Stability Analysis of Systems with Parameter Uncertainties under Fuzzy Logic Control

L. K. Wong[1] F. H. F. Leung[2] P. K. S. Tam[3]

Department of Electronic Engineering
The Hong Kong Polytechnic University
Hung Hom, Hong Kong

Abstract - Systems with parameter uncertainties are difficult to control. We propose a combined controller approach to tackle this problem. This approach involves designs of different controllers for different operation points. Then a fuzzy logic controller (FLC) is used to schedule these controllers according to the plant parameters. This FLC is different from the common FLCs in that plant parameters are used as the input variables in the premises of the fuzzy rules. Hence this FLC can tackle a large parameter change during operation. The resulting FLC will not only display good robustness but also lead to a stable closed-loop system. The stability will be proved by applying a new stability analysis method.

1. Introduction

One major problem in control systems is the parameter uncertainty in the plant. Many control techniques that can give excellent performance rely on the availability of an exact plant model. Unfortunately, plant models are often subject to approximations. Moreover, the plant parameters can be uncertain, i.e. the parameter may take any value inside a bounded range during operation. There are at lease two reasons. First, some of the plant parameters come with tolerance; for example, the mass and the elastic coefficient of the spring in a mass spring damper system; the resistance, inductance and capacitance of the components in an electronic circuit. Second, some plant parameters will be changing during operation, no matter the change is slow or sudden. Slow change may be due to temperature drift or component aging. Sudden change may be due to variations in physical conditions; for example, the number of passengers in a train.

To tackle the parameter uncertainties in a plant, many robust control techniques were developed. They can offer satisfactory performance and system stability under small variation of parameter values around a nominal operation point. However, when the variation of parameters is large, these techniques will eventually fail. To alleviate this problem, a multiple grid-point method was proposed by Leung [4]. This method uses more than one operating point (i.e. multiple grid-points) to describe the whole parameter space subject to disturbances, and a controller corresponding to each grid-point is designed. Once the parameters change, the grid-point as well as the controller changes accordingly. As a result, the closed-loop system is robust to large parameter changes.

If the actual parameters are near one grid-point in the parameter space, it is easy to select the grid-point and a good performance can be obtained by applying the corresponding controller. However, when the actual parameter values are near the middle of two or more grid-points, choosing any grid-point may not give a good solution. To tackle this problem, Ng [5] proposed a fuzzy scheduling approach and applied it to control a power converter. By combining the controllers corresponding to the relevant grid-points using a fuzzy logic controller (FLC), improvements on the transient responses can be seen. However, the system stability becomes another problem on applying this combined controller. Although the sub-system stability can be guaranteed when applying any one controller, the overall system stability cannot be directly ensured when the controllers are combined.

In view of this inadequacy, a stability analysis method for this kind of fuzzy logic control systems is proposed in this paper. It is developed from a stability analysis method proposed by Wong [1,2]. The method makes use of a common Lyapunov function to guarantee system stability. It had been shown that this approach can be applied to combine different controllers into a single fuzzy logic controller [1,3]. Since the control objective in this paper is to tackle systems with parameter changes, the premises of the fuzzy rules have to be modified. The input variables of the FLC become the parameter values instead of the system states, and the stability analysis is to be performed accordingly. The details will be given in Section 2. An illustrative example will then be given in Section 3. A conclusion will be drawn in Section 4.

[1] E-mail: enkin@encserver.en.polyu.edu.hk
[2] E-mail: enfrank@hkpucc.polyu.edu.hk
[3] E-mail: enptam@hkpucc.polyu.edu.hk

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2. Review of the stability analysis method

2.1 Fuzzy logic control system

Consider a single input n-th order non-linear system with parameter uncertainty of the following form:

$$\dot{x} = f(x,p) + b(x,p)u \qquad (1)$$

where $x = [x_1, x_2, \ldots\ldots, x_n]^T$ is the state-vector, $p = [p_1, p_2, \ldots\ldots, p_m]^T$ is the uncertain parameter vector, $f(x, p) = [f_1(x, p), f_2(x, p), \ldots\ldots, f_n(x, p)]^T$, $b(x, p) = [b_1(x, p), b_2(x, p), \ldots\ldots, b_n(x, p)]^T$ are functions describing the dynamics of the plant, and u is the control input of the plant that its value is determined by an FLC. The i-th **IF-THEN** rule in the fuzzy rule base of the FLC is of the following form:

