Linear Controllers for Fuzzy Systems Subject to Unknown Parameters: Stability Analysis and Design Based on Linear Matrix Inequality (LMI) Approach[1]

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Abstract

This paper presents a design approach of linear controllers for nonlinear systems with unknown parameters within known bounds. The plant is represented by a fuzzy model. Stability condition will be derived based on Lyapunov stability theory and formulated into an LMI (Linear Matrix Inequality) problem. The linear controller can be designed by solving the LMIs. To illustrate the merits and the design procedure of the proposed linear controller, an application example on stabilizing an inverted pendulum on a cart with unknown parameters is given.

I. INTRODUCTION

Control of nonlinear systems is difficult because we do not have systematic mathematical tools to help finding a necessary and sufficient condition to guarantee the stability and performance. The problem will become more complex if some of the parameters of the plant are unknown. By using a TSK fuzzy plant model [1-2, 7, 14] a nonlinear system can be expressed as a weighted sum of some simple sub-systems. This model gives a fixed structure to some of the nonlinear systems and thus facilitates the analysis of the systems. There are two ways to obtain the fuzzy plant model: 1) by performing system identification methods based on the input-output data of the plant [1-2, 7, 14], 2) deriving from the mathematical model of the nonlinear plant [5]. Stability of fuzzy model based systems has been investigated recently [4, 6-13]. A linear controller [13] was also proposed to control the plant. Most of the fuzzy controllers proposed are functions of the grades of membership of the fuzzy plant model. Hence, the membership functions of the fuzzy plant model must be known. It means that the parameters of the nonlinear plant must be known or be constant when the identification method is used to derive the fuzzy plant model. Practically, the parameters of many nonlinear plants will change during the operation, e.g., the load of a dc-dc converter, the number of passengers on board a train. In these cases, the robustness property of the fuzzy controller is an important concern.

A linear controller is proposed in this paper to tackle nonlinear plants represented by a fuzzy plant model of which the membership functions depends on some unknown plant parameters of known bounds. Stability of the closed-loop system will be analyzed based on the Lyapunov stability theory. It will be shown that the stability condition derived will be the same as that of the relaxed stability condition in [6] but the structure of the proposed linear controller is much simpler. The derived stability condition will be formulated into an LMI (Linear Matrix Inequality) problem [3]. By solving the LMIs using MATLAB, the parameters of the linear controller can be obtained.

II. FUZZY PLANT MODEL AND LINEAR CONTROLLER

An uncertain multivariable nonlinear system is to be controlled. The plant has the following form:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t))\mathbf{u}(t) \tag{1}$$

where $\mathbf{F}(\mathbf{x}(t)) \in \Re^{n\times n}$ and $\mathbf{B}(\mathbf{x}(t)) \in \Re^{n\times m}$ are the system matrix and input matrix respectively, both of them have known structure but subject to unknown parameters, $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The system of (1) is represented by a fuzzy plant model, which expresses the multivariable nonlinear system as a weighted sum of linear systems. A linear controller is to be designed to close the feedback loop.

A. Fuzzy Plant Model

Let p be the number of fuzzy rules describing the multivariable nonlinear plant of (1), the i-th rule is of the following format,

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i and … and $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \tag{2}$$

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$ in terms of the system states and unknown parameters of the nonlinear plant, α 1, …, Ψ, i 1, …, p, Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known system and input matrices respectively of the i-th rule sub-system. The system dynamics are described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\big), \tag{3}$$

$$\text{where } \sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1,\ w_i(\mathbf{x}(t)) \in [0 \quad 1] \text{ for all } i \tag{4}$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\Big(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\Big)} \tag{5}$$

is a nonlinear function of the system states and the unknown parameters. (Fuzzy modeling is discussed in [1-2, 7, 14].)

B. Linear Controller

A linear controller of the following form is employed to control the nonlinear plant of (1).

$$\mathbf{u}(t) = \left(\mathbf{G} - \sum_{j=1}^{p} m_j \mathbf{R}\mathbf{B}_j^{\mathrm{T}}\mathbf{P}\right)\mathbf{x}(t) \tag{6}$$

$$\text{where } m_j = 1 \text{ or } 0, j = 1, 2, \ldots, p \tag{7}$$

will be determined later. $\mathbf{G} \in \Re^{m\times n}$ is a feedback gain matrix, $\mathbf{R} \in \Re^{m\times m}$ and $\mathbf{P} \in \Re^{n\times n}$ are symmetric positive definite matrices to be designed, $(\cdot)^{\mathrm{T}}$ denotes the transpose of a matrix

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or vector. If all m_j are zero, it means the nonlinear system of (1) is open-loop stable. This can be seen in the later sections.

