

Modeling Nonlinearities of Ultrasonic Waves for Fatigue Damage Characterization: *Theory, Simulation, and Experimental Validation*

Ming Hong ^{a,b †}, Zhongqing Su ^{a,b *}, Qiang Wang ^c, Li Cheng ^{a,b}, and Xinlin Qing ^d

^a The Hong Kong Polytechnic University Shenzhen Research Institute
Shenzhen 518057, PR China

^b Department of Mechanical Engineering
The Hong Kong Polytechnic University, Kowloon, Hong Kong

^c College of Automation
Nanjing University of Posts and Telecommunications, Nanjing, P.R. China

^d Division of Aviation Health and Safety Management
Beijing Aeronautical Science and Technology Research Institute COMAC
Beijing, 100083, P.R. China

Submitted to *Ultrasonics*

(initially submitted on 4th May 2013; revised and resubmitted on 25 September 2013)

[†] Ph.D. candidate

* Corresponding author. Tel.: +852 2766 7818; fax: +852 2365 4703.
Email address: MMSU@polyu.edu.hk (Prof. Zhongqing Su, *Ph.D.*).

Abstract

A dedicated modeling technique for comprehending nonlinear characteristics of ultrasonic waves traversing in a fatigued medium was developed, based on a retrofitted constitutive relation of the medium by considering the nonlinearities originated from material, fatigue damage, as well as the “breathing” motion of fatigue cracks. Piezoelectric wafers, for exciting and acquiring ultrasonic waves, were integrated in the model. The extracted nonlinearities were calibrated by virtue of an acoustic nonlinearity parameter. The modeling technique was validated experimentally, and the results showed satisfactory consistency in between, both revealing: the developed modeling approach is able to faithfully simulate fatigue crack-incurred nonlinearities manifested in ultrasonic waves; a cumulative growth of the acoustic nonlinearity parameter with increasing wave propagation distance exists; such a parameter acquired via a sensing path is nonlinearly related to the offset distance from the fatigue crack to that sensing path; and neither the incidence angle of the probing wave nor the length of the sensing path impacts on the parameter significantly. This study has yielded a quantitative characterization strategy for fatigue cracks using embeddable piezoelectric sensor networks, facilitating deployment of structural health monitoring which is capable of identifying small-scale damage at an embryo stage and surveilling its growth continuously.

Keywords: modeling; fatigue crack characterization; nonlinearity of ultrasonic waves; Lamb waves; structural health monitoring; active sensing

1. Introduction

Effectual damage evaluation and continuous health monitoring are conducive to reliable service of engineering structures, and the risk of structural failure can accordingly be minimized. Taking advantage of appealing features including high sensitivity to structural damage, omnidirectional dissemination, fast propagation, and strong penetration through thickness, ultrasonic waves have been a subject of intense scrutiny over the years, with demonstrated compromise between conventional non-destructive evaluation (NDE) and emerging structural health monitoring (SHM) [1–7]. Predominantly, deployment of this group of techniques is often based on exploring changes in the linear wave scattering upon the interaction of incident probing waves with structural damage. These changes can be manifested in acquired ultrasonic wave signals, typified as delay in time-of-flight, wave attenuation, and mode conversion. These signal features, for example the delay in time-of-flight, show, to some extent, linear correlation with damage parameters such as the location, and are therefore referred to as *linear features*.

However, it is a corollary that linear features-based detection is fairly limited to evaluating damage with a size on the same order of the magnitude of the probing wavelength [8], presenting inefficiency in perceiving fatigue damage which often initiates at an unperceivable level much smaller than the probing wavelength. This is because the damage of small dimension is not anticipated to induce evident changes in linear features to be extracted from ultrasonic waves [9]. This situation has posed immediate urgency and entailed imperative needs for exploring other wave signal features that can be prominently modulated by small-scale damage, so as to endow the ultrasonic inspection with a capability of scrutinizing damage small in dimension and fatigue cracks in particular.

26 Nonlinear ultrasonic interrogation has emerged under such a demand. More specifically,
27 instead of extracting and canvassing linear signal properties, the nonlinear ultrasonic
28 inspection attempts to quantify the nonlinear distortion of probing waves due to the
29 damage, for instance the generation of higher-order harmonics. Such a detection
30 philosophy has ushered a new avenue of using ultrasonic waves to predict fatigue damage
31 at an embryo stage prior to the formation of gross damage detectable by linear techniques.

32

33 There has been a rich body of literature on the use of nonlinear features of ultrasonic waves
34 [9-16]. Most of existing research strength has been in a nature of experimental observation
35 on possible changes in nonlinear properties of probing waves, primarily Lamb waves (the
36 modality of elastic waves in thin plate- or shell-like structures) [10-13]. There are rather
37 limited studies devoted to the analytical investigation or numerical simulation of nonlinear
38 ultrasonic waves propagating in fatigued media. Among representative numerical methods
39 to simulate nonlinearities of media and/or damage are Local Interaction Simulation
40 Approach (LISA) [14], finite element method (FEM) [15], and Galerkin FEM [16].

