Dielectric response of graded composites having general power-law-graded cylindrical inclusions

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The dielectric response of graded composites having general power-law-graded cylindrical inclusions under a uniform applied electric field is investigated. The dielectric profile of the cylindrical inclusions is modeled by the equation $\varepsilon_i(r)=c(b+r)^k$ (where r is the radius of the cylindrical inclusions and c, b and k are parameters). Analytical solutions for the local electrical potentials are derived in terms of hypergeometric functions and the effective dielectric response of the graded composites is predicted in the dilute limit. Moreover, for a simple power-law dielectric profile $\varepsilon_i(r)=cr^k$ and a linear dielectric profile $\varepsilon_i(r)=c(b+r)$, analytical expressions of the electrical potentials and the effective dielectric response are derived exactly from our results by taking the limits $b\rightarrow 0$ and $k\rightarrow 1$, respectively. For a higher concentration of inclusions, the effective dielectric response is estimated by an effective-medium approximation. In addition, we have discussed the effective response of graded cylindrical composites with a more complex dielectric profile of inclusion, $\varepsilon_i(r)=c(b+r)^k e^{\beta r}$. *© 2005 American Institute of Physics*. [DOI: 10.1063/1.1947388]

I. INTRODUCTION

The transport properties of composites have attracted much attention because of their applications to engineering.[1–3] Recently, as a kind of inhomogeneous composites, the properties of graded composites are also investigated since they have many advantages over the traditional homogeneous composite materials.[4–12] In the laboratory, graded composites can be tailor-made for various specific needs by changing their microstructure or the spatial distribution and/or the physical properties of the inclusions. For example, the mechanical properties of a graded composite can be designed by varying their values along the radial direction of the cylindrical or spherical inclusions. As a result, the graded composite can have better mechanical properties in bonding strength, toughness, or wear resistance. Using different dielectric profiles of the inclusions, the distributions of local electric potential and electric field induced by an external electric field may result in some interesting optical properties.[13] In fact, more parameters in the grading profile offer more freedom in the control of the effective material properties. For instance, simple power-law, linear, and exponential profiles play different roles in controlling the overall properties of the graded composites.[14–16] In the science of functionally graded materials, because the main objective is to find a suitable grading profile for a given specific engineering application, we should study the properties of graded composites with general, complex profiles so that we can have more degrees of freedom in the design of new functionally graded materials. However, to the best of our knowledge, there are rarely any analytic results for the local potentials and electric fields in dielectric graded composites. Therefore, in this paper we will deal with a theoretical model of a kind of graded composite having graded cylindrical inclusions with the general power-law dielectric profile $\varepsilon_i(r)=c(b+r)^k$, where c, b, and k constants and r is the radius of the inclusions. Analytical potentials are exactly derived in terms of hypergeometric functions and the effective dielectric response is estimated by means of an effective-medium approximation. In addition, analytical potentials for the simple power law $\varepsilon_i(r)=cr^k$ and the linear profile $\varepsilon_i(r)=c(b+r)$ are also derived as limiting cases from our formulas. In addition, we have discussed the effective response of graded cylindrical composites with a more complex dielectric profile of inclusion, $\varepsilon_i(r)=c(b+r)^k e^{\beta r}$.

II. ELECTRIC POTENTIAL DISTRIBUTION

For a linear constitutive relationship $D=\varepsilon E$ between the electric displacement D and the electric field E, for both the host and the inclusion regions, we will investigate the potential distribution in a composite material having general power-law-graded cylindrical inclusions in an external uniform electric field. Under the quasielectrostatic condition, the corresponding governing equations are $\nabla\cdot D=0$ and $\nabla\times E=0$. The boundary conditions are the continuities of the local electric potentials and the normal components of the electric displacements at the interface between the inclusions and the host medium. At infinity the potential must match that of the external applied electric field.

