Design and Stability Analysis of Fuzzy Model-Based Nonlinear Controller for Nonlinear Systems Using Genetic Algorithm

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***Abstract*—This paper presents the stability analysis of fuzzy model-based nonlinear control systems, and the design of nonlinear gains and feedback gains of the nonlinear controller using genetic algorithm (GA) with arithmetic crossover and nonuniform mutation. A stability condition will be derived based on Lyapunov's stability theory with a smaller number of Lyapunov conditions. The solution of the stability conditions are also determined using GA. An application example of stabilizing a cart-pole typed inverted pendulum system will be given to show the stabilizability of the nonlinear controller.**

***Index Terms*—Fuzzy plant model, genetic algorithm (GA), nonlinear controller, nonlinear systems, stability.**

I. Introduction

FUZZY control has been a hot research topic. Despite the lack of a concrete theoretical basis, many successful applications on fuzzy control were reported in various areas such as sludge wastewater treatment [1], control of cement kiln [2], etc. However, without an in-depth analysis, the design may come with no guarantees of system stability and good system performance. Recently, stability analysis of fuzzy control systems based on a Takagi-Sugeno-Kang (TSK) fuzzy plant model [3], [7] was reported. The advantage of using the fuzzy model is that a nonlinear plant can be represented as a weighted sum of linear subsystems, so that some linear or nonlinear control theories can possibly be applied to design the controller. Different stability conditions for this class of fuzzy control systems were derived. In [4]–[7], [16], and [19], the Lyapunov stability theory was employed to analyze the system stability. Sliding mode theory was employed in [8] to help the analysis. In [13]–[15], the stability conditions were derived in terms of some matrix measures of the system matrices. A linear matrix inequality (LMI)-based design of fuzzy controllers can be found in [10]–[12]. A switching controller [17] and other controllers [16], [18] were also proposed to tackle nonlinear systems based on the TSK fuzzy plant model.

Genetic algorithm (GA) is a powerful random search technique to handle optimization problems [1]–[6], [17]. This is especially useful for complex optimization problems with a large number of parameters that make global analytical solutions difficult to obtain. It has been widely applied in different areas, such as fuzzy control [20], tuning of parameters of neural networks [21], eBook applications [22], load forecasting [23], etc.

In this paper, we focus on the system stability and present a stability analysis of fuzzy model-based nonlinear control systems. A nonlinear controller is proposed to control a system represented by a TSK fuzzy plant model [3]. The proposed controller, which takes the same form as that in [16], has a structure similar to that of the fuzzy controller reported in [6]. The main difference is that the weights in the proposed nonlinear controller are signed, but those of the controller in [6] must be positive (because they are the membership function values). Wang *et al.* derived a stability condition for TSK fuzzy model-based systems using Lyapunov stability theory [6]. A sufficient condition for the system stability is obtained by finding a common Lyapunov function for all the fuzzy subcontrol systems. For a TSK fuzzy plant model with p rules, a fuzzy controller with p rules (p subcontrollers) is used to close the feedback loop, and $p(p+1)/2$ Lyapunov conditions are required. In this paper, the number of subcontrollers of the nonlinear controller need not be the same as that of the TSK fuzzy plant model. By allowing both positive and negative weighting values in the proposed controller, the number of Lyapunov conditions can be reduced to p. We also provide a way of designing the nonlinear gains and the feedback gains of the nonlinear controller. The task of finding the common Lyapunov function can readily be formulated into an LMI problem [9]. GA with arithmetic crossover and nonuniform mutation [25] will be used to help find the solution of the derived stability conditions, and also determine the feedback gains of the subcontrollers. In this paper, GA is not only employed to solve the derived stability conditions in LMI form, but also used to obtain the controller gains that are included in the stability conditions of the proposed nonlinear controller. It is generally a difficult task to formulate the problems of solving both the solution of the stability conditions and the gains into a single LMI problem. By employing GA, this difficulty is removed. While other searching algorithms can be used, GA is one good method to obtain the design solution.

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II. Fuzzy Plant Model and Nonlinear Controller

We consider a multivariable nonlinear control system comprising a TSK fuzzy plant model and a nonlinear controller connected in closed-loop.

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A. TSK Fuzzy Plant Model

Let p be the number of fuzzy rules describing the nonlinear plant. The ith rule is of the following format:

$$\text{Rule } i\text{: IF} f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ and} \ldots \text{and } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i$$
$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \tag{1}$$

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t)), \alpha = 1, 2, \ldots, \Psi, i = 1, 2, \ldots, p, \Psi$ is a positive integer; $\mathbf{A}_i\Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known constant system and input matrices, respectively; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector; and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The system dynamics is described by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)) \tag{2}$$

where (3) and (4), shown at the bottom of the page, are known nonlinear functions, and $\mu_{M_\alpha^i}(x_\alpha(t)), \alpha = 1, 2, \ldots, n$ are known membership functions corresponding to the fuzzy terms M_α^i. (Thus, we assume that the TSK fuzzy plant model is known.)

