x}+tu)}{t^{1/p}} \le 0, & i \in I_0, \\ \lim\limits_{t\to 0+} \dfrac{h_j(\bar{x}+tu)}{t^{1/p}} = 0, & j \in J, \\ \lim\limits_{t\to 0+} \dfrac{G_k(\bar{x}+tu)}{t^{1/p}} = 0, & k \in \alpha, \\ \lim\limits_{t\to 0+} \dfrac{H_k(\bar{x}+tu)}{t^{1/p}} = 0, & k \in \gamma, \\ \lim\limits_{t\to 0+} \dfrac{\min\{G_k(\bar{x}+tu), H_k(\bar{x}+tu)\}}{t^{1/p}} = 0, & k \in \beta. \end{array} \right\}$$

First, we will show (i). Let $0.5 < p \leq 1$ be arbitrarily fixed. By definition, we have $KerD_+V^p(\bar{x}) \subset KerDV^p(\bar{x})$. According to Lemma 3.6(i) and (ii), we have $KerDV^p(\bar{x}) \subset T^{lin}_{MPCC}(\bar{x})$. Thus, to prove (i), it suffices to show $T^{lin}_{MPCC}(\bar{x}) \subset KerD_+V^p(\bar{x})$, which implies that $KerD_+V^p(\bar{x}) = KerDV^p(\bar{x}) = T^{lin}_{MPCC}(\bar{x})$. Following the proof of Proposition 2.4(i) (see [33]), it now remains to show that, for every $k \in \beta$,

$$\lim_{t \to 0+} \frac{\min\{G_k(\bar{x}+tu), H_k(\bar{x}+tu)\}}{t^{1/p}} = 0. \tag{27}$$

By a generalized Taylor expansion rule (see [7]), we have, for every $k \in \beta$,

$$\begin{aligned} &\frac{\min\{G_k(\bar{x}+tu), H_k(\bar{x}+tu)\}}{t^{\frac{1}{p}}} \\ &\leq t^{2-\frac{1}{p}} \min\left\{\frac{\nabla G_k(\bar{x})^T u}{t} + \frac{1}{2}G_k^{oo}(\bar{x}+t\theta u; u), \frac{\nabla H_k(\bar{x})^T u}{t} + \frac{1}{2}H_k^{oo}(\bar{x}+t\omega u; u)\right\}, \end{aligned} \tag{28}$$

where $0 < \theta < 1$ and $0 < \omega < 1$. Note that $x \to G_k^{oo}(x;u)$ and $x \to H_k^{oo}(x;u)$ are two upper semicontinuous functions with $G_k^{oo}(x;u)$ and $H_k^{oo}(x;u)$ finite when G_k and H_k are $\mathcal{C}^{1,1}$, and that $u \in T^{lin}_{MPCC}(\bar{x})$ implies $\min\{\nabla G_k(\bar{x})^T u, \nabla H_k(\bar{x})^T u\} = 0$. The min term on the right-hand side of (28) must be finite for sufficiently small $t > 0$. Since $t^{2-\frac{1}{p}} \to 0$ when $t \to 0+$, it then follows from (28) that (27) holds. Thus, we have obtained (i).

Next, we will show (ii). According to the definitions of the Dini upper directional derivative and the contingent derivative, and Lemma 3.6(i) and (ii), we have

$$KerD_+V^{0.5}(\bar{x}) = KerDV^{0.5}(\bar{x}) = T^{lin}_{MPCC}(\bar{x}) \Longleftrightarrow T^{lin}_{MPCC}(\bar{x}) \subset KerD_+V^{0.5}(\bar{x}).$$

Thus, it remains for us to show the equivalence of $T^{lin}_{MPCC}(\bar{x}) \subset KerD_+V^{0.5}(\bar{x})$ with conditions (a)–(d) and (e′)–(g′). To that end, we need not only limits in (18), (19), and (20), but some more limits for $\mathcal{C}^2$ functions $\varphi : R^n \to R$ and $\psi : R^n \to R$:

$$\begin{aligned} &\frac{\min\{\varphi(x+tu), \psi(x+tu)\}}{t^2} \\ &= \min\left\{\frac{1}{2}u^T\nabla^2\varphi(x)u + \frac{o(t^2)}{t^2}, \frac{\nabla\psi(x)^T u}{t} + \frac{1}{2}u^T\nabla^2\psi(x)u + \frac{o(t^2)}{t^2}\right\} \\ &\to \frac{1}{2}u^T\nabla^2\varphi(x)u \text{ as } t \to 0+, \end{aligned} \tag{29}$$

whenever $\varphi(x) = \psi(x) = 0$, $\nabla\varphi(x)^T u = 0$, and $\nabla\psi(x)^T u > 0$;