Let us consider long cylindrical inclusions with a general power-law dielectric function profile, $\varepsilon_i(r)=c(b+r)^k$, where r is the radius of the cylindrical inclusions randomly embed-

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ded in the $\hat{z}$ direction in a homogeneous and isotropic host having dielectric constant ε_h. The potential equations of this two-phase graded composite are

$$\nabla \varepsilon_\alpha(r) \cdot \nabla \Phi_\alpha + \varepsilon_\alpha(r)\nabla^2\Phi_\alpha = 0, \quad \text{in } \Omega_\alpha, \tag{1}$$

where Φ_α and Ω_α are the potentials and the domains of the α-type material, respectively, and the subscripts $\alpha=i,h$ denote the quantities in inclusion and host regions, respectively. If an external uniform electric field $E_a=E_0\hat{x}$ is applied to the composite along the $\hat{x}$ direction, in cylindrical coordinates, the potentials satisfy the following two-dimensional equations:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\varepsilon_\alpha(r) r \frac{\partial \Phi_\alpha(r,\varphi)}{\partial r}\right] + \frac{1}{r}\frac{\partial}{\partial \varphi}\left[\varepsilon_\alpha(r)\frac{1}{r}\frac{\partial \Phi_\alpha(r,\varphi)}{\partial \varphi}\right] = 0 \quad \text{in } \Omega_\alpha. \tag{2}$$

In the host region, the potential $\Phi_h(r,\varphi)$ can be found by using Eq. (2) and the boundary condition at infinity, $\Phi_h(r,\varphi)=-(r+Br^{-1})E_0\cos(\varphi)$. Using the separation of variables method, the potential in the inclusion region can be expressed as follows:

$$\Phi_i(r,\varphi) = \sum_{n=0}^{\infty} R_n(r)\cos(n\varphi). \tag{3}$$

Substituting Eq. (3) into Eq. (2), the equation for the radial part $R_n(r)$ of the potential in the inclusion region takes the form

$$r^2\frac{\partial^2 R_n(r)}{\partial r^2} + r[1 + kr/(b+r)]\frac{\partial R_n(r)}{\partial r} - n^2 R_n(r) = 0. \tag{4}$$

In order to solve Eq. (4) in terms of the hypergeometric functions, we introduce a variable transformation. Let $R_n(r) = z^s u_n(z)$, where $z=-r/b$, and substituting it into Eq. (4), we get the equation for the unknown function $u_n(z)$,

$$\frac{d^2u_n(z)}{dz^2} + [(2s+1)/z + k/(z-1)]\frac{du_n(z)}{dz} + [(s^2-n^2)/z^2 + sk/(z^2-z)]u_n = 0. \tag{5}$$

Clearly, if we let $s^2-n^2=0$ (i.e., $s=\pm n$), then Eq. (5) becomes the hypergeometric equation

$$z(1-z)\frac{d^2u_n}{dz^2} + [(2s+1) - z(2s+k+1)]\frac{du_n}{dz} - sku_n = 0. \tag{6}$$

The solution of Eq. (6) is the function $u_n(z) = F(\alpha_n,\beta_n,\gamma_n;z)$, where $F(\alpha_n,\beta_n,\gamma_n;z)$ is the hypergeometric function with $\alpha_{\pm n}=[\pm 2n+k-\sqrt{(\pm 2n+k)^2 \mp 4nk}]/2$, $\beta_{\pm n} = [\pm 2n+k+\sqrt{(\pm 2n+k)^2 \mp 4nk}]/2$, and $\gamma_{\pm n}=\pm 2n+1$. This hypergeometric function is analytic over the whole complex plane, except for some singular points.[17] The radial function $R_n(r)$ of the inclusion potential is thus

$$R_n(r) = A_n(-r/b)^n F(\alpha_n,\beta_n,\gamma_n;-r/b) + A_{-n}(-r/b)^{-n}F(\alpha_{-n},\beta_{-n},\gamma_{-n};-r/b). \tag{7}$$

Using the boundary conditions, the solutions of local potentials in the inclusion and host regions are then determined, respectively,

$$\Phi_h(r,\varphi) = -(r+Br^{-1})E_0\cos(\varphi), \quad r \geqslant a, \tag{8}$$

$$\Phi_i(r,\varphi) = -A_1(r/b)F(\alpha_1,\beta_1,\gamma_1;-r/b)E_0\cos(\varphi), \quad r \leqslant a, \tag{9}$$

where $\alpha_1=[2+k-\sqrt{4+k^2}]/2$, $\beta_1=[2+k+\sqrt{4+k^2}]/2$, and $\gamma_1 = 3$. The coefficients A_1 and B are given by

$$A_1 = -2a\varepsilon_h/[c\nu_1 + \varepsilon_h\nu_2], \quad B = a^2(A_1F/b - 1),$$

$$\xi = -a/b, \quad F = F(\alpha_1,\beta_1,\gamma_1;\xi),$$

$$F_1 = F(\alpha_1+1,\beta_1+1,\gamma_1+1;\xi),$$

$$\nu_1 = (b+a)^k(\xi F + \xi^2\alpha_1\beta_1\gamma_1^{-1}F_1), \text{ and } \nu_2 = \xi F.$$