B. Nonlinear Controller

A nonlinear controller consisting of c subcontrollers is proposed to close the feedback loop. The control output of the nonlinear controller is defined as

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \tag{5}$$

where $\mathbf{G}_j \in \Re^{m\times n}$, $j = 1, 2, \ldots, c$, are the feedback gain vectors that are to be designed, and

$$\sum_{j=1}^{c} m_j(\mathbf{x}(t)) = 1 \tag{6}$$

$$m_j(\mathbf{x}(t)) = \frac{\mu_{N^j}(\mathbf{x}(t))}{\sum_{k=1}^{c}\mu_{N^k}(\mathbf{x}(t))} \tag{7}$$

is a nonlinear function of $\mathbf{x}(t)$, and $\mu_{N^j}(\mathbf{x}(t))$, $j = 1, 2, \ldots, c$ are nonlinear gains to be designed. It should be noted that the nonlinear controller does not require $m_j(\mathbf{x}(t)) \in [0 \ \ 1]$ for all j.

III. Stability Analysis

A closed-loop system can be obtained by combining (2) and (5). Writing $w_i(\mathbf{x}(t))$ as w_i and $m_j(\mathbf{x}(t))$ as m_j, the fuzzy model-based nonlinear control system then becomes

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{H}_{ij}\mathbf{x}(t) \tag{8}$$

where

$$\mathbf{H}_{ij} = \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j. \tag{9}$$

To investigate the stability of the fuzzy model-based nonlinear control system of (8), we consider the following Lyapunov function in quadratic form:

$$V(\mathbf{x}(t)) = \frac{1}{2}\mathbf{x}(t)^T\mathbf{P}\mathbf{x}(t) \tag{10}$$

where $\mathbf{P} \in \Re^{n\times n}$ is a symmetric positive definite matrix. Then

$$\dot{V}(\mathbf{x}(t)) = \frac{1}{2}(\dot{\mathbf{x}}(t)^T\mathbf{P}\mathbf{x}(t) + \mathbf{x}(t)^T\mathbf{P}\dot{\mathbf{x}}(t)). \tag{11}$$

From (8), (11), and the property that $\sum_{i=1}^{p} w_i = \sum_{j=1}^{c} m_j = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j = 1$, we have

$$\begin{aligned}
\dot{V}&(\mathbf{x}(t)) \\
&= \frac{1}{2}\left[\left(\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{H}_{ij}\mathbf{x}(t)\right)^T \mathbf{P}\mathbf{x}(t) \right.\\
&\quad \left. + \mathbf{x}(t)^T\mathbf{P}\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{H}_{ij}\mathbf{x}(t)\right] \\
&= \frac{1}{2}\left\{\left[\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j (\mathbf{H}_{ij} + \mathbf{H}_m - \mathbf{H}_m)\mathbf{x}(t)\right]^T \mathbf{P}\mathbf{x}(t) \right.\\
&\quad \left. + \mathbf{x}(t)^T\mathbf{P}\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j (\mathbf{H}_{ij} + \mathbf{H}_m - \mathbf{H}_m)\mathbf{x}(t)\right\} \\
&= \frac{1}{2}\mathbf{x}(t)^T\left(\mathbf{H}_m^T\mathbf{P} + \mathbf{P}\mathbf{H}_m\right)\mathbf{x}(t) + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{x}(t)^T \\
&\quad \times [(\mathbf{H}_{ij} - \mathbf{H}_m)^T\mathbf{P} + \mathbf{P}(\mathbf{H}_{ij} - \mathbf{H}_m)]\mathbf{x}(t) \\
&= -\frac{1}{2}\mathbf{x}(t)^T\mathbf{Q}_m\mathbf{x}(t) - \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{x}(t)^T \\
&\quad \times (\mathbf{Q}_{ij} - \mathbf{Q}_m)\mathbf{x}(t)
\end{aligned} \tag{12}$$

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, \quad w_i(\mathbf{x}(t)) \in [0 \quad 1] \quad \text{for all } i \tag{3}$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{M_1^i}(x_1(t)) \times \mu_{M_2^i}(x_2(t)) \times \cdots \times \mu_{M_n^i}(x_n(t))}{\sum_{k=1}^{p}\left(\mu_{M_1^k}(x_1(t)) \times \mu_{M_2^k}(x_2(t)) \times \cdots \times \mu_{M_n^k}(x_n(t))\right)} \tag{4}$$

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where $\mathbf{H}_m \in \Re^{n\times n}$ is a stable symmetric matrix, which will be discussed later. $\mathbf{Q}_m \in \Re^{n\times n}$ is a symmetric positive definite matrix and $\mathbf{Q}_{ij} \in \Re^{n\times n}$ is a symmetric matrix. They are defined as

$$\mathbf{Q}_m = -\left(\mathbf{H}_m^T\mathbf{P} + \mathbf{P}\mathbf{H}_m\right) \quad (13)$$

$$\mathbf{Q}_{ij} = -\left(\mathbf{H}_{ij}^T\mathbf{P} + \mathbf{P}\mathbf{H}_{ij}\right), \quad i = 1,2,\ldots,p; \quad j = 1,2,\ldots,c. \quad (14)$$

