Algorithm for p^m-length Discrete Cosine Transform

Yuk-Hee CHAN and Wan-Chi SIU

Department of Electronic Engineering
Hong Kong Polytechnic
Hung Hom, Kowloon
Hong Kong

Abstract: A new algorithm is introduced such that we can realize a discrete cosine transform by correlations. This algorithm can be applied to any p^m-length DCT, where p is an odd prime, and is most suitable for VLSI implementation.

Introduction

Many fast algorithms for the computation of the discrete cosine transform(DCT) have been proposed since the introduction of the DCT by Ahmed et al.[1] in 1974. However, most of them were proposed for the realization of a 2^m-length DCT. Studies on the realization of DCT for lengths equal to prime squares or power of primes are rarely in progress. Recently, we found that it is possible to realize a prime length DCT through two half-length cyclic correlations, which results in a very regular structure. Hence, it is very suitable for VLSI implementation[2,3].

In this paper, we introduce a new general algorithm to realize a p^m-length DCT, where p is an odd prime, by using short correlations. This algorithm also results in an extremely regular structure and is most suitable for VLSI implementation.

Algorithm Derivation

The DCT[1] of a real data sequence $\{y(i):i=0,1,...N-1\}$ is defined as

$$Y(k) = \sum_{i=o}^{N-1} y(i) \cos\left(\frac{2\pi(2i+1)k}{4N}\right) \quad \text{for } k=0,1,...N-1 \quad (1)$$

By defining another sequence $\{x(i):i=0,1...N-1\}$ as

$$\begin{cases} x(N-1) = y(N-1) \\ x(i) \quad = y(i) - x(i+1) \end{cases} \quad \text{for } i=0,1...N-2 \quad (2)$$

we have $$Y(0) = \sum_{i=0}^{N-1} y(i) \quad (3)$$

and $$Y(k) = \{2T(k) + x(0)\} \cos\left(\frac{k\pi}{2N}\right) \quad \text{for } k=1,2...N-1 \quad (4)$$

where T(k) is defined as

$$T(k) = \sum_{i=1}^{N-1} x(i) \cos\left(\frac{\pi ik}{N}\right) \quad \text{for } k=1,2...N-1 \quad (5)$$

This basic formulation is then applied for the realization of the p^m-length DCT.

Realization of P^2-length DCT

If $N = p^2$ and p is an odd prime in eqn. 5, we have

$$T(k) = A(k) + B(k) \quad \text{for } k\in\Omega$$
$$T(k) = E(k) + F(k) \quad \text{for } k\in\Psi \quad (6)$$

where $\Omega = \{pn \mid n=1,2..p-1\}$

$\Psi = \{ n \mid n=1,2...N\text{-}1 \text{ and } n\notin\Omega \}$

$$A(k) = \sum_{i\in\Psi} x(i)\cos\left(\frac{\pi ik}{N}\right) \quad \text{for } k\in\Omega \ (7)$$

$$B(k) = \sum_{i\in\Omega} x(i)\cos\left(\frac{\pi ik}{N}\right) \quad \text{for } k\in\Omega \ (8)$$

$$E(k) = \sum_{i\in\Psi} x(i)\cos\left(\frac{\pi ik}{N}\right) \quad \text{for } k\in\Psi \ (9)$$

$$F(k) = \sum_{i\in\Omega} x(i)\cos\left(\frac{\pi ik}{N}\right) \quad \text{for } k\in\Psi \ (10)$$

For reference propose, we also define $\Psi_n = \{np+i \mid i=1,2..p-1\}$, where $n=0,1,..p-1$.

For B(k):

$$B(pk) = \sum_{i\in\Psi_o} s(ik)\,x(pi) \quad \text{for } k\in\Psi_o \ (11)$$

where $s(ik) = \begin{cases} 1 & \text{if } ik \text{ is even} \\ -1 & \text{if } ik \text{ is odd} \end{cases}$

Note that no multiplication is required.