For a linear dielectric profile $\varepsilon_i(r)=c(b+r)$, the local potential solutions can be obtained from Eqs. (8) and (9) by letting $k=1$. We have

$$\Phi_h(r,\varphi) = -(r+a^2Br^{-1})E_0\cos(\varphi), \tag{10}$$

$$\Phi_i(r,\varphi) = -A_1(r/b)F(\alpha_1,\beta_1,\gamma_1;-r/b)E_0\cos(\varphi), \tag{11}$$

where $\alpha_1=(3-\sqrt{5})/2$, $\beta_1=(3+\sqrt{5})/2$, $\gamma_1=3$, $A_1=-2a\varepsilon_h/[c(b+a)(\xi F+\xi^2\alpha_1\beta_1\gamma_1^{-1}F_1)+\varepsilon_h\xi F]$, and $B=A_1F/b-1$. We note that the coefficients ξ, A_1, and B of Eqs. (9) and (10) are the same as the coefficients, ξ', A_1, and D, of Eq. (18) in Refs. 14 and 18 if the radius of the cylindrical inclusion is unity, $a=1$. Therefore, we have demonstrated that our results reduce to that of having a linear power-law dielectric profile.

For a simple power-law profile $\varepsilon_i=cr^k$, we can also find the local potentials from Eqs. (8) and (9) by taking the limit $b\to 0$. In this case, using the properties of the hypergeometric function: $F(\alpha_1,\beta_1,\gamma_1;z)\to D(-z)^{-\alpha_1}$ for $z\to\infty$ (i.e., $b\to 0$) and $\beta_1 \geqslant \alpha_1$, where $D=\Gamma(\gamma_1)\Gamma(\beta_1-\alpha_1)/\Gamma(\gamma_1-\alpha_1)\Gamma(\beta_1)$, we obtained the following limits

$$\lim_{b\to 0} A_1(r/b)F(\alpha_1,\beta_1,\gamma_1;-r/b) = \lim_{b\to 0}\frac{2\varepsilon_h a^{1+(k-\sqrt{k^2+4})/2}}{\varepsilon_h + c(b+a)^k(1-c_1k/3)} r^{(\sqrt{k^2+4}-k)/2} = \frac{2\varepsilon_h a^{1+(k-\sqrt{k^2+4})/2}}{\varepsilon_h + ca^k(\sqrt{k^2+4}-k)/2} r^{(\sqrt{k^2+4}-k)/2},$$

$$\lim_{b\to 0} B = \lim_{b\to 0} a^2[A_1(1/b)F(\alpha_1,\beta_1,\gamma_1;-a/b) - 1] = \frac{2\varepsilon_h a^2}{\varepsilon_h + ca^k(\sqrt{k^2+4}-k)/2} - a^2,$$

where $c_1=\Gamma(\gamma_1+1)\Gamma(\beta_1)/\Gamma(\gamma_1)\Gamma(\beta_1+1)=\gamma_1/\beta_1$. Thus, the local potentials of graded composites having the simple power-law dielectric profile $\varepsilon_i=cr^k$ are obtained, respectively,

$$\Phi_h(r,\varphi) = -(r+a^2Br^{-1})E_0\cos(\varphi), \tag{12}$$

$$\Phi_i(r,\varphi) = -A_1 r^s E_0 \cos(\varphi), \quad (13)$$

where $A_1 = 2\varepsilon_h a^{1-s}/(\varepsilon_h + sca^k)$, $B = 2\varepsilon_h/(\varepsilon_h + sca^k) - 1$, and $s = (\sqrt{k^2+4} - k)/2$. For radius $a=1$, the potential solutions (12) and (13) are the same as Eq. (20) and (21) of Refs. 14 and 18 by comparing coefficients. Therefore, we have also demonstrated that our results reduce to that having a simple power-law profile.

