Stable and Robust Fuzzy Control for Uncertain Nonlinear Systems Based on a Grid-Point Approach[1]

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Abstract

This paper presents the stability and robustness analyses of uncertain nonlinear control systems. To proceed with the analysis, the uncertain nonlinear system is represented by a fuzzy model with uncertainties. Based on this model, under a single-grid-point (SGP) approach, a fuzzy controller is designed to close the feedback loop. A simple stability criterion is derived by estimating the norm of the system parameters and the matrix measure, and a robust area in the uncertain parameter space is defined. The design methodology of a multiple-grid-point (MGP) fuzzy controller for nonlinear systems subjected to large parameter uncertainties is developed based on the analysis results of the SGP approach.

1. Introduction

Our major concerns on fuzzy control systems are the system stability and a systematic design methodology. Many research works have been done in this area. Recently, the stability analysis is carried out based on a fuzzy model [1]. The stability condition for this class of fuzzy control systems was derived [2, 6], which involved a common positive definite matrix **P**. It is a difficult task to find such a **P**, especially when the number of rules of the fuzzy model is large. Moreover the result is valid only when the fuzzy model can exactly represent the plant. The robustness analysis and robust controller design for fuzzy control systems were also investigated [3, 5]. For the work in [5], the fuzzy control problem must be formulated into an H_∞ problem in advance. It may be a difficulty for some practical engineers or designers to understand the theories. More importantly, these works did not consider the situation when the uncertainties are too large to be handled by the designed fuzzy controller. The aims of this paper are to analyze the stability and robustness of uncertain fuzzy control systems, and to develop an easy-to-understand fuzzy stability theory and a simple design methodology for the fuzzy controllers.

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2. Fuzzy plant model and fuzzy controller

A general multivariable uncertain nonlinear control system can be represented as a fuzzy plant model with uncertainties and a fuzzy controller.

2.1. Fuzzy plant model with uncertainties

Let p be the number of fuzzy rules describing the uncertain nonlinear plant. The i-th rule is of the following format,

Rule i : IF x_1 is M_1^i and … and x_n is M_n^i

$$\text{THEN } \dot{\mathbf{x}} = (\mathbf{A}^i + \Delta\mathbf{A}^i)\mathbf{x} + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{u} \tag{1}$$

where M_k^i is a fuzzy term of rule i corresponding to the state x_k, $k = 1,2,\dots,n$, $i = 1,2,\dots,p$; $\Delta\mathbf{A}^i \in \Re^{n\times n}$ and $\Delta\mathbf{B}^i \in \Re^{n\times m}$ are the uncertainties of $\mathbf{A}^i \in \Re^{n\times n}$ and $\mathbf{B}^i \in \Re^{n\times m}$ respectively; $\mathbf{x} \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u} \in \Re^{m\times 1}$ is the input vector. The inferred system states are given by

$$\dot{\mathbf{x}} = \sum_{i=1}^{p} w^i \left((\mathbf{A}^i + \Delta\mathbf{A}^i)\mathbf{x} + (\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{u} \right) \tag{2}$$

where $\sum_{i=1}^{p} w^i = 1$, $w^i \in [0,\ 1]$ $\forall\, i$, w^i is the normalized weight of rule i and is a nonlinear function of $\mathbf{x}$.

2.2. Fuzzy controller

A fuzzy controller with c fuzzy rules is to be designed for the plant. The j-th rule is of the following format,

Rule j: IF x_1 is N_1^j and … and x_n is N_n^j

$$\text{THEN } \mathbf{u} = \mathbf{G}^j\mathbf{x} + \mathbf{r} \tag{3}$$

where N_l^j is a fuzzy term of rule j corresponding to the state x_l, $l = 1, \dots, n$, $j = 1, \dots, c$; $\mathbf{G}^j \in \Re^{m\times n}$ is the

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feedback gain of rule j, and $\mathbf{r} \in \Re^{m\times 1}$ is the reference input vector. The inferred output of the fuzzy controller is,

$$\mathbf{u} = \sum_{j=1}^{c} m^j\left(\mathbf{G}^j\mathbf{x}+\mathbf{r}\right) \tag{4}$$

where $\sum_{j=1}^{c} m^j = 1$, $m^i \in [0,\ 1]\ \forall j$, m^j is the normalized weight of rule j and is a nonlinear function of $\mathbf{x}$.

3. Stability and robustness analysis of single-grid-point approach

Stability and robustness of an uncertain fuzzy control system are to be analyzed in this section. Three design approaches closing the feedback loop are investigated.

3.1. General Design Approach (GDA)

General design approach allows differences in the number of rules and the rule antecedents between the fuzzy plant model and the fuzzy controller. From (1) to (4), the closed-loop fuzzy system is given by,

$$\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x}+\mathbf{B}_0\mathbf{r}+\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j((\mathbf{H}^{ij}+\Delta\mathbf{H}^{ij}-\mathbf{A}_0)\mathbf{x}+(\mathbf{B}^i-\mathbf{B}_0+\Delta\mathbf{B}^i)\mathbf{r}) \tag{5}$$

where $\mathbf{H}^{ij} = \mathbf{A}^i + \mathbf{B}^i\mathbf{G}^j$ (6)

$\Delta\mathbf{H}^{ij} = \Delta\mathbf{A}^i + \Delta\mathbf{B}^i\mathbf{G}^j$ (7)

The system of (5) deviates from the linear system of $\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x}+\mathbf{B}_0\mathbf{r}$ by the third term in the LHS of (5).