41

42 In general, a paramount challenge in analytically or numerically modeling nonlinear
43 ultrasonic waves in a fatigued medium is the comprehensive inclusion of all possible
44 sources of nonlinearities from both the medium itself and the damage, as well as the
45 interpretation on the modulation mechanism of fatigue damage on ultrasonic waves. Aimed
46 at a systematic comprehension of the nonlinear natures of ultrasonic waves in a medium
47 bearing fatigue damage, this study is dedicated to the establishment of a modeling
48 technique — supplemented with experimental validation — that is capable of producing
49 and interpreting nonlinearities in ultrasonic waves. Instead of using bulky wedge probes
50 that are commonly adopted in prevailing nonlinear ultrasonic interrogation, miniaturized

51 piezoelectric wafer sensors, which can be flexibly networked and permanently attached to
52 a structure under inspection, are utilized, benefiting extension of the approach to
53 embeddable SHM.

54

55 This paper is organized as follows: Section 2 discusses the origins of nonlinearities in an
56 elastic medium, serving as the cornerstone of the study, residing on which the modeling
57 technique for an ideally intact medium and its fatigued counterpart is developed. An
58 acoustic nonlinearity parameter is established to quantitatively calibrate the captured
59 nonlinearity. By integrating identified sources of nonlinearities, Section 3 models the
60 nonlinear properties manifested in ultrasonic waves traversing in a metallic medium
61 featuring introduced nonlinearities. Section 4 embraces implementation of the modeling
62 through finite element (FE) simulation, and signal processing for extracting nonlinearities
63 from ultrasonic wave signals. Case studies using the developed modeling technique are
64 presented in Section 5, investigating the dependence of the acoustic nonlinearity parameter
65 on wave propagation distance, on sensing path offset from a fatigue crack, and on wave
66 incidence angle and propagation distance. Finally, Section 6 renders concluding remarks.

67

68 **2. Modeling nonlinearities in elastic medium**

69 Consider an isotropic homogeneous solid with purely elastic behavior, the nonlinearities of
70 the medium that may contribute to nonlinear distortion of its guided ultrasonic waves can
71 originate from different sources, including mainly the material, the damage-driven
72 plasticity, the loading conditions, to name a few.

73

74 *2.1. Intact state*

75 When the medium is in an ideally intact state (no fatigue damage existent), two

76 nonlinearity sources are accountable: the inherent material nonlinearity and the geometric
77 nonlinearity, with the former from the intrinsic nonlinear elasticity of the medium (*viz.*, the
78 elasticity of lattices). Usually, lattice vibrations in a metallic medium are assumed to obey
79 simple harmonic motion and the material is assumed to be pristine (*i.e.*, no precipitates or
80 vacancies). This assumption is largely applicable for engineering applications in the
81 domain of linear elasticity. However, in reality there is always lattice anharmonicity
82 (referring to the crystal vibrations that do not follow the simple harmonic motion), and/or
83 there are precipitates and vacancies in the material. These nonlinearity effects, though
84 trivial, can be manifested by ultrasonic waves propagated in such a medium.

85

86 In the domain of nonlinear elasticity, the three-dimensional stress-strain relation for the
87 above solid medium can be depicted, with a second-order approximation, as follows [17]

$$88 \quad \sigma_{ij} = (C_{ijkl} + 1/2 M_{ijklmn} \varepsilon_{mn}) \varepsilon_{kl}, \quad (1)$$

89 where σ_{ij} is the stress tensor; ε_{mn} and ε_{kl} are the strain tensors; C_{ijkl} and such in its form in
90 the succeeding equations are the second-order elastic (SOE) tensors defined with Lamé
91 parameters λ_L and μ ; M_{ijklmn} is a tensor associated with the material and geometric
92 nonlinearities. If the second term in the parenthesis, $1/2 M_{ijklmn}$, is neglected, Eq. (1) reverts
93 to the three-dimensional Hooke's Law of linear elasticity.

94

95 In the meantime, the geometric nonlinearity is closely related to the material nonlinearity.
96 Generally, wave motion equations are written in Eulerian (special) coordinates, while
97 nonlinear elasticity in solids is formulated in Lagrangian (material) coordinates. For linear
98 elasticity, these two coordinate systems do not differ from each other; nevertheless, given
99 the material nonlinearity taken into account, a descriptive difference emerges, starting from
100 the second-order term of any physical quantity involved [18]. In simpler words, geometric

101 nonlinearity is induced mainly due to the mathematic transform between two coordinate
 102 systems. Hence, tensor M_{ijklmn} in Eq. (1) addresses both the material and geometric
 103 nonlinearities simultaneously, which can be expressed in terms of the notation by Landau
 104 and Lifshitz [19] as follows

$$105 \quad M_{ijklmn} = C_{ijklmn} + C_{ijln} \delta_{km} + C_{jnkl} \delta_{im} + C_{jlmn} \delta_{ik}, \quad (2)$$

106 where

$$107 \quad C_{ijklmn} = \frac{1}{2} \mathcal{A} (\delta_{ik} I_{jlmn} + \delta_{il} I_{jkmn} + \delta_{jk} I_{ilmn} + \delta_{jl} I_{ikmn}) \\ + 2\mathcal{B} (\delta_{ij} I_{klmn} + \delta_{kl} I_{mnij} + \delta_{mn} I_{ijkl}) + 2\mathcal{C} \delta_{ij} \delta_{kl} \delta_{mn}. \quad (3)$$