Also, we can show that Eqs. (8) and (9) reduce to the classical potential formula for homogeneous and isotropic composites with $\varepsilon_i = c$ if $k \to 0$. Using the limits, $\lim_{k\to 0} F(\alpha_1, \beta_1, \gamma_1; \xi) = 1$ and $\lim_{k\to 0} \alpha_1 \beta_1 \gamma_1^{-1} F_1 = 0$, we have $\lim_{k\to 0} A_1 = 2b\varepsilon_h/(c+\varepsilon_h)$ and $\lim_{k\to 0} B = a^2(\varepsilon_h - c)/(c+\varepsilon_h)$. Thus, the potentials for homogeneous composites are obtained

$$\Phi_h(r,\varphi) = -[r + a^2 r^{-1}(\varepsilon_h - c)/(c+\varepsilon_h)]E_0 \cos(\varphi),$$

$$\Phi_i(r,\varphi) = -2\varepsilon_h E_0 \cos(\varphi)/(c+\varepsilon_h).$$

The above formulas are the same as the well-known classical potentials for isotropic composites having cylindrical inclusions with inclusion dielectric constant $\varepsilon_i = c$.[19]

III. EFFECTIVE DIELECTRIC RESPONSE

Using the derived potential formulas, in the dilute limit, we can estimate the effective dielectric response of our graded composites. Historically, many approximation methods were proposed for the estimation of the effective properties of composites, such as the effective polarizability method, the mean-field method, the classical minimum-energy principle, etc.[20] In this section, Landau's formula is used to predict the effective dielectric response,[21]

$$\frac{1}{V}\int_V (D - \varepsilon_h E)dV = \bar{D} - \varepsilon_h \bar{E}, \quad (14)$$

where V is the volume of the composite. $\bar{D}$ and $\bar{E}$ denote the ensemble averages of the electric displacement and the electric field over the whole composite region, respectively. The effective dielectric response ε_e of the graded composites is defined by the equation $\bar{D} = \varepsilon_e \bar{E}$. Substituting the effective constitutive relationship into the right-hand side of Eq. (14) and using the constitutive equation $D_h = \varepsilon_h E_h$ in the host region, we have

$$\frac{1}{V}\int_{\Omega_i} [\varepsilon_i(r) - \varepsilon_h] E dV = (\varepsilon_e - \varepsilon_h)\bar{E}. \quad (15)$$

In the dilute limit, the effective dielectric response can be estimated by the potential of a single cylindrical inclusion in a uniform external electric field $\bar{E}$. From Eq. (15), we have

$$\varepsilon_e/\varepsilon_h = 1 + 2f_i A_1 (c\nu_3/\varepsilon_h - \nu_4), \quad (16)$$

where f_i is the volume fraction of the cylindrical inclusions and A_1 is given by Eq. (9).

To determine the coefficients ν_3 and ν_4, we consider the function $I(k,\alpha,\beta,\gamma;a/b) = \int_0^a (b+r)^k F(\alpha,\beta,\gamma;-r/b)dr$. Using the formula $dF(\alpha-1,\beta-1,\gamma-1;z)/dz = (\alpha-1)(\beta-1)(\gamma-1)^{-1}F(\alpha,\beta,\gamma;z)$ and integrating by parts, we have

$$I(k,\alpha,\beta,\gamma;a/b) = \int_0^a (b+r)^k F(\alpha,\beta,\gamma;-r/b)dr = \int_b^{a+b} r^k F(\alpha,\beta,\gamma;1-r/b)dr = \sum_{n=1}^{k+1} (-1)^{n-1} \tilde{\nu}(n,k,\alpha,\beta,\gamma;a/b),$$

where $\tilde{\nu}(n,k,\alpha,\beta,\gamma;a/b) = \nu(n,k,\alpha,\beta,\gamma;1-r/b)|_{r=a+b} - \nu(n,k,\alpha,\beta,\gamma;1-r/b)|_{r=b}$, $\nu(n,k,\alpha,\beta,\gamma;1-r/b) = (-b)^n (k)_{n-1} r^{k-n+1} F(\alpha-n,\beta-n,\gamma-n;1-r/b) \quad \Pi_{i=1}^n (\gamma-i)(\alpha-i) \times (\beta-i)$, $(k)_{n-1} = k(k-1)(k-2)\cdots(k-n+2)$, and $(k)_0 = 1$. The coefficients ν_3 and ν_4 can be found by using the above formulas and they are given below,