From (12)

$$\dot{V}(\mathbf{x}(t)) = -\frac{1}{2}\mathbf{x}(t)^T\mathbf{Q}_m\mathbf{x}(t) - \frac{1}{2}\sum_{j=1}^{c} m_j\mathbf{x}(t)^T \times \left(\sum_{i=1}^{p} w_i\mathbf{Q}_{ij} - \mathbf{Q}_m\right)\mathbf{x}(t) \quad (15)$$

and we set

$$m_j = \frac{\mathbf{x}(t)^T\left(\sum_{i=1}^{p} w_i\mathbf{Q}_{ij} - \mathbf{Q}_m\right)\mathbf{x}(t)}{\sum_{k=1}^{c}\left[\mathbf{x}(t)^T\left(\sum_{i=1}^{p} w_i\mathbf{Q}_{ik} - \mathbf{Q}_m\right)\mathbf{x}(t)\right]} \quad \text{for } j = 1,2,\ldots,c. \quad (16)$$

By comparing (16) to (7), (16) gives the design of $m_j, j = 1,2,\ldots,c$, such that $\mu_{N^j}(\mathbf{x}(t)) = \mathbf{x}(t)^T(\sum_{i=1}^{p} w_i\mathbf{Q}_{ij} - \mathbf{Q}_m)\mathbf{x}(t)$, and (16) satisfies the condition of (6). Considering the denominator at the right-hand side (RHS) of (16), we have

$$\sum_{k=1}^{c}\left[\mathbf{x}(t)^T\left(\sum_{i=1}^{p} w_i\mathbf{Q}_{ik} - \mathbf{Q}_m\right)\mathbf{x}(t)\right] = \sum_{i=1}^{p} w_i\left[\mathbf{x}(t)^T\left(\sum_{k=1}^{c}\mathbf{Q}_{ik} - c\mathbf{Q}_m\right)\mathbf{x}(t)\right]. \quad (17)$$

We choose $\mathbf{Q}_{ik}$ and $\mathbf{Q}_m$ such that

$$\sum_{k=1}^{c}\mathbf{Q}_{ik} - c\mathbf{Q}_m > 0 \quad \text{for } i = 1,2,\ldots,p. \quad (18)$$

As $w_i(\mathbf{x}(t)) \in [0 \;\; 1]$ for all i, and at least one of the $w_i \neq 0$ (a property of the TSK fuzzy plant model), (18) implies that (17) will always be greater than or equal to zero. It is equal to zero only when $\mathbf{x}(t) = \mathbf{0}$. Under this condition, the output of the nonlinear controller of (5) should be zero and we choose $m_j = (1/c)$ for satisfying the condition of (6). From (15) and (16)

$$\dot{V}(\mathbf{x}(t)) = -\frac{1}{2}\mathbf{x}(t)^T\mathbf{Q}_m\mathbf{x}(t) - \frac{\sum_{j=1}^{c}\left[\mathbf{x}(t)^T\left(\sum_{i=1}^{p} w_i\mathbf{Q}_{ij} - \mathbf{Q}_m\right)\mathbf{x}(t)\right]^2}{2\sum_{k=1}^{c}\left[\mathbf{x}(t)^T\left(\sum_{i=1}^{p} w_i\mathbf{Q}_{ik} - \mathbf{Q}_m\right)\mathbf{x}(t)\right]}. \quad (19)$$

As the second term at the right side of (19) is semi-positive definite, we have

$$\dot{V}(\mathbf{x}(t)) \leq -\frac{1}{2}\mathbf{x}(t)^T\mathbf{Q}_m\mathbf{x}(t) \leq 0. \quad (20)$$

Hence, we can conclude that the fuzzy model-based nonlinear control system is asymptotically stable. The problem remaining is how to determine $\mathbf{Q}_m$. Considering (18), if $\sum_{k=1}^{c}\mathbf{Q}_{ik} > 0$ for $i = 1,2,\ldots,p$, it can be shown that there exists a $\mathbf{Q}_m$ such that $\sum_{k=1}^{c}\mathbf{Q}_{ik} - c\mathbf{Q}_m > 0$ for $i = 1,2,\ldots,p$ using the following theorem.

Theorem 1 (Spectral Shift) [24]: Let $\lambda_1, \lambda_2, \ldots, \lambda_\eta$ be the eigenvalues of a matrix $\mathbf{A} \in \Re^{n\times n}$. The eigenvalues of $\mathbf{A} - \varepsilon\mathbf{I}$ are $\lambda_1 - \varepsilon, \lambda_2 - \varepsilon, \ldots, \lambda_\eta - \varepsilon$, where ε is a scalar.