108 In Eqs. (2) and (3), δ_{km} and such in its form with different index orders are the Kronecker
 109 deltas; I_{jlmn} and such in its form are the fourth-order identity tensors. C_{ijklmn} is the
 110 third-order elastic (TOE) tensor describing the material nonlinearity, and the last three
 111 terms in Eq. (2) all together address the geometric nonlinearity. As shown in Eq. (3), C_{ijklmn}
 112 is determined by three TOE constants \mathcal{A} , \mathcal{B} and \mathcal{C} , which can be regarded as the
 113 inherent properties of the material, to be measured experimentally [20,21]. C_{ijklmn} can
 114 further be expressed explicitly with Voigt notation in terms of the three TOE constants, as

$$115 \quad \begin{cases} c_{111} = 2\mathcal{A} + 6\mathcal{B} + 2\mathcal{C} \\ c_{112} = 2\mathcal{B} + 2\mathcal{C} \\ c_{123} = 2\mathcal{C} \\ c_{144} = 1/2\mathcal{A} + \mathcal{B} \\ c_{155} = \mathcal{B} \\ c_{456} = 1/4\mathcal{A}, \end{cases} \quad (4)$$

116 where $c_{IJK} = C_{ijklmn}$ ($I, J, K \in \{1, 2, \dots, 6\}$). For example, the cases that $I = 1, 2, \dots, 6$ are
 117 corresponding to those when $ij = 11, 22, 33, 12, 23, 31$, respectively, and any other scalar
 118 components of C_{ijklmn} fall into the six cases defined by Eq. (4).

119

120 For generality, first consider a one-dimensional medium, such as a rod, which can be

121 governed by the one-dimensional nonlinear stress-strain equation as follows

$$122 \quad \sigma = (E + E_2 \varepsilon) \varepsilon, \quad (5)$$

123 where σ , ε , E , and E_2 are the stress, strain, and the first- and second-order Young's moduli
124 of the medium, respectively. E reflects the linear properties, whereas E_2 introduces the
125 nonlinear effects [22] to the medium. Solving Eqs. (1) through (3) yields

$$126 \quad E_2 = -\frac{1}{2}(3E + 2\mathcal{A} + 6\mathcal{B} + 2\mathcal{C}). \quad (6)$$

127 Further, take the ratio of the two Young's moduli

$$128 \quad \beta_g = \frac{E_2}{E} = -\frac{1}{2}\left(3 + \frac{2\mathcal{A} + 6\mathcal{B} + 2\mathcal{C}}{E}\right). \quad (7)$$

129 In this study, β_g defined by Eq. (7) is referred to as the *global nonlinearity parameter* of the
130 medium in its intact state. Obviously, β_g is dominated by four material property parameters
131 (Young's modulus and three TOE constants) provided the medium contains, ideally, no
132 fatigue damage or plastic deformation. Thus β_g can be regarded as a material property to
133 quantify the intrinsic nonlinearities of the medium.

134

135 *2.2. Fatigued state*

136 For a medium bearing fatigue damage, there are additional sources of nonlinearities in the
137 vicinity of the damage besides β_g . Fatigue damage initially appears in the form of
138 microstructure defects such as dislocations. Under repetitious loading, dislocations
139 accumulate and form persistent slip bands (PSBs), which, at the grain boundaries, nucleate
140 microcracks on the scale of millimeters. Finally, microcracks coalesce and grow into
141 macroscopic cracks that propagate through the material. Because the number of loading
142 cycles it takes to produce a macroscopic crack leading to structural failure is much smaller
143 than the number to initiate microcracks, it is of vital importance to detect fatigue cracks
144 before a microcrack forms. In what follows, we limit the discussion to the nonlinearities

145 presented in the early stage of fatigue damage up to the point when PSB walls emerge.

146

147 For low-cycle fatigue (LCF), since the loading stress is larger than the yield strength of the
148 material, bulk plastic deformation takes place, which is not homogeneous; instead it is
149 expected to locally concentrate in the vicinity of the fatigue damage [23]. On the other
150 hand, for high-cycle fatigue (HCF), with primarily elastic deformation, localized
151 micro-plastic deformation also exists in the region of crack initiation. Therefore, for both
152 LCF and HCF, localized plasticity is present in the vicinity of crack initiation due to
153 dislocations and slips, both distorting guided waves and introducing nonlinear properties to
154 wave signals.

155

156 To reflect the localized plasticity above, a *local nonlinearity parameter* β_l is defined, in
157 contrast to β_g , to account for the plasticity-induced nonlinearity incurred by fatigue damage.
158 β_l reads, for a dislocation dipole setting [24], as

$$159 \quad \beta_l = \frac{16\pi\Omega R^2 \Lambda h^3 (1-\nu)^2 (\lambda_L + 2\mu)^2}{\mu^2 b}, \quad (8)$$

160 where Ω and R are two factors of conversions from dislocation to longitudinal, and from
161 longitudinal to shear displacements, respectively; Λ is the dipole dislocation density
162 associated with the plastic strain of the medium, h the dipole height, ν the Poisson's ratio,
163 and b the Burgers vector. Usually, the nonlinearity due to micro-plastic deformation
164 contributes more significantly to the nonlinear distortion of ultrasonic waves than intrinsic
165 material nonlinearity does [12, 13].