$$\begin{aligned}\nu_3 &= a^{-2}b^{-1}\int_0^a (b+r)^k \left[F(\alpha_1,\beta_1,\gamma_1;-r/b) + \frac{1}{2}r\frac{dF(\alpha_1,\beta_1,\gamma_1;-r/b)}{dr}\right] rdr \\ &= a^{-2}b^{-1}[I(k+1,\alpha_1,\beta_1,\gamma_1;a/b) - bI(k,\alpha_1,\beta_1,\gamma_1;a/b)] - (ab)^{-2}\alpha_1\beta_1(2\gamma_1)^{-1} \\ &\quad \times [I(k+2,\alpha_1+1,\beta_1+1,\gamma_1+1;a/b) - 2bI(k+1,\alpha_1+1,\beta_1+1,\gamma_1+1;a/b) \\ &\quad + b^2 I(k,\alpha_1+1,\beta_1+1,\gamma_1+1;a/b)],\end{aligned} \quad (17)$$

$$\begin{aligned}\nu_4 &= a^{-2}b^{-1}\int_0^a \left[F(\alpha_1,\beta_1,\gamma_1;-r/b) + \frac{1}{2}r\frac{dF(\alpha_1,\beta_1,\gamma_1;-r/b)}{dr}\right] rdr \\ &= b^{-1}a^{-2}[I(1,\alpha_1,\beta_1,\gamma_1;a/b) - bI(0,\alpha_1,\beta_1,\gamma_1;a/b)] - (ab)^{-2}\alpha_1\beta_1(2\gamma_1)^{-1}[I(2,\alpha_1+1,\beta_1+1,\gamma_1+1;a/b) \\ &\quad - 2bI(1,\alpha_1+1,\beta_1+1,\gamma_1+1;a/b) + b^2 I(0,\alpha_1+1,\beta_1+1,\gamma_1+1;a/b)].\end{aligned} \quad (18)$$

In Eqs. (17) and (18), note that $I(0,\alpha_1,\beta_1,\gamma_1;a/b) = a$ because $F(\alpha_1,\beta_1,\gamma_1;z) = 1$ when $k=0$ (i.e., $\alpha_1 = 0$).

Considering the case $k \to 0$, we can demonstrate that the well-known Maxwell's formula of effective response of composites can be derived from Eq. (16). In this case, we can easily obtain the following limits: $\lim_{k\to 0}\nu_2/\nu_1 = 1$ and $\lim_{k\to 0}\nu_3/\nu_1 = \lim_{k\to 0}\nu_4/\nu_1 = -1/2a$. With these limits, Eq. (16) takes the form $\varepsilon_e/\varepsilon_h = 1 + 2f_i(c-\varepsilon_h)/(c+\varepsilon_h)$, which is the classical Maxwell's formula.[19]

Finally, note that the effective response of graded composites with linear dielectric profiles $\varepsilon_i(r) = c(b+r)$ and simple power-law profile $\varepsilon_i(r) = cr^k$ can also be obtained from Eq. (16) by setting $k=1$ and taking the limit $b \to 0$, respectively. In fact, in Sec. II, we have already demonstrated that analytical expressions for the potentials can be obtained from the potential equations (8) and (9) for graded composites with our general dielectric profile $\varepsilon_i(r) = c(b+r)^k$. Moreover, the local electric fields of inclusion regions along the $\hat{x}$ direction can also be obtained from the potential equation (9) [or Eqs. (11) and (13)]. Thus, the effective response of these graded composites can be directly derived from Eq. (16).

(a) For dielectric profile $\varepsilon_i(r)=c(b+r)$, in the dilute limit, we have

$$\varepsilon_e/\varepsilon_h = 1 + 2f_iA_1(c\nu_5/\varepsilon_h - \nu_6), \quad (19)$$

where A_1 is given in Eq. (11) and ν_5 and ν_6 are obtained from ν_3 and ν_4 of Eqs. (17) and (18) by letting $k=1$, respectively. Therefore, Eq. (19) is the same as that of Eq. (22) of Ref. 18 by comparing coefficients.