3.2. Parallel Design Approach (PDA)

Parallel design approach uses the same number of rules and rule antecedents of the plant model in the fuzzy controller. The closed-loop fuzzy system is then given by,

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}_0\mathbf{x}+\mathbf{B}_0\mathbf{r}+\sum_{i=1}^{p} w^i(w^i(\mathbf{H}^{ii}+\Delta\mathbf{H}^{ii}-\mathbf{A}_0)\mathbf{x} \\ &+(\mathbf{B}^i+\Delta\mathbf{B}^i-\mathbf{B}_0)\mathbf{r})+2\sum_{i<j}^{p} w^i w^j(\mathbf{J}^{ij}+\Delta\mathbf{J}^{ij}-\mathbf{A}_0)\mathbf{x}\end{aligned} \tag{8}$$

where $\mathbf{J}^{ij} = \dfrac{\mathbf{H}^{ij}+\mathbf{H}^{ji}}{2}$, $\Delta\mathbf{J}^{ij} = \dfrac{\Delta\mathbf{H}^{ij}+\Delta\mathbf{H}^{ji}}{2}$ (9)

$\mathbf{H}^{ij} = \mathbf{A}^i + \mathbf{B}^i\mathbf{G}^j$, $\Delta\mathbf{H}^{ij} = \Delta\mathbf{A}^i + \Delta\mathbf{B}^i\mathbf{G}^j$ (10)

3.3. Simplified Design Approach (SDA)

Simplified design approach requires the subsystem in each rule of the fuzzy plant model possesses a common input matrix $\mathbf{B}=\mathbf{B}_0$, and the fuzzy controller has the same number of rules with the same antecedents as the fuzzy plant model. The closed-loop fuzzy system is given by,

$$\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x}+\mathbf{B}\mathbf{r}+\sum_{j=1}^{p} w^j((\mathbf{H}^j+\Delta\mathbf{H}^j-\mathbf{A}_0)\mathbf{x}+\Delta\mathbf{B}\mathbf{r}) \tag{11}$$

where $\mathbf{H}^j = \mathbf{A}^j + \mathbf{B}\mathbf{G}^j$, (12)

$\Delta\mathbf{H}^j = \Delta\mathbf{A}^j + \Delta\mathbf{B}\mathbf{G}^j$ (13)

It should be noted that in order to simplify (5) to (11), either one of the following conditions should hold:

$$\mathbf{B}^i = \mathbf{B},\ \Delta\mathbf{B}^i = \Delta\mathbf{B}, \tag{14}$$

$$\sum_{i=1}^{p} w^i\mathbf{B}^i = \mathbf{B},\ \sum_{i=1}^{p} w^i\Delta\mathbf{B}^i = \Delta\mathbf{B} \tag{15}$$

If (14) holds, a linear system can be obtained by feedback compensation (i.e. pole placement technique); otherwise, it is obtained by feedback linearization with respect to linear sub-systems satisfying (15).

3.4. Stability and robustness analysis

In the following paragraph, we proceed to the stability and robustness analysis with reference to GDA. The analysis procedures for PDA and SDA are similar to those of GDA, and the results will be given without proof. Consider the Taylor's series

$$\mathbf{x}(t+\Delta t) = \mathbf{x}(t)+\dot{\mathbf{x}}(t)\Delta t+\mathbf{o}(\Delta t) \tag{16}$$

where $\mathbf{o}(\Delta t)$ is the higher order terms and $\Delta t > 0$,

$$\lim_{\Delta t\to 0^+}\frac{\|\mathbf{o}(\Delta t)\|}{\Delta t} = 0 \tag{17}$$

From (5) and (16),

$$\begin{aligned}\|\mathbf{x}(t+\Delta t)\| &\le \|(\mathbf{I}+\mathbf{A}_0\Delta t)\mathbf{x}(t)\|+\|(\mathbf{B}_0\mathbf{r}+\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j((\mathbf{H}^{ij} \\ &+\Delta\mathbf{H}^{ij}-\mathbf{A}_0)\mathbf{x}(t)+(\mathbf{B}^i-\mathbf{B}_0)\mathbf{r}+\Delta\mathbf{B}^i\mathbf{r}))\Delta t\|+\|\mathbf{o}(\Delta t)\|\end{aligned} \tag{18}$$

where $\|\cdot\|$ denotes the l_2 norm for vectors and l_2 induced norm for matrices. From (18),

$$\begin{aligned}\lim_{\Delta t\to 0^+}\frac{\|\mathbf{x}(t+\Delta t)\|-\|\mathbf{x}(t)\|}{\Delta t} &\le \lim_{\Delta t\to 0^+}\{(\|\mathbf{I}+\mathbf{A}_0\Delta t\|-1)\|\mathbf{x}(t)\| \\ &+\|[\mathbf{B}_0\mathbf{r}+\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j((\mathbf{H}^{ij}+\Delta\mathbf{H}^{ij}-\mathbf{A}_0)\mathbf{x}(t)+ \\ &(\mathbf{B}^i-\mathbf{B}_0)\mathbf{r}+\Delta\mathbf{B}^i\mathbf{r})]\Delta t\|+\|\mathbf{o}(\Delta t)\|\}/\Delta t\end{aligned} \tag{19}$$