III. Stability and Linear Controller Design

To design the linear controller, the values of $m_j, j = 1, 2, \ldots, p$, are determined under the consideration of the system stability. From (3), (6), and writing $w_i(\mathbf{x}(t))$ as w_i,

$$\mathbf{x}(t) = \sum_{i=1}^{p} w_i \left(\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \left(\mathbf{G} - \sum_{j=1}^{p} m_j \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \right) \mathbf{x}(t) \right)$$
$$= \sum_{i=1}^{p} w_i \left(\overline{\mathbf{A}}_i \mathbf{x}(t) - \mathbf{B}_i \sum_{j=1}^{p} m_j \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \right) \quad (8)$$

where $\overline{\mathbf{A}}_i = \mathbf{A}_i + \mathbf{B}_i \mathbf{G}$, $i = 1\ 2, \ldots, p$ (9)

From (8),

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \left(\overline{\mathbf{A}}_i - \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \right) \mathbf{x}(t) - \sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t)$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \left(\overline{\mathbf{A}}_i - \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \right) \mathbf{x}(t) + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i (w_j - m_j) \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \quad (10)$$

It can be observed from (10) that $\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i \overline{\mathbf{A}}_i \mathbf{x}(t)$ if $\mathbf{R} = \mathbf{0}$. This system is stable if there exists a common positive definite matrix $\mathbf{P}_A \in \Re^{n \times n}$ such that all $\overline{\mathbf{A}}_i^{\mathrm{T}} \mathbf{P}_A + \mathbf{P}_A \overline{\mathbf{A}}_i$ are negative definite (*Theorem 2.1* in [5]). In [6], Wang *et al.* derived a relaxed stability condition for the following fuzzy control law:

$$\mathbf{u}(t) = \sum_{j=1}^{p} w_j(\mathbf{x}(t)) \mathbf{G}_j \mathbf{x}(t) \quad (11)$$

where $\mathbf{G}_j \in \Re^{n \times m}$, $j = 1, 2, \ldots, p$, are feedback gain matrices. The fuzzy controller is function of $w_j(\mathbf{x}(t))$, implying that $w_j(\mathbf{x}(t))$ must be known. Practically, $w_j(\mathbf{x}(t))$ is related to the plant parameters and are unknown. A linear controller of (6), which does not involve $w_j(\mathbf{x}(t))$, is used instead such that the relaxed stability condition can still be achieved. Comparing (6) to (11), the structure of (6) is simpler. To investigate the system stability of (10), we consider the following quadratic Lyapunov function,

$$V = \frac{1}{2} \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \quad (12)$$

$$\dot{V} = \frac{1}{2} \left(\dot{\mathbf{x}}(t)^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \dot{\mathbf{x}}(t) \right) \quad (13)$$

From (10) and (13),

$$\dot{V} = \frac{1}{2} \left(\begin{array}{l} \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \left(\overline{\mathbf{A}}_i - \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \right) \mathbf{x}(t) \\ + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i (w_j - m_j) \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \end{array} \right)^{\mathrm{T}} \mathbf{P} \mathbf{x}(t)$$
$$+ \frac{1}{2} \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \left(\begin{array}{l} \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \left(\overline{\mathbf{A}}_i - \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \right) \mathbf{x}(t) \\ + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i (w_j - m_j) \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \end{array} \right)$$

$$= \frac{1}{2} \mathbf{x}(t)^{\mathrm{T}} \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{Q}_{ij} \mathbf{x}(t) + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i (w_j - m_j) \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \quad (14)$$

where,

$$\mathbf{Q}_{ij} = \overline{\mathbf{A}}_i^{\mathrm{T}} \mathbf{P} + \mathbf{P} \overline{\mathbf{A}}_i - \mathbf{P} \mathbf{B}_j \mathbf{R} \mathbf{B}_i^{\mathrm{T}} \mathbf{P} - \mathbf{P} \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \quad (15)$$