For E(k):

We split eqn. 9 into even and odd sequences

$$E(2k) = \sum_{i\in\Psi} x(i)\cos\left(\frac{2\pi ik}{N}\right) \quad k\in\{k \mid 2k\in\Psi \text{ and } k\in\Psi\} \quad (12)$$

$$E(N\text{-}2k) = \sum_{i\in\Psi} e(i)\cos\left(\frac{2\pi ik}{N}\right) \quad k\in\{k \mid N\text{-}2k\in\Psi \text{ and } k\in\Psi\} \quad (13)$$

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where $e(i) = (-1)^i x(i)$

If p is an odd prime, Ψ is cyclic under multiplication modulo N and so is isomorphic to the field $Z_{p(p-1)}$. Let g be a primitive element such that $\{<g^n>_N \mid n=1,2...p(p-1)\} = \Psi$, then, after expanding the domain of E(2k) and E(N-2k) to $k\in\Psi$, we have

$$E'(k) = \sum_{i=1}^{p(p-1)} x'(i)\, C(i+k) \qquad \text{for } k=1,2...p(p-1) \quad (14)$$

$$E''(k) = \sum_{i=1}^{p(p-1)} e'(i)\, C(i+k) \qquad \text{for } k=1,2...p(p-1) \quad (15)$$

where $C(n) = \cos[(2\pi/N)<g^n>_N]$ for n is integer

$$x'(i) = x(<g^i>_N)$$

$$e'(i) = e(<g^i>_N)$$

$$E'(k) = E(2<g^k>_N)$$

$$E''(k) = E(N-2<g^k>_N)$$

As $C(p(p-1)/2 + n) = C(n)$, we have $E'(p(p-1)/2+k) = E'(k)$ and $E''(p(p-1)/2+k) = E''(k)$. Hence, eqns. 14 and 15 can be further simplified to

$$E'(k) = \sum_{i=1}^{p(p-1)/2} \{x'(i)+x'(p(p-1)/2+i)\}\, C(i+k) \qquad \text{for } k=1,2...p(p-1)/2 \quad (16)$$

$$E''(k) = \sum_{i=1}^{p(p-1)/2} \{e'(i)+e'(p(p-1)/2+i)\}\, C(i+k) \qquad \text{for } k=1,2...p(p-1)/2 \quad (17)$$

It shows the fact that two p(p-1)/2-length cyclic correlations are required to compute $\{E(k):k\in\Psi\}$.

For A(k):

Eqn. 7 can be rewritten as

$$A(pk) = \sum_{n=0}^{p-1} \sum_{i\in\Psi_n} x(i) \cos\left(\frac{\pi ik}{p}\right) \qquad \text{for } k\in \Psi_o \quad (18)$$

We can also split it into even and odd sequences

$$A(2pk) = \sum_{n=0}^{p-1} \sum_{i\in\Psi_n} x(i) \cos\left(\frac{2\pi ik}{p}\right) \qquad \text{for } k \in \{k \mid 2k\in\Psi_o \text{ and } k\in\Psi_o\} \quad (19)$$

$$A(N-2pk) = \sum_{n=0}^{p-1} \sum_{i\in\Psi_n} (-1)^i x(i) \cos\left(\frac{2\pi ik}{p}\right) \qquad \text{for } k \in \{k \mid p\text{-}2k\in\Psi_o \text{ and } k\in\Psi_o\} \quad (20)$$

As $\cos[2\pi k(np+i)/p] = \cos[2\pi ki/p]$, we have

$$A(2pk) = \sum_{i\in\Psi_o} \left\{ \sum_{n=0}^{p-1} x(np+i) \right\} \cos\left(\frac{2\pi ik}{p}\right) \qquad \text{for } k \in \{k \mid 2k\in\Psi_o \text{ and } k\in\Psi_o\} \quad (21)$$

$$A(N-2pk) = \sum_{i\in\Psi_o} \left\{ \sum_{n=0}^{p-1} (-1)^{n+i} x(np+i) \right\} \cos\left(\frac{2\pi ik}{p}\right) \qquad \text{for } k \in \{k \mid p\text{-}2k\in\Psi_o \text{ and } k\in\Psi_o\} \quad (22)$$