Proof: Let $\mathbf{Q}_i = \sum_{k=1}^{c}\mathbf{Q}_{ik} > 0, i = 1,2,\ldots,p$. By using the spectral shift property of Theorem 1, it can be seen that $\sum_{k=1}^{c}\mathbf{Q}_{ik} - \varepsilon\mathbf{I} = \sum_{k=1}^{c}(\mathbf{Q}_{ik} - (\varepsilon/c)\mathbf{I}) > 0$ if $\min_i \lambda_{\min}(\mathbf{Q}_i) > \varepsilon > 0$, where $\min_i \lambda_{\min}(\mathbf{Q}_i)$ denotes the smallest eigenvalues among $\mathbf{Q}_i, \mathbf{I}$ is the identity matrix. By comparing $\sum_{k=1}^{c}\mathbf{Q}_{ik} - c\mathbf{Q}_m > 0$ of (18) with $\sum_{k=1}^{c}\mathbf{Q}_{ik} - \varepsilon\mathbf{I} > 0$ term by term, we have $c\mathbf{Q}_m = \varepsilon\mathbf{I} \Rightarrow \mathbf{Q}_m = (\varepsilon/c)\mathbf{I} > 0$. Consequently, we can conclude that if $\sum_{k=1}^{c}\mathbf{Q}_{ik} > 0$, there must exist a positive definite matrix $\mathbf{Q}_m$ such that $\sum_{k=1}^{c}\mathbf{Q}_{ik} - c\mathbf{Q}_m > 0$. In the stability analysis, we need a stable matrix $\mathbf{H}_m$ to guarantee the system stability. The existence of $\mathbf{H}_m$ will be as follows. By multiplying $\mathbf{P}^{-1}$ to both sides of (13), we have $-\mathbf{P}^{-1}\mathbf{Q}_m\mathbf{P}^{-1} = \mathbf{P}^{-1}\mathbf{H}_m^T + \mathbf{H}_m\mathbf{P}^{-1}$. As $\mathbf{Q}_m = (\varepsilon/c)\mathbf{I}$, $-((\varepsilon\mathbf{P}^{-1}\mathbf{P}^{-1})/c) = \mathbf{P}^{-1}\mathbf{H}_m^T + \mathbf{H}_m\mathbf{P}^{-1} \Rightarrow -((\varepsilon\mathbf{P}^{-1}\mathbf{P}^{-1})/c) = (-\mathbf{P}^{-1})(-\mathbf{H}_m^T) + (-\mathbf{H}_m)(-\mathbf{P}^{-1})$. Let $\bar{\mathbf{Q}} = ((\varepsilon\mathbf{P}^{-1}\mathbf{P}^{-1})/c)$ which is a symmetric positive definite matrix, and $\bar{\mathbf{P}} = \bar{\mathbf{P}}^T = -\mathbf{P}^{-1}$ which is a symmetric negative definite matrix and $\bar{\mathbf{H}}_m = \bar{\mathbf{H}}_m^T = -\mathbf{H}_m$, we have a Lyapunov equation $-\bar{\mathbf{Q}} = \bar{\mathbf{P}}^T\bar{\mathbf{H}}_m + \bar{\mathbf{H}}_m\bar{\mathbf{P}}$. Once $\bar{\mathbf{P}}$ is known, a stable matrix $\mathbf{H}_m$ can be solved. **QED**

From the above, we obtain $\mathbf{Q}_m = (\varepsilon/c)\mathbf{I} > 0$ and prove the existence of $\mathbf{H}_m$. The stable matrix $\mathbf{H}_m$ in (13) is not necessary to be known as the nonlinear controller of (16) depends on $\mathbf{Q}_m$ but not $\mathbf{H}_m$.

A sufficient condition for the stability of the fuzzy model-based nonlinear control system can be summarized by the following lemma.

Lemma 1: A fuzzy model-based nonlinear control system of (8) is guaranteed to be stable if we choose the nonlinear gains of the nonlinear controller of (5) as (see the equation at the bottom of the page) $\min_i \lambda_{\min}(\sum_{k=1}^{c}\mathbf{Q}_{ik}) > \varepsilon > 0$, and there is a common solution of $\mathbf{P}$ for the following p linear matrix inequalities:

$$\sum_{k=1}^{c}\mathbf{Q}_{ik} > 0 \quad \text{for all } i = 1,2,\ldots,p$$

$$\begin{cases} \mu_{\mathrm{N}^j}(\mathbf{x}(t)) = \mathbf{x}(t)^T\left(\sum_{i=1}^{p} w_i\mathbf{Q}_{ij} - \frac{\varepsilon}{c}\mathbf{I}\right)\mathbf{x}(t), & \text{when } \mathbf{x}(t) \neq \mathbf{0} \\ \mu_{\mathrm{N}^j}(\mathbf{x}(t)) = \frac{1}{c}, & \text{when } \mathbf{x}(t) = \mathbf{0} \end{cases} \quad \text{for } j = 1,2,\ldots,c$$

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where

$$\mathbf{Q}_{ij} = -\left(\mathbf{H}_{ij}^T\mathbf{P} + \mathbf{P}\mathbf{H}_{ij}\right) \quad \text{for } i = 1,2,\ldots,p \quad j = 1,2,\ldots,c$$
$$\mathbf{H}_{ij} = \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j.$$

Lemma 1 states the way of choosing the nonlinear gains of the nonlinear controller. The number of subcontrollers is not necessarily the same as that of the TSK fuzzy plant model. This gives a flexibility of designing the nonlinear controller. With a smaller number of subcontrollers, the nonlinear controller is simpler in structure and lower in cost. The number of linear matrix inequalities is p, instead of $p(p+1)/2$, as stated in [6].