Rule i: **IF** *<premise i>* **THEN** $u = u_i$ (2)

where *<premise i>* is the premise of rule i with input variable z. This input variable will be the uncertain parameter p in this paper (but it can also be the state x as in the common FLC). $u = u_i$ is the control output of rule i. It can be a single value or a function of the state x. The shape of the membership functions associated with the input fuzzy levels, the method of fuzzification, and the algorithm of rule inference can be arbitrary because these do not affect the stability analysis. A degree of membership $\mu_i \in [0, 1]$ is obtained for each rule i. It is assumed that for any z in the input universe of discourse Z, there exists at least one μ_i among all rules that is non-zero. By applying the weighted sum defuzzification method, the overall output of the FLC is given by:

$$u = \frac{\sum_{i=1}^{k} \mu_i u_i}{\sum_{i=1}^{k} \mu_i} \qquad (3)$$

where k is the total number of rules. Here we need to define the following terms: active/inactive fuzzy rules, and active region of a fuzzy rule.

Definition 2.1: For any input $z_o \in Z$, if the degree of membership μ_i corresponding to fuzzy rule i is zero, this fuzzy rule i is called an *inactive fuzzy rule* for the input z_o; otherwise, it is called an *active fuzzy rule*. An *active region* of a fuzzy rule is defined as a region $Z_r \subset Z$ such that its degree of membership μ_i is non-zero for all $z_o \in Z_r$.

It should be noted that for any input z_o, an inactive fuzzy rule will not affect the controller output u. Hence, (3) can be re-written so as to consider all active fuzzy rules (where $\mu_i \neq 0$ for $z = z_o$) only,

$$u = \frac{\sum_{i=1,\mu_i \neq 0}^{k} \mu_i u_i}{\sum_{i=1,\mu_i \neq 0}^{k} \mu_i} \qquad (4)$$

Now, among all the control output u_i for $z = z_o$ of the sub-systems corresponding to the active fuzzy rules, there exists a maximum value $u_{\max}$ and a minimum value $u_{\min}$. Then,

$$\frac{\sum_{i=1,\mu_i \neq 0}^{k} \mu_i u_{\min}}{\sum_{i=1,\mu_i \neq 0}^{k} \mu_i} \leq \frac{\sum_{i=1,\mu_i \neq 0}^{k} \mu_i u_i}{\sum_{i=1,\mu_i \neq 0}^{k} \mu_i} \leq \frac{\sum_{i=1,\mu_i \neq 0}^{k} \mu_i u_{\max}}{\sum_{i=1,\mu_i \neq 0}^{k} \mu_i}$$

$$\Rightarrow \quad u_{\min} \frac{\sum_{i=1,\mu_i \neq 0}^{k} \mu_i}{\sum_{i=1,\mu_i \neq 0}^{k} \mu_i} \leq u \leq u_{\max} \frac{\sum_{i=1,\mu_i \neq 0}^{k} \mu_i}{\sum_{i=1,\mu_i \neq 0}^{k} \mu_i}$$

$$\Rightarrow \quad u_{\min} \leq u \leq u_{\max}, \qquad (5)$$

equality holds when $u_i = u_{\min} = u_{\max}$

In conclusion, the overall FLC output is bounded by $u_{\max}$ and $u_{\min}$ among the rules if the weighted sum defuzzification method is employed to derive u for any $z_o \in Z$.

