Stability analysis of T-S fuzzy-model-based control systems using fuzzy Lyapunov function

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Abstract— **This paper investigates the system stability of T-S fuzzy-model-based control systems based on an improved fuzzy Lyapunov function. Various non-PDC fuzzy controllers are proposed to close the feedback loop. The characteristic of T-S fuzzy model is considered to facilitate the stability analysis. Under a particular case, the time-derivative information of the membership functions vanishes, which simplifies the stability analysis and leads to relaxed stability analysis results. A general case is then considered. An improved non-PDC fuzzy controller is proposed based on the properties of the T-S fuzzy model. The improved non-PDC fuzzy controller exhibits a favourable property to relax the stability conditions. Based on the fuzzy Lyapunov function, stability conditions in terms of linear matrix inequalities are derived to guarantee the system stability. Simulation examples are given to illustrate the effectiveness of the proposed non-PDC fuzzy control schemes.**

I. INTRODUCTION

Fuzzy-model-based control approach offers a systematic way to tackle nonlinear systems. Based on the T-S fuzzy model [1]-[2], a nonlinear plant can be represented as an average weighted sum of linear systems. This particular form offers a general framework to represent the nonlinear plant and provides an effective platform to facilitate the stability analysis and controller synthesis.

Fuzzy-model-based control systems have been extensively investigated in the past decade. Basic stability conditions derived based on Lyapunov stability theory were given in [3]-[4] in terms of linear matrix inequalities (LMI) [5]. Based on the parallel distribution compensation (PDC) design technique [4], the information of the membership functions is employed for the design of fuzzy controller. It has been reported in [6]-[13] that the information facilitates the stability analysis and offers less conservative stability analysis results.

In the work of [6]-[13], a parameter-independent Lyapunov function (PILF) was employed to investigate the system stability. In [14], a fuzzy Lyapunov function (FLF), which is a kind of parameter-dependent Lyapunov function (PDLF), was proposed. As further information is considered in the FLF, it has shown a great potential to further relax the stability analysis result compared to that of PILF. However, for the continuous-time case, the FLF will produce the time derivatives of membership functions, making the stability analysis tedious and difficult. It has been shown that the time derivatives of membership functions will disappear under a particular type of fuzzy models [15]. Nevertheless, when the PDC fuzzy controller is considered [14]-[15], the stability conditions cannot be formulated in terms of LMIs. As a result, the solution to the stability conditions cannot be simply solved numerically by some convex programming techniques such as MATLAB LMI toolbox. To hurdle the difficulty, a completing square technique was employed in [14] to formulate the stability conditions in terms of LMIs. However, as redundant matrix terms are introduced in the stability analysis, conservativeness is introduced to the stability conditions. In [16]-[17], to further utilize the advantage of the FLF, a non-PDC fuzzy controller was proposed. Comparing to the PDC fuzzy controller, the non-PDC fuzzy controller [16]-[17] offers greater design flexibility and shows greater potential to relax the stability conditions under the consideration of FLF. Moreover, with the non-PDC fuzzy controller, the stability conditions can be formulated as LMIs. The FLF was also employed to investigate the system stability of discrete-time fuzzy-model-based control systems [18]-[19].

In this paper, we further extend our fundamental work in [17]. An improved FLF is proposed to investigate the fuzzy-model-based control system. An improved non-PDC fuzzy controller is proposed to take advantage of the FLF to produce relaxed stability analysis results. The characteristic of the fuzzy model is considered to facilitate the stability analysis. A particular case of fuzzy model is first considered of which the time derivatives of membership functions are not required for the stability analysis. A general fuzzy model is then considered. An improved non-PDC fuzzy controller is proposed to close the feedback loop. As further information is considered by the non-PDC fuzzy controller, more relaxed stability conditions can be achieved. Based on the FLF, stability conditions in terms of LMIs are derived to guarantee the system stability of the fuzzy-model-based control systems.

II. FUZZY MODEL AND NON-PDC FUZZY CONTROLLER

A fuzzy-model-based control system, comprising a nonlinear plant represented by a fuzzy model and a non-PDC fuzzy controller connected in a closed loop, is considered.

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A. Fuzzy Model

Let p be the number of fuzzy rules describing the behaviour of the nonlinear plant. The i-th rule is of the following format:

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND … AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

$$\text{THEN } \mathbf{x}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \tag{1}$$

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$, α = 1, 2, ..., Ψ; i = 1, 2, ..., p; Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known constant system and input matrices respectively; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The system dynamics are described by,

$$\mathbf{x}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\big) \tag{2}$$

where,

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1,\ w_i(\mathbf{x}(t)) \in \begin{bmatrix}0 & 1\end{bmatrix} \text{ for all } i \tag{3}$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t)))\times \quad \times\mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\Big(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t)))\times \quad \times\mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\Big)} \tag{4}$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$, α = 1, 2, …, Ψ, is the grade of membership corresponding to the fuzzy term of M_α^i.

B. Non-PDC Fuzzy Controller

A non-PDC fuzzy controller is proposed to control the nonlinear plant in the form of (2) is proposed. The output of the non-PDC fuzzy controller is governed by the fuzzy rules in the following format.

Rule j: IF $f_1(\mathbf{x}(t))$ is M_1^j AND … AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^j

$$\text{THEN } \mathbf{u}(t) = \mathbf{F}_j\mathbf{P}(\mathbf{x}(t))^{-1}\mathbf{x}(t),\ j = 1, 2, ..., p \tag{5}$$

where $\mathbf{F}_j \in \Re^{m\times n}$ is the constant feedback gain of rule j and $\mathbf{P}(\mathbf{x}(t)) = \mathbf{P}(\mathbf{x}(t))^{\mathrm{T}} \in \Re^{n\times n} > 0$ is a nonlinear matrix function to be determined. The inferred non-PDC fuzzy controller is defined as,

$$\mathbf{u}(t) = \sum_{j=1}^{p} w_j(\mathbf{x}(t))\mathbf{F}_j\mathbf{P}(\mathbf{x}(t))^{-1}\mathbf{x}(t) \tag{6}$$

III. Stability Analysis

The system stability of the fuzzy-model-based control systems is considered in this section. Various non-PDC fuzzy controllers are proposed to handle the nonlinear plant in the form of (2). For brevity, $w_i(\mathbf{x}(t))$ and $\mathbf{P}(\mathbf{x}(t))$ are denoted as w_i and $\mathbf{P}$ respectively. The equality that $\sum_{i=1}^{p} w_i = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j = 1$ given by the property of the membership functions is used during the system analysis.

A. Special Case

A special case of fuzzy model facilitating the stability analysis is considered. From (2) and (6), the fuzzy-model-based control system is defined as follows.

$$\mathbf{x}(t) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\big(\mathbf{A}_i + \mathbf{B}_i\mathbf{F}_j\mathbf{P}^{-1}\big)\mathbf{x}(t) \tag{7}$$

The system stability of the fuzzy-model-based control system of (7) is investigated. Compared to [16]-[17], the following improved FLF is considered.

$$V(t) = \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{x}(t) \tag{8}$$

where $\mathbf{P} = \mathbf{P}^{\mathrm{T}} = \sum_{k=1}^{p}\sum_{l=1}^{p} w_k w_l \mathbf{P}_{k,l} > 0$ and $\mathbf{P}_{k,l} \in \Re^{n\times n}$, k, l = 1, 2, ..., p. The condition for $\mathbf{P} > 0$ will be considered later.