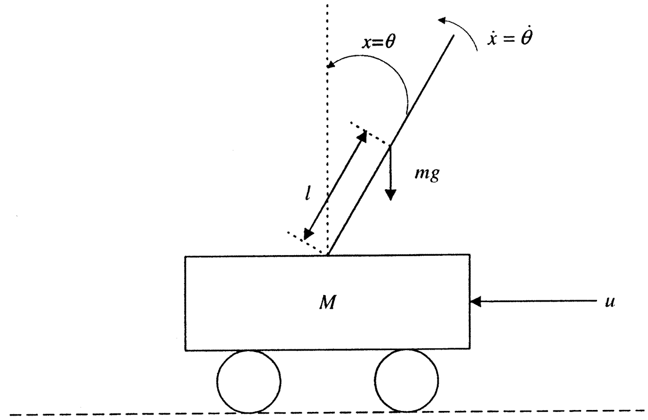

Fig. 1. Cart-pole typed inverted pendulum system.

IV. Solving the Stability Conditions and Obtaining the Feedback Gains

In this section, the problems of solving the stability conditions derived in the previous section and obtaining the feedback gains of the proposed nonlinear controller will be tackled using GA with arithmetic crossover and nonuniform mutation [25]. From Lemma 1, the closed-loop control system formed by (2) and (5) is stable if there exists a transformation matrix $\mathbf{P}$ and $\mathbf{G}_j$, $j = 1,2,\ldots,p$, satisfying the following condition:

$$-\sum_{j=1}^{c}[(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)^T\mathbf{P} + \mathbf{P}(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)] > 0 \quad \text{for } i = 1,2,\ldots,p. \quad (21)$$

Using GA, we can find

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}$$

and

$$\mathbf{G}_j = \begin{bmatrix} G_{11}^j & G_{12}^j & \cdots & G_{1n}^j \\ G_{21}^j & G_{22}^j & \cdots & G_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ G_{m1}^j & G_{m2}^j & \cdots & G_{mn}^j \end{bmatrix}$$

such that the conditions of (21) are satisfied. In order to make $\mathbf{P}$ to be symmetric, we let $P_{ij} = P_{ji}, i = 1,2,\ldots n; j = 1,2,\ldots,n$. The fitness function is defined as follows:

$$\text{fitness} = \sum_{i=1}^{p} n_i\lambda_{\max}\left(\sum_{j=1}^{c}[(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)^T \times \mathbf{P} + \mathbf{P}(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)]\right) \quad (22)$$

where $n_i > 0, i = 1,2,\ldots,p$, is a variable to be adjusted and $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of the argument. The problems of finding $\mathbf{P}$ and $\mathbf{G}_j$ are now formulated into a minimization problem. The aim is to minimize the fitness function of (22) with $\mathbf{P}$ and $\mathbf{G}_j$ using GA with arithmetic crossover and nonuniform mutation [25]. As $\mathbf{P}$ and $\mathbf{G}_j$ are the variables of the fitness function of (22), they are used to form the genes of the chromosomes. Finding the solution to this minimization problem, however, does not imply that the conditions of (21) are immediately satisfied. Hence, different $n_i, i = 1,2,\ldots,p$ may need to be used to weight the terms of (22) in order to change the significance of different terms on the RHS of (22). For instance, if one of the terms in (22) is very negative, the conditions of (21) may not be satisfied because the fitness value has been dominated by the effect of that term. A small value of n_i corresponding to that term can be used to attenuate the effect of that term in the fitness function. The function of n_i of (22) is to make the stability conditions of (21) satisfied easier.

The procedure for finding the nonlinear controller can be summarized as follows.

Step 1) Obtain the mathematical model of the nonlinear plant to be controlled.

Step 2) Obtain the TSK fuzzy plant model for the system stated in Step 1 by means of a fuzzy modeling method (e.g., the method proposed in [3] and [7]).

Step 3) Determine the number of subcontrollers of the nonlinear controller. Take the elements of $\mathbf{P}, \mathbf{G}_j$ and n_i as the genes to form the chromosome. Define the boundaries of each gene. Determine the number of iterations for searching and the parameters (probabilities of crossover and mutation, and the shaping value [25] for the nonuniform mutation) for the GA process. Solve $\mathbf{P}, \mathbf{G}_j$, and $n_i, j = 1,2,\ldots,c; i = 1,2,\ldots,p$ with the fitness function defined in (22) using GA.