2.2 Stability analysis method

The premise of the stability criterion in this paper is that on applying each rule to the plant individually, the closed-loop sub-system formed is stable in the sense of Lyapunov (ISL) in the rule's active region, and each rule shares a common quadratic Lyapunov function $V(x) = x^T P x$ such that

i) $V(x)$ is positive definite and continuously differentiable,
ii) $\dot{V}(x) \leq 0$ in the rule's active region (6)

For an input $z_o \in Z$, let the maximum and minimum control signals among all active fuzzy rules be $u_{\max}$ and $u_{\min}$ respectively. From (6), we have the sub-systems formed by these two rules satisfying the following conditions

$$\dot{V}(x) \leq 0 \text{ for } z = z_o,\ u = u_{\max} \qquad (7)$$

$$\dot{V}(x) \leq 0 \text{ for } z = z_o,\ u = u_{\min} \qquad (8)$$

Lemma 2.1: If a system in the form of (1) satisfies the premise of the stability criterion of (6), for all $z_o \in Z$, we have $\dot{V}(x) \leq 0$ for $u \in [u_{\min}, u_{\max}]$.

Proof:

$$V(x) = x^T P x \qquad (9)$$

$$\Rightarrow \quad \dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} \qquad (10)$$

From (1),

$$\dot{V}(x) = (f(x,p) + b(x,p)u)^T P x + x^T P (f(x,p) + b(x,p)u)$$
$$= F(x,p) + B(x,p)u \qquad (11)$$

where $F(x, p) = f(x, p)^T P x + x^T P f(x, p)$,
$B(x, p) = b(x, p)^T P x + x^T P b(x, p)$.

Note that both $F(x, p)$ and $B(x, p)$ are scalars. Then, two cases should be considered: $B(x, p)$ is positive and $B(x, p)$ is negative for $z = z_o$.

Case 1: $B(x, p)$ is positive

By using condition (7), for $z = z_o$ and $u = u_{\max}$

$$\begin{aligned} \dot{V}(x)\big|_{u=u_{\max}} &= F(x,p) + B(x,p)u_{\max} \le 0 \\ \Rightarrow \quad \dot{V}(x) &= F(x,p) + B(x,p)u \le 0 \quad \forall\, u \le u_{\max} \end{aligned} \tag{12}$$

$$\Rightarrow \dot{V}(x) \le 0 \text{ for } z = z_o \text{ and } u \in [u_{\min}, u_{\max}]$$

Case 2: $B(x, p)$ is negative

By using condition (8), for $z = z_o$ and $u = u_{\min}$

$$\begin{aligned} \dot{V}(x)\big|_{u=u_{\min}} &= F(x,p) + B(x,p)u_{\min} \le 0 \\ \Rightarrow \quad \dot{V}(x) &= F(x,p) + B(x,p)u \le 0 \quad \forall\, u \ge u_{\min} \end{aligned} \tag{13}$$

$$\Rightarrow \dot{V}(x) \le 0 \text{ for } z = z_o \text{ and } u \in [u_{\min}, u_{\max}] \tag{14}$$

From (12) and (13), the lemma is proved.

QED

Theorem 2.1: Consider an FLC as described in Section 2.1, if every rule of the FLC applying to the plant of (1) individually gives a stable sub-system ISL in the active region of the fuzzy rule subject to a common Lyapunov function, and the defuzzification method is realized as given by (3), the whole fuzzy logic control system is stable ISL.

Proof: It has been shown in (5) that for an arbitrary input $z_o \in Z$, the control output of an FLC is bounded by $u_{\min}$ and $u_{\max}$ if the weighted sum defuzzification method is employed. Hence, if all sub-systems satisfy $\dot{V}(x) \le 0$ in the active regions of the rules, by *Lemma 2.1*, $\dot{V}(x) \le 0$ for all $z_o \in Z$ and the closed-loop system is stable ISL under the control of the FLC.

QED

3. Illustrative Example

In this section, the above stability analysis method will be applied to a plant with parameter uncertainty under the control of an FLC. This FLC is used to schedule two PI controllers. The block diagram of the system is shown in Fig. 1. The state equation and output equation of the plant are as follows:

$$\begin{aligned} \dot{x} &= \mathbf{A}x + \mathbf{B}u \\ y &= \mathbf{C}x \end{aligned} \tag{15}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the state vector; $\mathbf{A} = \begin{bmatrix} -5 & p_2 \\ 1 & -2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 1 + \dfrac{1}{p_2} \end{bmatrix}$ and $\mathbf{C} = [1 \;\; 0]$ define the plant parameters.