From (14),

$$\dot{V} = \frac{1}{4} \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{x}(t)^{\mathrm{T}} \mathbf{J}_{ij} \mathbf{x}(t) + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i (w_j - m_j) \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \quad (16)$$

where,

$$\mathbf{J}_{ij} = \mathbf{Q}_{ij} + \mathbf{Q}_{ji} \quad (17)$$

From (17), let,

$$\mathbf{J}_{ij} < 0 \text{ for all } i \text{ and } j \quad (18)$$

$$m_j = \frac{1 + \mathrm{sgn}\left(\sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \right)}{2} \text{ for all } j = 1, 2, \ldots, p \quad (19)$$

$$\mathrm{sgn}(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ -1 & \text{otherwise} \end{cases} \quad (20)$$

From (16) to (19), we have,

$$\dot{V} = \frac{1}{4} \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{x}(t)^{\mathrm{T}} \mathbf{J}_{ij} \mathbf{x}(t) + \sum_{j=1}^{p} \left[\left(w_j - \frac{1}{2} \right) + \frac{1}{2} - \frac{1 + \mathrm{sgn}\left(\sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \right)}{2} \right] \times \sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t)$$
$$\leq \frac{1}{4} \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{x}(t)^{\mathrm{T}} \mathbf{J}_{ij} \mathbf{x}(t) + \sum_{j=1}^{p} \left| w_j - \frac{1}{2} \right| \left| \sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \right| - \sum_{j=1}^{p} \frac{1}{2} \left| \sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \right|$$
$$= \frac{1}{4} \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{x}(t)^{\mathrm{T}} \mathbf{J}_{ij} \mathbf{x}(t) - \sum_{j=1}^{p} \left(\frac{1}{2} - \left| w_j - \frac{1}{2} \right| \right) \left| \sum_{i=1}^{p} w_i \mathbf{x}(t)^{\mathrm{T}} \mathbf{P} \mathbf{B}_i \mathbf{R} \mathbf{B}_j^{\mathrm{T}} \mathbf{P} \mathbf{x}(t) \right| \quad (21)$$

As $w_i - \frac{1}{2} \in \left[-\frac{1}{2} \ \ \frac{1}{2} \right]$, $i = 1, 2, \ldots, p$, from the property of the fuzzy plant model, we have $\frac{1}{2} - \left| w_j - \frac{1}{2} \right| \geq 0$, $j = 1, 2, \ldots, p$. Therefore, (21) implies,

$$\dot{V} \leq \frac{1}{4} \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{x}(t)^{\mathrm{T}} \mathbf{J}_{ij} \mathbf{x}(t) \leq 0 \quad (22)$$

Equality holds when $\mathbf{x}(t) = \mathbf{0}$. Hence, the closed-loop system of (10) is asymptotically stable, i.e., $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$. From (19), to make the values of $m_j, j = 1, 2, \ldots p$, to be either 0 or 1, a sufficient condition is given by,

$$m_j = \begin{cases} 0 & \text{if } \mathbf{PB}_i\mathbf{RB}_j^{\mathrm{T}}\mathbf{P} \le \mathbf{0} \text{ for } i = 1, 2, \cdots, p; j = 1, 2, \cdots, p \\ 1 & \text{if } \mathbf{PB}_i\mathbf{RB}_j^{\mathrm{T}}\mathbf{P} \ge \mathbf{0} \text{ for } i = 1, 2, \cdots, p; j = 1, 2, \cdots, p \end{cases} \quad (23)$$