From (17) and (19),

$$\begin{aligned}\frac{d\|\mathbf{x}(t)\|}{dt} &\le \mu[\mathbf{A}_0]\|\mathbf{x}(t)\|+\|\mathbf{B}_0\mathbf{r}+\sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j((\mathbf{H}^{ij} \\ &+\Delta\mathbf{H}^{ij}-\mathbf{A}_0)\mathbf{x}(t)+(\mathbf{B}^i-\mathbf{B}_0)\mathbf{r}+\Delta\mathbf{B}^i\mathbf{r})\|\end{aligned} \tag{20}$$

where

$$\mu[\mathbf{A}_0] = \lim_{\Delta t\to 0^+}\frac{\|\mathbf{I}+\mathbf{A}_0\Delta t\|-1}{\Delta t} = \lambda_{\max}\left(\frac{\mathbf{A}_0+\mathbf{A}_0^*}{2}\right) \tag{21}$$

is the corresponding matrix measure of the induced matrix norm $\|\mathbf{A}_0\|$; $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of the matrix; $\mathbf{A}_0^*$ denotes the conjugate transpose of $\mathbf{A}_0$. From (20),

$$\frac{\mathrm{d}\|\mathbf{x}(t)\|}{\mathrm{d}t} \le \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j (\mu[\mathbf{A}_o] + \|\mathbf{H}^{ij} - \mathbf{A}_o\| + \|\Delta\mathbf{H}^{ij}\|)\|\mathbf{x}(t)\| + \sum_{i=1}^{p} w^i \|(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\| \tag{22}$$

Let $\mu[\mathbf{A}_o]$ be designed such that

$$\mu[\mathbf{A}_o] \le -(\|\mathbf{H}^{ij} - \mathbf{A}_o\| + \|\Delta\mathbf{H}^{ij}\|_{\max}) - \varepsilon \text{ for all } i, j \tag{23}$$

where $\|\Delta\mathbf{H}^{ij}\|_{\max}$ is the maximum value of the uncertainty $\|\Delta\mathbf{H}^{ij}\|$, and ε is a designed positive constant. Increasing the value of ε will usually result in a system with improved performance but degraded robustness. From (22) and (23).

$$\frac{\mathrm{d}\|\mathbf{x}(t)\|}{\mathrm{d}t} \le -\varepsilon\|\mathbf{x}(t)\| + \sum_{i=1}^{p} w^i \|(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\|$$

$$\Rightarrow \left(\frac{\mathrm{d}\|\mathbf{x}(t)\|}{\mathrm{d}t} + \varepsilon\|\mathbf{x}(t)\|\right)e^{\varepsilon(t-t_o)} \le \sum_{i=1}^{p} w^i \|(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\| e^{\varepsilon(t-t_o)}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\|\mathbf{x}(t)\| e^{\varepsilon(t-t_o)}) \le \sum_{i=1}^{p} w^i \|(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\| e^{\varepsilon(t-t_o)} \tag{24}$$

where t_o is an arbitrary initial time. Based on (24), there are two cases to prove the system stability: $\mathbf{r} = \mathbf{0}$ and $\mathbf{r} \neq \mathbf{0}$.

Proof. For $\mathbf{r} = \mathbf{0}$, from (24),

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\mathbf{x}(t)\| e^{\varepsilon(t-t_o)}) \le 0$$

$$\Rightarrow \|\mathbf{x}(t)\| e^{\varepsilon(t-t_o)} \le \|\mathbf{x}(t_o)\|$$

$$\Rightarrow \|\mathbf{x}(t)\| \le \|\mathbf{x}(t_o)\| e^{-\varepsilon(t-t_o)} \tag{25}$$

Since ε is a positive value, $\|\mathbf{x}(t)\| \to 0$ as $t \to \infty$.