166

167 Taking both the global and local nonlinearity parameters into account, a *hybrid acoustic*
168 *nonlinearity parameter* β is constructed in this study, up to the point when PSB walls with
169 plastic strains are produced, as

$$\beta = \begin{cases} \beta_g, & \text{Intact state} \\ \beta_g + \beta_l. & \text{Damaged state due to local plasticity} \end{cases} \quad (9)$$

171 Once a localized plasticity is introduced to the structure, β is increased owing to β_l only in
 172 the vicinity of the fatigue damage, while in the rest of the material it remains unchanged as
 173 β_g . This local change provides a theoretical basis for quantitative characterization of fatigue
 174 damage in virtue of captured β .

175

176 2.3. Contact acoustic nonlinearity

177 In addition to β , as the fatigue deterioration continues to the point when microcracks are
 178 physically nucleated with sufficient accumulation of dislocations, the *contact acoustic*
 179 *nonlinearity* (CAN) also needs to be addressed: when ultrasonic waves traverse the
 180 interface between two surfaces of a microcrack, the “breathing” motion pattern of the crack
 181 under cyclic loading closes the gap during wave compression, in which compressive and
 182 shear stresses are transmitted, according to the “breathing crack model” [25]; while when
 183 tension opens the crack, tensile stress cannot be transmitted. Since various stress-strain
 184 relations exist across the crack depending on whether the microcrack is open or closed,
 185 their superposition imposes an additional localized nonlinearity, CAN, to the signals of
 186 ultrasonic waves guided by the medium [26]. Moreover, other factors such as roughness of
 187 the crack surfaces, friction, thermal effects, and partial closure, may also present, more or
 188 less, nonlinearities and contribute to changes in CAN. With such complexity, there has
 189 been a lack of means to theoretically relate CAN to the ultrasonic wave responses in terms
 190 of β [26]. However, CAN could be conveniently modeled to approach the reality, to be
 191 discussed in Section 4.

192

193 3. Modeling nonlinearities in ultrasonic waves

194 *3.1. Longitudinal waves*

195 Continuing discussion on the above one-dimensional medium, we consider the longitudinal
 196 waves in the medium first. By neglecting dispersion and attenuation, the motion equation
 197 for the guided waves in the Lagrangian coordinates is formulated as

$$198 \quad \rho \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \quad (10)$$

199 where ρ is the density of the medium, $u(x,t)$ the particle displacement at x along
 200 propagation direction at instant t (abbreviated as u in what follows). Having discussed the
 201 nonlinear stress-strain relation in a one-dimensional isotropic homogeneous solid, wave
 202 responses in this medium can be obtained by solving Eqs. (5) and (10) simultaneously in
 203 virtue of a perturbation theory [22]. The yielded particle displacement u has a zero-order
 204 solution at the excitation frequency ω , plus a first-order perturbation solution at the double
 205 frequency 2ω , as

$$206 \quad u = A_1 \cos(kx - \omega t) + A_2 \cos(2kx - 2\omega t), \quad (11)$$

207 where k is the wavenumber, ω the angular frequency of excitation, and A_1 and A_2 the
 208 magnitudes of the fundamental wave mode at ω , and the second harmonic wave mode at
 209 2ω , respectively. The two magnitudes are linked to each other by $A_2 = \frac{\hat{\beta}}{8} A_1^2 k^2 x$, where

210 $\hat{\beta}$ denotes the *acoustic nonlinearity parameter*, to be obtained via either experimentation
 211 or simulation. It reads, after rearrangement,

$$212 \quad \hat{\beta} = \frac{8}{k^2 x} \frac{A_2}{A_1^2}. \quad (12)$$

213 It is noteworthy that $\hat{\beta}$ is not always equal to the ideally modeled β defined by Eq. (9)
 214 under all circumstances, particularly when there are physical fatigue cracks presented. That
 215 is because some types of nonlinearities, such as CAN, are not analytically included in the
 216 stress-strain model as discussed in Section 2, which, nonetheless, also contribute to the

217 generation of higher-order harmonics and can be reflected by the measured $\hat{\beta}$. Thus, the
218 actual change in $\hat{\beta}$ due to local plasticity and emerging microcracks, determined via
219 either experimentation or simulation, should in general be larger than the value of β_l as
220 analytically obtained via Eq. (8), in which only micro-plastic deformation is considered.
221 Yet, previous works [9,12,13] have demonstrated that an increase in β due to the presence
222 of plasticity-driven fatigue damage can be adequately, if not exactly, represented by the
223 change in $\hat{\beta}$.