Remark 1: $\mathbf{P} > 0$ implies non-singularity. If we have $\mathbf{P} > 0$, the inverse of $\mathbf{P}$ must exist. Furthermore, $\mathbf{P} > 0$ implies $\mathbf{P}^{-1} > 0$.

In the following analysis, for simplicity and without loss of generality, it is assumed that the following design conditions are fulfilled:

1) The number of rules of the fuzzy controller, p, is even. For an odd p, this condition can be easily achieved by adding an identical rule to the rule base.
2) The membership functions of the fuzzy model satisfy $w_k = -w_{k+1}$, k = 1, 3, ..., p – 1.
3) The FLF is chosen such that $\mathbf{P}_{k,l} = \mathbf{P}_{k+1,l} = \mathbf{P}_{k,l+1} = \mathbf{P}_{k+1,l+1}$, k, l = 1, 3, ..., p – 1.

Under the above conditions, we have the following equalities.

$$\sum_{k=1}^{p}\sum_{l=1}^{p} w_k w_l \mathbf{P}_{k,l} = \sum_{k=1,3,\ ,p}\ \sum_{l=1}^{p}\big(w_k - w_{k+1}\big)w_l\mathbf{P}_{k,l} = \mathbf{0} \tag{9}$$

$$\sum_{k=1}^{p}\sum_{l=1}^{p} w_k w_l \mathbf{P}_{k,l} = \sum_{k=1}^{p}\ \sum_{l=1,3,\ ,p} w_k\big(w_l - w_{l+1}\big)\mathbf{P}_{k,l} = \mathbf{0} \tag{10}$$

Remark 2: Condition 2) can be further generalized for a fuzzy model with the membership functions exhibiting the property of $w_i = \xi_i - \sum_{r=i+1}^{c} w_r$ where $0 \le \xi_i \le 1$ and $c \le p$. For instance, we consider a fuzzy model with membership functions of w_i, i = 1, 2, ..., 9 which exhibit the property that $w_1 = 0.25 - w_2$, $w_3 = 0.25 - w_4 - w_5$ and $w_6 = 0.5 - w_7 - w_8 - w_9$. Hence, we have $w_1 = -w_2$, $w_3 = -w_4 - w_5$ and $w_6 = -w_7 - w_8 - w_9$. By choosing properly $\mathbf{P}_{k,l}$, i.e., $\mathbf{P}_{k,l} = \mathbf{P}_{1,1}$, k, l = 1, 2; $\mathbf{P}_{k,l} = \mathbf{P}_{3,3}$, k, l = 3, 4, 5; $\mathbf{P}_{k,l} = \mathbf{P}_{6,6}$, k, l = 6, 7, 8, 9, we have $\sum_{k=1}^{9}\sum_{l=1}^{9} w_k w_l \mathbf{P}_{k,l} = \mathbf{0}$ and $\sum_{k=1}^{9}\sum_{l=1}^{9} w_k w_l \mathbf{P}_{k,l} = \mathbf{0}$.

The equalities of (9) and (10) facilitate the stability analysis based on FLF as the time-derivative information of the membership functions vanishes. From (8), and based on the above design conditions, the time derivative of $V(t)$ is as follows.

$$V(t) = \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{x}(t) - \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{P}\mathbf{P}^{-1}\mathbf{x}(t) \tag{11}$$

From (9) and (10), we have

$$\mathbf{P} = \sum_{k=1}^{p}\sum_{l=1}^{p} w_k w_l \mathbf{P}_{k,l} + \sum_{k=1}^{p}\sum_{l=1}^{p} w_k w_l \mathbf{P}_{k,l} = \mathbf{0} \tag{12}$$

From (7), (11), (12), we have

$$V(t) = \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{x}(t)$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{P}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{P} + \mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{F}_j\right)\mathbf{z}(t) \tag{13}$$

where $\mathbf{z}(t) = \mathbf{P}^{-1}\mathbf{x}(t)$. Recalling that $\mathbf{P} = \mathbf{P}^{\mathrm{T}} = \sum_{k=1}^{p}\sum_{l=1}^{p} w_k w_l \mathbf{P}_{k,l} > 0$, from (13), we have

$$V(t) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i w_j w_k \mathbf{z}(t)^{\mathrm{T}}\begin{pmatrix}\mathbf{P}_{j,k}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{P}_{j,k} \\ +\mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{F}_j\end{pmatrix}\mathbf{z}(t) \tag{14}$$

Denote $\mathbf{\Xi} = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i w_j w_k \mathbf{\Xi}_{ijk}$ where $\mathbf{\Xi}_{ijk} = \mathbf{P}_{j,k}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{P}_{j,k} + \mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{F}_j$, $i, j, k = 1, 2, ..., p$. It can be seen that asymptotic stability of the fuzzy-model-based control system of (7) is implied by $V(t) \le 0$ (equality holds when $\mathbf{z}(t) = \mathbf{0}$). $V(t) \le 0$ can be achieved when $\mathbf{\Xi} < 0$. By following the similar analysis procedure in [16], we have,

$$\mathbf{\Xi} = \sum_{i=1}^{p} w_i^3 \mathbf{\Xi}_{iii} + \sum_{i=1}^{p}\sum_{\substack{j=1\\ j\ne i}}^{p} w_i^2 w_j\left(\mathbf{\Xi}_{iij} + \mathbf{\Xi}_{iji} + \mathbf{\Xi}_{jii}\right) + \sum_{i=1}^{p-2}\sum_{j=i+1}^{p-1}\sum_{k=j+1}^{p} w_i w_j w_k \begin{pmatrix}\mathbf{\Xi}_{ijk} + \mathbf{\Xi}_{ikj} + \mathbf{\Xi}_{jik} \\ +\mathbf{\Xi}_{jki} + \mathbf{\Xi}_{kij} + \mathbf{\Xi}_{kji}\end{pmatrix} \tag{15}$$

Define $\mathbf{Y}_{ijk} \in \Re^{n\times n}$, $\mathbf{Y}_{iii} = \mathbf{Y}_{iii}^{\mathrm{T}}$, $i = 1, 2, ..., p$, $\mathbf{Y}_{iij} = \mathbf{Y}_{jii}^{\mathrm{T}}$, $i, j = 1, 2, ..., p$; $i \ne j$, $\mathbf{Y}_{ijk} = \mathbf{Y}_{ikj}^{\mathrm{T}}$, $\mathbf{Y}_{jik} = \mathbf{Y}_{jki}^{\mathrm{T}}$ and $\mathbf{Y}_{kij} = \mathbf{Y}_{kji}^{\mathrm{T}}$, $i = 1, 2, ..., p-2$; $j = 1, 2, ..., p-1$; $k = 1, 2, ..., p$.