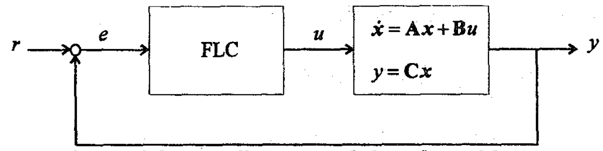

Fig. 1. A block diagram of the closed-loop system

The parameter p_2 is not fixed such that $2 \le p_2 \le 9$. Furthermore, define

$$e = x_r - x \tag{16}$$

$$v = \int \mathbf{C} e\, dt \tag{17}$$

where $e = [e_1 \;\; e_2]^{\mathrm{T}}$ is the error state vector, $x_r = [x_{r1} \;\; x_{r2}]^{\mathrm{T}}$ is the reference state vector, v is the integral state. To carry out the proof of stability, we introduce a new state e_v such that

$$e_v = v_r - v \tag{18}$$

where v_r is the reference value of v. The value of v_r is governed by the following condition:

$$\mathbf{A}\, x_r + \mathbf{B}\, k_i\, v_r = \mathbf{0} \tag{19}$$

where k_i is an integral gain of the PI controller. The reference state x_{r1} equals to the reference input r in Fig. 1. It is not needed to know the values of e_2, x_{r2}, e_v and v_r because they will not be used in deriving the control output of the FLC. From (15) to (18),

$$\begin{bmatrix} \dot{e}_v \\ \dot{e} \end{bmatrix} = \begin{bmatrix} 0 & -\mathbf{C} \\ 0 & \mathbf{A} \end{bmatrix} \begin{bmatrix} e_v \\ e \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathbf{B} \end{bmatrix} u - \begin{bmatrix} 0 \\ \mathbf{A}x_r \end{bmatrix} \tag{20}$$

To tackle the parameter uncertainty in p_2, we design two PI controllers corresponding to p_2 being equal to 2 and 9 respectively. Then these two controllers is combined into an FLC. The two fuzzy rules of the FLC are:

Rule 1: IF p_2 is SM

THEN $u = u_1 = k_{p1}\, e_1 + k_{i1}\, v = 20\, e_1 + 20\, v$

Rule 2: IF p_2 is LR

THEN $u = u_2 = k_{p2}\, e_1 + k_{i2}\, v = 3\, e_1 + v$

where SM and LR are fuzzy sets. Their membership functions are shown in Fig. 2. Also a Lyapunov function is defined as follows:

$$V = [e_v \;\; e]\,\mathbf{P} \begin{bmatrix} e_v \\ e \end{bmatrix} \tag{21}$$

where $\mathbf{P} = \begin{bmatrix} 3.1341 & -0.0985 & -0.6166 \\ -0.0985 & 0.8481 & -0.5319 \\ -0.6166 & -0.5319 & 0.7949 \end{bmatrix}$.

To verify the system stability under the control of the FLC, we should show that Rule 1 and Rule 2 can give $\dot{V} \le 0$ for all p_2 in the active region of the corresponding rule.

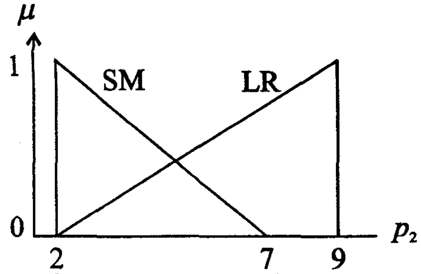

Fig. 2. Membership functions

Consider Rule 1, from (20)

$$\begin{bmatrix}\dot{e}_v\\ \dot{e}\end{bmatrix}=\begin{bmatrix}0 & -\mathbf{C}\\ 0 & \mathbf{A}\end{bmatrix}\begin{bmatrix}e_v\\ e\end{bmatrix}+\begin{bmatrix}0\\ -\mathbf{B}\end{bmatrix}(k_{p1}e_1+k_{i1}v)-\begin{bmatrix}0\\ \mathbf{A}x_r\end{bmatrix}$$