Step 4) Design the nonlinear gains of the nonlinear controller based on Lemma 1.

V. Application Example

An application example on stabilizing a cart-pole typed inverted pendulum system [6] is given in this section. A nonlinear controller will be used to control the plant. Simulation results will be given. We shall see that the number of LMIs involved is p. The nonlinear controllers will be designed based on the procedure given in Section IV.

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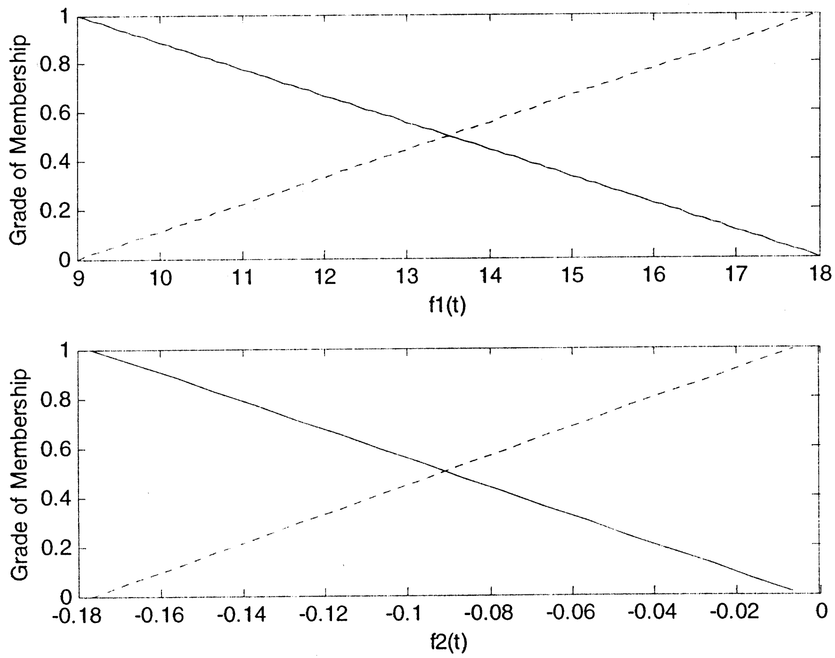

Fig. 2. Membership functions for the fuzzy plant model.

Step 1: Fig. 1 shows the diagram of the inverted pendulum system. Its dynamic equation is given by

$$\ddot{\theta}(t)=\frac{g\sin(\theta(t))-aml\dot{\theta}(t)^2\sin(2\theta(t))/2-a\cos(\theta(t))u(t)}{4l/3-aml\cos^2(\theta(t))} \tag{23}$$

where

θ	angular displacement of the pendulum;
$g=9.8$ m/s^2	acceleration due to gravity;
$m=2$ kg	mass of the pendulum;
$a=1/(m+M), M=8$ kg	mass of the cart;
$2l=1$ m	length of the pendulum;
u	force applied to the cart.

The objective of this application example is to design a fuzzy controller to close the feedback loop of (23) such that $\theta=0$ at steady state.

Step 2: The nonlinear plant can be represented by a fuzzy model with four fuzzy rules. The ith rule is given by

$$\begin{aligned}&\text{Rule } i\text{: IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND } f_2(\mathbf{x}(t)) \text{ is } M_2^i\\ &\qquad\text{THEN } \dot{\mathbf{x}}(t)=\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_iu(t) \quad \text{for } i=1,2,3,4\end{aligned} \tag{24}$$

so that the system dynamics is described by

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{4}w_i(\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_iu(t)) \tag{25}$$

where

$$\mathbf{x}(t)=[x_1(t) \quad x_2(t)]^T=[\theta(t) \quad \dot{\theta}(t)]^T$$

$$\theta(t)\in[\theta_{\min} \quad \theta_{\max}]=\left[-\frac{22\pi}{45} \quad \frac{22\pi}{45}\right]$$

and

$$\dot{\theta}(t)\in[\dot{\theta}_{\min} \quad \dot{\theta}_{\max}]=[-5 \quad 5]$$

$$f_1(\mathbf{x}(t))=\frac{g-amlx_2(t)^2\cos(x_1(t))}{4l/3-aml\cos^2(x_1(t))}\left(\frac{\sin(x_1(t))}{x_1(t)}\right)$$

and

$$f_2(\mathbf{x}(t))=-\frac{a\cos(x_1(t))}{4l/3-aml\cos^2(x_1(t))}$$

$$\mathbf{A}_1=\mathbf{A}_2=\begin{bmatrix}0 & 1\\ f_{1_{\min}} & 0\end{bmatrix}$$

and

$$\mathbf{A}_3=\mathbf{A}_4=\begin{bmatrix}0 & 1\\ f_{1_{\max}} & 0\end{bmatrix}$$