After expanding the domain of A(2pk) and A(N-2pk) to $k\in\Psi_o$, we have

$$A'(k) = \sum_{i\in\Psi_o} \xi'(i)\; C_o(i+k) \qquad \text{for } k\in\Psi_o \quad (23)$$

$$A''(k) = \sum_{i\in\Psi_o} \zeta'(i)\; C_o(i+k) \qquad \text{for } k\in\Psi_o \quad (24)$$

where $C_o(n) = \cos[(2\pi/p)<g_o^n>_p]$ for n is an integer

$$A'(k) = A(2p<g_o^k>_p)$$

$$A''(k) = A(N-2p<g_o^k>_p)$$

$$\xi'(i) = \xi(<g_o^i>_p)$$

$$\zeta'(i) = \zeta(<g_o^i>_p)$$

$$\xi(i) = \sum_{n=0}^{p-1} x(np+i)$$

$$\zeta(i) = \sum_{n=0}^{p-1} (-1)^{n+i} x(np+i) \quad \text{and}$$

$$\{<g_o^i>_p \mid i\in\Psi_o\} = \Psi_o.$$

One may observe that $C_o((p-1)/2 + n) = C_o(n)$ for $n=1,2...(p-1)/2$. Hence, we have $A'(k) = A'((p-1)/2+k)$ and $A''(k) = A''((p-1)/2+k)$. In such case, eqns. 23 and 24 can be further simplified to

$$A'(k) = \sum_{i=1}^{(p-1)/2} \{\xi'(i)+\xi'((p-1)/2+i)\}\; C_o(i+k) \qquad \text{for } k=1,2...(p-1)/2 \quad (25)$$

$$A''(k) = \sum_{i=1}^{(p-1)/2} \{\zeta'(i)+\zeta'((p-1)/2+i)\}\; C_o(i+k) \qquad \text{for } k=1,2...(p-1)/2 \quad (26)$$

These show that two (p-1)/2-length cyclic correlations are required to compute $\{A(k):k\in\Omega\}$.

For F(k):

Eqn. 10 can be rewritten as

$$F(k) = \sum_{i\in\Psi_o} x(pi)\; \cos\left(\frac{\pi ik}{p}\right) \qquad \text{for } k\in\Psi \quad (27)$$

$$\text{As} \quad F(2np+k) = F(k) \quad (28)$$

$$\text{and} \quad F((2n+1)p+k) = \sum_{i\in\Psi_o} (-1)^i x(pi)\; \cos\left(\frac{\pi ik}{p}\right) \quad (29)$$

for $k\in\Psi_o$ and n is positive integer

$$\text{only} \quad F(k) = \sum_{i\in\Psi_o} x(pi)\; \cos\left(\frac{\pi ik}{p}\right) \qquad \text{for } k\in\Psi_o \quad (30)$$

$$F_o(k) = \sum_{i\in\Psi_o} (-1)^i x(pi) \cos\left(\frac{\pi ik}{p}\right) \quad \text{for } k\in\Psi_o \quad (31)$$

are required. Besides, it has the property that

$$F(k) = F_o(p-k) \quad \text{for } k\in\Psi_o \quad (32)$$

Hence, we need to deal with the sequence $\{F(k):k\in\Psi_o\}$ only. After expanding the domain of F(2k) and F(p-2k) to $k\in\Psi_o$, we have

$$F'(k) = \sum_{i\in\Psi_o} \alpha'(i)\ C_o(i+k) \quad \text{for } k\in\Psi_o \quad (33)$$

$$F''(k) = \sum_{i\in\Psi_o} \beta'(i)\ C_o(i+k) \quad \text{for } k\in\Psi_o \quad (34)$$

where $F'(k) = F(2<g_o^k>_p)$

$F''(k) = F(p-2<g_o^k>_p)$

$\alpha'(i) = \alpha(<g_o^i>_p)$

$\beta'(i) = \beta(<g_o^i>_p)$

$\alpha(i) = x(pi)$

$\beta(i) = (-1)^i x(pi)$ and

$\{<g_o^i>_p \mid i\in\Psi_o\} = \Psi_o$.