$$\begin{aligned} &\frac{\min\{\varphi(x+tu), \psi(x+tu)\}}{t^2} \\ &= \min\left\{\frac{1}{2}u^T\nabla^2\varphi(x)u + \frac{o(t^2)}{t^2}, \frac{1}{2}u^T\nabla^2\psi(x)u + \frac{o(t^2)}{t^2}\right\} \\ &\to \min\left\{\frac{1}{2}u^T\nabla^2\varphi(x)u, \frac{1}{2}u^T\nabla^2\psi(x)u\right\} \quad \text{as } t \to 0+, \end{aligned} \tag{30}$$

whenever $\varphi(x) = \psi(x) = 0$ and $\nabla\varphi(x)^T u = \nabla\psi(x)^T u = 0$. In view of (26) when $p = 0.5$, and the limits in (18), (19), (20), (29), and (30), the equivalence of $T^{lin}_{MPCC}(\bar{x}) \subset KerD_+V^{0.5}(\bar{x})$ with conditions (a)–(d) and (e′)–(g′) follows easily and deserves no more detailed proof. This completes the proof. ∎

The following lemma and proposition are helpful for establishing some relationships between penalty functions $\mathcal{G}_p$ and $\mathcal{H}_p$, and between the various stationarity indication sets defined previously.

LEMMA 3.9. *Let $a, b \in R$, $\sigma_1 \geq \max\{a+1, b+1, 2-a, 2-b\}$, and $\sigma_2 \geq \max\{\sqrt{|a|}, \sqrt{|b|}, 1\}$. Then,*

$$\frac{(-a)_+ + (-b)_+ + |ab|}{\sigma_1} \leq |\min\{a,b\}| \leq \sigma_2 \sqrt{(-a)_+ + (-b)_+ + |ab|}. \tag{31}$$

Proof. Noting that a and b are symmetrical, we need to consider only three cases: (i) $0 \leq b \leq a$, (ii) $b < 0 < a$, (iii) $b \leq a \leq 0$. For case (i), we have

$$(-a)_+ + (-b)_+ + |ab| = ab \leq (a+1)b \leq \sigma_1 b = \sigma_1 |\min\{a,b\}|$$

and

$$|\min\{a,b\}| = b = \sqrt{b^2} \leq \sqrt{ab} \leq \sigma_2 \sqrt{(-a)_+ + (-b)_+ + |ab|}.$$

For case (ii), we have

$$(-a)_+ + (-b)_+ + |ab| = -b(a+1) = (a+1)|\min\{a,b\}| \leq \sigma_1 |\min\{a,b\}|$$

and

$$|\min\{a,b\}| = -b \leq -b\sqrt{a+1} = \sqrt{-b}\sqrt{-b(a+1)} \leq \sigma_2 \sqrt{(-a)_+ + (-b)_+ + |ab|}.$$

For case (iii), we have

$$(-a)_+ + (-b)_+ + |ab| = -a - b + ab \leq -2b + ab = (2-a)|\min\{a,b\}| \leq \sigma_1 |\min\{a,b\}|$$

and

$$|\min\{a,b\}| = -b \leq -b\sqrt{-a+1} \leq \sqrt{-b}\sqrt{-a-b+ab} \leq \sigma_2 \sqrt{(-a)_+ + (-b)_+ + |ab|}.$$

This completes the proof. ∎

PROPOSITION 3.10. *Let $\delta > 0$ and $y \in R^n$. Then, there exist $\theta > 0$ and $\eta > 0$ such that*

$$\frac{1}{\theta} U(x) \leq V(x) \leq \eta \sqrt{U(x)} \quad \forall x \in B_\delta(y), \tag{32}$$

which implies that

$$KerDU^{\frac{p}{2}}(\bar{x}) \subset KerDV^p(\bar{x}) \subset KerDU^p(\bar{x}), \tag{33}$$

where $0 \leq p \leq 1$.

Proof. We need only show (32) since (33) follows readily from (32). Let $\theta = \max_{k \in K} \theta_k$, where

$$\theta_k = \max_{x \in B_\delta(y)} \max\{G_k(x)+1, H_k(x)+1, 2-G_k(x), 2-H_k(x)\}.$$