$\mathbf{PB}_i\mathbf{RB}_j^{\mathrm{T}}\mathbf{P}$, $i = 1, 2, \ldots, p; j = 1, 2, \ldots, p$, are symmetric. This happens under a sufficient condition that there exist matrices $\mathbf{S}_j \in \Re^{m \times m}$, $j = 1, 2, \ldots, p$, such that $\mathbf{B}_i = \mathbf{B}_i\mathbf{S}_j$ or $\mathbf{B}_i = -\mathbf{B}_i\mathbf{S}_j$, and $\mathbf{RS}_j^{\mathrm{T}} = \mathbf{S}_j\mathbf{R}^{\mathrm{T}} \ge \mathbf{0}$, $i = 1, 2, \ldots, p; j = 1, 2, \ldots, p$. Under the condition of $\mathbf{B}_i = \mathbf{B}_j\mathbf{S}_j$, it can be seen that $\mathbf{x}(t)^{\mathrm{T}}\mathbf{PB}_i\mathbf{RB}_j^{\mathrm{T}}\mathbf{Px}(t) = \mathbf{x}(t)^{\mathrm{T}}\mathbf{PB}_i\mathbf{RS}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}\mathbf{Px}(t)$. As $\mathbf{RS}_j^{\mathrm{T}} = \mathbf{S}_j\mathbf{R}^{\mathrm{T}} \ge \mathbf{0}$, $j = 1, 2, \ldots, p$, is symmetric, there must exists a symmetric matrix $\mathbf{W}_j \in \Re^{m \times m}$ such that $\mathbf{W}_j\mathbf{W}_j^{\mathrm{T}} = \mathbf{RS}_j^{\mathrm{T}} = \mathbf{S}_j\mathbf{R}^{\mathrm{T}} \ge \mathbf{0}$. Then $\mathbf{x}(t)^{\mathrm{T}}\mathbf{PB}_i\mathbf{RS}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}\mathbf{Px}(t) = \mathbf{x}(t)^{\mathrm{T}}\mathbf{PB}_i\mathbf{W}_j\mathbf{W}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}\mathbf{Px}(t)$. Let $\mathbf{y}(t)^{\mathrm{T}} = \mathbf{W}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}\mathbf{Px}(t)$, then $\mathbf{x}(t)^{\mathrm{T}}\mathbf{PB}_i\mathbf{RB}_j^{\mathrm{T}}\mathbf{Px}(t) = \mathbf{y}(t)^{\mathrm{T}}\mathbf{y}(t) \ge 0$. Similarly, under the condition of $\mathbf{B}_i = -\mathbf{B}_i\mathbf{S}_j$, it can be shown that $-\mathbf{x}(t)^{\mathrm{T}}\mathbf{PB}_i\mathbf{B}_j^{\mathrm{T}}\mathbf{Px}(t) = -\mathbf{y}(t)^{\mathrm{T}}\mathbf{y}(t) \le 0$. We can state the design of (6) and the stability condition of (18) (the finding of **R** and **P**) into an LMI problem. From (15) and (19),

$$\overline{\mathbf{A}}_i^{\mathrm{T}}\mathbf{P} + \mathbf{P}\overline{\mathbf{A}}_i + \overline{\mathbf{A}}_j^{\mathrm{T}}\mathbf{P} + \mathbf{P}\overline{\mathbf{A}}_j - 2\mathbf{P}\left(\mathbf{B}_j\mathbf{RB}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{RB}_j^{\mathrm{T}}\right)\mathbf{P} < 0$$
$$\Rightarrow \mathbf{P}^{-1}\begin{bmatrix} \overline{\mathbf{A}}_i^{\mathrm{T}}\mathbf{P} + \mathbf{P}\overline{\mathbf{A}}_i + \overline{\mathbf{A}}_j^{\mathrm{T}}\mathbf{P} + \mathbf{P}\overline{\mathbf{A}}_j \\ -2\mathbf{P}\left(\mathbf{B}_j\mathbf{RB}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{RB}_j^{\mathrm{T}}\right)\mathbf{P} \end{bmatrix}\mathbf{P}^{-1} < 0$$
$$\Rightarrow \mathbf{P}^{-1}\overline{\mathbf{A}}_i^{\mathrm{T}} + \overline{\mathbf{A}}_i\mathbf{P}^{-1} + \mathbf{P}^{-1}\overline{\mathbf{A}}_j^{\mathrm{T}} + \overline{\mathbf{A}}_j\mathbf{P}^{-1} - 2\left(\mathbf{B}_j\mathbf{RB}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{RB}_j^{\mathrm{T}}\right) < 0 \quad (24)$$

It can be seen that (24) involves $\frac{p(p+1)}{2}$ LMIs as some of them are the same. The solutions of **R** and **P** will both be symmetric. The results are summarized as follows.

Lemma 1: *The closed-loop control system of* (8) *is guaranteed to be asymptotically stable if the parameters of the linear controller,* **R** *and* **P** *are symmetric positive definite and satisfy the following LMIs,*

$$\mathbf{P}^{-1}\overline{\mathbf{A}}_i^{\mathrm{T}} + \overline{\mathbf{A}}_i\mathbf{P}^{-1} + \mathbf{P}^{-1}\overline{\mathbf{A}}_j^{\mathrm{T}} + \overline{\mathbf{A}}_j\mathbf{P}^{-1} - 2\left(\mathbf{B}_j\mathbf{RB}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{RB}_j^{\mathrm{T}}\right) < 0$$

for all i and j,
and

$$m_j = \begin{cases} 0 & \text{if } \mathbf{PB}_i\mathbf{RB}_j^{\mathrm{T}}\mathbf{P} \le \mathbf{0} \text{ for } i = 1, 2, \cdots, p; j = 1, 2, \cdots, p \\ 1 & \text{if } \mathbf{PB}_i\mathbf{RB}_j^{\mathrm{T}}\mathbf{P} \ge \mathbf{0} \text{ for } i = 1, 2, \cdots, p; j = 1, 2, \cdots, p \end{cases}$$

$\mathbf{PB}_i\mathbf{RB}_j^{\mathrm{T}}\mathbf{P}$ *is necessary to be symmetric.*

The design procedure is summarized as follows.