224

225 Based on Eq. (12), the degree of nonlinearity associated with the material and the fatigue
226 damage can be determined by probing the magnitudes of the fundamental and its
227 corresponding second harmonic modes. In order to detect the damage, one may be more
228 interested in the change in $\hat{\beta}$ rather than its absolute value; therefore, at a fixed wave
229 propagation distance with certain wavenumber, the acoustic nonlinearity parameter can be
230 normalized with respect to its initial value in the intact state of the medium, yielding a
231 *relative acoustic nonlinearity parameter*, β' , to quantify the degree of *change* of the
232 nonlinearities of ultrasonic waves, as

233
$$\beta' = \frac{A_2}{A_1^2}. \quad (13)$$

234 This relatively defined nonlinearity parameter is proportional to β and contains essential
235 information reflecting the nonlinear properties of wave propagation. Thus, it can serve as a
236 primary index to be monitored for quantitative characterization of fatigue damage.

237

238 3.2. Lamb waves

239 Lamb waves are highly dispersive and multimodal, with multiple modes propagating
240 simultaneously at different velocities through all excitation frequencies and at higher

241 frequencies in particular. The dispersive nature of Lamb waves makes the generation of
242 higher-order harmonics and their measurements a challenging task, compared to the case
243 for longitudinal waves aforementioned. Nevertheless, their second harmonics can become
244 significant and cumulative when particular fundamental modes are excited at deliberately
245 selected frequencies, which simultaneously meet the conditions of *synchronism* and
246 *non-zero power flux* [10]. In particular, synchronism is referred to as the phase velocity
247 matching and group velocity matching between the excited fundamental mode and the
248 accordingly generated second harmonic mode, respectively and concurrently [12,13]. This
249 matching warrants power transfer from the fundamental to the second harmonic mode of
250 the same mode type (either both are symmetrical or both are antisymmetric) smoothly,
251 resulting in cumulative generation of the second harmonic mode along wave propagation.

252

253 For illustration, Fig. 1 displays the dispersion curves calculated using DISPERSE[®] for
254 Lamb waves in an aluminum plate from zero through 10 MHz·mm. Mode pair (S_1 , S_2) is
255 an option to accommodate both synchronism and non-zero power flux, hence called a
256 *synchronized mode pair* hereinafter, of which S_1 , the first-order symmetric Lamb mode at
257 the frequency-thickness product of 3.57 MHz·mm, and S_2 , the second-order symmetric
258 Lamb mode at 7.14 MHz·mm, propagate at the same phase and group velocities,
259 respectively. Therefore, when S_1 mode is excited in a plate of uniform thickness (along
260 with other modes excited at that frequency), S_2 mode is expected to be generated at twice
261 the excitation frequency, being the corresponding second harmonic of S_1 . Owing to the
262 same phase and group velocities, generation of the second harmonic mode is cumulative
263 along wave propagation.

264

265 Extending the discussion from the one-dimensional medium to a plate-like medium
266 guiding Lamb waves, the acoustic nonlinearity parameter for longitudinal waves can be

267 redefined for Lamb waves by multiplying Eq. (12) with a scaling factor γ , which is a
268 function depending on Lamb wave factors and medium properties [27], as

$$269 \quad \hat{\beta}_{Lamb} = \frac{8}{k^2 x} \frac{A_2}{A_1^2} \gamma. \quad (14)$$

270 $\hat{\beta}_{Lamb}$ signifies the acoustic nonlinearity parameter for Lamb waves. Allowing for the fact
271 that a plate has an unchanged γ before and after the occurrence of fatigue damage, and
272 following the normalization process of $\hat{\beta}$ as detailed earlier for the one-dimensional
273 medium, γ can be eliminated and thus the relative nonlinearity parameter β' , from Eq. (13),
274 alone is sufficient to characterize fatigue damage in the plate.

275

276 **4. Implementation of modeling**

277 Based upon the above discussions on nonlinearities in the medium (when it is in intact and
278 fatigued states) and on nonlinear guided ultrasonic waves, a dedicated modeling technique
279 is developed and realized in conjunction with the commercial FEM software
280 Abaqus[®]/EXPLICIT. All possible sources of nonlinearities including CAN, as introduced
281 in the above, are taken into account in the modeling.

282

283 The modeling of material, geometric, and plasticity-driven nonlinearities is achieved by
284 introducing a modified nonlinear stress-strain relation to Abaqus[®]/EXPLICIT through user
285 subroutine VUMAT, which is specifically used for material definition. For the material and
286 geometric nonlinearities that exist homogeneously in the medium, material constants
287 including density, Young's modulus, and TOE constants are passed in the modeling; for the
288 plasticity-driven nonlinearity, an extra local nonlinearity parameter is added to the global
289 nonlinearity parameter in the vicinity of the fatigue damage (by calling a second set of
290 property values in the same VUMAT subroutine). As the simulation is dynamic in nature,

291 *Forward Euler* procedure (explicit integration) is used in VUMAT to transform the
292 constitutive rate equation, defined according to Eq. (1), to an incremental equation. A
293 two-state architecture is adopted, so that variables such as the strain increment and stress
294 increment have their initial values in old arrays and updated values in new arrays in each
295 iteration. Taking advantage of the explicit integration, these variables are all vectorized to
296 facilitate computing process. In order to maintain the simulation stability, time increment is
297 controlled to warrant convergence of calculation. Moreover, considering the magnitude of
298 deformation that can be induced by a piezoelectric lead zirconate titanate (PZT) wafer (to
299 be used in succeeding experimental validation), double precision is specified in simulation
300 to ensure computation accuracy.