$$\mathbf{B}_1=\mathbf{B}_3=\begin{bmatrix}0\\ f_{2_{\min}}\end{bmatrix}$$

and

$$\mathbf{B}_2=\mathbf{B}_4=\begin{bmatrix}0\\ f_{2_{\max}}\end{bmatrix}$$

$$f_{1_{\min}}=9 \quad \text{and} \quad f_{1_{\max}}=18$$

$$f_{2_{\min}}=-0.1765 \quad \text{and} \quad f_{2_{\max}}=-0.0052$$

$$w_i=\frac{\mu_{M_1^i}(f_1(\mathbf{x}(t)))\times\mu_{M_2^i}(f_2(\mathbf{x}(t)))}{\sum_{l=1}^{4}\left(\mu_{M_1^l}(f_1(\mathbf{x}(t)))\times\mu_{M_2^l}(f_2(\mathbf{x}(t)))\right)}$$

$$\mu_{M_1^\beta}(f_1(\mathbf{x}(t)))=\frac{-f_1(\mathbf{x}(t))+f_{1_{\max}}}{f_{1_{\max}}-f_{1_{\min}}} \quad \text{for } \beta=1,2$$

$$\mu_{M_1^\delta}(f_1(\mathbf{x}(t)))=1-\mu_{M_1^1}(f_1(\mathbf{x}(t))) \quad \text{for } \delta=3,4$$

$$\mu_{M_2^\kappa}(f_2(\mathbf{x}(t)))=\frac{-f_2(\mathbf{x}(t))+f_{2_{\max}}}{f_{2_{\max}}-f_{2_{\min}}} \quad \text{for } \kappa=1,3$$

and

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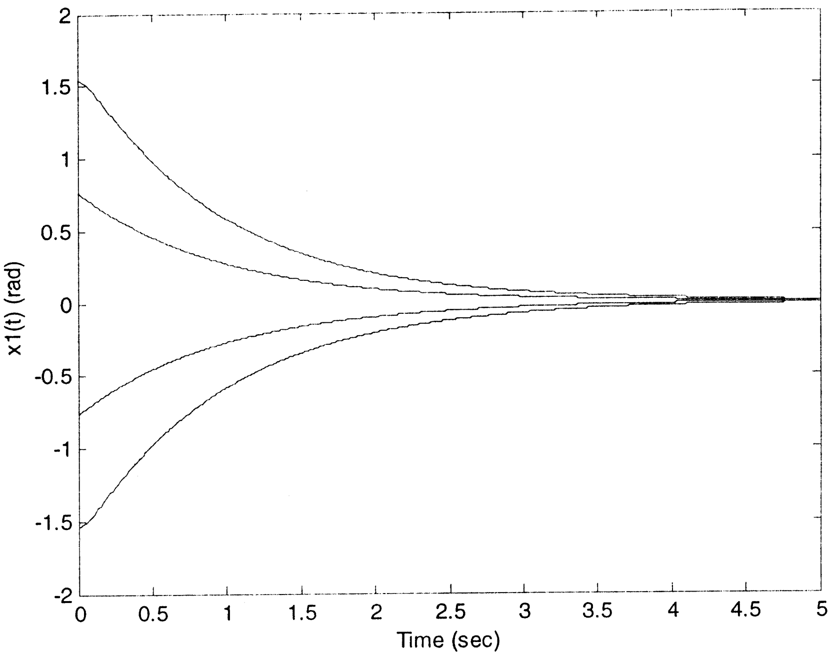

Fig. 3. Responses of $x_1(t)$ of the inverted pendulum system.

$$\mu_{M_2^\phi}(f_2(\mathbf{x}(t))) = 1 - \mu_{M_2^1}(f_2(\mathbf{x}(t))) \quad \text{for } \phi = 2, 4$$

are the membership functions, as shown in Fig. 2. (Details about the derivation of the TSK fuzzy plant model for the inverted pendulum system can be found in [5].)

Step 3: When a nonlinear controller having four subcontrollers is designed for the plant of (25), we have

$$u(t) = \sum_{j=1}^{4} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t). \qquad (26)$$

In order to guarantee the closed-loop system stability and obtain the feedback gains of the nonlinear controller of (26), from (22), we have to solve the $\mathbf{P}$ and $\mathbf{G}_j, j = 1, 2, 3, 4; i = 1, 2, 3, 4$, using GA with the following fitness function:

$$\text{fitness} = \sum_{i=1}^{4} n_i \lambda_{\max}\left[\sum_{j=1}^{4}[(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)^{\mathrm{T}} \times \mathbf{P} + \mathbf{P}(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)]\right]. \qquad (27)$$