(b) For dielectric profile $\varepsilon_i(r)=cr^k$, in the dilute limit, we have

$$\varepsilon_e/\varepsilon_h = 1 + f_iA_1a^{s-1}[ca^k(s+1)/\varepsilon_h(s+k+1) - 1], \quad (20)$$

where A_1 and s are given in Eq. (13). Hence, Eq. (20) is the same as Eq. (23) of Ref. 18 or Eq. (17) of Ref. 16.

For higher concentration of the inclusions, adopting the spirit of the Maxwell-Garnett approximation, we can overcome the difficulty of estimating the average electric field over the whole composite region by considering a typical inclusion with the unknown effective dielectric response ε_e embedded in a host medium with dielectric constant ε_h.[22–24] As an effective-medium approximation, the average electric field over the inclusion region can be regarded as that of the whole original composite region. Therefore, we get[21]

$$\bar{E} = 2\varepsilon_h E_a/(\varepsilon_e + \varepsilon_h). \quad (21)$$

Substituting Eq. (21) into the right-hand side of Eq. (15), we found the effective dielectric response of the graded composite for higher volume fractions of the inclusions.,

$$\varepsilon_e/\varepsilon_h = 1 + 2f_iH/(1 - f_iH), \quad (22)$$

where $H=A_1(c\nu_3/\varepsilon_h-\nu_4)$. A_1 and ν_3 and ν_4 are given in Eqs. (9) and (16), respectively. It is clear that Eq. (22) reduces to Eq. (16) if we expand the power of the volume fraction up to the first order as long as the volume fraction satisfies the condition $|f_iH| \ll 1$. Thus Eq. (22) is a correct expression of effective dielectric response in the dilute limit.

For the case $k\to 0$, we will show that Eq. (22) reduces exactly to the famous Maxwell-Garnett approximation for homogeneous composites. In this case, we have the following limits: $\lim_{k\to 0}\alpha_1=0, \lim_{k\to 0}F(\alpha_1,\beta_1,\gamma_1;-r/b)=1, \lim_{k\to 0}A_1=2b\varepsilon_h/(c+\varepsilon_h)$, and $\lim_{k\to 0}\nu_3=\lim_{k\to 0}\nu_4=1/(2b)$. With these limits, we have $\lim_{k\to 0}H=(c-\varepsilon_h)/(c+\varepsilon_h)$. Therefore, Eq. (22) reduces to the following equation:

$$\varepsilon_e/\varepsilon_h = [c(1+f_i) + \varepsilon_h(1-f_i)]/[c(1-f_i) + \varepsilon_h(1+f_i)]. \quad (23)$$

Equation (23) is the well-known Maxwell-Garnett approximation.[22] At higher concentration of inclusions, for the linear dielectric profile $\varepsilon_i(r)=c(b+r)$ and the simple power-law dielectric profile $\varepsilon_i(r)=cr^k$, the effective response of graded composites can be derived.

(a) For dielectric profile $\varepsilon_i(r)=c(b+r)$, we have

$$\varepsilon_e/\varepsilon_h = 1 + 2f_iH/(1 - f_iH), \quad (24)$$

where $H=A_1(c\nu_5/\varepsilon_h-\nu_6)$ and A_1 is given by Eq. (11).

(b) For dielectric profile $\varepsilon_i(r)=cr^k$, we get

$$\varepsilon_e/\varepsilon_h = 1 + 2f_iH/(1 - f_iH), \quad (25)$$

where $H=\frac{1}{2}A_1a^{s-1}[c(s+1)a^k/\varepsilon_h(s+k+1)-1]$ and A_1 given in Eq. (13).