Let

$$\mathbf{Y}_{iii} > \mathbf{\Xi}_{iii},\ i = 1, 2, ..., p \tag{16}$$

$$\mathbf{Y}_{iij} + \mathbf{Y}_{iji} + \mathbf{Y}_{jii} \ge \mathbf{\Xi}_{iij} + \mathbf{\Xi}_{iji} + \mathbf{\Xi}_{jii},\ i, j = 1, 2, ..., p;\ i \ne j \tag{17}$$

$$\mathbf{Y}_{ijk} + \mathbf{Y}_{ikj} + \mathbf{Y}_{jik} + \mathbf{Y}_{jki} + \mathbf{Y}_{kij} + \mathbf{Y}_{kji} \ge \begin{matrix}\mathbf{\Xi}_{ijk} + \mathbf{\Xi}_{ikj} + \mathbf{\Xi}_{jik} \\ +\mathbf{\Xi}_{jki} + \mathbf{\Xi}_{kij} + \mathbf{\Xi}_{kji}\end{matrix},\ i = 1, 2, ..., p-2;\ j = 1, 2, ..., p-1;\ k = 1, 2, ..., p \tag{18}$$

From (15) to (18), we have,

$$\mathbf{\Xi} \le \sum_{i=1}^{p} w_i^3 \mathbf{Y}_{iii} + \sum_{i=1}^{p}\sum_{\substack{j=1\\ j\ne i}}^{p} w_i^2 w_j\left(\mathbf{Y}_{iij} + \mathbf{Y}_{iji} + \mathbf{Y}_{jii}\right) + \sum_{i=1}^{p-2}\sum_{j=i+1}^{p-1}\sum_{k=j+1}^{p} w_i w_j w_k\left(\mathbf{Y}_{ijk} + \mathbf{Y}_{ikj} + \mathbf{Y}_{jik} + \mathbf{Y}_{jki} + \mathbf{Y}_{kij} + \mathbf{Y}_{kji}\right) \tag{19}$$

From (14) and (19), we have,

$$V(t) = \sum_{k=1}^{p} w_k \mathbf{r}(t)^{\mathrm{T}}\mathbf{Y}_k\mathbf{r}(t) \tag{20}$$

where $\mathbf{r}(t) = \begin{bmatrix} w_1\mathbf{z}(t) \\ w_2\mathbf{z}(t) \\ \vdots \\ w_p\mathbf{z}(t)\end{bmatrix}$ and $\mathbf{Y}_k = \begin{bmatrix}\mathbf{Y}_{1k1} & \mathbf{Y}_{1k2} & \cdots & \mathbf{Y}_{1kp} \\ \mathbf{Y}_{2k1} & \mathbf{Y}_{2k2} & \cdots & \mathbf{Y}_{2kp} \\ \vdots & & & \vdots \\ \mathbf{Y}_{pk1} & \mathbf{Y}_{pk2} & \cdots & \mathbf{Y}_{pkp}\end{bmatrix}$.

It can be seen that $V(t) \le 0$ is achieved when $\mathbf{Y}_k < 0$, $k = 1, 2, ..., p$. Nevertheless, the stability analysis based on FLF of (8) requires that $\mathbf{P} = \mathbf{P}^{\mathrm{T}} = \sum_{k=1}^{p}\sum_{l=1}^{p} w_k w_l \mathbf{P}_{k,l} > 0$. With condition 3), designing that $\mathbf{P}_{k,k} = \mathbf{P}_{k,k}^{\mathrm{T}}$, k = 1, 3, ..., p − 1 and $\mathbf{P}_{k,l} = \mathbf{P}_{l,k}^{\mathrm{T}}$, j = 1, 3, ..., p − 1; $k < l$, we have

$$\sum_{k=1}^{p}\sum_{l=1}^{p} w_k w_l \mathbf{P}_{k,l} = \sum_{k=1}^{p} w_k^2 \mathbf{P}_{k,k} + \sum_{l=1}^{p}\sum_{k<l} w_k w_l\left(\mathbf{P}_{k,l} + \mathbf{P}_{l,k}\right) \tag{21}$$

It can be seen from (21) that $\sum_{k=1}^{p}\sum_{l=1}^{p} w_k w_l \mathbf{P}_{k,l} > 0$ can be achieved by $\mathbf{P}_{k,k} > 0$, $k = 1, 2, ..., p$ and $\mathbf{P}_{k,l} + \mathbf{P}_{l,k} \ge 0$, l = 1, 2, ..., p; $k < l$. The analysis result is summarized in the following theorem.

Theorem 1: *The fuzzy-model-based control system, formed by the nonlinear plant in the form* (2) *and the non-PDC fuzzy controller in the form of* (6) *connected in a closed loop, is asymptotically stable if conditions* 1) *to* 3) *are satisfied and there exists* $\mathbf{P}_{k,k} = \mathbf{P}_{k,k}^{\mathrm{T}} \in \Re^{n\times n}$, *k = 1, 3, ..., p – 1,* $\mathbf{P}_{k,l} = \mathbf{P}_{l,k}^{\mathrm{T}} \in \Re^{n\times n}$ *l = 1, 3, ...,p – 1; k < l,* $\mathbf{Y}_{ijk} \in \Re^{n\times n}$, $\mathbf{Y}_{iii} = \mathbf{Y}_{iii}^{\mathrm{T}}$, *i = 1, 2, ..., p,* $\mathbf{Y}_{iij} = \mathbf{Y}_{jii}^{\mathrm{T}}$, *i, j = 1, 2, ..., p; i ≠ j,* $\mathbf{Y}_{ijk} = \mathbf{Y}_{ikj}^{\mathrm{T}}$, $\mathbf{Y}_{jik} = \mathbf{Y}_{jki}^{\mathrm{T}}$ *and* $\mathbf{Y}_{kij} = \mathbf{Y}_{kji}^{\mathrm{T}}$, *i = 1, 2, ..., p – 2; j = 1, 2, ..., p – 1; k = 1, 2, ..., p such that the following LMIs are satisfied.*

$\mathbf{P}_{k,k} > 0$, *k = 1, 2, ..., p;* $\mathbf{P}_{k,l} + \mathbf{P}_{l,k} \ge 0$, *l = 1, 2, ..., p; k < l;* (16) *to* (18) *and* $\mathbf{Y}_k < 0$, *k = 1, 2, ..., p where* $\mathbf{\Xi}_{ijk} = \mathbf{P}_{j,k}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{P}_{j,k} + \mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{F}_j$.

Remark 3: The non-PDC fuzzy controller can be made simpler by considering $\mathbf{P}_{k,l} = \mathbf{P}_k = \mathbf{P}_k^{\mathrm{T}} \in \Re^{n\times n}$ for all k and l. In this case, we have $\mathbf{P} = \mathbf{P}^{\mathrm{T}} = \sum_{k=1}^{p}\sum_{l=1}^{p} w_k w_l \mathbf{P}_k = \sum_{k=1}^{p} w_k \mathbf{P}_k$. $\mathbf{P} > 0$ is implied by $\mathbf{P}_k > 0$ for all k. In this case, the FLF of (8) is reduced to that in [16]-[17].

Remark 4: Comparing to the analysis in [16]-[17], under the particular case of the fuzzy model, the time-derivative information of the membership functions, $w_i(\mathbf{x}(t))$, is not required. This makes the stability analysis easy and simplifies the implementation of the non-PDC fuzzy controller.

Remark 5: The stability conditions in Theorem 1 are reduced to those in [11] when $\mathbf{P}_{j,k}$ are all common for all j and k.