301

302 To model a fatigue crack in the medium, a seam crack definition is imposed on each
303 surface of the crack, as illustrated in Fig. 2, to enable the breathing behavior when waves
304 traverse the crack. Furthermore, a contact-pair interaction and associated properties are
305 defined on the crack interface to supplement the modeling of CAN. Figures 3a through 3d
306 demonstrate the process of ultrasonic Lamb waves traversing a breathing crack. It can be
307 clearly seen that the stress induced by Lamb wave propagation is continuously transmitted
308 through the crack when it is closed (Figs. 3b and 3c), while on the contrary an open crack
309 interrupts the transmission process (Fig. 3d).

310

311 Using the above modeling technique, case studies are performed and analyzed (to be
312 detailed in the next section). As a representative, Fig. 4 shows the time-domain
313 presentation of a typical Lamb wave signal. In order to extract the required signal features
314 to construct β' , short-time Fourier transform (STFT) is performed on the time-domain
315 signal, with the obtained spectrogram shown in Fig. 5. The window size of STFT is

316 carefully chosen such that sufficient details of both time and frequency information of the
317 signal could be retained. From the spectrogram, one can extract the amplitude profiles at
318 fundamental and double frequencies as a respective function of time. In accordance with
319 Fig. 1, it is obvious that both the fundamental mode S_1 and its corresponding second
320 harmonic mode S_2 are the first arrivals at their corresponding frequencies. Thus, the first
321 peak value of the amplitude profile at the fundamental frequency is identified as A_1 (of S_1
322 mode), and the first peak value of the amplitude profile at the second harmonic frequency
323 as A_2 (of S_2 mode), as illustrated in Fig. 6. Consequently, β' associated with this particular
324 signal can be calculated using Eq. (13).

325

326 **5. Validation and discussions**

327 The proposed modeling technique was verified by case studies, followed with experimental
328 validation.

329

330 *5.1. β' vs. Propagation distance*

331 The change in β' subject to wave propagation distance in an undamaged plate medium was
332 interrogated first. The rationale of this test is: when a synchronized mode pair is excited, β'
333 extracted from captured ultrasonic Lamb wave signals is expected to be proportional to
334 wave propagation distance, provided the medium remains in an ideally healthy state.

335

336 Using the developed modeling approach, a 6061 aluminum plate ($400 \times 380 \times 4.5$ mm³),
337 shown schematically in Fig. 7a, was modeled. The three-dimensional nonlinear elastic
338 stress-strain relation described by Eq. (1) was recalled to define material properties through
339 VUMAT, introducing both the material and geometric nonlinearities. The SOE constants
340 (Lamé parameters λ_L and μ) can be calculated from the material's Young's modulus (68.9

341 GPa in this case) and Poisson's ratio (0.33). The TOE constants of the aluminum used in all
342 case studies are: $\mathcal{A} = -320$ GPa, $\mathcal{B} = -200$ GPa and $\mathcal{C} = -190$ GPa [20].
343 Three-dimensional eight-node brick elements (C3D8R) in Abaqus[®] were used to mesh the
344 plate through a structured meshing algorithm. All the elements were equally sized at 0.25
345 mm in the in-plane dimensions to ensure at least 20 elements per wavelength of the
346 fundamental wave mode S_1 , and 10 per wavelength of S_2 , to warrant simulation precision;
347 nine elements were assigned along the plate thickness. Nine circular PZT wafers were
348 assumed to be on the same side of the plate, each of which had a nominal diameter of 8
349 mm and a thickness of 0.5 mm. These PZT wafers served, respectively, as a wave actuator
350 (A) and eight sensors ($S_I - S_{VIII}$), hence configuring eight actuator-sensor paths. Deliberate
351 positioning of the nine PZT wafers led to a wave propagation distance d_i ($i = I, II \dots VIII$)
352 between A and S_i ranging from 60 to 200 mm in an increment of 20 mm, with
353 actuator-sensor path A – S_I being the shortest (60 mm) and A – S_{VIII} the longest (200 mm),
354 as shown in Fig. 7a. PZT wafers were modeled by creating a disc-like object comprising
355 four elements. Along each actuator-sensor path, five-cycle Hann-windowed sinusoidal tone
356 bursts at 800 kHz were excited by imposing uniform in-plane radial displacements on the
357 nodes along periphery of the actuator model, as illustrated in Fig. 7b. The excitation
358 frequency of 800 kHz enabled the generation of S_1 mode as the fundamental wave mode in
359 accordance with Fig. 1 (considering the plate thickness is 4.5 mm). Dynamic simulation
360 was performed, and wave signals were acquired at locations of eight sensors in the form of
361 in-plane elemental strains.