$$=\begin{bmatrix}0 & -\mathbf{C}\\ 0 & \mathbf{A}\end{bmatrix}\begin{bmatrix}e_v\\ e\end{bmatrix}+\begin{bmatrix}0\\ -\mathbf{B}\end{bmatrix}(k_{p1}e_1+k_{i1}v_r-k_{i1}e_v)-\begin{bmatrix}0\\ \mathbf{A}x_r\end{bmatrix}$$

$$\begin{bmatrix}\dot{e}_v\\ \dot{e}\end{bmatrix}=\begin{bmatrix}0 & -\mathbf{C}\\ \mathbf{B}k_{i1} & \mathbf{A}-\mathbf{B}k_{p1}\mathbf{C}\end{bmatrix}\begin{bmatrix}e_v\\ e\end{bmatrix}-\begin{bmatrix}0\\ \mathbf{A}x_r+\mathbf{B}k_{i1}v_r\end{bmatrix} \quad (22)$$

From (19),

$$\begin{bmatrix}\dot{e}_v\\ \dot{e}\end{bmatrix}=\begin{bmatrix}0 & -\mathbf{C}\\ \mathbf{B}k_{i1} & \mathbf{A}-\mathbf{B}k_{p1}\mathbf{C}\end{bmatrix}\begin{bmatrix}e_v\\ e\end{bmatrix}=\mathbf{A}_{c1}\begin{bmatrix}e_v\\ e\end{bmatrix} \quad (23)$$

From (21),

$$\dot{V}=\begin{bmatrix}e_v & e\end{bmatrix}(\mathbf{A}_{c1}{}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{A}_{c1})\begin{bmatrix}e_v\\ e\end{bmatrix}$$

$$=\begin{bmatrix}e_v & e\end{bmatrix}\mathbf{Q}_1\begin{bmatrix}e_v\\ e\end{bmatrix}$$

To ensure the sub-system under the control of Rule 1 is stable, we should show that $\dot{V}$ is negative definite for the active region of Rule 1, i.e. $2 \le p_2 \le 7$. Hence, the real parts of the eigenvalues of $\mathbf{Q}_1$ when p_2 increases from 2 to 7 are plotted as shown in Fig. 3. Since the eigenvalues are negative, we can conclude that $\dot{V}$ is negative definite, and this sub-system is stable.

Similarly, for the sub-system under the control of Rule 2, the same procedure can be taken with k_{p1} and k_{i1} changed into k_{p2} and k_{i2}. By using the same Lyapunov function in (21), we have

$$\dot{V}=\begin{bmatrix}e_v & e\end{bmatrix}(\mathbf{A}_{c2}{}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{A}_{c2})\begin{bmatrix}e_v\\ e\end{bmatrix}$$

$$=\begin{bmatrix}e_v & e\end{bmatrix}\mathbf{Q}_2\begin{bmatrix}e_v\\ e\end{bmatrix}$$

where $\mathbf{A}_{c2}=\begin{bmatrix}0 & -\mathbf{C}\\ \mathbf{B}k_{i2} & \mathbf{A}-\mathbf{B}k_{p2}\mathbf{C}\end{bmatrix}$.

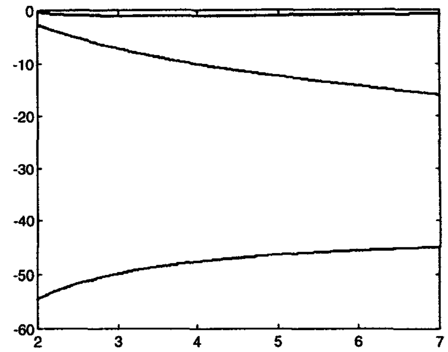

Fig. 3. Real parts of eigenvalues of $\mathbf{Q}_1$ when p_2 is from 2 to 7

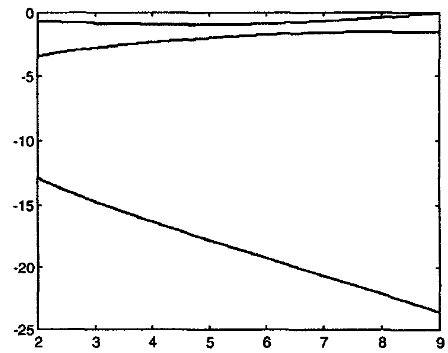

Fig. 4. Real parts of eigenvalues of $\mathbf{Q}_2$ when p_2 is from 2 to 9

Similarly, the real parts of the eigenvalues of $\mathbf{Q}_2$ when p_2 increases from 2 to 9 are shown in Fig. 4. It can be seen that $\dot{V}$ is also negative definite. Hence this sub-system is also stable. From *Theorem 2.1*, the closed-loop system under the control of the FLC is stable.