B. Non-PDC Fuzzy Controller with Time-Derivative Information

In the following, we consider a general fuzzy model, i.e. conditions (1) to (3) are not required to be satisfied. A non-PDC fuzzy controller+ in the following form is proposed.

$$\mathbf{u}(t)=\sum_{j=1}^{p}w_j\mathbf{F}_j\mathbf{P}^{-1}\mathbf{x}(t)+\sum_{j=1}^{p}\overline{w}_j\mathbf{G}_j\mathbf{P}^{-1}\mathbf{x}(t) \qquad (22)$$

where $\mathbf{G}_j\in\Re^{m\times n}$ is a feedback gain to be determined, and $\overline{w}_j=w_j+\rho\dot{w}_j+\sigma_j>0$ and ρ and σ_j are scalars to be determined. The non-PDC fuzzy controller of (22) can be rewritten in the form of $\mathbf{u}(t)=\sum_{j=1}^{p}w_j\left(\mathbf{F}_j+\mathbf{G}_j+\sum_{k=1}^{p}\sigma_k\mathbf{G}_k\right)\mathbf{P}^{-1}\mathbf{x}(t)+\sum_{j=1}^{p}\dot{w}_j\rho\mathbf{G}_j\mathbf{P}^{-1}\mathbf{x}(t)$. It is equivalent to the non-PDC fuzzy controller proposed in [16]-[17]. From (2) and (22), we have the following fuzzy-model-based control system.

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{p}w_i\left(\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\left(\sum_{j=1}^{p}w_j\mathbf{F}_j\mathbf{P}^{-1}\mathbf{x}(t)+\sum_{j=1}^{p}\overline{w}_j\mathbf{G}_j\mathbf{P}^{-1}\mathbf{x}(t)\right)\right)$$
$$=\sum_{i=1}^{p}\sum_{j=1}^{p}w_iw_j\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_j\mathbf{P}^{-1}\right)\mathbf{x}(t)+\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\overline{w}_j\mathbf{B}_i\mathbf{G}_j\mathbf{P}^{-1}\mathbf{x}(t) \qquad (23)$$

To investigate the system stability of (23), we consider the Lyapunov function candidate in (8). From (8) and (23), recalling $\mathbf{z}(t)=\mathbf{P}^{-1}\mathbf{x}(t)$, we have,

$$\dot{V}(t)=\dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{x}(t)+\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}^{-1}\dot{\mathbf{x}}(t)-\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}^{-1}\dot{\mathbf{P}}\mathbf{P}^{-1}\mathbf{x}(t)$$
$$=\sum_{i=1}^{p}\sum_{j=1}^{p}w_iw_j\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{P}\mathbf{A}_i^{\mathrm{T}}+\mathbf{A}_i\mathbf{P}+\mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{F}_j\right)\mathbf{z}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\overline{w}_j\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j\right)\mathbf{z}(t)$$
$$-\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\left(\rho\dot{w}_j+w_j-w_j+\sigma_j-\sigma_j\right)\mathbf{z}(t)^{\mathrm{T}}\frac{1}{\rho}\left(\mathbf{P}_{i,j}+\mathbf{P}_{j,i}\right)\mathbf{z}(t)$$
$$=\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}w_iw_jw_k\mathbf{z}(t)^{\mathrm{T}}\begin{pmatrix}\mathbf{P}_{j,k}\mathbf{A}_i^{\mathrm{T}}+\mathbf{A}_i\mathbf{P}_{j,k}\\+\mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{F}_j\\+\frac{1}{\rho}\left(\mathbf{P}_{j,k}+\mathbf{P}_{k,j}\right)\\+\sum_{r=1}^{p}\frac{\sigma_r}{\rho}\left(\mathbf{P}_{i,r}+\mathbf{P}_{r,i}\right)\end{pmatrix}\mathbf{z}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\overline{w}_j\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j-\frac{1}{\rho}\left(\mathbf{P}_{i,j}+\mathbf{P}_{j,i}\right)\right)\mathbf{z}(t)$$
$$=\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}w_iw_jw_k\mathbf{z}(t)^{\mathrm{T}}\begin{pmatrix}\mathbf{P}_{j,k}\left(\mathbf{A}_i+\frac{1}{\rho}\mathbf{I}\right)^{\mathrm{T}}\\+\left(\mathbf{A}_i+\frac{1}{\rho}\mathbf{I}\right)\mathbf{P}_{j,k}\\+\mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{F}_j\\+\sum_{r=1}^{p}\frac{\sigma_r}{\rho}\left(\mathbf{P}_{i,r}+\mathbf{P}_{r,i}\right)\end{pmatrix}\mathbf{z}(t) \qquad (24)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\overline{w}_j\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j-\frac{1}{\rho}\left(\mathbf{P}_{i,j}+\mathbf{P}_{j,i}\right)\right)\mathbf{z}(t)$$

From (3), we have $\sum_{i=1}^{p}w_i=1$ which leads to $\sum_{i=1}^{p}\dot{w}_i=0$. This property can bring about some slack matrices to facilitate the stability analysis. Considering an arbitrary matrix $\mathbf{\Lambda}_i=\mathbf{\Lambda}_i^{\mathrm{T}}\in\Re^{n\times n}$, we have $\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\dot{w}_j\mathbf{\Lambda}_i=\sum_{i=1}^{p}w_i\mathbf{\Lambda}_i\sum_{j=1}^{p}\dot{w}_j=\mathbf{0}$. From (24), we have,

$$\dot{V}(t)=\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}w_iw_jw_k\mathbf{z}(t)^{\mathrm{T}}\begin{pmatrix}\mathbf{P}_{j,k}\left(\mathbf{A}_i+\frac{1}{\rho}\mathbf{I}\right)^{\mathrm{T}}\\+\left(\mathbf{A}_i+\frac{1}{\rho}\mathbf{I}\right)\mathbf{P}_{j,k}\\+\mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{F}_j\\+\sum_{r=1}^{p}\frac{\sigma_r}{\rho}\left(\mathbf{P}_{i,r}+\mathbf{P}_{r,i}\right)\end{pmatrix}\mathbf{z}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\overline{w}_j\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j-\frac{1}{\rho}\left(\mathbf{P}_{i,j}+\mathbf{P}_{j,i}\right)\right)\mathbf{z}(t)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\dot{w}_j\mathbf{z}(t)^{\mathrm{T}}\mathbf{\Lambda}_i\mathbf{z}(t)$$
$$=\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}w_iw_jw_k\mathbf{z}(t)^{\mathrm{T}}\begin{pmatrix}\mathbf{P}_{j,k}\left(\mathbf{A}_i+\frac{1}{\rho}\mathbf{I}\right)^{\mathrm{T}}\\+\left(\mathbf{A}_i+\frac{1}{\rho}\mathbf{I}\right)\mathbf{P}_{j,k}\\+\mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{F}_j\\+\sum_{r=1}^{p}\frac{\sigma_r}{\rho}\left(\mathbf{P}_{i,r}+\mathbf{P}_{r,i}\right)\\-\left(\frac{1}{\rho}+\sum_{r=1}^{p}\frac{\sigma_r}{\rho}\right)\mathbf{\Lambda}_i\end{pmatrix}\mathbf{z}(t) \qquad (25)$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{p}w_i\overline{w}_j\mathbf{z}(t)^{\mathrm{T}}\begin{pmatrix}\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j\\-\frac{1}{\rho}\left(\mathbf{P}_{i,j}+\mathbf{P}_{j,i}\right)+\frac{1}{\rho}\mathbf{\Lambda}_i\end{pmatrix}\mathbf{z}(t)$$