IV. DISCUSSIONS

For a more complex dielectric profile of the graded cylindrical inclusion, $\varepsilon_i(r)=c(b+r)^k e^{\beta r}$, the power-series method can be used to derive for the exact local potentials and the effective dielectric response of graded cylindrical composites. As a discussion, we give here the main results for the local potentials of graded cylindrical composites with radius a. The general solutions for potentials in inclusion and host regions are

$$\Phi_i(r,\varphi) = A_1 r\left(\sum_{i=0}^{\infty} a_i r^i\right)\cos(\varphi)E_0, \quad r \le a < |b|, \quad (26)$$

$$\Phi_h(r,\varphi) = -(r + Br^{-1})\cos(\varphi)E_0, \quad r > a, \quad (27)$$

where $A_1=-2\varepsilon_h/[c(b+a)^k e^{a\beta}\Sigma_{i=0}^{\infty}(i+1)a_ia^i+\varepsilon_h\Sigma_{i=0}^{\infty}a_ia^i]$ and $B=-a^2(1+A_1\Sigma_{i=0}^{\infty}a_ia^i)$.

$$a_{i+2} = \{[1 - i(i+1) - d(i+2) - 2(i+1)]a_{i+1} - \beta(1+i)a_i\}/b[(i+3)^2 - 1], \quad (i=0,1,2\ldots) \quad (28)$$

where $a_1=(1-d)a_0/(3b)$, $a_0=1$, and $d=\beta b+k+1$. From Eqs. (26) and (27), we can obtain the local potentials of graded cylindrical composites with the dielectric profiles $\varepsilon_i(r)=c(b+r)^k$ and $\varepsilon_i(r)=ce^{\beta r}$.

(a) For the general power-law profile $\varepsilon_i(r)=c(b+r)^k$, taking the limit $\beta\to 0$ in Eqs. (26) and (27), we obtain the following limits: $a_i=(-1/b)^i(\alpha_1)_i(\beta_1)_i/(\gamma_1)_i i!$ (where $\alpha_1=(2+k-\sqrt{k^2+4})/2$, $\beta_1=(2+k+\sqrt{k^2+4})/2$, and $\gamma_1=3$, $\Sigma_{i=0}^{\infty}a_ia^i=F(\alpha_1,\beta_1,\gamma_1;-a/b)$, $\Sigma_{i=0}^{\infty}a_ir^i=F(\alpha_1,\beta_1,\gamma_1;-r/b)$, and $\Sigma_{i=0}^{\infty}(i+1)a_ia^i=F(\alpha_1,\beta_1,\gamma_1;-a/b)-(a/b)(\alpha_1\beta_1/\gamma_1)F(\alpha_1+1,\beta_1+1,\gamma_1+1;-a/b)$, where $F(\alpha_1,\beta_1,\gamma_1;z)$ is the hypergeometric function. With these limits, we have

$$\Phi_i(r,\varphi) = -A_1(r/b)F(\alpha_1,\beta_1,\gamma_1;-r/b)\cos(\varphi)E_0, \quad r \le a, \quad (29)$$

$$\Phi_h(r,\varphi) = -(r + Br^{-1})\cos(\varphi)E_0, \quad r > a, \quad (30)$$

where $A_1=-2a\varepsilon_h/[c\nu_1+\varepsilon_h\nu_2]$, $B=a^2(A_1F/b-1)$, $\xi=-a/b$, $F=F(\alpha_1,\beta_1,\gamma_1;\xi)$, $F_1=F(\alpha_1+1,\beta_1+1,\gamma_1+1;\xi)$, $\nu_1=(b+a)^k(\xi F+\xi^2\alpha_1\beta_1\gamma_1^{-1}F_1)$, and $\nu_2=\xi F$. So Eqs. (29) and (30) are the same as Eqs. (8) and (9).

(b) For exponential profile $\varepsilon_i(r)=ce^{\beta r}$, we get the following formulas if we take the limit $k\to 0$ to Eqs. (26) and (27): $a_{i+1}=-\beta a_i/(i+3)$ [or $a_{i+1}=(-\beta)^{i+1}a_0/(3)_{i+1}$, where $(a)_i=a(a+1)(a+2)\cdots(a+i-1),(a)_0=1$], $\Sigma_{i=0}^{\infty}a_ia^i=F(1,3,-\beta a)\Sigma_{i=0}^{\infty}(i+1)a_ia^i=F(2,3,-\beta a)$, and