Similarly, as $C_o((p-1)/2 + n) = C_o(n)$, eqns. 33 and 34 can be further simplified to

$$F'(k) = F'((p-1)/2+k) = \sum_{i=1}^{(p-1)/2} \left\{\alpha'(i)+\alpha'((p-1)/2+i)\right\} C_o(i+k) \quad \text{for } k=1,2...(p-1)/2 \quad (35)$$

$$F''(k) = F''((p-1)/2+k) = \sum_{i=1}^{(p-1)/2} \left\{\beta'(i)+\beta'((p-1)/2+i)\right\} C_o(i+k) \quad \text{for } k=1,2...(p-1)/2 \quad (36)$$

Hence it is clear that two (p-1)/2-length cyclic correlations are required for the computation of $\{F(k):k\in\Psi\}$.

Realization of P^n-length DCT

We have shown that a p^2-length DCT can be realized by correlations. Actually, this method can also be applied to p^n-length DCT. We first select all integers that contain a factor p from the set $\Theta^n = \{1,2,...,p^n-1\}$ to form the set Ω^n. Those elements left form another set Ψ^n. Then eqn. 5 can be realized through the realization of

$$A_n(k) = \sum_{i\in\Psi^n} x(i) \cos\left(\frac{\pi ik}{N}\right) \quad \text{for } k\in\Omega^n \quad (37)$$

$$B_n(k) = \sum_{i\in\Omega^n} x(i) \cos\left(\frac{\pi ik}{N}\right) \quad \text{for } k\in\Omega^n \quad (38)$$

$$E_n(k) = \sum_{i\in\Psi^n} x(i) \cos\left(\frac{\pi ik}{N}\right) \quad \text{for } k\in\Psi^n \quad (39)$$

$$F_n(k) = \sum_{i\in\Omega^n} x(i) \cos\left(\frac{\pi ik}{N}\right) \quad \text{for } k\in\Psi^n \quad (40)$$

where $\Theta^n = \{1,2...p^n-1\}$

$\Omega^n = \{pm \mid m\in\Theta^{n-1}\}$

$\Psi^n = \{m \mid m\in\Theta^n \text{ and } m\notin\Omega^n\}$

and, for the sake of simplicity, we call these formulations as $A_n(k)$, $B_n(k)$, $E_n(k)$ and $F_n(k)$ with order n respectively for future reference. Note that the input sequence is not necessary to be $\{x(i)\}$ when these structures are referred.

As Ψ^n forms a cyclic group with $p^{n-1}(p-1)$ elements under multiplication modulo p^n, we can realize $E_n(2k)$ and $E_n(p^n-2k)$ by correlations using exactly the same technique used in p^2 case.

On the other hand, it is noted that $A_n(k)$ and $F_n(k)$ can be expressed in the form of $A_{n-1}(k)$, $E_{n-1}(k)$ and $F_{n-1}(k)$ accordingly. This fact can be easily observed by the following deduction:

For $A_n(k)$,

$$A_n(pk) = \sum_{i\in\Psi^{n-1}} \left(\sum_{d\in\Gamma_e} x(dp^{n-1}+i)\right) \cos\left(\frac{\pi ik}{p^{n-1}}\right) + (-1)^k \sum_{i\in\Psi^{n-1}} \left(\sum_{d\in\Gamma_o} x(dp^{n-1}+i)\right) \cos\left(\frac{\pi ik}{p^{n-1}}\right) \quad \text{for } k\in\Theta^{n-1} \quad (41)$$

For $F_n(k)$,

$$F_n(k+dp^{n-1}) = \sum_{i\in\Theta^{n-1}} x(pi) \cos\left(\frac{\pi ik}{p^{n-1}}\right) \quad \text{for } d\in\Gamma_e,\ k\in\Psi^{n-1} \quad (42)$$

$$F_n(k+dp^{n-1}) = \sum_{i\in\Theta^{n-1}} (-1)^i x(pi) \cos\left(\frac{\pi ik}{p^{n-1}}\right) \quad \text{for } d\in\Gamma_o,\ k\in\Psi^{n-1} \quad (43)$$

where $\Gamma_o = \{1,3,5..p-2\}$ and $\Gamma_e = \{0,2,4..p-1\}$

These equations involve four $E_{n-1}(k)$, two $F_{n-1}(k)$ and two $A_{n-1}(k)$ structures. As $E_{n-1}(k)$ can be realized through a correlation, this decomposition technique suggests an approach for us to realize $F_n(k)$ and $A_n(k)$. One can decompose $F_n(k)$ and $A_n(k)$ to $F_2(k)$ and $A_2(k)$ by using this technique recursively and then realize them with the same technique mentioned in p^2 case.