Denote

$$\mathbf{\Xi}_{ijk}=\mathbf{P}_{j,k}\left(\mathbf{A}_i+\frac{1}{\rho}\mathbf{I}\right)^{\mathrm{T}}+\left(\mathbf{A}_i+\frac{1}{\rho}\mathbf{I}\right)\mathbf{P}_{j,k}+\mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{F}_j+\sum_{r=1}^{p}\frac{\sigma_r}{\rho}\left(\mathbf{P}_{i,r}+\mathbf{P}_{r,i}\right)-\left(\frac{1}{\rho}+\sum_{r=1}^{p}\frac{\sigma_r}{\rho}\right)\mathbf{\Lambda}_i,\ i,\ j,\ k=1,2,\ldots,p \qquad (26)$$

which satisfy (16) to (18). Let $\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}+\mathbf{B}_i\mathbf{G}_j-\frac{1}{\rho}\left(\mathbf{P}_{i,j}+\mathbf{P}_{j,i}\right)+\frac{1}{\rho}\mathbf{\Lambda}_i<0$ for all i and j. From (16) to (18), (26),

$$\dot{V}(t)\le\sum_{i=1}^{p}w_i\mathbf{r}(t)^{\mathrm{T}}\mathbf{Y}_i\mathbf{r}(t) \qquad (27)$$

It can be seen from (27) that $\dot{V}(t)\le 0$ (equality holds when $\mathbf{z}(t)$ = $\mathbf{0}$) is implied by $\mathbf{Y}_i<0$ and

$\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{G}_j - \frac{1}{\rho}(\mathbf{P}_{i,j} + \mathbf{P}_{j,i}) + \frac{1}{\rho}\mathbf{\Lambda}_i < 0$ which further imply the asymptotic stability of the fuzzy-model-based control system of (23). Referring to (8), the above analysis is valid for $\mathbf{P} = \sum_{k=1}^{p}\sum_{l=1}^{p} w_k w_l \mathbf{P}_{k,l} > 0$ which can be written as $\mathbf{P} = \begin{bmatrix} w_1\mathbf{I} \\ w_2\mathbf{I} \\ \vdots \\ w_p\mathbf{I} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \cdots & \mathbf{P}_{1,p} \\ \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \cdots & \mathbf{P}_{2,p} \\ \vdots & & & \vdots \\ \mathbf{P}_{p,1} & \mathbf{P}_{p,2} & \cdots & \mathbf{P}_{p,p} \end{bmatrix} \begin{bmatrix} w_1\mathbf{I} \\ w_2\mathbf{I} \\ \vdots \\ w_p\mathbf{I} \end{bmatrix} > 0$. It can be seen that $\mathbf{P} > 0$ is guaranteed by $\begin{bmatrix} \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \cdots & \mathbf{P}_{1,p} \\ \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \cdots & \mathbf{P}_{2,p} \\ \vdots & & & \vdots \\ \mathbf{P}_{p,1} & \mathbf{P}_{p,2} & \cdots & \mathbf{P}_{p,p} \end{bmatrix} > 0$. The analysis result is summarized in the following theorem.

Theorem 2: *The fuzzy-model-based control system, formed by the nonlinear plant in the form* (2) *and the non-PDC fuzzy controller in the form of* (22) *connected in a closed loop, is asymptotically stable if* $w(\mathbf{x}(t)) + \rho w(\mathbf{x}(t)) + \sigma_i > 0$, *i = 1, 2, ..., p where* ρ *and* σ_i *are scalars and there exists* $\mathbf{P}_{k,l} = \mathbf{P}_{l,k}^{\mathrm{T}} \in \Re^{n\times n}$, *k, l = 1, 2, ..., p;* $\mathbf{\Lambda}_i = \mathbf{\Lambda}_i^{\mathrm{T}} \in \Re^{n\times n}$, *i = 1, 2, ..., p;* $\mathbf{Y}_{iii} = \mathbf{Y}_{iii}^{\mathrm{T}}$, *i = 1, 2, ..., p,* $\mathbf{Y}_{iij} = \mathbf{Y}_{jii}^{\mathrm{T}}$, *i, j = 1, 2, ..., p; i ≠ j,* $\mathbf{Y}_{ijk} = \mathbf{Y}_{ikj}^{\mathrm{T}}$, $\mathbf{Y}_{jik} = \mathbf{Y}_{jki}^{\mathrm{T}}$ *and* $\mathbf{Y}_{kij} = \mathbf{Y}_{kji}^{\mathrm{T}}$, *i = 1, 2, ..., p – 2; j = 1, 2, ..., p – 1; k = 1, 2, ..., p, such that the following LMIs are satisfied:* $\begin{bmatrix} \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \cdots & \mathbf{P}_{1,p} \\ \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \cdots & \mathbf{P}_{2,p} \\ \vdots & & & \vdots \\ \mathbf{P}_{p,1} & \mathbf{P}_{p,2} & \cdots & \mathbf{P}_{p,p} \end{bmatrix} > 0$, (16) *to* (18), $\mathbf{Y}_i < 0$, *i = 1, 2, ..., p and* $\mathbf{G}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{G}_j - \frac{1}{\rho}(\mathbf{P}_{i,j} + \mathbf{P}_{j,i}) + \frac{1}{\rho}\mathbf{\Lambda}_i < 0$, *i, j = 1, 2, ..., p. In* (16) *to* (18),

$$\mathbf{\Xi}_{ijk} = \mathbf{P}_{j,k}\left(\mathbf{A}_i + \frac{1}{\rho}\mathbf{I}\right)^{\mathrm{T}} + \left(\mathbf{A}_i + \frac{1}{\rho}\mathbf{I}\right)\mathbf{P}_{j,k} + \mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{F}_j + \sum_{r=1}^{p}\frac{\sigma_r}{\rho}(\mathbf{P}_{i,r} + \mathbf{P}_{r,i}) - \left(\frac{1}{\rho} + \sum_{r=1}^{p}\frac{\sigma_r}{\rho}\right)\mathbf{\Lambda}_i .$$

C. An Improved Non-PDC Fuzzy Controller

An improved non-PDC fuzzy controller in the following form is proposed:

$$\mathbf{u}(t) = \sum_{j=1}^{p} w_j \mathbf{F}_j \mathbf{P}^{-1}\mathbf{x}(t) + \sum_{j=1}^{p}\sum_{k=1}^{p} \overline{w}_j w_k \mathbf{H}_{jk}\mathbf{P}^{-1}\mathbf{x}(t) \tag{28}$$

where $\mathbf{H}_{jk} \in \Re^{m\times n}$ is the feedback gains to be determined. Comparing to (22), the proposed non-PDC fuzzy controller of (28) includes w_k in the second term to facilitate the stability analysis. From (2) and (28), we have the following fuzzy-model-based control system.