Clearly, $\theta \geq \frac{3}{2}$. By Lemma 3.9 and the definitions of U and V, we have

$$U(x) \leq \theta V(x) \quad \forall x \in B_\delta(y),$$

which gives the first inequality in (32). Now, let $\eta = (|K|+1)\tilde{\eta}$, where $\tilde{\eta} = \max\{\max_{k\in K}\eta_k, S_{max}\}$, $\eta_k = \max_{x\in B_\delta(y)}\max\{\sqrt{|G_k(x)|}, \sqrt{|H_k(x)|}, 1\}$, and $S_{max} = \max_{x\in B_\delta(y)}\sqrt{S(x)}$. By Lemma 3.9 and the definition of η_k, we have for each $x \in B_\delta(y)$ and each $k \in K$

$$|\min\{G_k(x), H_k(x)\}| \le \eta_k\sqrt{(-G_k(x))_+ + (-H_k(x))_+ + |G_k(x)H_k(x)|}. \tag{34}$$

By the definition of S_{max}, we have for each $x \in B_\delta(y)$

$$S(x) \le S_{max}\sqrt{S(x)}. \tag{35}$$

Then it follows from (34), (35), and the definitions of U and V that, for each $x \in B_\delta(y)$,

$$\begin{aligned} V(x) &\le \tilde{\eta}\left\{\sqrt{S(x)} + \sum_{k\in K}\sqrt{(-G_k(x))_+ + (-H_k(x))_+ + |G_k(x)H_k(x)|}\right\} \\ &\le (|K|+1)\tilde{\eta}\sqrt{S(x) + \sum_{k\in K}[(-G_k(x))_+ + (-H_k(x))_+ + |G_k(x)H_k(x)|]} \\ &= \eta\sqrt{U(x)}, \end{aligned}$$

where the second inequality follows from Lemma 4.1 in [15]. Therefore, we have shown that (32) holds. This completes the proof. □

Remark 3.4. By (32), $\mathcal{H}_p$ is exact at $\bar{x}$ if $\mathcal{G}_p$ is exact at $\bar{x}$, and $\mathcal{G}_{\frac{p}{2}}$ is exact at $\bar{x}$ if $\mathcal{H}_p$ is exact at $\bar{x}$. Due to (33) and the definitions of the stationarity indication sets, we have

$$(2\Pi_m(U,\bar{x}) \cap [0,1]) \subset \Pi_m(V,\bar{x}) \subset \Pi_m(U,\bar{x}).$$

Remark 3.5. Let all data be given as in Lemma 3.1, and in addition let $\bar{x} \in E$ be any local minimizer of (MPACC). By Lemma 3.1 and Proposition 3.10, there exist $\delta > 0$ and $\tau > 0$ such that for all $x \in B_\delta(\bar{x})$,

(36)
$$d(x,E) \le \tau\sqrt{(\|(Ax+a)_+\| + \|(-Bx-b)_+\| + \|(-Cx-c)_+\| + \|(Bx+b)^T(Cx+c)\|}.$$

Clearly, (36) implies that the $l_{0.5}$ penalty function $\mathcal{G}_{0.5}$ is exact at $\bar{x}$, and (11) implies that the l_1 penalty function $\mathcal{H}_1$ is exact at $\bar{x}$. It follows easily from Propositions 3.5(ii) and 3.8(ii) that for (MPACC)

$$0.5 \in \Pi_m(U,\bar{x}) \cap \Pi_m(V,\bar{x}).$$

Therefore, both Theorems 3.4 and 3.7 can be applied to deduce that any local minimizer of (MPACC) is a Mordukhovich stationary point. This result has been revealed in [34] and [12].

Appendix. Proof of Lemma 3.1.

Let $\psi(x) = \max\{\|(Ax+a)_+\|_\infty, \|\min\{Bx+b, Cx+c\}\|_\infty\}$. Since all norms in finite spaces are equivalent, it is easy to see that (11) amounts to

$$d(x,E) \le \tau\psi(x). \tag{37}$$

It is rather straightforward to reformulate ψ as

$$\psi(x) = \max_{1\le j\le r, 1\le i\le s}\{\psi_{1j}(x), \psi_{2i}(x), \psi_{3i}(x), \psi_{4i}(x)\},$$

where $\psi_{1j}(x) = A_j x + a_j$, $\psi_{2i}(x) = -B_i x - b_i$, $\psi_{3i}(x) = -C_i x - c_i$, $\psi_{4i}(x) = \min\{B_i x + b_i, C_i x + c_i\}$, and A_j, B_i, C_i are rows of A, B, C, respectively. Clearly, these functions are all directionally differentiable. Therefore, ψ is also directionally differentiable and for any $x, u \in R^n$,

$$\psi'(x; u) = \max_{1 \le k \le 4, i \in I_k(x)} \psi'_{ki}(x; u),$$

where

$$I_1(x) = \{j \mid 1 \le j \le r, \psi_{1j}(x) = \psi(x)\}$$

and

$$I_k(x) = \{i \mid 1 \le i \le s, \psi_{ki}(x) = \psi(x)\} \quad \forall\, 2 \le k \le 4.$$