Similarly, $B_n(k)$ can be decomposed into formulations with lower order and then realized through cyclic correlations by applying the following steps accordingly:

1. To partition Θ^n to Ω^n and Ψ^n,
2. To map Ω^n to Θ^{n-1} bijectively by the mapping function $f(i) = i/p$,

3. To partition Ψ^n to p sets $\{dp^{n-1}+i \mid i\in\Psi^{n-1}\}$ for $d\in\{0,1,2..p-1\}$.

In such case, a p^m-length DCT can be realized through correlations.

Example

Let us use a length-25 DCT to clarify our proposal. In such case, N = 25, p = 5 and we have

$$\Psi_0 = \{1,2,3,4\}, \quad \Psi_1 = \{6,7,8,9\},$$
$$\Psi_2 = \{11,12,13,14\}, \quad \Psi_3 = \{16,17,18,19\},$$
$$\Psi_4 = \{21,22,23,24\},$$
$$\Psi = \Psi_0 \cup \Psi_1 \cup \Psi_2 \cup \Psi_3 \cup \Psi_4,$$
$$\Omega = \{5,10,15,20\}.$$

We choose $g = g_0 = 3$, then we can use eqns. 28, 29, 32, 35 and 36 to compute $\{F(k):k\in\Psi\}$,

$$\begin{bmatrix}F'(1)\\F'(2)\end{bmatrix} = \begin{bmatrix}F'(3)\\F'(4)\end{bmatrix} = \begin{bmatrix}F(4)\\F(2)\end{bmatrix} = \begin{bmatrix}C_o(2) & C_o(3)\\C_o(3) & C_o(2)\end{bmatrix} \begin{bmatrix}x(15)+x(10)\\x(20)+x(5)\end{bmatrix}$$

$$\begin{bmatrix}F''(1)\\F''(2)\end{bmatrix} = \begin{bmatrix}F''(3)\\F''(4)\end{bmatrix} = \begin{bmatrix}F(1)\\F(3)\end{bmatrix} = \begin{bmatrix}C_o(2) & C_o(3)\\C_o(3) & C_o(2)\end{bmatrix} \begin{bmatrix}-x(15)+x(10)\\x(20)-x(5)\end{bmatrix}$$

$$\begin{bmatrix}F(1)\\F(2)\\F(3)\\F(4)\end{bmatrix} = \begin{bmatrix}F(11)\\F(12)\\F(13)\\F(14)\end{bmatrix} = \begin{bmatrix}F(21)\\F(22)\\F(23)\\F(24)\end{bmatrix} = \begin{bmatrix}F(9)\\F(8)\\F(7)\\F(6)\end{bmatrix} = \begin{bmatrix}F(19)\\F(18)\\F(17)\\F(16)\end{bmatrix}$$

For the computation of $\{E(k):k\in\Psi\}$, we can use eqns. 16 and 17

$$\begin{bmatrix}E'(1)\\E'(2)\\E'(3)\\E'(4)\\E'(5)\\E'(6)\\E'(7)\\E'(8)\\E'(9)\\E'(10)\end{bmatrix} = \begin{bmatrix}E'(11)\\E'(12)\\E'(13)\\E'(14)\\E'(15)\\E'(16)\\E'(17)\\E'(18)\\E'(19)\\E'(20)\end{bmatrix} = \begin{bmatrix}E(6)\\E(18)\\E(4)\\E(12)\\E(14)\\E(8)\\E(24)\\E(22)\\E(16)\\E(2)\end{bmatrix} = \mathbf{C} \begin{bmatrix}x(3)+x(22)\\x(9)+x(16)\\x(2)+x(23)\\x(6)+x(19)\\x(18)+x(7)\\x(4)+x(21)\\x(12)+x(13)\\x(11)+x(14)\\x(8)+x(17)\\x(24)+x(1)\end{bmatrix}$$