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^{p} w_i \left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i \left(\sum_{j=1}^{p} w_j \mathbf{F}_j \mathbf{P}^{-1}\mathbf{x}(t) + \sum_{j=1}^{p}\sum_{k=1}^{p} \overline{w}_j w_k \mathbf{H}_{jk}\mathbf{P}^{-1}\mathbf{x}(t) \right)\right) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \left(\mathbf{A}_i + \mathbf{B}_i\mathbf{F}_j\mathbf{P}^{-1}\right)\mathbf{x}(t) \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i \overline{w}_j w_k \mathbf{B}_i\mathbf{H}_{jk}\mathbf{P}^{-1}\mathbf{x}(t)\end{aligned} \tag{29}$$

We consider the Lyapunov function candidate in (8) to investigate the system stability of (29). From (8) and (29), recalling $\mathbf{z}(t) = \mathbf{P}^{-1}\mathbf{x}(t)$ and following the same line of logic of derivation as in the previous analysis, we have,

$$\begin{aligned}\dot{V}(t) &= \dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}^{-1}\dot{\mathbf{x}}(t) - \mathbf{x}(t)^{\mathrm{T}}\mathbf{P}^{-1}\dot{\mathbf{P}}\mathbf{P}^{-1}\mathbf{x}(t) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i w_j w_k \mathbf{z}(t)^{\mathrm{T}} \begin{pmatrix} \mathbf{P}_{j,k}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{P}_{j,k} \\ + \mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{F}_j \\ + \frac{1}{\rho}(\mathbf{P}_{j,k} + \mathbf{P}_{k,j}) \\ + \sum_{r=1}^{p}\frac{\sigma_r}{\rho}(\mathbf{P}_{i,r} + \mathbf{P}_{r,i}) \end{pmatrix} \mathbf{z}(t) \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i \overline{w}_j w_k \mathbf{z}(t)^{\mathrm{T}} \begin{pmatrix} \mathbf{H}_{jk}^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{H}_{jk} \\ - \frac{1}{\rho}(\mathbf{P}_{i,j} + \mathbf{P}_{j,i}) \end{pmatrix} \mathbf{z}(t)\end{aligned} \tag{30}$$

Considering an arbitrary matrix $\mathbf{\Lambda}_{ik} = \mathbf{\Lambda}_{ki}^{\mathrm{T}} \in \Re^{n\times n}$, we have $\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i \dot{w}_j w_k \mathbf{\Lambda}_{ik} = \sum_{i=1}^{p}\sum_{k=1}^{p} w_i w_k \mathbf{\Lambda}_{ik} \overbrace{\sum_{j=1}^{p} \dot{w}_j}^{0} = \mathbf{0}$. From (30), we have,

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i w_j w_k \mathbf{z}(t)^{\mathrm{T}} \begin{pmatrix} \mathbf{P}_{j,k}\left(\mathbf{A}_i + \frac{1}{\rho}\mathbf{I}\right)^{\mathrm{T}} \\ + \left(\mathbf{A}_i + \frac{1}{\rho}\mathbf{I}\right)\mathbf{P}_{j,k} \\ + \mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{F}_j \\ + \sum_{r=1}^{p}\frac{\sigma_r}{\rho}(\mathbf{P}_{i,r} + \mathbf{P}_{r,i}) \end{pmatrix} \mathbf{z}(t) \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i \overline{w}_j w_k \mathbf{z}(t)^{\mathrm{T}} \begin{pmatrix} \mathbf{H}_{jk}^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{H}_{jk} \\ - \frac{1}{\rho}(\mathbf{P}_{i,j} + \mathbf{P}_{j,i}) \end{pmatrix} \mathbf{z}(t) \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i \left(\rho \dot{w}_j + w_j - w_j + \sigma_j - \sigma_j\right) w_k \mathbf{z}(t)^{\mathrm{T}} \frac{1}{\rho}\mathbf{\Lambda}_{ik}\mathbf{z}(t)\end{aligned}$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i w_j w_k \mathbf{z}(t)^{\mathrm{T}} \begin{pmatrix} \mathbf{P}_{j,k}\left(\mathbf{A}_i + \frac{1}{\rho}\mathbf{I}\right)^{\mathrm{T}} \\ +\left(\mathbf{A}_i + \frac{1}{\rho}\mathbf{I}\right)\mathbf{P}_{j,k} \\ +\mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{F}_j \\ +\sum_{r=1}^{p}\frac{\sigma_r}{\rho}\left(\mathbf{P}_{i,r} + \mathbf{P}_{r,i}\right) \\ -\left(\frac{1}{\rho} + \sum_{r=1}^{p}\frac{\sigma_r}{\rho}\right)\mathbf{\Lambda}_{ik} \end{pmatrix} \mathbf{z}(t) \quad (31)$$

$$+ \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i \overline{w}_j w_k \mathbf{z}(t)^{\mathrm{T}} \begin{pmatrix} \mathbf{H}_{jk}^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{H}_{jk} \\ -\frac{1}{\rho}\left(\mathbf{P}_{i,j} + \mathbf{P}_{j,i}\right) + \frac{1}{\rho}\mathbf{\Lambda}_{ik} \end{pmatrix} \mathbf{z}(t)$$

Denote

$$\mathbf{\Xi}_{ijk} = \mathbf{P}_{j,k}\left(\mathbf{A}_i + \frac{1}{\rho}\mathbf{I}\right)^{\mathrm{T}} + \left(\mathbf{A}_i + \frac{1}{\rho}\mathbf{I}\right)\mathbf{P}_{j,k} + \mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{F}_j + \sum_{r=1}^{p}\frac{\sigma_r}{\rho}\left(\mathbf{P}_{i,r} + \mathbf{P}_{r,i}\right) - \left(\frac{1}{\rho} + \sum_{r=1}^{p}\frac{\sigma_r}{\rho}\right)\mathbf{\Lambda}_{ik}, \quad i, j, k = 1, 2, ..., p \quad (32)$$

which satisfy (16) to (18). Let

$$\mathbf{S}_{iji} > \mathbf{H}_{ji}^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{H}_{ji} - \frac{1}{\rho}\left(\mathbf{P}_{i,j} + \mathbf{P}_{j,i}\right) + \frac{1}{\rho}\mathbf{\Lambda}_{ii}, \quad i, j = 1, 2, ..., p \quad (33)$$

$$\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^{\mathrm{T}} \geq \mathbf{H}_{jk}^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{H}_{jk} - \frac{1}{\rho}\left(\mathbf{P}_{i,j} + \mathbf{P}_{j,i}\right) + \frac{1}{\rho}\mathbf{\Lambda}_{ik} + \mathbf{H}_{ji}^{\mathrm{T}}\mathbf{B}_k^{\mathrm{T}} + \mathbf{B}_k\mathbf{H}_{ji} - \frac{1}{\rho}\left(\mathbf{P}_{k,j} + \mathbf{P}_{j,k}\right) + \frac{1}{\rho}\mathbf{\Lambda}_{ki}, \quad j, k = 1, 2, ..., p; \ i < k \quad (34)$$

where $\mathbf{S}_{ijk} = \mathbf{S}_{kji}^{\mathrm{T}} \in \Re^{n \times n}$. From (16) to (18), (33) and (34), (31) becomes,