For $\mathbf{r} \neq \mathbf{0}$, from (24),

$$\|\mathbf{x}(t)\| e^{\varepsilon(t-t_o)} \le \|\mathbf{x}(t_o)\| + \int_{t_o}^{t} \sum_{i=1}^{p} w^i \|(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\| e^{\varepsilon(\tau-t_o)} \mathrm{d}\tau$$

$$\Rightarrow \|\mathbf{x}(t)\| e^{\varepsilon(t-t_o)} \le \|\mathbf{x}(t_o)\| + \|(\tilde{\mathbf{B}} + \Delta\tilde{\mathbf{B}})\mathbf{r}\| \int_{t_o}^{t} e^{\varepsilon(\tau-t_o)} \mathrm{d}\tau$$

where

$$\|(\tilde{\mathbf{B}} + \Delta\tilde{\mathbf{B}})\mathbf{r}\| \ge \max_i \|(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\|_{\max} \ge \|(\mathbf{B}^i + \Delta\mathbf{B}^i)\mathbf{r}\|$$

$$\Rightarrow \|\mathbf{x}(t)\| \le \|\mathbf{x}(t_o)\| e^{-\varepsilon(t-t_o)} + \frac{\|(\tilde{\mathbf{B}} + \Delta\tilde{\mathbf{B}})\mathbf{r}\|}{\varepsilon}(1 - e^{-\varepsilon(t-t_o)}) \tag{26}$$

Since the right hand side of (26) is finite if $\mathbf{r}$ is bounded, the system states are also bounded. **QED**

Hence, condition (23) provides a sufficient criterion of stability for the system of (5). The stability criterion and the robust area in the uncertain parameter space of the closed-loop fuzzy systems can be summarized by the following two theorems.

Theorem 1. *Under GDA, the fuzzy control system as given by* (5) *without uncertainty, i.e.* $\|\Delta\mathbf{H}^{ij}\| = 0$, *is stable if* $\mathbf{A}_o$ *is designed such that* $\mu[\mathbf{A}_o]$ *as defined in* (21) *has the property:*

$$\mu[\mathbf{A}_o] \le -\|\mathbf{H}^{ij} - \mathbf{A}_o\| - \varepsilon \text{, for all } i \text{ and } j. \tag{27}$$

Under PDA, the fuzzy control system as given by (8) *without uncertainty, i.e.* $\|\Delta\mathbf{H}^{ii}\| = 0$ *and* $\|\Delta\mathbf{J}^{ij}\| = 0$, *is stable if* $\mathbf{A}_o$ *is designed such that* $\mu[\mathbf{A}_o]$ *as defined in* (21) *has the property:*

$$\mu[\mathbf{A}_o] \le \|\mathbf{H}^{ii} - \mathbf{A}_o\| - \varepsilon \text{, for all } i. \tag{28}$$

and

$$\mu[\mathbf{A}_o] \le \|\mathbf{J}^{ij} - \mathbf{A}_o\| - \varepsilon \text{, for all } i<j \tag{29}$$

Under SDA, the fuzzy control system of (11) *without uncertainty, i.e.* $\|\Delta\mathbf{H}^{j}\| = 0$, *is stable if* $\mathbf{A}_o$ *is designed such that* $\mu[\mathbf{A}_o]$ *as defined in* (21) *has the property:*

$$\mu[\mathbf{A}_o] \le \|\mathbf{H}^{j} - \mathbf{A}_o\| - \varepsilon \text{, for all } j. \tag{30}$$

Theorem 2. *The robust area of a fuzzy control system is defined as the area in the parameter space inside which uncertainties are allowed to exist without affecting the system stability.*

Under GDA, with the uncertain fuzzy control system given by (5), *the robust area is governed by,*

$$\|\Delta\mathbf{H}^{ij}\|_{\text{Robust area}} \le -\mu[\mathbf{A}_o] - \|\mathbf{H}^{ij} - \mathbf{A}_o\| - \varepsilon \text{ for all } i, j \tag{31}$$

The uncertain fuzzy control system is stable if the uncertainty $\|\Delta\mathbf{H}^{ij}\|$, *with* $\|\Delta\mathbf{H}^{ij}\|_{\max}$ *as its maximum value, satisfies the following condition:*

$$\|\Delta\mathbf{H}^{ij}\| \le \|\Delta\mathbf{H}^{ij}\|_{\max} \le \|\Delta\mathbf{H}^{ij}\|_{\text{Robust area}} \text{, for all } i, j \tag{32}$$

Under PDA, with the uncertain fuzzy control system given by (8), *the robust area is governed by,*

$$\|\Delta\mathbf{H}^{ii}\|_{\text{Robust area}} \le -\mu[\mathbf{A}_o] - \|\mathbf{H}^{ii} - \mathbf{A}_o\| - \varepsilon \text{, for all } i \tag{33}$$

and

$$\|\Delta\mathbf{J}^{ij}\|_{\text{Robust area}} \le -\mu[\mathbf{A}_o] - \|\mathbf{J}^{ij} - \mathbf{A}_o\| - \varepsilon \text{, for all } i<j \tag{34}$$

The uncertain fuzzy control system is stable if the uncertainties $\|\Delta\mathbf{H}^{ii}\|$ and $\|\Delta\mathbf{J}^{ij}\|$, *with* $\|\Delta\mathbf{H}^{ii}\|_{\max}$ and $\|\Delta\mathbf{J}^{ij}\|_{\max}$ *as their maximum values respectively, satisfy the following conditions:*

$$\|\Delta\mathbf{H}^{ii}\| \le \|\Delta\mathbf{H}^{ii}\|_{\max} \le \|\Delta\mathbf{H}^{ii}\|_{\text{Robust area}} \text{, for all } i \tag{35}$$

and
$$\|\Delta\mathbf{J}^{ij}\| \le \|\Delta\mathbf{J}^{ij}\|_{\max} \le \|\Delta\mathbf{J}^{ij}\|_{\text{Robust area}} \text{, for all } i<j \tag{36}$$