To show (37), it suffices according to Theorem 1.17 of [2] to show the existence of $\delta > 0$ and $\mu > 0$ such that for all $x \in B_\delta(\bar{x}) \backslash E$, there exists $u_x \in R^n$ with $\|u_x\| = 1$ satisfying

$$\psi'(x; u_x) \le -\mu,$$

or equivalently,

$$\psi'_{ki}(x; u_x) \le -\mu \quad \forall\, 1 \le k \le 4,\ i \in I_k(x). \tag{38}$$

Fix arbitrarily $\delta > 0$ such that $I_k(x) \subset I_k(\bar{x})$ for all $x \in B_\delta(\bar{x})$ and $1 \le k \le 4$. Since $I_k(\bar{x})$ is finite for any $1 \le k \le 4$, there must exist a finite subset T of $B_\delta(\bar{x}) \backslash E$ such that each $x \in B_\delta(\bar{x}) \backslash E$ corresponds to a unique $y \in T$ such that

$$I_k(x) = I_k(y) \quad \forall\, 1 \le k \le 4.$$

Let $x \in B_\delta(\bar{x}) \backslash E$, $1 \le k \le 4$ and let $i \in I_k(x)$ be fixed. Then, there is a unique $y \in T$ such that $I_k(x) = I_k(y)$ for all $1 \le k \le 4$. Moreover, we have $\psi_{ki}(x), \psi_{ki}(y) > 0$ and $\psi_{ki}(\bar{x}) = 0$, since $i \in I_k(x) = I_k(y)$ and $\psi(x) > 0$. Set

$$u_x = \frac{\bar{x} - y}{\|\bar{x} - y\|}.$$

Clearly, $\|u_x\| = 1$. In the rest of the proof, we will show that there is a positive constant $\mu(y, k, i)$ such that

$$\psi'_{ki}(x; u_x) \le -\mu(y, k, i), \tag{39}$$

which implies (38) because there are only finitely many such positive constants $\mu(y, k, i)$. Before proceeding to show (39), we assume without loss of generality that $B\bar{x} + b = C\bar{x} + c = 0$. In fact, if $B_i\bar{x} + b_i > C_i\bar{x} + c_i = 0$ for some $1 \le i \le s$, then the complementarity constraint $B_i x + b_i \ge 0$, $C_i x + c_i \ge 0$, $(B_i x + b_i)(C_i x + c_i) = 0$ or the constraint $\min\{B_i x + b_i, C_i x + c_i\} = 0$ is equivalent locally around $\bar{x}$ to two inequalities $C_i x + c \le 0$ and $-(C_i x + c) \le 0$, which can be treated in the same way as we will treat inequalities $A_j x + a_j \le 0$. In what follows, we will show (39) by considering four different values of k. When $k = 1$, we have

$$\psi'_{ki}(x; u_x) = A_i \frac{\bar{x} - y}{\|\bar{x} - y\|} = \frac{-a_i - A_i y}{\|\bar{x} - y\|} = \frac{-\psi_{ki}(y)}{\|\bar{x} - y\|} =: -\mu(y, k, i) < 0,$$

which shows (39). When $k = 2$, we have

$$\psi'_{ki}(x; u_x) = -B_i \frac{\bar{x} - y}{\|\bar{x} - y\|} = \frac{b_i + B_i y}{\|\bar{x} - y\|} = \frac{-\psi_{ki}(y)}{\|\bar{x} - y\|} =: -\mu(y, k, i) < 0,$$

which shows (39). When $k = 3$, (39) can be obtained in the same way as in the case when $k = 2$. When $k = 4$, we have $B_i\bar{x} + b_i = C_i\bar{x} + c_i = 0$ by assumption. If $C_i x + c_i > B_i x + b_i > 0$, we have

$$\psi'_{ki}(x; u_x) = B_i \frac{\bar{x} - y}{\|\bar{x} - y\|} = \frac{-b_i - B_i y}{\|\bar{x} - y\|} \leq \frac{-\psi_{ki}(y)}{\|\bar{x} - y\|} =: -\mu(y, k, i) < 0,$$

which shows (39). If $B_i x + b_i > C_i x + c_i > 0$, (39) can be obtained in the same way as in the case when $C_i x + c_i > B_i x + b_i > 0$. If $B_i x + b_i = C_i x + c_i > 0$, we have

$$\psi'_{ki}(x; u_x) = \min\{B_i \frac{\bar{x} - y}{\|\bar{x} - y\|}, C_i \frac{\bar{x} - y}{\|\bar{x} - y\|}\} = \frac{-\psi_{ki}(y)}{\|\bar{x} - y\|} =: -\mu(y, k, i) < 0,$$

which shows (39). This completes the proof. ∎

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