$$V(t) \leq \sum_{i=1}^{p} w_i \mathbf{r}(t)^{\mathrm{T}}\mathbf{Y}_i\mathbf{r}(t) + \sum_{i=1}^{p}\sum_{j=1}^{p}\overline{w}_j w_i^2 \mathbf{z}(t)^{\mathrm{T}}\mathbf{S}_{iji}\mathbf{z}(t) + \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{i<k}\overline{w}_j w_i w_k \mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^{\mathrm{T}}\right)\mathbf{z}(t)$$
$$= \sum_{i=1}^{p} w_i \mathbf{r}(t)^{\mathrm{T}}\mathbf{Y}_i\mathbf{r}(t) + \sum_{i=1}^{p}\overline{w}_i \mathbf{r}(t)^{\mathrm{T}}\mathbf{S}_i\mathbf{r}(t) \quad (35)$$

where $\mathbf{S}_i = \begin{bmatrix} \mathbf{S}_{1i1} & \mathbf{S}_{1i2} & & \mathbf{S}_{1ip} \\ \mathbf{S}_{2i1} & \mathbf{S}_{2i2} & & \mathbf{S}_{2ip} \\ & & & \\ \mathbf{S}_{pi1} & \mathbf{S}_{pi2} & & \mathbf{S}_{pip} \end{bmatrix}$. It can be seen from (35) that $V(t) \leq 0$ (equality holds when $\mathbf{z}(t) = \mathbf{0}$) is implied by $\mathbf{Y}_i < 0$ and $\mathbf{S}_i < 0$ which further imply the asymptotic stability of the fuzzy-model-based control system of (30). The analysis result is summarized in the following theorem.

Theorem 3: *The fuzzy-model-based control system, formed by the nonlinear plant in the form* (2) *and the non-PDC fuzzy controller in the form of* (28) *connected in a closed loop, is asymptotically stable if* $w(\mathbf{x}(t)) + \rho w(\mathbf{x}(t)) + \sigma_i > 0$, *i = 1, 2, ..., p where* ρ *and* σ_i *are scalars and there exists* $\mathbf{P}_{k,l} = \mathbf{P}_{l,k}^{\mathrm{T}} \in \Re^{n \times n}$, *k, l = 1, 2, ...,p;* $\mathbf{S}_{ijk} = \mathbf{S}_{kji}^{\mathrm{T}} \in \Re^{n \times n}$, *j, k = 1, 2, ..., p; i < k,* $\mathbf{\Lambda}_{ik} = \mathbf{\Lambda}_{ki}^{\mathrm{T}} \in \Re^{n \times n}$, *i, k = 1, 2, ..., p;* $\mathbf{Y}_{iii} = \mathbf{Y}_{iii}^{\mathrm{T}}$, *i = 1, 2, ..., p,* $\mathbf{Y}_{iij} = \mathbf{Y}_{jii}^{\mathrm{T}}$, *i, j = 1, 2, ..., p; i ≠ j,* $\mathbf{Y}_{ijk} = \mathbf{Y}_{ikj}^{\mathrm{T}}$, $\mathbf{Y}_{jik} = \mathbf{Y}_{jki}^{\mathrm{T}}$ *and* $\mathbf{Y}_{kij} = \mathbf{Y}_{kji}^{\mathrm{T}}$, *i = 1, 2, ..., p – 2; j = 1, 2, ..., p – 1; k = 1, 2, ..., p, such that the following LMIs are satisfied:*
$\begin{bmatrix} \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & & \mathbf{P}_{1,p} \\ \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & & \mathbf{P}_{2,p} \\ & & & \\ \mathbf{P}_{p,1} & \mathbf{P}_{p,2} & & \mathbf{P}_{p,p} \end{bmatrix} > 0$, (16) *to* (18), (33) *to* (34), *and* $\mathbf{Y}_i < 0$ *and* $\mathbf{S}_i < 0$, *i = 1, 2, ..., p. In* (16) *to* (18),

$$\mathbf{\Xi}_{ijk} = \mathbf{P}_{j,k}\left(\mathbf{A}_i + \frac{1}{\rho}\mathbf{I}\right)^{\mathrm{T}} + \left(\mathbf{A}_i + \frac{1}{\rho}\mathbf{I}\right)\mathbf{P}_{j,k} + \mathbf{F}_j^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} + \mathbf{B}_i\mathbf{F}_j + \sum_{r=1}^{p}\frac{\sigma_r}{\rho}\left(\mathbf{P}_{i,r} + \mathbf{P}_{r,i}\right) - \left(\frac{1}{\rho} + \sum_{r=1}^{p}\frac{\sigma_r}{\rho}\right)\mathbf{\Lambda}_i$$

IV. Simulation Examples

To investigate the effectiveness of the proposed non-PDC fuzzy controllers, two simulation examples are considered to investigate the stability regions of the proposed and some published stability conditions.

A. Simulation Example 1

In this simulation example, the fuzzy model satisfying conditions 1) to 3) is considered. The stability region given by the stability conditions in Theorem 1 is investigated through the fuzzy model with the following rules.

Rule i: IF $x_1(t)$ is M_1^i

THEN $\mathbf{x}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)$, $i = 1, 2, 3, 4$ (36)

where $\mathbf{x}(t) = [x_1(t) \quad x_2(t)]^{\mathrm{T}}$; $\mathbf{A}_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}$, $\mathbf{A}_3 = \begin{bmatrix} -0.02 & -2.11 \\ 1.28 & -0.11 \end{bmatrix}$ and $\mathbf{A}_4 = \begin{bmatrix} -a & -4.33 \\ 0.1 & 0.05 \end{bmatrix}$; $\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{B}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{B}_3 = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $\mathbf{B}_4 = \begin{bmatrix} -b+2 \\ -1 \end{bmatrix}$; $1 \leq a \leq 12$, $1 \leq b \leq 6$. The membership functions are defined as

$w_1(x_1(t)) = \mu_{M_1^1}(x_1(t)) = 1 - \dfrac{1}{1 + e^{-\frac{x_1(t)+2.5}{10}}}$,

$w_2(x_1(t)) = \mu_{M_1^2}(x_1(t)) = 1 - w_1(x_1(t))$,

$w_3(x_1(t)) = \mu_{M_1^3}(x_1(t)) = 1 - \frac{1}{1+e^{-\frac{x_1(t)-2.5}{10}}}$ and $w_4(x_1(t)) = \mu_{M_1^4}(x_1(t)) = 1 - w_3(x_1(t))$. It can be seen that the membership functions satisfy $\dot{w}_1(x_1(t)) = -\dot{w}_2(x_1(t))$ and $\dot{w}_3(x_1(t)) = -\dot{w}_4(x_1(t))$. The non-PDC fuzzy controller of (6) is employed to handle the control problem. Stability conditions in Theorem 1 are employed to examine the stability region which is shown in Fig. 1 denoted by "o". For comparison purpose, stability conditions in [11] are employed to obtain the stability region which is shown in Fig. 1 denoted by "×". It can be seen from Fig. 1 that the proposed stability conditions offer a larger stability region.

B. Simulation Example 2

A simulation example is considered to show the effectiveness of the proposed stability conditions for general fuzzy models. Consider the fuzzy model with the following rules.