Under SDA, with the uncertain fuzzy control system given by (11), *the robust area is governed by*

$$\left\|\Delta\mathbf{H}^{j}\right\|_{\text{Robust area}} \le -\mu[\mathbf{A}_0] - \left\|\mathbf{H}^{j} - \mathbf{A}_0\right\| - \varepsilon \text{, for all } j \quad (37)$$

The uncertain fuzzy control system is stable if the uncertainty $\left\|\Delta\mathbf{H}^{j}\right\|$, *with* $\left\|\Delta\mathbf{H}^{j}\right\|_{\max}$ *as its maximum values, satisfies the following condition:*

$$\left\|\Delta\mathbf{H}^{j}\right\| \le \left\|\Delta\mathbf{H}^{j}\right\|_{\max} \le \left\|\Delta\mathbf{H}^{j}\right\|_{\text{Robust area}} \text{, for all } j \quad (38)$$

4. Design methodology of multiple-grid-point fuzzy controller

Based on the analysis result of the last section, a systematic MGP design methodology is introduced to deal with nonlinear systems subjected to large parameter uncertainties that cannot be handled by employing the SGP approach. The proposed MGP approach is to design a number of SGP fuzzy controllers which give good performance in local regions of the parameter space. The SGP fuzzy controllers are combined so that the union of their robust areas fully covers the whole uncertain parameter space. The situation is depicted in Figure 1. In this Figure 1, the circles denote the robust areas of the grid-points (which are the centers of the circles denoting the nominal system parameters of the local systems). The dotted rectangle is the operating uncertain parameter space. During the operation, an SGP fuzzy controller is chosen if the parameter are inside its robust area.

On designing controllers in other grid-points, it can be assumed that the nominal parameters at grid-point *G0* are shifted to grid-point *Gq* by $\Delta\mathbf{A}_{Gq} \in \Re^{n\times n}$ and $\Delta\mathbf{B}_{Gq} \in \Re^{n\times m}$ which are constants. A compensated feedback gain , $\Delta\mathbf{G}_{Gq} \in \Re^{n\times m}$ corresponding to grid-point *Gq* is needed to compensate the uncertainties to keep the system stable. The closed-loop systems in (5) (GDA) and (11) (SDA) which deviate from grid-point *G0* by $\Delta\mathbf{A}_{Gq}$ and $\Delta\mathbf{B}_{Gq}$ are given as follows. (The design procedures for PDA is similar to that of GDA).

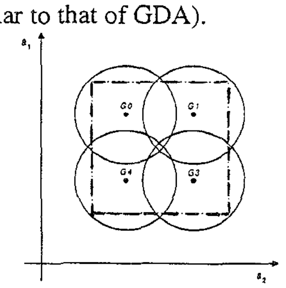

Figure 1. The idea of the multiple-grid-point approach.

For the case without common input matrix (**B**) (GDA),

$$\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x} + \mathbf{B}_0\mathbf{r} + \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j ((\mathbf{H}^{ij} + \Delta\mathbf{H}^{ij} + \Delta\mathbf{A}_{Gq} + \Delta\mathbf{B}_{Gq}(\mathbf{G}_{G0}^{j} + \Delta\mathbf{G}_{Gq}) + (\mathbf{B}^i + \Delta\mathbf{B}^i)\Delta\mathbf{G}_{Gq} - \mathbf{A}_0)\mathbf{x} + (\mathbf{B}^i - \mathbf{B}_0)\mathbf{r} + (\Delta\mathbf{B}^i + \Delta\mathbf{B}_{Gq})\mathbf{r}) \quad (39)$$

For the case with a common input matrix (**B**) (SDA),

$$\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x} + \mathbf{B}\mathbf{r} + \sum_{j=1}^{p} w^j ((\mathbf{H}^{j} + \Delta\mathbf{H}^{j} + \Delta\mathbf{A}_{Gq} + \Delta\mathbf{B}_{Gq}(\mathbf{G}_{G0}^{j} + \Delta\mathbf{G}_{Gq}) + (\mathbf{B} + \Delta\mathbf{B})\Delta\mathbf{G}_{Gq} - \mathbf{A}_0)\mathbf{x} + (\Delta\mathbf{B} + \Delta\mathbf{B}_{Gq})\mathbf{r}) \quad (40)$$