Step I). Obtain the fuzzy plant model by means of, e.g., fuzzy modeling methods [1-2, 7, 14] or other ways.

Step II). Find the gain **G** and obtain **P** and **R** by solving the LMIs in Lemma I.

Step III). Determine m_j, $j = 1, 2, \ldots, p$, according to Lemma 1. Obtain the control law according to (6).

IV. Application Example

An application example of cart-pole type inverted pendulum system [6] (Fig. 1) will be given in this section to show the design procedure of the linear controller.

Step I).The dynamic equation of the inverted pendulum is

$$\ddot{\theta}(t) = \frac{g\sin(\theta(t)) - aml\dot{\theta}(t)^2\sin(2\theta(t))/2 - a\cos(\theta(t))u(t)}{4l/3 - aml\cos^2(\theta(t))} \quad (25)$$

where θ is the angular displacement of the pendulum, g = 9.8m/s^2 is the acceleration due to gravity, m = 2kg is the mass of the pendulum, $a = 1/(m+M)$, $M \in [8 \quad 80]kg$ is the mass of the cart, $2l$ = 1m is the length of the pendulum, and u is the force applied to the cart. We design a linear controller such that $\theta = 0$ at steady state. (25) can be modeled by a fuzzy plant model having four rules.

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND $f_2(\mathbf{x}(t))$ is M_2^i

THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)$ for $i = 1, 2, 3, 4$ (26)

so that the system dynamics is described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{4} w_i\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)\right) \quad (27)$$

where $\mathbf{x}(t) = [x_1(t) \quad x_2(t)]^{\mathrm{T}} = [\theta(t) \quad \dot{\theta}(t)]^{\mathrm{T}}$, $\theta(t) \in [\theta_{\min} \quad \theta_{\max}] = \left[-\frac{7\pi}{18} \quad \frac{7\pi}{18}\right]$ and $\dot{\theta}(t) \in [\dot{\theta}_{\min} \quad \dot{\theta}_{\max}] = [-5 \quad 5]$;

$$f_1(\mathbf{x}(t)) = \frac{g - amlx_2(t)^2\cos(x_1(t))}{4l/3 - aml\cos^2(x_1(t))}\left(\frac{\sin(x_1(t))}{x_1(t)}\right) \text{ and}$$

$$f_2(\mathbf{x}(t)) = -\frac{a\cos(x_1(t))}{4l/3 - aml\cos^2(x_1(t))}; \quad \mathbf{A}_1 = \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ f_{1_{\min}} & 0 \end{bmatrix}$$

and $\mathbf{A}_3 = \mathbf{A}_4 = \begin{bmatrix} 0 & 1 \\ f_{1_{\max}} & 0 \end{bmatrix}$; $\mathbf{B}_1 = \mathbf{B}_3 = \begin{bmatrix} 0 \\ f_{2_{\min}} \end{bmatrix}$ and $\mathbf{B}_2 = \mathbf{B}_4 = \begin{bmatrix} 0 \\ f_{2_{\max}} \end{bmatrix}$; $f_{1_{\min}} = 9$ and $f_{1_{\max}} = 20$, $f_{2_{\min}} = -0.1765$ and $f_{2_{\max}} = -0.0052$;

$$w_i = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t)))}{\sum_{l=1}^{4}\left(\mu_{\mathrm{M}_1^l}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^l}(f_2(\mathbf{x}(t)))\right)};$$

$\mu_{\mathrm{M}_1^\beta}(f_1(\mathbf{x}(t))) = \frac{-f_1(\mathbf{x}(t)) + f_{1_{\max}}}{f_{1_{\max}} - f_{1_{\min}}}$ for β = 1, 2 and $\mu_{\mathrm{M}_1^\delta}(f_1(\mathbf{x}(t))) = 1 - \mu_{\mathrm{M}_1^1}(f_1(\mathbf{x}(t)))$ for δ = 3, 4; $\mu_{\mathrm{M}_2^\varepsilon}(f_2(\mathbf{x}(t))) = \frac{-f_2(\mathbf{x}(t)) + f_{2_{\max}}}{f_{2_{\max}} - f_{2_{\min}}}$ for ε = 1, 3 and $\mu_{\mathrm{M}_2^\phi}(f_2(\mathbf{x}(t))) = 1 - \mu_{\mathrm{M}_2^1}(f_2(\mathbf{x}(t)))$ for ϕ = 2, 4 are the membership functions.