The start-up responses of the closed-loop system are shown in Fig. 5 to Fig. 7. Fig. 5 shows the responses for $p_2 = 4.5$. It can be seen that if u_1 is used, an overshoot exists. On the other hand, if u_2 is used, the rise time is slow. The FLC schedules the two controllers so that a better result can be obtained. Fig. 6 shows the responses for $p_2 = 2$. The responses under the control of u_1 and the FLC are the same. It is because only u_1 is active at this value of p_2 as can be seen in Fig. 2. If u_2 is used in this case, the rise time is very slow. Similarly, the responses for $p_2 = 9$ is shown in Fig. 7. The response obtained by applying the FLC and u_2 are the same while that by applying u_1 has a large overshoot. Hence, it can be seen that the FLC can tackle a larger parameter variation than a single PI controller. The system stability is also guaranteed.

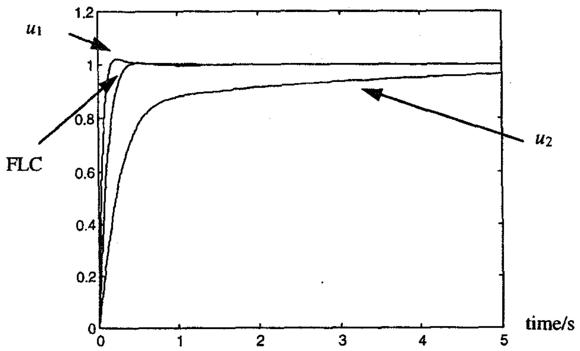

Fig. 5. Start-up output response when p_2 = 4.5

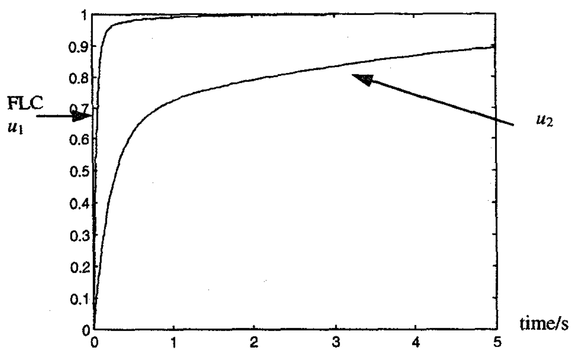

Fig. 6. Start-up output response when p_2 = 2

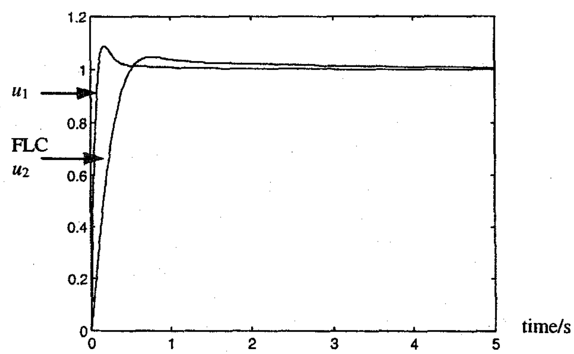

Fig. 7. Start-up output response when p_2 = 9

4. Conclusion

A stability analysis for systems with parameter uncertainties under fuzzy logic control is addressed. This is achieved by applying a proposed stability analysis method. Different from the commonly used FLCs, the premises of the fuzzy rules in this paper use the uncertain parameter as an input variable instead of the states to tackle the robust control problem. Modifications to a previously proposed method have been made so that the stability analysis of such a kind of FLCs can be carried out. An illustrative example has been given to show the merits and ability of the proposed FLC. Under a wide range of parameter variation, the closed-loop system has been proved to be stable and good transient responses are also obtained.

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