The stability criterion for (39) becomes

$$\left\|\Delta\mathbf{H}^{ij} + \Delta\mathbf{A}_{Gq} + \Delta\mathbf{B}_{Gq}(\mathbf{G}_{G0}^{j} + \Delta\mathbf{G}_{Gq}) + (\mathbf{B}^i + \Delta\mathbf{B}^i)\Delta\mathbf{G}_{Gq}\right\|_{\max} \le -\mu[\mathbf{A}_0] - \left\|\mathbf{H}^{ij} - \mathbf{A}_0\right\| - \varepsilon \quad (41)$$

and that for (40) becomes

$$\left\|\Delta\mathbf{H}^{j} + \Delta\mathbf{A}_{Gq} + \Delta\mathbf{B}_{Gq}(\mathbf{G}_{G0}^{j} + \Delta\mathbf{G}_{Gq}) + (\mathbf{B} + \Delta\mathbf{B})\Delta\mathbf{G}_{Gq}\right\| \le -\mu[\mathbf{A}_0] - \left\|\mathbf{H}^{j} - \mathbf{A}_0\right\| - \varepsilon \quad (42)$$

The resultant feedback gain $\mathbf{G}^{j}_{Gq}$ is defined as

$$\mathbf{G}_{Gq}^{j} = \mathbf{G}_{G0}^{j} + \Delta\mathbf{G}_{Gq} \quad (43)$$

Based on (41) and (42), there are two cases to be considered for the design of the feedback gains on other grid-points.

Case I: Common input matrices (**B**) and $\Delta\mathbf{B} = \mathbf{0}$

This is the simplest case. The trick is to make the system of (42) become (23) with $\Delta\mathbf{B} = \Delta\mathbf{B}_{Gq} = \mathbf{0}$. It occurs when,

$$\Delta\mathbf{A}_{Gq} + \mathbf{B}\Delta\mathbf{G}_{Gq} = \mathbf{0} \quad (44)$$

The solution of (42) is,

$$\Delta\mathbf{G}_{Gq} = -\mathbf{B}^{-1}\Delta\mathbf{A}_{Gq} \text{ if } \mathbf{B} \text{ is a square matrix} \quad (45)$$

$$\Delta\mathbf{G}_{Gq} = -(\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\Delta\mathbf{A}_{Gq} \text{ if } \mathbf{B} \text{ is not a square matrix} \quad (46)$$

Note that (46) is a solution only when $\Delta\mathbf{A}_{Gq} - \mathbf{B}(\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\Delta\mathbf{A}_{Gq} = \mathbf{0}$. If it can not be satisfied, the design methodology given in case II should be used. The value of $\Delta\mathbf{A}_{Gq}$ is given by

$$\Delta\mathbf{A}_{Gq} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \cdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & & \alpha_{nn} \end{bmatrix} \times |\Delta a| \quad (47)$$

where α_{kl} is an integer corresponding to the number of shifts for the row-*k* column-*l* parameter of $\Delta\mathbf{A}_{Gq}$ from $\mathbf{A}_{Go}$, *k* and *l* =1,2,...,*n*. The distance of this element from that of *G0* is $\alpha_{kl}|\Delta a|$. $|\Delta a|$ is the bound of each uncertain element of $\Delta\mathbf{A}^{i}$ and is defined as

$$|\Delta a| = \frac{\min_i \left\| \Delta \mathbf{A}^i \right\|_{\text{Robust area}}}{n} \tag{48}$$

By choosing the bound of $|\Delta a|$ as (48), $\left\|\Delta \mathbf{H}^i\right\|_{\text{Robust area}} = \left\|\Delta \mathbf{A}^i\right\|_{\text{Robust area}} \geq \left\|\Delta \mathbf{A}^i\right\|_{\max} \geq \left\|\Delta \mathbf{A}^i\right\|$ which means the stability criterion is always satisfied. The proof can be obtained from Appendix A by assuming $|\Delta b| = 0$. From (44) to (47), (42) becomes (23), implying that the system is stable by using the control law of (43) when the parameters are shifted by $\Delta \mathbf{A}_{Gq}$. In this case, the size of robust area for each grid-point is the same as that of *G0*.

Case II: $\Delta \mathbf{B} \neq \mathbf{0}$

In this case, the design of the feedback gain on other grid-points becomes difficult. As in case I, we want the sum of the terms with subscript *Gq* in (41) and (42) to be equal to zero. However, there are no solutions for both cases. One way to design the controllers for grid-points other than *G0* is by choosing another $\mathbf{A_o}$ and redesign the fuzzy controller such that the inequalities of (41) and (42) hold. In this way, the robust area for each controller may be different. The design can be summarized as follows. Assume that the set of control laws is

$$\begin{matrix} \{(\mathbf{G}_{G11},(\Delta\mathbf{A}_{G11},\Delta\mathbf{B}_{G11})), & (\mathbf{G}_{G12},(\Delta\mathbf{A}_{G11},\Delta\mathbf{B}_{G12})), & \cdots & (\mathbf{G}_{G1r},(\Delta\mathbf{A}_{G11},\Delta\mathbf{B}_{G1r})), \\ \vdots & \vdots & \vdots & \vdots \\ (\mathbf{G}_{Gs1},(\Delta\mathbf{A}_{Gs1},\Delta\mathbf{B}_{Gs1})), & (\mathbf{G}_{Gs2},(\Delta\mathbf{A}_{Gs1},\Delta\mathbf{B}_{Gs2})), & \cdots & (\mathbf{G}_{Gst},(\Delta\mathbf{A}_{Gs1},\Delta\mathbf{B}_{Gst}))\} \end{matrix} \tag{49}$$