362

363 The same configuration was used in the experiment to validate the above modeling. Nine
364 PZT wafers were individually surface-mounted through a thin adhesive layer on a
365 damage-free 6061 aluminum plate as shown in Fig. 7c, and then connected to a signal

366 generation and acquisition system developed on a *VXI* platform [28]. Shielded wires and
367 standard BNC connectors were adopted to reduce measurement noise. The same excitation
368 signal as in the simulation was generated by Agilent E1441 waveform generator, and then
369 amplified by a US-TXP-3 linear power amplifier to $100 V_{p-p}$ before it was applied on the
370 PZT actuator. Lamb wave signals were acquired by the eight PZT sensors through Agilent
371 E1438 signal digitizer at a sampling rate of 100 MHz.

372

373 STFT-based signal processing as described in Section 4 was recalled to obtain the relative
374 acoustic nonlinearity parameter β' associated with each sensing path, in accordance with
375 Eq. (13). The relationships of β' vs. propagation distance obtained from the modeling and
376 the experiment are displayed in Figs. 8a and 8b, respectively. Note that β' features a unit of
377 strain in simulation, but V^{-1} in the experiment. It can be seen that both relationships reveal
378 a quasi-linear cumulative growth of β' over propagation distance, confirming that the
379 nonlinearity manifested in a captured ultrasonic Lamb wave signal is essentially from the
380 intrinsic nonlinearity of the medium as anticipated previously. The consistency between
381 the two relationships in Fig. 8 has validated the developed modeling technique.

382

383 *5.2. β' vs. Sensing path offset*

384 Ultrasonic Lamb waves propagating in an aluminum plate with a fatigue crack was
385 examined using the developed modeling technique, complemented with experimental
386 validation. Since a fatigue crack was introduced, the plasticity-driven nonlinearity and
387 CAN were taken into consideration in the modeling, as explained in Section 2. Through
388 this investigation, it is desired to establish a quantitative relation between β' of each
389 sensing path and its offset distance from the fatigue crack.

390

391 A 6061 aluminum plate ($400 \times 240 \times 4.5 \text{ mm}^3$) was modeled, as shown schematically in Fig.
392 9a. A triangular notch of 15 mm by 25 mm was presumed at the center of the upper edge,
393 consistent with the following experiment configuration, to serve as a fatigue crack initiator.
394 A 4-mm long surface crack with 2.25 mm in depth and a uniform initial clearance of zero
395 between the two crack surfaces (*viz.*, the two surfaces of the fatigue crack were initially in
396 contact with each other) was modeled, running in parallel with the plate width from the tip
397 of the notch. Ten circular PZT wafers, each with a 5-mm nominal diameter and a 0.5-mm
398 thickness, were allocated on the same side of the plate. Five sensing paths $A_i - S_i$ ($i = I,$
399 II, \dots, V), each 200-mm long, were hence configured, all of which were perpendicular to
400 the crack orientation. The offset distance d_o (defined in Fig. 9b) from the middle point of
401 the fatigue crack to each sensing path was 0, 10, 20, 30, and 40 mm, respectively.

402

403 In analogy with the first case study, modeling procedures including meshing, material
404 definition, signal acquisition, and dynamic simulation were recalled. For the material
405 definition associated with a certain degree of plastic deformation in the neighborhood of
406 the crack, the following representative values were used in the simulation, with a reference
407 to the data given in [29], to estimate the local nonlinearity parameter using Eq. (8): $\Omega = R$
408 $= 0.33$, $b = 0.286 \text{ nm}$, $\Lambda = 1 \times 10^{15} \text{ m}^{-2}$, and $h = 5.4 \text{ nm}$. In addition, CAN was added in the
409 modeling because of the inclusion of the physical crack. Sixteen-cycle Hann-windowed
410 sinusoidal tone bursts of 800 kHz were excited (considering the plate thickness). This
411 prolonged excitation duration, compared with the five cycles used previously, was aimed at
412 achieving a compromise between clear separation of different wave packets within the
413 propagation distance of 200 mm and explicit recognition of second harmonic generation
414 (as the cycle number of tone bursts increase, the wave bandwidth is reduced, the signal
415 energy is more concentrated near the excitation frequency, the peak amplitude increases,

416 and accordingly wave dispersion is minimized, all together benefiting the recognition of
417 second harmonic [30]).

418

419 In the experiment, a 6061 aluminum plate of the same dimensions and the same PZT
420 configuration as those in modeling was tested. The specimen had first undergone a HCF
421 test on a MTS fatigue platform, subject to a sinusoidal tensile load at 5 Hz with a
422 magnitude of 10 kN. After approximately 200,000 cycles, a barely visible fatigue crack
423 was produced, measuring 4 mm in length and half of the plate thickness in depth, running
424 from the notch tip and roughly paralleling the plate width. Ten PZT wafers of 5 mm
425 diameter were attached to the plate in line with Fig. 9a, only after completion of the fatigue
426 testing, to avoid possible degradation in adhesive layers during the fatigue processing. The
427 test was carried out in accordance with the experimental procedure described in the
428 previous case study.

429

430 Upon signal processing using STFT, β' for each sensing path was calculated and
431 normalized with respect to the value acquired from path $A_I - S_I$, as displayed in Fig. 10,
432 where the normalized β' against the offset distance from modeling and from the
433 experiment are compared. In order to extend the discussion applicable to general scenarios
434 of arbitrary frequency-thickness products, d_o is normalized with respect to the probing
435 wavelength λ . The two curves in Fig. 10 agree with one another quantitatively, with the
436 greatest error less than 9%, corroborating the effectiveness of the modeling approach.