Rule i: IF $x_3(t)$ is M_1^i

THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)$, i = 1, 2, 3 (37)

where $\mathbf{x}(t) = [x_1(t) \quad x_2(t) \quad x_3(t)]^T$;

$$\mathbf{A}_1 = \begin{bmatrix} 1.59 & -7.29 & 0 \\ 0.01 & 0 & 0 \\ 0 & -0.17 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0.02 & -4.64 & 0 \\ 0.35 & 0.21 & 0 \\ 0 & -0.78 & 0 \end{bmatrix} \text{ and}$$

$$\mathbf{A}_3 = \begin{bmatrix} -a & -4.33 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & -0.21 \end{bmatrix}; \quad \mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} \text{ and}$$

$$\mathbf{B}_3 = \begin{bmatrix} -b+6 \\ -1 \\ 0 \end{bmatrix}; \ 2 \le a \le 12,\ 2 \le b \le 20.$$

The membership functions are defined as $w_1(x_3(t)) = \mu_{M_1^1}(x_3(t)) = 1 - \frac{1}{1+e^{-\frac{x_3(t)+1}{0.8}}}$, $w_2(x_3(t)) = \mu_{M_1^2}(x_3(t)) = 1 - w_2(x_3(t)) - w_3(x_3(t))$ $w_3(x_3(t)) = \mu_{M_1^3}(x_3(t)) = 1 - \frac{1}{1+e^{-\frac{x_3(t)-1}{0.8}}}$. It is assumed that the fuzzy model works in the operating region of $x_2(t) \in [-2 \quad 2]$ and $x_3(t) \in [-2 \quad 2]$. Based on the defined membership functions, we have $\dot{w}_1(x_3(t)) = -\frac{1.25\left(e^{-\frac{x_3(t)+1}{0.8}}\right)\dot{x}_3(t)}{\left(1+e^{-\frac{x_3(t)+1}{0.8}}\right)^2}$, $\dot{w}_2(x_3(t)) = -\dot{w}_2(x_3(t)) - \dot{w}_3(x_3(t))$ and $\dot{w}_3(x_3(t)) = \frac{1.25\left(e^{-\frac{x_3(t)-1}{0.8}}\right)\dot{x}_3(t)}{\left(1+e^{-\frac{x_3(t)-1}{0.8}}\right)^2}$ where $\dot{x}_3 = \left(-0.17w_1(x_3(t)) - 0.78w_2(x_3(t))\right)x_2(t) - 0.21w_3(x_3(t))x_3(t)$. It follows that $\overline{w}_i(x_3(t)) = w_i(x_3(t)) + \rho\dot{w}_i(x_3(t)) + \sigma_i > 0$, i = 1, 2, 3, are satisfied for ρ = 1.2, σ_1 = 0.0222, σ_2 = –0.0717 and σ_3 = 0.0282.

The proposed non-PDC fuzzy controllers in (22) and (28) are employed to close the feedback loop of the fuzzy model. Stability conditions in Theorem 2 and Theorem 3 are employed respectively to examine the stability regions. Fig. 2 shows the stability regions corresponding to the non-PDC fuzzy controllers of (22) and (28) denoted by "×" and "o" respectively. It can be seen from Fig. 2 that the stability conditions in Theorem 3 offer a larger stability region. However, the stronger stabilization ability of non-PDC fuzzy controller of (28) is due to the extra feedback gains are included, which increases the structural complexity of the controller. Consequently, it is suggested to employ the non-PDC fuzzy controller of (22) for the fuzzy model with parameters chosen from the overlapping stability region in Fig. 2 to lower the structural complexity of the controller. For comparison purpose, the stability conditions in [16]-[17] are employed to check the stability regions which is shown in Fig. 3 denoted by "×" and "o" respectively. It can be seen that the proposed stability conditions offer larger stability regions as compared with those in [16]-[17].

V. Conclusion

The system stability of fuzzy-model-based control systems with non-PDC fuzzy controller has been investigated based on an improved fuzzy Lyapunov function. Special and general cases of fuzzy models have been considered in this paper. Under the special case, the time derivatives of the membership functions vanish in the stability analysis. Consequently, the stability analysis can be made easy and lead to less conservative stability analysis result as compared to the traditional PDC approach. A general fuzzy model has been considered. An improved non-PDC fuzzy controller, which includes more feedback gains, has been proposed to close the feedback loop. The improved non-PDC fuzzy controller has exhibited a favourable property to further relax the stability conditions. Based on the fuzzy Lyapunov function, stability conditions in terms of linear matrix inequalities have been derived to help designing a stable fuzzy-model-based control system. Simulation results have been given to illustrate the effectiveness of the proposed non-PDC fuzzy control approach.

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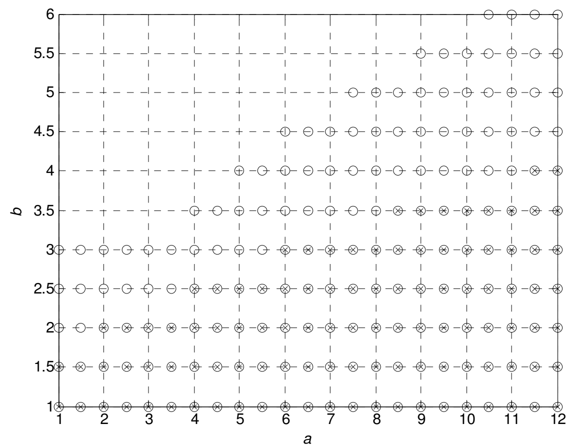

Fig. 1. Stability regions of stability conditions in Theorem 1 denoted by "o" and [11] denoted by "×".

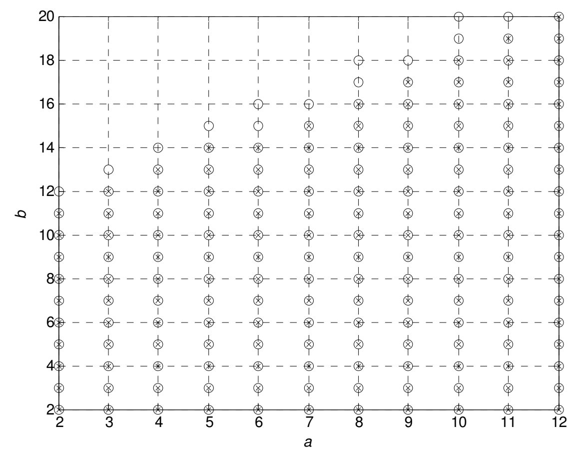

Fig. 2. Stability regions of stability conditions in Theorem 2 denoted by "×" and Theorem 3 denoted by "o".

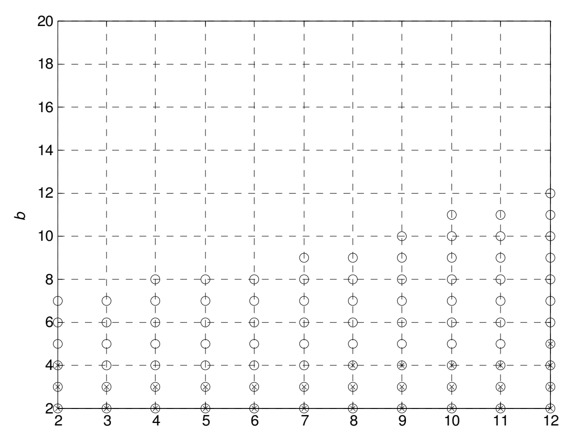

Fig. 3. Stability regions of stability conditions in [16] denoted by "×" and [17] denoted by "o".