where *r*, *s* and *t* are positive integers, $\mathbf{G}_{Guv}$ is a matrix containing all $\mathbf{G}^j_{Guv}$ is the control laws corresponding to the parameters shifted by $(\Delta \mathbf{A}_{Guw}, \Delta \mathbf{B}_{Guv})$. They serve the same function of $\Delta \mathbf{A}_{Gq}$ and $\Delta \mathbf{B}_{Gq}$ as in case I. Corresponding to each pair, a control law, $\mathbf{G}_{Guv}$, is designed. For each fuzzy controller with control law $\mathbf{G}_{Guv}$, there are robust areas existed in the **A** and **B** parameter spaces. Hence, stable and robust fuzzy controllers for each local plant which has system parameters of $\mathbf{A}^i_{G0} + \Delta \mathbf{A}_{Gjk}$ and $\mathbf{B}^i_{G0} + \Delta \mathbf{B}_{Gjk}$ can be designed. For each designed fuzzy controller, it has its robust area in the **A** parameter space and **B** parameter space. These robust areas must be designed such that they overlap with the adjacent ones and the whole operation spaces in **A** and **B** parameter spaces are fully covered. The bounds of the elements of the uncertain parameters for the local system with a given robust area are given in Appendix A.

5. Selection of control law and its effect to the overall system stability

When the robust area of a SGP fuzzy controller cannot cover the whole operation parameter spaces, changing of the fuzzy controller from one grid-point to another is necessary to keep the system stable. The choice of the SGP fuzzy controller is based on the information of the parameter uncertainties. Intuitively, if either one of the following conditions occurs, the current control law must be changed to stabilize the system.

$$\left|\Delta a_{jk}\right| = \sum_{i=1}^{n} w^i \Delta a^i_{jk} > |\Delta a| \quad \text{for } j,\, k=1,2,\ldots,n \tag{50}$$

$$\left|\Delta b_{jl}\right| = \sum_{i=1}^{n} w^i \Delta b^i_{jl} > |\Delta b| \quad \text{for } j=1,2,\ldots,n,\ l=1,2,\ldots,m \tag{51}$$

where Δa^i_{jk} and Δb^i_{jl} (measured or estimated parameter uncertainties), are the row-*j* column-*k* and row-*j* column-*l* elements of $\Delta \mathbf{A}^i$ and $\Delta \mathbf{B}^i$ respectively. The control law corresponding to a grid-point is used if the distance between the measured or estimated parameters and the grid-point is the shortest and the robust area of that grid-point covers the uncertainties.

Due to the changing of the fuzzy controllers to cope with the unexpected contingencies, it seems that there is a possibility to cause instability. This case is revealed in Figure 2. In this figure, the left and the middle diagrams are two stable local systems. However, when these two systems switch at the instants during the phase trajectory hits the horizontal axis of the left diagram or the vertical axis of the middle diagram, the resultant phase trajectory becomes the one in the right diagram. This is an unstable system as the system states move away from the origin. Still, this case never happens in our MGP approach, as the norm of the system states is exponential decaying which is governed by the time constant, $1/\varepsilon$ ((25) and (26)). Hence. the systems states of a MGP control systems must always approach the origin for (25) (globally exponentially stable) or be bounded (26).

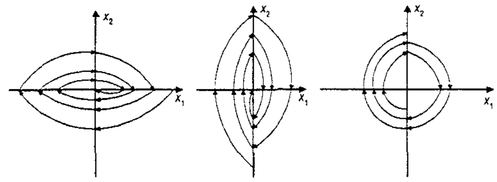

Figure 2. Phase planes of stable and unstable systems

6. Accessibility of uncertainties and its effect to the overall system stability

The fuzzy controller is selected based on the information of the uncertain parameters. For measurable uncertain parameters, the measured information is then used directly for choosing the control law. For unmeasurable uncertain parameters, a parameter estimator can be applied to estimate the uncertain parameters and use them to choose the control law.

The overall system stability is guaranteed for the latter case by the following proof (assuming that the problem mentioned in section 5 is solved). Consider an uncertain fuzzy control system with a parameter estimator applying GDA (Similar results can be obtained for PDA and SDA),

$$\dot{\mathbf{x}} = \sum_{i=1}^{p}\sum_{j=1}^{c} w^i m^j ((\mathbf{A}^i + \Delta\hat{\mathbf{A}}^i)\mathbf{x} + (\mathbf{B}^i + \Delta\hat{\mathbf{B}}^i)\mathbf{u} + \mathbf{f}(\dot{\mathbf{e}},\mathbf{e})) \quad (52)$$

where $\Delta\hat{\mathbf{A}}^i \in \Re^{n\times n}$ and $\Delta\hat{\mathbf{B}}^i \in \Re^{n\times m}$ are the estimated parameters, $\mathbf{e} \in \Re^{n\times 1}$ is the difference between the actual and estimated system states, and $\mathbf{f}(\dot{\mathbf{e}},\mathbf{e}) \in \Re^{n\times 1}$ is the error function which satisfies the following condition,

$$\lim_{t\to\infty} \mathbf{f}(\dot{\mathbf{e}},\mathbf{e}) = \mathbf{0} \quad (53)$$