$$\begin{bmatrix}E''(1)\\E''(2)\\E''(3)\\E''(4)\\E''(5)\\E''(6)\\E''(7)\\E''(8)\\E''(9)\\E''(10)\end{bmatrix} = \begin{bmatrix}E''(11)\\E''(12)\\E''(13)\\E''(14)\\E''(15)\\E''(16)\\E''(17)\\E''(18)\\E''(19)\\E''(20)\end{bmatrix} = \begin{bmatrix}E(19)\\E(7)\\E(21)\\E(13)\\E(11)\\E(17)\\E(1)\\E(3)\\E(9)\\E(23)\end{bmatrix} = \mathbf{C} \begin{bmatrix}-x(3)+x(22)\\-x(9)+x(16)\\x(2)-x(23)\\x(6)-x(19)\\x(18)-x(7)\\x(4)-x(21)\\x(12)-x(13)\\-x(11)+x(14)\\x(8)-x(17)\\x(24)-x(1)\end{bmatrix}$$

where $\mathbf{C}$ is the correlation 10*10 matrix of C(n)

For the computation of $\{A(k):k\in\Omega\}$, we can make use of eqns 25 and 26:

$$\begin{bmatrix}A'(1)\\A'(2)\end{bmatrix} = \begin{bmatrix}A'(3)\\A'(4)\end{bmatrix} = \begin{bmatrix}A(20)\\A(10)\end{bmatrix} = \begin{bmatrix}C_o(2) & C_o(3)\\C_o(3) & C_o(2)\end{bmatrix} \begin{bmatrix}a(1)\\a(2)\end{bmatrix}$$

where $a(1) = x(2)+x(7)+x(12)+x(17)+x(22)$
$\quad +x(3)+x(8)+x(13)+x(18)+x(23)$

$a(2) = x(1)+x(6)+x(11)+x(16)+x(21)$
$\quad +x(4)+x(9)+x(14)+x(19)+x(24)$

$$\begin{bmatrix}A''(1)\\A''(2)\end{bmatrix} = \begin{bmatrix}A''(3)\\A''(4)\end{bmatrix} = \begin{bmatrix}A(5)\\A(15)\end{bmatrix} = \begin{bmatrix}C_o(2) & C_o(3)\\C_o(3) & C_o(2)\end{bmatrix} \begin{bmatrix}b(1)\\b(2)\end{bmatrix}$$

where $b(1) = x(2)-x(7)+x(12)-x(17)+x(22)$
$\quad -x(3)+x(8)-x(13)+x(18)-x(23)$

$b(2) = -x(1)+x(6)-x(11)+x(16)-x(21)$
$\quad +x(4)-x(9)+x(14)-x(19)+x(24)$

and, finally, for the computation of $\{B(k):k\in\Omega\}$, we can use eqn. 11

$$\begin{bmatrix}B(5)\\B(10)\\B(15)\\B(20)\end{bmatrix} = \begin{bmatrix}-x(5) & +x(10) & -x(15) & +x(20)\\x(5) & +x(10) & +x(15) & +x(20)\\-x(5) & +x(10) & -x(15) & +x(20)\\x(5) & +x(10) & +x(15) & +x(20)\end{bmatrix}$$

Then, $\{T(k):k = 1,2..24\}$ can be determined by eqn. 6. By using this technique, a length-25 DCT can be realized through cyclic correlations.

Conclusions

We have seen that a p^2-length DCT can be realized through four (p-1)/2-length and two (p-1)p/2-length cyclic correlations with a cost of N-1 multiplications. This formulation is potentially useful for the realization of discrete cosine transform using VLSI techniques especially if the distributed arithmetic technique is going to be used. It is also interesting that an N length cyclic correlation can theoretically be realized by 2N-d multiplications[4], where d is the number of factorial polynomials of z^N-1, when it is realized with the Largrange interpolation formula. As the minimum number of factorial polynomials of both polynomial $z^{(p-1)p/2}$-1 and polynomial $z^{(p-1)/2}$-1 are 2, the number of multiplications required to realize a p^2-length DCT can be given by $3p^2+2p-17$. The number of multiplications per point is therefore given by $3+2p^{-1}-17p^{-2}$, which is not larger than 52/17. Hence, it would imply theoretically that the number of multiplications per point of a p^2-length DCT is about 3 irrespective to the value of p.

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