437

438 The relationship of β' vs. d_o/λ implies that the largest value of β' takes place when the
439 sensing path just pass through the fatigue crack (*i.e.*, $A_I - S_I$); while it drops rather quickly,
440 as d_o/λ increases to 1.8 (corresponding to $d_o = 10$ mm), and then the decrease slows down

441 as d_o/λ reaches 3.6 (corresponding to $d_o = 20$ mm). Beyond this point, β' decreases
442 gradually to approximately 40% of its largest magnitude and no considerable change can
443 be observed after d_o/λ reaches 5.5 ($d_o = 30$ mm).

444

445 *5.3. Varied incidence angle and wave propagation distance*

446 Note that the relationship of β' vs. d_o/λ presented in Fig. 10 was determined when the
447 sensing path was perpendicular to the crack with a fixed path length of 200 mm. Pursuant
448 to this, β' , subject to varied incidence angles of probing waves and distinct propagation
449 distances, was interrogated in FE simulation.

450

451 Dynamic simulation was conducted using the developed modeling technique, with
452 additional PZT wafers introduced to the model as indicated by actuators A_A , A_B , and
453 sensors S_{VI} , S_{VII} , S_{VIII} , S_{IX} in Fig. 9a, which formed another set of sensing paths of different
454 incidence angles. The angle was varied from 90° (e.g., path $A_I - S_I$) to 84.3° (e.g., path $A_B -$
455 S_{II}), and further to 78.7° (e.g., path $A_A - S_{III}$). For each incidence angle, the path lengths and
456 the offset increment were kept consistent at approximately 200 mm and 10 mm,
457 respectively, as in the previous case. Fig. 11 displays the relation of β' vs. d_o/λ obtained
458 accordingly. It is interesting to note that no substantial variations can be observed in terms
459 of the trend of the curves when the incident angle varies. All these curves show that in the
460 nearest region of the crack, there is a 40–60% drop in β' ; beyond the point of $d_o/\lambda = 3.6$,
461 the change in β' can be regarded as insignificant, in line with the observations in Fig. 10.

462

463 In addition to the effect of incidence angle, the dependence of β' on the length of the
464 sensing path was also investigated. The path lengths were varied in the modeling from 100
465 to 400 mm in an increment of 100 mm, and their results are compared in Fig. 11. Once
466 again, there are no considerable variations among the results: the largest β' always occurs

467 at $d_o = 0$, and the decrease tends to slow down once beyond the point where $d_o/\lambda = 3.6$.
468 This finding echoes the previous statement that local nonlinearity due to fatigue damage (β_l)
469 is more dominant in introducing nonlinear distortion to ultrasonic waves, compared to its
470 global (material) counterpart (β_g).

471

472 The curves shown in Figs. 10 and 11 collectively corroborate a reliable relationship
473 between the relative acoustic nonlinearity parameter and the offset distance of sensing
474 paths in a sensor network. With an up to approximately $\pm 10^\circ$ change from orthogonality in
475 incidence angle and a wide range of propagation distance, β' depends exclusively on the
476 offset distance of the sensing path from the fatigue crack. This conclusion can be especially
477 helpful in locating fatigue cracks in conjunction with the use of a sensor network.

478

479 **6. Concluding Remarks**

480 Given the importance of fatigue crack characterization using nonlinear features of
481 ultrasonic waves, a modeling technique to interpret the nonlinearities associated with
482 material and fatigue damage is established. The modeling technique attempts to
483 comprehend material, geometric, plasticity-driven, and contact acoustic nonlinearities that
484 may contribute to the nonlinear distortion of ultrasonic wave propagation into the modeling.
485 Results from the simulation and the experiments show good consistency, both revealing
486 that the defined relative acoustic nonlinearity parameter β' has a linear cumulative growth
487 with wave propagation distance due to the material and geometric nonlinearities. It is also
488 discovered that β' of a particular sensing path in the sensor network can be quantitatively
489 related to the its offset from the crack within a wide range of sensing path lengths and
490 angles of incidence. Meanwhile, it is important to realize that, with the quantitative relation
491 of β' vs. d_o/λ obtained, the modeling methodology established in this study shows a great

492 potential to accommodate the demands of embeddable structural health monitoring. In
493 practice, this technique would not need a baseline signal, and is not affected by extra
494 nonlinearities such as those from instrumentation, making it tempting for the localization
495 of fatigue cracks using a PZT sensor network.

496

497 **Acknowledgements**

498 This project is supported by National Natural Science Foundation of China (Grant No.
499 51375414, 11272272 and 11202107). This project is also supported by the Hong Kong
500 Research Grants Council via a General Research Fund (GRF) (No. 523313). Qiang Wang
501 is grateful for the Doctoral Program of Higher Education (Grant No. 20113223120008) and
502 Natural Science Foundation of Jiangsu Higher Education Institutions of China (Grant No.
503 11KJB130002).