Step II) A linear controller is designed for (25) such that,

$$\mathbf{u}(t) = \left(\mathbf{G} - \sum_{j=1}^{4} m_j\mathbf{RB}_j^{\mathrm{T}}\mathbf{P}\right)\mathbf{x}(t) \quad (28)$$

Choose $\mathbf{G} = \frac{\mathbf{G}_1 + \mathbf{G}_2 + \mathbf{G}_3 + \mathbf{G}_4}{2} = [3441.4886 \quad 786.6260]$ such that the eigenvalues of $\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i$, i = 1, 2, 3, 4, are −2 and −2. By solving the LMIs given in the Lemma 1, we have

$\mathbf{R} = 1885.8000$ and $\mathbf{P} = \begin{bmatrix} 19.0182 & 4.1711 \\ 4.1711 & 1.1205 \end{bmatrix}$.

Step III) According to Lemma 1, $m_j = 1$ for j = 1, 2, 3, 4. Hence, the linear control law is given by: $\mathbf{u}(t) = [6300.0384 \quad 1552.5546]\mathbf{x}(t)$

Fig. 2 to Fig. 3 show the responses of the system states with M = 8kg (solid line) and M = 80kg (dotted line) under the initial condition of $\mathbf{x}(0) = \begin{bmatrix} \frac{5\pi}{18} & 0 \end{bmatrix}^T$, $\mathbf{x}(0) = \begin{bmatrix} \frac{7\pi}{18} & 0 \end{bmatrix}^T$ and $\mathbf{x}(0) = \begin{bmatrix} \frac{22\pi}{45} & 0 \end{bmatrix}^T$. The plant used in this paper is the same as that used in [6]. It can be observed that the responses provided by our linear controller, which has a simpler structure, are similar to those provided by the fuzzy controller in [6].

V. Conclusion

A design of linear controller for nonlinear plants with unknown parameters based on an LMI approach has been given. The plant is represented by a TSK model. An application example on stabilizing a cart-pole type inverted pendulum system has been given to illustrate the design procedure and the merits of the proposed linear controller.

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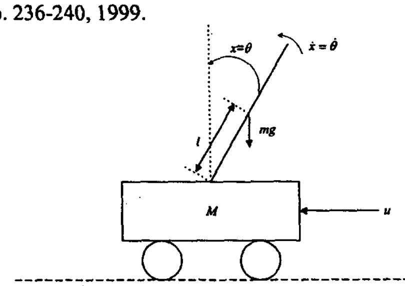

Fig. 1. Cart-pole type inverted pendulum system.

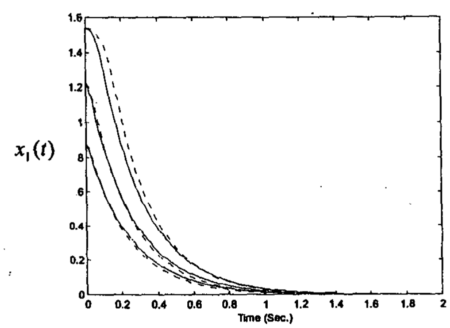

Fig. 2. Responses of $x_1(t)$ under M = 8kg (solid line) and M = 80kg (dotted line).

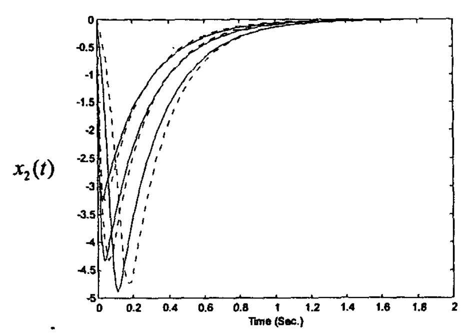

Fig. 3. Responses of $x_2(t)$ under M = 8kg (solid line) and M = 80kg (dotted line).