$\Sigma_{i=0}^{\infty} a_i r^i = F(1,3,-\beta r)$, where $F(\alpha,\gamma,z)$ is the Kummer function.[17] Substituting these limits into Eqs. (26) and (27), we get

$$\Phi_i(r,\varphi) = A_1 r F(1,3,-\beta r)\cos(\varphi)E_0, \quad r \leqslant a, \tag{31}$$

$$\Phi_h(r,\varphi) = -(r + Br^{-1})\cos(\varphi)E_0, \quad r > a, \tag{32}$$

where $A_1 = -2\varepsilon_h/(ce^{a\beta}\nu_1 + \varepsilon_h \nu_2)$, $B = a^2(\varepsilon_h\nu_2 - ce^{a\beta}\nu_1)/(ce^{a\beta}\nu_1 + \varepsilon_h\nu_2)$, $\nu_1 = F(2,3,-a\beta)$, and $\nu_2 = F(1,3,-a\beta)$. Clearly, Eqs. (31) and (32) are the same as Eqs. (13) and (14) of Ref. 18 at $c=1$.

(c) The effective dielectric response of the graded cylindrical composites with a more complex dielectric profile of inclusion, $\varepsilon_i(r) = c(b+r)^k e^{\beta r}$, can be derived by using Eq. (15)

$$\varepsilon_e = \varepsilon_h + f_i A_1(\varepsilon_h H_1 - ca^{-2} H_2), \tag{33}$$

where f_i is the volume fraction of the inclusions, A_1 is given in Eq. (26), $H_1 = \Sigma_{i=0}^{\infty}[(i+1/i+3)a_{i+1}a^{i+1} + (2/i+2)a_i a^i]$, and $H_2 = \Sigma_{i=0}^{\infty}[(i+1)a_{i+1}I_{i+2} + 2a_i I_{i+1}]$ where $I_i = \int_0^a r^i (b+r)^k e^{\beta r} dr$ ($i = 1,2,3,\ldots$). In order to calculate I_i, we introduce a linear transformation $x = r+b$ to I_i so that $I_i = e^{-b\beta}\int_b^{a+b}(x-b)^i x^k e^{\beta x} dx$, where, $i = 1,2,3,\ldots$. Thus, we have $I_i = e^{-b\beta}\int_b^{a+b}[\Sigma_{m=0}^{i} C_i^m x^{i-m+k} e^{\beta x}(-b)^m]dx = e^{-b\beta}\Sigma_{m=0}^{i} C_i^m (-b)^m T_{m,k}$ and $T_{m,k} = \int_b^{a+b} x^{i-m+k} e^{\beta x} dx$. Here, one should note that the effective responses of graded cylindrical composites with the dielectric profiles of inclusions, $\varepsilon_i(r) = c(b+r)^k$ and $\varepsilon_i(r) = ce^{\beta r}$, can also be derived from Eq. (33) by taking the limits $\beta \to 0$ and $k \to 0$, respectively.

V. CONCLUSIONS

For graded composites having general power-law dielectric profile cylindrical inclusions, we have derived analytical expressions for the local potentials in terms of a hypergeometric function and the effective response by means of the average field method in the dilute limit. For higher concentrations of inclusions, an effective-medium approximation is employed. In our model for the general power-law dielectric profile, three freedom parameters are included, and we find that the effective property of the general power-law profile is quite different from that of the simple power law $\varepsilon_i(r) = cr^k$ and the linear profile $\varepsilon_i(r) = c(b+r)$. It is demonstrated that our results reduce to those of the simple power-law and the linear profiles by taking the limits $b \to 0$ and $k \to 1$, respectively. In addition, we have given the effective response of graded cylindrical composites with complex dielectric profile of inclusion $\varepsilon_i(r) = c(b+r)^k e^{\beta r}$. The results of our models can be used to test the differential effective dipole approximation, which is valid for arbitrary gradient profiles.[15,25] Actually, the general power-law profile and the complex dielectric profile have advantages over the simple power-law and the linear profiles. The analytical potential expressions and the effective response of the graded composites are much more useful in controlling the response of functionally graded materials due to having more controllable parameters in the inclusion physical property profile. In fact, it is expected that the three-dimensional problem of graded composites having spherical inclusions with the general power-law dielectric and the complex dielectric profiles can be solved on the basis of the method of this paper, and it is also possible to derive nonlinear effective response of weakly nonlinear graded composites with general power-law profile by the perturbation method.[26,27] Of course, our results can also be used to estimate the effective electric conductivity and the effective thermal conductivity of graded composites having general and complex power-law graded cylindrical inclusions with perfect boundary conditions at the interface between the inclusions and the host.

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