The minimum and maximum values of each element $\mathbf{P}$ are chosen to be -1 and 1, respectively. The minimum and maximum values of each element of $\mathbf{G}_1$–$\mathbf{G}_4$ are chosen to be 0 and 4500, respectively. The minimum and maximum values of n_i are chosen to be 0 and 10, respectively. The population size is 10 and the initial values of $\mathbf{P}, \mathbf{G}_j$, and n_i are randomly generated. After applying the GA process, we obtain

$$\mathbf{P} = \begin{bmatrix} 0.9877 & 0.0678 \\ 0.0678 & 0.0666 \end{bmatrix}$$

and

$$\mathbf{G}_1 = [4176.4868 \quad 4438.2388]$$
$$\mathbf{G}_2 = [4200.1314 \quad 3710.2710]$$
$$\mathbf{G}_3 = [4223.6645 \quad 3631.4680]$$

and

$$\mathbf{G}_4 = [4308.6079 \quad 4053.5941].$$

Step 4: According to Lemma 1, the nonlinear gains are designed as (28), shown at the bottom of the page.

As $\min_i \lambda_{\min}(\sum_{k=1}^{c} \mathbf{Q}_{ik}) = 1.3982 > \varepsilon > 0$, we choose $\varepsilon = 0.1$. Figs. 3 and 4 show the responses of the system states under the initial conditions of

$$\mathbf{x}(0) = \begin{bmatrix} \frac{22\pi}{45} & 0 \end{bmatrix}^T, \quad \mathbf{x}(0) = \begin{bmatrix} \frac{11\pi}{45} & 0 \end{bmatrix}^T$$
$$\mathbf{x}(0) = \begin{bmatrix} -\frac{11\pi}{45} & 0 \end{bmatrix}^T, \quad \text{and} \quad \mathbf{x}(0) = \begin{bmatrix} -\frac{22\pi}{45} & 0 \end{bmatrix}^T.$$

From this example, it can be seen that the number of LMIs is fixed to be four (the number of rules of the TSK fuzzy plant model), which will not be affected by the number of subcontrollers of the nonlinear controller.

$$\begin{cases} \mu_{\mathrm{N}^j}(\mathbf{x}(t)) = \mathbf{x}(t)^T \left(\displaystyle\sum_{i=1}^{4} w_i \mathbf{Q}_{ij} - \frac{\varepsilon}{4}\mathbf{I}\right)\mathbf{x}(t), & \text{when } \mathbf{x}(t) \neq \mathbf{0} \\ \mu_{\mathrm{N}^j}(\mathbf{x}(t)) = \dfrac{1}{4}, & \text{when } \mathbf{x}(t) = \mathbf{0} \end{cases} \quad \text{for } j = 1, 2, 3, 4 \qquad (28)$$

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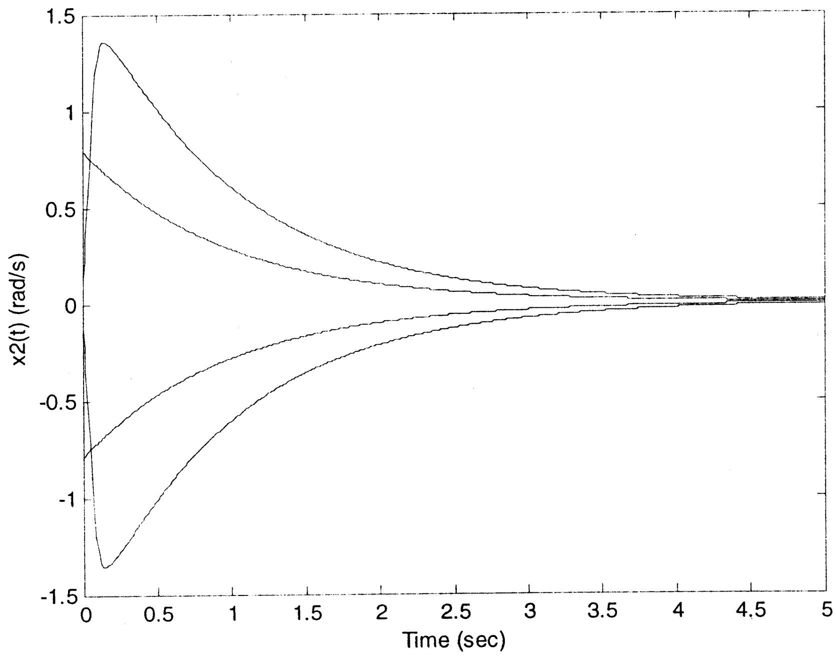

Fig. 4. Responses of $x_2(t)$ of the inverted pendulum system.

VI. Conclusion

The stability analysis and design of TSK fuzzy model-based nonlinear control systems have been discussed. A stability criterion has been derived. This criterion involves p linear matrix inequalities irrespective of the number of the subcontrollers, where p is the number of rules of the TSK fuzzy plant model. The number of subcontrollers of the nonlinear controller need not be the same as that of the TSK fuzzy plant model. A design on the nonlinear gains of the nonlinear controller has been presented. GA has been used to find the solution to the stability conditions and determine the feedback gains of the subcontrollers. An application example has been used to illustrate the stabilizability of the proposed nonlinear controllers and the design procedure.

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