Refer to (52), the system can be viewed as subjecting to estimated uncertain parameters with $\mathbf{f}(\dot{\mathbf{e}},\mathbf{e})$ as an extra input. When $\mathbf{f}(\dot{\mathbf{e}},\mathbf{e}) \neq \mathbf{0}$, it can be shown that the norm of the system states are bounded by following the steps in section 3.4. Hence, the overall system is stable. When $\mathbf{f}(\dot{\mathbf{e}},\mathbf{e}) = \mathbf{0}$, (52) is reduced to the system of (5), and (25) or (26) can be obtained. Hence, the overall system is still stable if the control law is applied and the estimated uncertain parameters are inside its robust area. This is an important issue which has not been mentioned in [4].

7. Conclusions

The stability and robustness of a general multivariable uncertain fuzzy control system are analyzed. The stability criterion and robust area with respect to a single-grid-point in the parameter space are derived. By using the simple and easy-to-understand stability theory derived, a stable and robust fuzzy controller can be designed easily and systematically. Based on the analysis results on the SGP approach, a systematic design methodology for uncertain nonlinear systems subjected to large uncertainties using a multiple-grid-point approach is presented.

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Appendix A

The bounds of the elements of the parameter uncertainties of $\Delta\mathbf{A}^i \in \Re^{n\times n}$ and $\Delta\mathbf{B}^i \in \Re^{n\times m}$ are defined in this appendix such that the stability criteria in theorem 1 and 2 are always satisfied. we start with

$$\left\|\Delta\mathbf{H}^{ij}\right\| \le \left\| \begin{bmatrix} |\Delta a| & \cdots & |\Delta a| \\ \vdots & \cdots & \vdots \\ |\Delta a| & \cdots & |\Delta a| \end{bmatrix} + \begin{bmatrix} |\Delta b| & \cdots & |\Delta b| \\ \vdots & \cdots & \vdots \\ |\Delta b| & \cdots & |\Delta b| \end{bmatrix} \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \cdots & \vdots \\ g_{m1} & \cdots & g_{mn} \end{bmatrix} \right\| \quad (A1)$$

where $\mathbf{G} = \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \cdots & \vdots \\ g_{m1} & \cdots & g_{mn} \end{bmatrix}$ is the feedback gain among all $\mathbf{G}^{ij}$ such that the right hand side of (A1) is maximum. (How it can be determined will be discussed later.) From (A1),

$$\left\|\Delta\mathbf{H}^{ij}\right\| \le \left\| \begin{bmatrix} |\Delta a| + |\Delta b||g_{11}+\cdots+g_{m1}| & \cdots & |\Delta a| + |\Delta b||g_{1n}+\cdots+g_{mn}| \\ \vdots & \cdots & \vdots \\ |\Delta a| + |\Delta b||g_{11}+\cdots+g_{m1}| & \cdots & |\Delta a| + |\Delta b||g_{1n}+\cdots+g_{mn}| \end{bmatrix} \right\|$$

$$= \sqrt{n}\left\| \begin{bmatrix} |\Delta a| + |\Delta b||g_{11}+\cdots+g_{m1}| & \cdots & |\Delta a| + |\Delta b||g_{1n}+\cdots+g_{mn}| \end{bmatrix} \right\|$$

$$= \sqrt{n}\sqrt{n|\Delta a|^2 + 2|\Delta a||\Delta b| \sum_{j=1}^{n}\left|\sum_{i=1}^{m} g_{ij}\right| + |\Delta b|^2 \sum_{j=1}^{n}\left|\sum_{i=1}^{m} g_{ij}\right|^2} \quad (A2)$$

$\mathbf{G}$ should be chosen if it maximizes (A2). Let (A2) $= \min_{i,j}\left\|\Delta\mathbf{H}^{ij}\right\|_{\text{Robust area}}$, then,

$$\sum_{j=1}^{n}\left|\sum_{i=1}^{m} g_{ij}\right|^2 |\Delta b|^2 + 2|\Delta a||\Delta b|\sum_{j=1}^{n}\left|\sum_{i=1}^{m} g_{ij}\right| + n|\Delta a|^2 - \frac{\min_{i,j}\left\|\Delta\mathbf{H}^{ij}\right\|^2_{\text{Robust area}}}{n} = 0 \quad (A3)$$

By choosing $|\Delta a|$ and $|\Delta b|$ with (A3), the following condition holds.

$$\min_{i,j}\left\|\Delta\mathbf{H}^{ij}\right\|_{\text{Robust Area}} \ge \left\|\Delta\mathbf{H}^{ij}\right\|_{\max} \ge \left\|\Delta\mathbf{H}^{ij}\right\| \quad (A4)$$

For a SGP approach and a given robust area, if $|\Delta a|$ and $|\Delta b|$ are chosen using (A3), then (A4) will always be satisfied.